Communications in Mathematical Analysis
Volume 11, Number 1, pp. 121–136 (2011) www.math-res-pub.org/cma
ISSN 1938-9787

Factorization of Generalized Lamé and Heun's Differential Equations

M. N. Hounkonnou *
International Chair of Mathematical Physics and Applications (ICMPA-UNESCO Chair)
University of Abomey-Calavi
072 B.P. 50 Cotonou, Republic of Benin

A. Ronveaux†
Department of Mathematics
Catholic University of Louvain
2 Chemin du Cyclotron, *B*-1348 Louvain-la-Neuve, Belgium

(Communicated by Michal Fečkan)

Abstract

This paper addresses new results on the factorization of the general Heun's operator, extending the investigations performed in previous works [*Applied Mathematics and Computation* **141** (2003), 177 - 184 and **189** (2007), 816 - 820]. Both generalized $k-$Lamé and $k-$Heun's second order differential equations are considered.

AMS Subject Classification:4A05; 34A30.

Keywords: Factorization, Fuchsian differential's equations, Heun's differential equation, Lamé's differential equation.

1 Introduction

Fuchsian differential equations of second order and their confluent forms in general, and in particular those derived from the Ince techniques [10], play a major role in the investigations of partial differential equations of mathematical physics like Laplace, Helmholtz or Schrödinger's equations.

Assuming that the $k+3$ regular single singularities are located at $x=0$, $x=1$, $x=a_i$, $i=\overline{1,k}$ and $x=\infty$, then, after an appropriate change of function, one of the indices at

*E-mail address: norbert.hounkonnou@cipma.uac.bj
†E-mail address: andre.ronveaux@student.uclouvain.be

each finite singularity can be shifted to zero and the Fuchsian equation can be expressed as [11, 16]:

$$y''(x)+\left(\frac{\gamma}{x}+\frac{\delta}{x-1}+\sum_{i=1}^{k}\frac{\varepsilon_i}{x-a_i}\right)y'(x)+\frac{\alpha\beta x^k+\sum_{i=1}^{k}\rho_i x^{k-i}}{x(x-1)\prod_{i=1}^{k}(x-a_i)}y(x)=0. \tag{1.1}$$

The two indices at each singularity a_i are $(0,1-\varepsilon_i)$ and $(0,1-\gamma)$, $(0,1-\delta)$, (α,β) at $x=0$, $x=1$, and $x=\infty$, respectively, taking into account the Fuchsian relation:

$$\alpha+\beta+1=\gamma+\delta+\sum_{i=1}^{k}\varepsilon_i. \tag{1.2}$$

The case $\varepsilon_i\equiv 0$, $(i=\overline{1,k})$, gives the hypergeometric equation while $k=1$ corresponds to the Heun's equation [17].

(1.1) can be rewritten more compactly as:

$$y''(x)+\left(\sum_{i=1}^{k+2}\frac{\varepsilon_i}{x-a_i}\right)y'(x)+\frac{\alpha\beta x^k+\sum_{i=1}^{k}\rho_i x^{k-i}}{\prod_{i=1}^{k+2}(x-a_i)}y(x)=0. \tag{1.3}$$

The character *reducible* or *irreducible* of linear ordinary differential equation (O.D.E) is important in relation with the monodromy group [11]. A reducible (resp. irreducible) second order O.D.E will be called here one which admits [resp. does not admit] a non trivial solution satisfying a first order linear equation. This definition, used by many authors [5, 7, 15] with or without additional conditions, allows to set a simple criterion of reducibility for a differential equation with polynomial coefficients, from a factorized form. Indeed, for instance if we can transform (1.1) into the following factorized form

$$\begin{aligned}\mathcal{H}_k[y(x)] &\equiv \left[Q_{k+2}(x)D^2+Q_{k+1}D+Q_k\right]y(x)\\ &= [L(x)D+M(x)]\left[\overline{L}(x)D+\overline{M}(x)\right]y(x),\end{aligned} \tag{1.4}$$

where $D=\frac{\mathrm{d}}{\mathrm{d}x}$ and Q_j are polynomials of degree j, (i.e. given the polynomials Q_{k+2}, Q_{k+1}, Q_k, we can find L, $\overline{L}$, M, $\overline{M}$), then we say that the equation $\mathcal{H}_k[y(x)]=0$ is reducible. Such a study has been performed in previous works for the Heun's equation [18, 19]

$$y''(x)+\left[\frac{\gamma}{x}+\frac{\delta}{x-1}+\frac{\varepsilon}{x-a}\right]y'(x)+\frac{\alpha\beta x-q}{x(x-1)(x-a)}y(x)=0 \tag{1.5}$$

and for its four confluent equations [8]. The factorization of the Heun's equation has given rise to 6 non trivial situations in accordance with the known $F-$homotopic transformations [1]. In [12], R. S Maier noticed that, for the Lamé equation ($k=1$ with $\gamma=\delta=\varepsilon=1/2$)

$$y''(x)+\frac{1}{2}\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x-a}\right)y'(x)-\frac{l(l+1)x+4q}{4x(x-1)(x-a)}y(x)=0, \tag{1.6}$$

the factorization is reduced only to the two cases $l=1$, $l=\frac{1}{2}$.

To our best knowledge of the literature, and as mentioned in [12], the factorization of the general Lamé equation with $k+3$ singularities, is still lacking. Such a gap will be fulfilled in this paper. Further, even if some algorithms have been elaborated in order to factorize any ODE with rational function coefficients [4, 20, 22, 23], the factorization of an arbitrary k-Heun equation remains unsolved, except for the cases of $k = 1$ and 2 which have been recently treated in [9].

The aim of this work is to investigate both generalized k-Heun and k-Lamé equations [14], which are important in lattice statistics. These equations have also invested a large number of applications in physics. See [6] (and references therein) for a nice review on their interest in physical applications. To mention a few, retrieved from the indicated works, where for instance Heun's equations as well as their solutions impose their usefulness, it is worthy of attention to outline their importance in the description of quasi-modes of near extremal black branes, in the hyperspherical harmonics with applications in three-body systems, in the elaboration of a method of calculation of propagators for the case of a massive spin 3/2 field for arbitrary space-time dimensions and mass, in parametric resonance after inflation, as well as in the separation of variables for the Schr*ö*dinger equation in a large number of problems, typically for the radial coordinate, and in non-linear formulation involving Painlevé type equations. A number of known equations of mathematical physics, like the Lamé, spheroidal wave and Mathieu equations, also pertain to particular cases of Heun equations.

The equation (1.4) can be expanded to give:

$$\begin{aligned} \mathcal{H}_k[y(x)] &= (LD+M)\left(\overline{L}D+\overline{M}\right)y(x) \\ &= \left\{L\overline{L}D^2+\left[L\left(\overline{L}'+\overline{M}\right)+M\overline{L}\right]D+\left(L\overline{M}'+M\overline{M}\right)\right\}y(x) \end{aligned} \tag{1.7}$$

generating the following 3 basic relations:

$$\begin{cases} Q_{k+2} &= L\overline{L}, \\ Q_{k+1} &= L\left(\overline{L}'+\overline{M}\right)+M\overline{L}, \\ Q_k &= L\overline{M}'+M\overline{M}. \end{cases} \tag{1.8}$$

The computation and properties of the polynomials $M \equiv M(x)$, $\overline{M} \equiv \overline{M}(x)$, $L \equiv L(x)$, $\overline{L} \equiv \overline{L}(x)$ are explicitly given in the sequel for different situations.

2 Preliminary remarks

1) $Q_{k+2}(x)$ contains $k+2$ distinct linear factors combined in the product $L(x)\overline{L}(x)$ in different ways. For instance, if j factors appear in L and $k+2-j$ in $\overline{L}$, the number of decompositions is given by the classical binomial formula:

$$\sum_{j=0}^{k+2}\binom{k+2}{j} = 2^{k+2}, \tag{2.1}$$

excluding the extremal cases

$$\begin{aligned} L(x) &= Q_{k+2}(x), \overline{L} = 1;\ Q_k = 0, \\ \mathcal{H}_k &= LD(D+\overline{M}) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} L(x) &= 1, \overline{L} = Q_{k+2}(x), M = 0 \text{ (integrable case} \overline{M} = Q_{k+1} - Q'_{k+2}, \\ Q_k &= \overline{M}'), \ \mathcal{H}_k = D(\overline{L}D + \overline{M}) \end{aligned} \tag{2.3}$$

which are irrelevant or trivial in this study. Selecting a pair $L, \overline{L}$ inside the $2^{k+2} - 2$ possibilities, the computation of the polynomials $M(x)$ and $\overline{M}(x)$ of degrees m and $\overline{m}$, respectively, can be done in the two following steps:

i) Use the coefficients in $Q_{k+1}(x)$ and $Q_{k+2}(x)$ to compute all the $(m + \overline{m} \leq k)$ coefficients of $M(x)$ and $\overline{M}(x)$ from the second and third relations of (1.8).
ii) Use the third relation of (1.8) to find the values of the parameters ρ_i, $(i = \overline{1,k})$ and the product $\alpha\beta$, exploiting the expressions of polynomials M and $\overline{M}$ computed in i).

2) The general $k-$Lamé's equation can be retrieved from the $k-$ Heun's equation as a particular case corresponding to $\gamma = \delta = \varepsilon = 1/2$, $(i = \overline{1,k})$. Moreover, the indices at each finite singularity are equal to (0, 1/2) and the Fuchsian relation (1.2) is reduced to $\alpha + \beta = k/2$. Thus, the general $k-$Lamé's equation takes the form

$$\mathcal{L}_k[y] \equiv Q_{k+2}y'' + Q_{k+1}y' + \left(\alpha\beta x^k + \sum_{i=1}^{k} \rho_i x^{k-i}\right) y = 0 \tag{2.4}$$

with now

$$Q_{k+1} = \frac{1}{2}(Q_{k+2})' = \frac{1}{2}\left[L\overline{L}\right]', \tag{2.5}$$

where

$$Q_{k+2} = x(x-1)\prod_{j=1}^{k}(x - a_j) = L\overline{L}. \tag{2.6}$$

The property (2.5) and the second relation in (1.8) then lead to:

$$\frac{1}{2}\left(\frac{L}{\overline{L}}\right)' = \frac{L\overline{M} + M\overline{L}}{\overline{L}^2}. \tag{2.7}$$

The number of situations to be investigated in the decomposition of $L\overline{L}$ can be reduced using the symmetry property between polynomials $M, \overline{M}$ and $M^\star, \overline{M}^\star$, as the product $L\overline{L}$ can be realized in different ways permuting L and $\overline{L}$, to give the factorization:

$$\mathcal{L}_k[y] = \left[\overline{L}D + M^\star\right]\left[LD + \overline{M}^\star\right] y(x). \tag{2.8}$$

In this case, the relation corresponding to (2.7) becomes

$$\frac{1}{2}\left(\frac{\overline{L}}{L}\right)' = \frac{\overline{L}\,\overline{M}^\star + M^\star L}{L^2}. \tag{2.9}$$

From (2.7) and (2.9) we deduce

$$L\left(\overline{M} + M^\star\right) + \overline{L}\left(M + \overline{M}^\star\right) = 0 \tag{2.10}$$

which is satisfied if, for example, $M = -\overline{M}^\star$ and $\overline{M} = -M^\star$.

3 Factorization of Lamé's equation

3.1 Case $\mathcal{L}_1[y]$, $k = 1$

Let us consider equation (1.5) with $\gamma = \delta = \varepsilon = \frac{1}{2}$. Even if this case has been already solved in [17], as a subcase of the full Heun factorization, we can easily recover the polynomials M and $\overline{M}$ from the procedure described in Section 2 for the factorization terms $L = x(x-1)$, $\overline{L} = x - a$ as example. The unknown polynomials M and $\overline{M}$ being written as:

$$M = Ax + B, \overline{M} = \overline{A} \tag{3.1}$$

with

$$\begin{aligned} Q_3(x) &= L\overline{L} = x^3 - x^2(a+1) + xa, & (3.2)\\ Q_2(x) &= L(\overline{L}' + \overline{M}) + M\overline{L} = \frac{3}{2}x^2 - x(a+1) + \frac{a}{2} & (3.3)\\ Q_1(x) &= M\overline{M}, & (3.4) \end{aligned}$$

we find

$$A = 1, B = -\frac{1}{2}, \overline{A} = -\frac{1}{2}. \tag{3.5}$$

Then, the third relation in (1.8) gives:

$$\alpha\beta = -\frac{1}{2}, q = -\frac{1}{4} \tag{3.6}$$

and the Fuschian relation yields

$$\alpha + \beta = \frac{1}{2}. \tag{3.7}$$

From the previous section, it is immediate that $M^\star = -\overline{M}$ and $\overline{M}^\star = -M$ already solve this factorization problem with $L = x - a, \overline{L} = x(x-1)$.

A surprising result, consequence of a remark by Maier [12], is that for $L = x$, or $(x-1)$, or $(x-a)$, $\big(\overline{L} = (x-1)(x-a), x(x-a), x(x-1)\big)$, the polynomials M are the same. From the symmetry property $M^\star = -\overline{M}, \overline{M}^\star = -M$, this result also works for the 3 other factorizations: $L = (x-1)(x-a), x(x-a), x(x-1); \overline{L} = x, (x-1), (x-a)$. All results for Lamé's equation factorization are summarized in table 1, where we have introduced a full symmetry with respect to the poles, now located in $a_i, (i = 1,2,3)$.

3.2 Case $\mathcal{H}_1(y)$

In the same way the data for the operateur $\mathcal{H}_1(y)$ are given in tables 2 and 3, where the degrees r and $\bar{r}$ of polynomials L and $\bar{L}$, respectively, are denoted by the couple $(r, \bar{r})$.

Table 1. Factorization of the Lame's operator $\mathcal{L}_1[y]$

$$\begin{aligned} \mathcal{L}_1 &= (x-a_1)(x-a_2)(x-a_3)D^2+[\varepsilon_1(x-a_2)(x-a_3)+\varepsilon_2(x-a_1)(x-a_3) \\ &+ \varepsilon_3(x-a_1)(x-a_2)]D+(\alpha\beta x+\rho_1)I_d \\ &= (LD+M)(\overline{L}D+\overline{M}), \varepsilon_1=\varepsilon_2=\varepsilon_3=\frac{1}{2}. \end{aligned}$$

L	$\overline{L}$	M	$\overline{M}$
$x-a_1$	$(x-a_2)(x-a_3)$	$\frac{1}{2}$	$-x+\frac{1}{2}(a_2+a_3)$
$(x-a_1)(x-a_2)$	$x-a_3$	$x-\frac{1}{2}(a_1+a_2)$	$-\frac{1}{2}$

α	β	ρ_1
$\frac{3}{2}$	-1	$\frac{1}{4}(a_2+a_3)+a_1$
$-\frac{1}{2}$	1	$\frac{1}{4}(a_1+a_2)$

Table 2. Factorization of the Heun's operator $\mathcal{H}_1[y]$: $(r,\bar{r})=(1,2)$.

$$\mathcal{H}_1[y]=Q_3(x)y''+Q_2(x)y'+(\alpha\beta x+\rho_1)y=(LD+M)(\overline{L}D+\overline{M})y,$$

$$M=\varepsilon_1,\ \overline{M}=(\varepsilon_2+\varepsilon_3-2)x+\overline{B},\ \alpha=\varepsilon_2+\varepsilon_3-2,\ \beta=\varepsilon_1+1.$$

L	$\overline{L}$	$\overline{B}$	ρ_1
$x-a_1$	$(x-a_2)(x-a_3)$	$a_2(1-\varepsilon_3)+a_3(1-\varepsilon_2)$	$a_1(2-\varepsilon_2-\varepsilon_3)$ $+a_2\varepsilon_1(1-\varepsilon_3)+a_3\varepsilon_1(1-\varepsilon_2)$

Table 3. Factorization of the Heun's operator $\mathcal{H}_1[y]$: $(r,\bar{r})=(2,1)$.

$$\mathcal{H}_1[y]=Q_3(x)y''+Q_2(x)y'+(\alpha\beta x+\rho_1)y=(LD+M)(\overline{L}D+\overline{M})y,$$

$$M=(\varepsilon_1+\varepsilon_2)x+B,\quad \overline{M}=\varepsilon_3-1,\ \alpha=\varepsilon_3-1,\ \beta=\varepsilon_1+\varepsilon_2.$$

L	$\overline{L}$	B	ρ_1
$(x-a_1)(x-a_2)$	$x-a_3$	$-(a_1\varepsilon_2+a_2\varepsilon_1)$	$(1-\varepsilon_3)(a_1\varepsilon_2+a_2\varepsilon_1)$

3.3 Case $\mathcal{L}_2[y]$, $k=2$

Setting $k=2$, $a_1=a, a_2=b$ and $\gamma=\delta=\varepsilon_1=\varepsilon_2=\frac{1}{2}$ in (1.1), we arrive at the Lamé equation

$$\mathcal{L}_2[y]=Q_4(x)y''+Q_3(x)y'+Q_2(x)y=0, \tag{3.8}$$

where

$$\begin{cases} Q_4(x) &= x(x-1)(x-a)(x-b) \\ &= x^4-x^3(1+a+b)+x^2(a+b+ab)-abx, \\ Q_3(x) &= \frac{1}{2}Q_4'(x)=\frac{1}{2}\left[4x^3-3x^2(1+a+b)+2x(a+b+ab)-ab\right], \\ Q_2(x) &= \alpha\beta x^2+\rho_1 x+\rho_2, \\ \alpha+\beta &= 1. \end{cases}$$

First letting $L=x(x-1)$ and $\overline{L}=(x-a)(x-b)$, M and $\overline{M}$ are first degree polynomials

$$M=Ax+B, \overline{M}=\overline{A}x+\overline{B}. \tag{3.9}$$

The four equations satisfied by A, B, $\overline{A}$, $\overline{B}$, determined by identifying the second relation in (1.8) with the expression of $Q_3(x)$ in (3.9), give the solutions

$$A=1, B=-\frac{1}{2}, \overline{A}=-1, \overline{B}=\frac{a+b}{2}. \tag{3.10}$$

Furthermore, the comparison of third relation in (1.8) with the expression of $Q_2(x)$ in (3.9) yields

$$\alpha\beta=-2 \tag{3.11}$$

and

$$\rho_1 = \frac{a+b+3}{2}, \tag{3.12}$$

$$\rho_2 = -\frac{a+b}{4}. \tag{3.13}$$

These results as well as those corresponding to the other choices of L and $\overline{L}$ are reproduced in tables 4 to 6 for $\gamma=\delta=\varepsilon_1=\varepsilon_2=\frac{1}{2}$ as in the case $k=1$, in which appear essentially 2 situations [12]; the case $k=2$ generates only 3 situations with $\alpha\beta=-2,\ -3/4,\ -15/4$. Symmetry in the poles a_i is also respected here.

3.4 Case $\mathcal{L}_k[y]$, arbitrary k

A first particular solution $y_1(x)$ of the Lamé's equation can be expressed, from the factorization form, by

$$\overline{L}(x)y_1'(x)+\overline{M}(x)y_1(x)=0 \tag{3.14}$$

yielding for all decompositions L, $\overline{L}$ ($k=1,2$, see tables)

$$y_1(x)=\sqrt{\overline{L}(x)}. \tag{3.15}$$

Table 4. Factorization of the Lame's operator $\mathcal{L}_2[y]$: $(r,\bar{r}) = (2,2)$.

$$\mathcal{L}_2[y] = Q_4(x)y'' + Q_3(x)y' + (\alpha\beta x^2 + \rho_1 x + \rho_2)y = (LD+M)(\overline{L}D+\overline{M})y,$$

$$M = x + B, \quad \overline{M} = -x + \overline{B}.$$

L	$\overline{L}$	B	$\overline{B}$
$(x-a_1)(x-a_2)$	$(x-a_3)(x-a_4)$	$-\frac{1}{2}(a_1+a_2)$	$\frac{1}{2}(a_3+a_4)$

α	β	ρ_1	ρ_2
-1	2	$\frac{3}{2}(a_1+a_2)+\frac{1}{2}(a_3+a_4)$	$-\frac{1}{4}(a_1+a_2)(a_3+a_4)-a_1a_2$

Table 5. Factorization of the Lame's operator $\mathcal{L}_2[y] : (r,\bar{r}) = (3,1)$.

$$\mathcal{L}_2[y] = Q_4(x)y'' + Q_3(x)y' + (\alpha\beta x^2 + \rho_1 x + \rho_2)y = (LD+M)(\overline{L}D+\overline{M})y$$

$$M = \frac{3}{2}x^2 + Bx + C,\ \overline{M} = -\frac{1}{2},\ \alpha = \frac{3}{2},\ \beta = -\frac{1}{2}.$$

L	$\overline{L}$	B	C
$(x-a_1)(x-a_2)(x-a_3)$	$x-a_4$	$-(a_1+a_2+a_3)$	$\frac{1}{2}(a_1a_2+a_1a_3+a_2a_3)$

ρ_1	ρ_2
$\frac{1}{2}(a_1+a_2+a_3)$	$-\frac{1}{4}(a_1a_2+a_1a_3+a_2a_3)$

Table 6. Factorization of the Lame's operator $\mathcal{L}_2[y] : (r,\bar{r}) = (1,3)$.

$$\mathcal{L}_2[y] = Q_4(x)y'' + Q_3(x)y' + (\alpha\beta x^2 + \rho_1 x + \rho_2)y = (LD+M)(\overline{L}D+\overline{M})y,$$

$$\overline{M} = -\frac{3}{2}x^2 + \overline{B}x + \overline{C},\ M = \frac{1}{2},\ \alpha = -\frac{3}{2},\ \beta = \frac{5}{2}.$$

L	$\overline{L}$	$\overline{B}$	$\overline{C}$
$x-a_1$	$(x-a_2)(x-a_3)(x-a_4)$	$a_2+a_3+a_4$	$-\frac{1}{2}(a_2a_3+a_2a_4+a_3a_4)$

ρ_1	ρ_2
$\frac{3}{2}(2a_1+a_2+a_3+a_4)$	$-a_1(a_2+a_3+a_4)-\frac{1}{4}(a_2a_3+a_2a_4+a_3a_4)$

The second solution can be readily determined as

$$y_2(x) = \sqrt{\overline{L}(x)} \int \frac{dx}{\overline{L}(x)\sqrt{L(x)\overline{L}(x)}}. \tag{3.16}$$

The solution $y_1(x)$ is deducible from a more general solution, the so-called pseudo-Lamé's polynomials of the form [24]:

$$y(x) = \sqrt{\overline{L}(x)} P_N(x), \tag{3.17}$$

where $P_N(x)$ is a polynomial of degree N which cannot be explicitly written, existing only in a determinantal form. Here, P_N is a constant ($N=0$) and $y_1(x) = \sqrt{\overline{L}(x)}$. But now, working in opposite way, the solution $y_1 = \sqrt{\overline{L}(x)}$ can allow to determine first $\overline{M}$ and then M as follows: from $\overline{M} = -\overline{L}'/2$, $(y_1'/y_1 = -\overline{M}/\overline{L})$ and $Q_{k+1} = (L\overline{L})'/2 = L\overline{L}' + L\overline{M} + M\overline{L}$, we deduce $M = L'/2$. Therefore the corresponding general factorization gives, for any decomposition $(L,\overline{L})$

$$\mathcal{L}_k[y(x)] = [(LD+M)(\overline{L}D+\overline{M})]y(x) = [(LD+\frac{L'}{2})(\overline{L}D-\frac{\overline{L}'}{2})]y(x) = 0, \tag{3.18}$$

and after expansion:

$$y'' + \frac{1}{2}\frac{(L\overline{L})'}{L\overline{L}}y' - \frac{1}{2}\left(\frac{\overline{L}''}{\overline{L}} + \frac{1}{2}\frac{\overline{L}'L'}{\overline{L}L}\right)y = 0.$$

From this representation and our definition on the *reducibility* (see section 1), we get the following result:

Theorem 3.1. *Let L and $\overline{L}$ be two polynomials such that*

$$L(x)\overline{L}(x) = \prod_{i=1}^{k+2}(x-a_i). \tag{3.19}$$

Then, all differential operators of the type

$$D^2 + \frac{1}{2}\left(\sum_{j=1}^{k+2}\frac{1}{x-a_j}\right)D + \frac{\sum_{l=0}^{k}\rho_l x^{k-l}}{\prod_{i=1}^{k+2}(x-a_i)} \tag{3.20}$$

are reducible if the last term in (3.20) can be written as

$$-\frac{1}{2}\left[\frac{1}{2}u(x)v(x) + v'(x) + v^2(x)\right], \tag{3.21}$$

where $u(x) = \frac{L'(x)}{L(x)}$ and $v(x) = \frac{\overline{L}'(x)}{\overline{L}(x)}$.

4 Factorization of the Heun's equation

4.1 Case $\mathcal{H}_2[y]$, $k=2$

The first extension of Heun's equation with $k=2$ can be written as $(y(x)\equiv y)$

$$y''+\left[\frac{\gamma}{x}+\frac{\delta}{x-1}+\frac{\varepsilon_1}{x-a}+\frac{\varepsilon_2}{x-b}\right]y'+\frac{\alpha\beta x^2+\rho_1 x+\rho_2}{x(x-1)(x-a)(x-b)}y=0 \tag{4.1}$$

with the parameters linked by the Fuchs's condition $\alpha+\beta+1=\gamma+\delta+\varepsilon_1+\varepsilon_2$. In polynomial form, this gives

$$\mathcal{H}_2[y]=Q_4(x)y''+Q_3(x)y'+Q_2(x)y=0, \tag{4.2}$$

where

$$\begin{aligned} Q_4(x) &= x(x-1)(x-a)(x-b) \\ &= x^4-x^3(1+a+b)+x^2(a+b+ab)-xab; && (4.3)\\ Q_3(x) &= \gamma(x-1)(x-a)(x-b)+\delta x(x-a)(x-b); \\ &+ \varepsilon_1 x(x-1)(x-b)+\varepsilon_2 x(x-1)(x-a) && (4.4)\\ Q_2(x) &= \alpha\beta x^2+\rho_1 x+\rho_2. && (4.5)\end{aligned}$$

Its factorized operator reads

$$\begin{aligned}\mathcal{H}_2[y] &= (LD+M)\left(\overline{L}D+\overline{M}\right)\\ &= L\overline{L}D^2+\left(L\overline{L}'+L\overline{M}+M\overline{L}\right)D+\left(L\overline{M}'+M\overline{M}\right).\end{aligned} \tag{4.6}$$

Let us consider a peculiar simple situation

$$L=x(x-1), \qquad \overline{L}=(x-a)(x-b). \tag{4.7}$$

The third relation in (1.8), i.e. $L\overline{M}'+M\overline{M}=Q_2(x)$, constrains M and $\overline{M}$ to be of degree 1. Let $M=xA+B$, $\overline{M}=x\overline{A}+\overline{B}$. The unknowns A, B, $\overline{A}$, $\overline{B}$ are given using the second relation in (1.8). From $Q_2(x)=\overline{A}x(x-1)+(xA+B)(x\overline{A}+\overline{B})$, we get

$$\begin{cases} \overline{A}+A\overline{A} = \alpha\beta, \\ -\overline{A}+\overline{B}A+B\overline{A} = \rho_1. \end{cases} \tag{4.8}$$

In general, as for the Lamé's equation, some kind of symmetry appears when permuting L and $\overline{L}$ in the factorization form, allowing to reduce the possible number of factorizations by a factor 2. The use of the adjoint operator $\mathcal{H}^\star$ of $\mathcal{H}$ is now useful. Indeed, the Lagrange adjoint $\mathcal{H}^\star$ of $\mathcal{H}$ is given by [10, 3]

$$\mathcal{H}^\star=Q^\star_{k+2}D^2+Q^\star_{k+1}D+Q^\star_k \tag{4.9}$$

with

$$Q^\star_{k+2}=Q_{k+2},\;\; Q^\star_{k+1}=2Q'^\star_{k+2}-Q_{k+1},\;\; Q^\star_k=Q''_{k+2}-Q'_{k+1}+Q_k. \tag{4.10}$$

It is not very easy to write in general $\mathcal{H}^\star$ in the form of the equation (2.4) with star parameters $\alpha^\star$, $\beta^\star$, $\rho_i^\star$, as the corresponding equations are quadratic. But, from the factorized form of $\mathcal{H}$, the computation of polynomials $M^\star$ and $\overline{M}^\star$ becomes trivial from the relations:

$$\mathcal{H} = (LD+M)(\overline{L}D+\overline{M}), \;\; \mathcal{H}^\star = (\overline{L}D+\overline{M})^\star(LD+M)^\star \tag{4.11}$$

and, from equation (4.10), we get

$$\begin{aligned} \mathcal{H}^\star &= (-\overline{L}D+\overline{L}'-\overline{M})(-LD+L'-M) \\ &= (\overline{L}D+\overline{M}^\star)(LD+M^\star) \end{aligned} \tag{4.12}$$

with

$$\overline{M}^\star = \overline{M}-\overline{L}',\; M^\star = M-L' \tag{4.13}$$

generalizing the relation (2.8).

In tables 7 to 9, we give the values of M, $\overline{M}$, α, β, ρ_1 and ρ_2 as well as those corresponding to the other choices of L and $\overline{L}$. Of course, the symmetry in the factorization of $\mathcal{H}_2$ is not so rich as in the case of $\mathcal{L}_2$. Nevertheless, a quick sight in the tables allows to classify families having the same representation for ρ_1 and ρ_2 in terms of A, $\overline{A}$, B, $\overline{B}$. See tables 5 - 7. The symmetry between the poles a_i are also kept in all tables. See the case of symmetry between L and $\overline{L}$ in [9].

Table 7. Factorization of the Heun's operator $\mathcal{H}_2[y] : (r,\bar{r}) = (2,2)$.

$$\mathcal{H}_2[y] = Q_4(x)y'' + Q_3(x)y' + (\alpha\beta x^2 + \rho_1 x + \rho_2)y = (LD+M)(\overline{L}D+\overline{M})y.$$

$$\overline{M} = (\varepsilon_3+\varepsilon_4-2)x+\overline{B},\; M = (\varepsilon_1+\varepsilon_2)x+B,\; \alpha = \varepsilon_1+\varepsilon_2+1,\; \beta = \varepsilon_3+\varepsilon_4-2.$$

L	$\overline{L}$	B	$\overline{B}$
$(x-a_1)(x-a_2)$	$(x-a_3)(x-a_4)$	$-(a_1\varepsilon_2+a_2\varepsilon_1)$	$a_3(1-\varepsilon_4)+a_4(1-\varepsilon_3)$

ρ_1
$(2-\varepsilon_3-\varepsilon_4)[a_1(1+\varepsilon_2)+a_2(1+\varepsilon_1)]+(\varepsilon_1+\varepsilon_2)[a_3(1-\varepsilon_4)+a_4(1-\varepsilon_3)]$

ρ_2
$a_1a_2(\varepsilon_3+\varepsilon_4-2)+(a_2\varepsilon_1+a_1\varepsilon_2)[a_4(\varepsilon_3-1)+a_3(\varepsilon_4-1)]$

4.2 Case $\mathcal{H}_k[y]$, arbitrary k

The technique used in the previous section for $k-$Lamé equation can be generalized for the factorization of the $k-$Heun's equation, written now with all poles located at $x = a_i$, $(i = 1,2,\ldots,k+2)$ as

$$y'' + \left(\sum_{i=1}^{k+2}\frac{\varepsilon_i}{x-a_i}\right)y' + \frac{\alpha\beta x^k + \sum_{j=1}^{k}\rho_j x^{k-j}}{\prod_{i=1}^{k+2}(x-a_i)}y = 0. \tag{4.14}$$

Table 8. Factorization of the Heun's operator $\mathcal{H}_2[y] : (r, \bar{r}) = (3, 1)$.

$$\mathcal{H}_2[y] = Q_4(x)y'' + Q_3(x)y' + (\alpha\beta x^2 + \rho_1 x + \rho_2)y = (LD + M)(\overline{L}D + \overline{M})y,$$

$$\overline{M} = \varepsilon_4 - 1,\ M = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)x^2 + Bx + C,\ \alpha = \varepsilon_1 + \varepsilon_2 + \varepsilon_3,\ \beta = \varepsilon_4 - 1.$$

L	$\overline{L}$	B
$(x - a_1)(x - a_2)(x - a_3)$	$(x - a_4)$	$-a_1(\varepsilon_2 + \varepsilon_3) - a_2(\varepsilon_1 + \varepsilon_3) - a3(\varepsilon_1 + \varepsilon_2)$

C	ρ_1
$a_1a_3\varepsilon_2 + a_1a_2\varepsilon_3 + a_2a_3\varepsilon_1$	$(1 - \varepsilon_4)[a_1(\varepsilon_2 + \varepsilon_3) + a_2(\varepsilon_1 + \varepsilon_3) + a_3(\varepsilon_1 + \varepsilon_2)]$

ρ_2
$(\varepsilon_4 - 1)(a_1a_2\varepsilon_3 + a_1a_3\varepsilon_2 + a_2a_3\varepsilon_1)$

Table 9. Factorization of the Heun's operator $\mathcal{H}_2[y] : (r, \bar{r}) = (1, 3)$.

$$\mathcal{H}_2[y] = Q_4(x)y'' + Q_3(x)y' + (\alpha\beta x^2 + \rho_1 x + \rho_2)y = 0 = (LD + M)(\overline{L}D + \overline{M})y,$$

$$\overline{M} = (\varepsilon_2 + \varepsilon_3 + \varepsilon_4 - 3)x^2 + \overline{B}x + \overline{C},\ M = \varepsilon_1,\ \alpha = \varepsilon_2 + \varepsilon_3 + \varepsilon_4 - 3,\ \beta = \varepsilon_1 + 2.$$

L	$\overline{L}$	$\overline{B}$
$x - a_1$	$(x - a_2)(x - a_3)(x - a_4)$	$a_2(2 - \varepsilon_3 - \varepsilon_4) + a_3(2 - \varepsilon_2 - \varepsilon_4) + a_4(2 - \varepsilon_2 - \varepsilon_3)$

$\overline{C}$	ρ_1
$a_2a_3(\varepsilon_4 - 1) + a_2a_4(\varepsilon_3 - 1)$ $+a_3a_4(\varepsilon_2 - 1)$	$6a_1 + 2(1 - a_1)(\varepsilon_2 + \varepsilon_3 + \varepsilon_4) + 2\varepsilon_1(a_2 + a_3 + a_4)$ $-(1 + \varepsilon_1)[a_2(\varepsilon_3 + \varepsilon_4) + a_3(\varepsilon_2 + \varepsilon_4) + a_4(\varepsilon_2 + \varepsilon_3)]$

ρ_2
$a_1a_2(\varepsilon_3 + \varepsilon_4 - 2) + a_1a_4(\varepsilon_2 + \varepsilon_3 - 2) + a_1a_3(\varepsilon_2 + \varepsilon_4 - 2)$ $+\varepsilon_1[a_2a_3(\varepsilon_4 - 1) + a_2a_4(\varepsilon_3 - 1) + a_3a_4(\varepsilon_2 - 1)]$

Let $\overline{L} = \prod_{i=1}^{p}(x-a_i)$, $1 \leq p < k+2$. The first solution $y_1(x)$ of

$$\overline{L}y_1' + \overline{M}y_1 = 0 \tag{4.15}$$

is the generalization of $\sqrt{L}$ in the Lamé's case (see tables):

$$y_1(x) = \prod_{i=1}^{p}(x-a_i)^{1-\varepsilon_i} \tag{4.16}$$

giving now

$$\frac{y_1'}{y_1} = (\ln y_1)' = \sum_{i=1}^{p}\frac{1-\varepsilon_i}{x-a_i} = -\frac{\overline{M}}{\overline{L}}, \tag{4.17}$$

from which we deduce

$$\overline{M}(x) = \sum_{i=1}^{p}(\varepsilon_i - 1)\prod_{\substack{r=1\\r\neq i}}^{p}(x-a_r). \tag{4.18}$$

The knowledge of $\overline{M}$ from (4.17) allows to compute M, as before, from:

$$\begin{aligned}\frac{Q_{k+1}}{Q_{k+2}} &= \frac{L\overline{L}' + L\overline{M} + M\overline{L}}{L\overline{L}} = \sum_{i=1}^{k+2}\frac{\varepsilon_i}{x-a_i}\\ \frac{M}{L} &= \sum_{i=1}^{k+2}\frac{\varepsilon_i}{x-a_i} + \sum_{i=1}^{p}\frac{1-\varepsilon_i}{x-a_i} - \sum_{i=1}^{p}\frac{1}{x-a_i} = \sum_{i=p+1}^{k+2}\frac{\varepsilon_i}{x-a_i},\end{aligned} \tag{4.19}$$

affording

$$M(x) = \sum_{i=p+1}^{k+2}\varepsilon_i\prod_{\substack{r=p+1\\r\neq i}}^{k+2}(x-a_r). \tag{4.20}$$

We now arrive at the following result:

Theorem 4.1. *Consider the more general k-Heun equation,* $k \in \mathbb{N}^*$,

$$y''(x) + \left(\sum_{j=1}^{k+2}\frac{\varepsilon_j}{x-a_j}\right)y'(x) + \frac{\alpha\beta x^k + \sum_{i=1}^{k}\rho_i x^{k-i}}{\prod_{j=1}^{k+2}(x-a_j)}y(x) = 0. \tag{4.21}$$

Let L and $\bar{L}$ be two polynomials defined by (4.22) and (4.23), where $0 \leq p \leq k+1$ *and* $a_{i_n}, a_{j_m} \in \{a_j,\, j = 1,\ldots,k+2\}$:

$$L(x) = 1 \; if \; p = 0, \quad L(x) = \prod_{n=1}^{p}(x-a_{i_n}) \; if \; p \neq 0 \quad and \tag{4.22}$$

$$\bar{L}(x) = \prod_{m=1}^{k-p+2}(x-a_{j_m}) \tag{4.23}$$

such that $L(x)\bar{L}(x) = \prod_{j=1}^{k+2}(x-a_j)$.

Let $Q_k(x) = \alpha\beta x^k + \sum_{i=1}^{k}\rho_i x^{k-i}$.

Then, the equations of the form (4.21) are factorizable for polynomials Q_k such that: ($\delta_{0,p}$ is the Kronecker's symbol and $\varepsilon_{i_n}, \varepsilon_{j_m} \in \{\varepsilon_j,\, j = 1, \ldots, k+2\}$)

$$\begin{aligned} Q_k(x) &= \left(L(x)\bar{L}'(x) + (1-\delta_{0,p})\, L(x)\bar{L}(x) \sum_{n=1}^{p} \frac{\varepsilon_{i_n}}{x-a_{i_n}} \right) \sum_{m=1}^{k-p+2} \frac{\varepsilon_{j_m}-1}{x-a_{j_m}} \\ &+ L(x)\bar{L}(x) \sum_{m=1}^{k-p+2} \frac{1-\varepsilon_{j_m}}{(x-a_{j_m})^2}. \end{aligned} \tag{4.24}$$

A particular solution of the equation (4.21) is given by:

$$y(x) = \prod_{m=1}^{k-p+2} (x-a_{j_m})^{1-\varepsilon_{j_m}}. \tag{4.25}$$

5 Concluding remarks

Some important features deserve to be pointed out.

1. The factorization of all k-Heun's equations allows to determine all reducible Fuchsian equations of second order, given the poles a_i and the indices ε_i. The polynomials $Q_k = Q_k(a_i, \varepsilon_i; x)$ in Theorem 4.1 fixes all the reducible equations.

2. Heine and Stieltjes [21] studied the following problem starting from equation (1.3) (up to a change of function): determine all polynomials Q_k generating solutions of type $y(x) = A(x)P_N(x)$, where $A(x)$ is an algebraic function, like $(x-a_i)^{1-\varepsilon_i}$ and P_N a polynomial of degree N. Many results are known about the interlacing of the zeros of Q_{k+2} with the zeros of P_N, but explicit representation of P_N is not known, even in the simplest case $k = 1$. The factorization performed here gives a set of Heun's equations for which interlacing is impossible ($N = 0$).

3. The factorization can be extended easily, as in [8], to all possible confluent variants of the general k-Heun's equation.

4. The given tables for $k = 1, 2, 3$ allow to emphasize the different symmetries of poles and indices permutation, as already noticed by Maier for L, see subsection 3.1.

Acknowledgments

The authors are thankful to Dr Alain Moussiaux from the Facultés Universitaires Notre Dame de la Paix (FUNDP), Namur, Belgium, for helpful discussions. They also express their gratefulness to Dr Yves Justum and all the staff of the Laboratoire de Chimie Physique of the University of Paris Sud XI, (Scientific Centre of Orsay, France), for their hospitality.

References

[1] F. M. Arscott, *Heun's differential Eequations*, Oxford University Press, Oxford and New York 1995, (1995), pp 17-18 and pp 41-42.

[2] D. Basic and H. Schmid, Heun equation, Teukolsky equation and type-D metrics, *J. Math. Phys.* **48** (2007), 042502.

[3] G. Birkhoff and G-R. Rota, *Ordinary differential equations*, Ginn and Company 1962, (1962).

[4] M. Bronstein, *An improved algorithm for factoring linear ordinary differential operators*, Proceedings ISSAC'94, Oxford, UK, ACM Press 1994, (1994).

[5] F. Bureau, *Sur les équations différentielles linéaires du second ordre*, Mémoires de la Société Royale des Sciences de Liège, 3^{ie} tome.

[6] E. S. Cheb-Terrab, New closed form solutions in terms of ${}_pF_q$ for families of the general, confluent and bi-confluent Heun differential equations, *J. Phys. A: Math.Gen.* **37** (2004), 9923

[7] G. H. Halphen, Mémoire sur la réduction des équations différentielles linéaires aux formes intégrales, *Acad. Sc. Paris,* 2^{ie} *serie* t. $XXVIII$ (1984).

[8] M. N. Hounkonnou, A. Ronveaux and K. Sodoga, Factorization of some confluent Heun's differential equations, *Appl. Math. Comput.* **189** (2007), pp 816-820.

[9] M. N. Hounkonnou, A. Ronveaux, *Generalized Heun and Lamé's equations: factorization*, http://arxiv1: 0902/0902.2991v1 [math-ph].

[10] E. L. Ince, *Ordinary differential equation*, Dover Publication Inc 1999, (1999).

[11] T. Kimura, On Fuchsian differential equations reducible to hypergeometric equation by linear transformations, *Funkcial. Ekvac.* **13** (1970), pp 213-217.

[12] R. S. Maier, Private communication (2002).

[13] G. Petiau, *Les classes de solutions de Lamé*, International Congress of Mathematics, Helsinki, August 1978, (1978).

[14] A. Pham Ngoc Dinh, Sur les fonctions généralisées de Lamé d'ordre $(n-2)$ associées à une représentation induite de $E(n)$, *Bull. Sc. Math.* 2^e*, série* **104** (1980), pp 113-134.

[15] M. Picard, *Traité d'analyse* T. III, Gauthier-Villars, Paris 1903, (1903).

[16] E. G. C. Poole, *Introduction of linear differential equations*, Oxford University Press, Oxford Clarenton Press, 1936, (1936).

[17] A. Ronveaux, Editor, *Heun's differential equations*, Oxford University Press, Oxford and New York 1995, (1995).

[18] A. Ronveaux, Factorisation de l'opérateur de Heun, *Rev. Questions Sci.* **172** (2001), pp 409-416.

[19] A. Ronveaux, Factorization of the Heun's differential equation, *Appl. Math. Comput.* **141** (2003), pp 177-184.

[20] F. Schwarz, *A factorization algorithm for linear ordinary differential equations*, Proceedings of ISSAC'89, Oxford, ACM Press 1989, (1989).

[21] G. Szego, *Orthogonal polynomials*, Amer. Math. Soc. 1939, (1939).

[22] S. P. Tsarev, *An algorithm for complete enumeration of all factorization of a linear ordinary differential operator*, Proceedings of ISSAC'89, ACM Press 1996, (1996), pp 176-189.

[23] M. Van Hoeij, Factorization of differential operators with rational functions Coeficients, *J. Symbolic Computation* **24** (1997), pp 537-561.

[24] E. T. Whittaker and G. N. Watson, *A course of modern analysis*, Cambridge University Press 1927, (1927).