CHAPTER XIII.

ON RADICAL FUNCTIONS.

240. THE reader is already familiar with the fact that if $\operatorname{sn} u$ represent the ordinary Jacobian elliptic function, the square root of $1-\operatorname{sn}^2 u$ may be treated as a single-valued function of u. Such a property is possessed by other square roots. Thus for instance we have*

$$\sqrt{(1-\operatorname{sn} u)(1-k\operatorname{sn} u)}$$
$$= M\sin\frac{\pi}{4K}(K-u)\prod_m \frac{\left[1-2q^m\sin\frac{\pi u}{2K}+q^{2m}\right]\left[1-2q^{m-\frac{1}{2}}\sin\frac{\pi u}{2K}+q^{2m-1}\right]}{1-2q^{2m-1}\cos\frac{\pi u}{K}+q^{4m-2}},$$

where M is a certain constant, and, as usual, $q=e^{-\pi K'/K}$. The single-valuedness of the function $\sqrt{(1-\operatorname{sn} u)(1-k\operatorname{sn} u)}$ can be immediately seen to follow from the fact that *each of the zeros and poles of the function* $(1-\operatorname{sn} u)(1-k\operatorname{sn} u)$ *is of the second order.* It is manifest that we can easily construct other functions having the same property. If now we write $u=u^{x,a}$ and consider the square root on the dissected elliptic Riemann surface, we shall thereby obtain a single-valued function of the place x, whose values on the two sides of either period loop will have a ratio, constant along that loop, which is equal to ± 1.

Ex. Prove that the function

$$\sqrt{(\sqrt{\wp u-e_1}-\sqrt{e_2-e_1})(\sqrt{\wp u-e_1}-\sqrt{e_3-e_1})}$$

is a single-valued function of u.

Further we have, in Chapter XI., in dealing with the hyperelliptic case associated with an equation of the form

$$y^2=(x-a_1)\dots(x-a_{2p})(x-c),$$

* Cf. Cayley, *Elliptic Functions* (1876), Chap. XI. The function may be regarded as a doubly periodic function, with $8K$, $2iK'$ as its fundamental periods. It is of the fourth order, with K, $5K$, $K+iK'$, $5K+iK'$ as zeros, and iK', $2K+iK'$, $4K+iK'$, $6K+iK'$ as poles.

been led to the consideration of functions of the form $\sqrt{(c-x_1)\ldots(c-x_p)}$, which are expressible by theta functions with arguments $u, = u^{x_1, a_1} + \ldots\ldots + u^{x_p, a_p}$. These functions are not only single-valued functions of the arguments u, but, when the Riemann surface is dissected in the ordinary way, also of every one of the places $x_1, \ldots, x_p$. In fact the square root $\sqrt{c-x}$ is a single-valued function of the place x because, c being a branch place, $x-c$ vanishes to the second order at the place, and the point at infinity being a branch place, $x-c$ is there infinite to the second order. The values of the square root $\sqrt{c-x}$ on the two sides of any period loop will have a ratio, constant along that loop, which is equal to ± 1.

241. More generally it may be proved, for any Riemann surface, that if Z be a rational function such that each of its zeros and poles is of the mth order, the mth root, $\sqrt[m]{Z}$, is a single-valued function of position on the dissected surface, with factors at the period loops which are mth roots of unity. And it is easy to prove this in another way by obtaining an expression for such a function. For let $\alpha_1, \ldots, \alpha_r$ be the distinct poles of Z, and $\beta_1, \ldots, \beta_r$ its distinct zeros, so that the function is of order mr. Let $\Pi_{z, c}^{x, a}$ be the normal elementary integral of the third kind and $v_1^{x, a}, \ldots, v_p^{x, a}$ the normal integrals of the first kind. Then when the paths are restricted not to cross the period loops we have* equations

$$m\,(v_i^{\beta_1, \alpha_1} + \ldots\ldots + v_i^{\beta_r, \alpha_r}) = k_i + k_1'\tau_{i, 1} + \ldots\ldots + k_p'\tau_{i, p}, \qquad (i = 1, 2, \ldots, p),$$

wherein $k_1, \ldots, k_p, k_1', \ldots, k_p'$ are certain integers independent of i. Hence the expression

$$e^{m\,[\Pi_{\beta_1, \alpha_1}^{x, a} + \ldots\ldots + \Pi_{\beta_r, \alpha_r}^{x, a}] - 2\pi i k_1' v_1^{x, a} - \ldots\ldots - 2\pi i k_p' v_p^{x, a}},$$

wherein a is an arbitrary fixed place, represents the rational function Z, save for an arbitrary constant; and we have

$$\sqrt[m]{Z} = A e^{\Pi_{\beta_1, \alpha_1}^{x, a} + \ldots\ldots + \Pi_{\beta_r, \alpha_r}^{x, a} - \frac{2\pi i}{m}(k_1' v_1^{x, a} + \ldots\ldots + k_p' v_p^{x, a})},$$

where A is a certain constant. This expression defines $\sqrt[m]{Z}$ on the dissected surface as a single-valued function of position. More accurately it defines one branch of $\sqrt[m]{Z}$, the other $m-1$ branches being obtained by multiplying A by mth roots of unity. So defined, the function $\sqrt[m]{Z}$ is affected, at the period loop α_i, with a factor $e^{-\frac{2\pi i}{m}k_i'}$, and, at the period loop α_i', with the factor $e^{\frac{2\pi i}{m}k_i}$.

242. We have, in chapters X., XI., been concerned with other functions, namely the theta functions which also have the property of being single-

* Chap. VIII. § 155.

valued on the dissected Riemann surface, but affected with a factor for each period loop. They are also simpler than rational functions, in that they do not possess poles. It is therefore of interest to express such functions as $\sqrt[m]{Z}$ by means of theta functions; and the expression has an importance arising from the fact that the theory of the theta functions may be established independently of the theory of the algebraic integrals. To explain this mode of representation consider the quotient

$$\psi(u) = \frac{\vartheta(u-e;\ q)\,\vartheta(u-f;\ r)\,......}{\vartheta(u-E;\ Q)\,\vartheta(u-F;\ R)\,......},$$

where the numerator and denominator contain the same number of factors, $\vartheta(u, q)$ denotes the function (Chap. X. § 189) given by

$$\Sigma\Sigma\,......\,e^{au^2+2hu(n+q')+b(n+q')^2+2\pi iq(n+q')},$$

$q, r, \ldots, Q, R, \ldots$ denote any characteristics, and $e, f, \ldots, E, F, \ldots$ denote any arguments.

Then by the formula (§ 190)

$$\vartheta(u+\Omega_M;\ q) = e^{\lambda_M(u)+2\pi i(Mq'-M'q)}\,\vartheta(u;\ q),$$

where M, M' denote integers, we have $\psi(u+\Omega_M)/\psi(u) = e^L$, where L is

$$\lambda_M(u-e)+\lambda_M(u-f)+\,......\,-\lambda_M(u-E)-\lambda_M(u-F)-\,......$$
$$+\,2\pi iM(q'+r'+\,......\,-Q'-R'-\ldots)-2\pi iM'(q+r+\,......\,-Q-R-\ldots),$$

namely, is

$$-\lambda_M(e+f+\,......\,-E-F-\ldots)+2\pi iM(q'+r'+\,......\,-Q'-R'-\ldots)$$
$$-\,2\pi iM'(q+r+\,......\,-Q-R-\ldots).$$

Thus if

$$e_i+f_i+\,......\,=E_i+F_i+\,......,$$

and

$$q_i+r_i+\,......\,-(Q_i+R_i+\ldots)=\frac{1}{m}K_i,\quad (i=1, 2, \ldots, p),$$

$$q_i'+r_i'+\,......\,-(Q_i'+R_i'+\ldots)=\frac{1}{m}K_i',$$

where K_i, K_i' are integers and m is an integer, it follows, for integral values of M, M', that

$$[\psi(u+\Omega_M)/\psi(u)]^m = 1.$$

If now we take $b = i\pi\tau$, as in § 192, and put $u^{x,a}$ for u, $\vartheta(u-e;\ q)$ becomes a single-valued function of x whose zeros are (§§ 190 (L), 179) the places $x_1, \ldots, x_p$, given by

$$e-\Omega_q \equiv u^{x_1, a_1}+\,......\,+u^{x_p, a_p},$$

where $a_1, \ldots, a_p$ are p places determined from the place a, just as in § 179 the places $m_1, \ldots, m_p$ were determined from the place m; hence, in this case, $\psi(u)$ is the mth root of a rational function, having for zeros places

$$x_1, \ldots, x_p, z_1, \ldots, z_p, \ldots,$$

each m times repeated, and for poles places

$$X_1, \ldots, X_p, Z_1, \ldots, Z_p, \ldots,$$

each m times repeated, these places being subject only to the conditions expressed by the equations

$$u^{x_1, X_1} + \ldots\ldots + u^{x_p, X_p} + u^{z_1, Z_1} + \ldots\ldots + u^{z_p, Z_p} + \ldots\ldots \equiv -\frac{1}{m}\Omega_{K, K'}, \quad \text{(A)}.$$

In this representation we have obtained a function of which the number of m times repeated zeros is a multiple of p, and also the number of m times repeated poles is a multiple of p. It is easy however to remove this restriction by supposing a certain number of the places $x_1, \ldots, x_p, z_1, \ldots, z_p$ to coincide with places of the set $X_1, \ldots, X_p, Z_1, \ldots, Z_p, \ldots\ldots$

243. A rational function on the Riemann surface is characterised by the facts that it is a single-valued function of position, such that itself and its inverse have no infinities but poles, which has, moreover, the same value at the two sides of any period loop. The functions we have described may clearly be regarded as generalisations of the rational functions, the one new property being that the values of the function at the two sides of any period loop have a ratio, constant along that loop, which is a root of unity. For these functions there holds a theorem, expressed by the equations (A) above, which may be regarded as a generalisation of Abel's theorem for integrals of the first kind; and, when the poles of such a function are given, the number of zeros that can be arbitrarily assigned is the same as for a rational function having the same poles, being in general all but p of them; this follows from the theory of the solution of Jacobi's inversion problem (Chap. IX.; cf. also §§ 37, 93). It will be seen in the course of the following chapter that we can also consider functions of a still more general kind, having constant factors at the period loops which are not roots of unity, and possessing, beside poles, also essential singularities; such functions may be called *factorial* functions. The particular functions so far considered may be called *radical* functions; it is proper to consider them first, in some detail, on account of their geometrical interpretation and because they furnish a convenient method of expressing the solution of several problems connected with Jacobi's inversion problem.

244. The most important of the radical functions are those which are square roots of rational functions, and in view of the general theory developed in the next chapter it will be sufficient to confine ourselves to these functions.

In dealing with these we shall adopt the invariant representation by means of ϕ-polynomials, which has already been described*. An integral polynomial of the rth degree in the p fundamental ϕ-polynomials, $\phi_1, \ldots, \phi_p$, will be denoted by $\Phi^{(r)}$, or $\Psi^{(r)}$, when its $2r(p-1)$ zeros are subject to no condition. When all the zeros are of the second order, and fall therefore, in general, at $r(p-1)$ distinct places, the polynomial will be denoted by $X^{(r)}$ or $Y^{(r)}$; we have† already been concerned with such polynomials, $X^{(1)}$, of the first degree in $\phi_1, \ldots, \phi_p$.

It is to be shewn now that the square root $\sqrt{X^{(r)}}$ can properly be associated with a certain characteristic of $2p$ half-integers; and for this purpose it is convenient to utilise the places $m_1, \ldots, m_p$, arising from an arbitrary place m, which have already‡ occurred in the theory of the theta functions. These places are§ such that if a non-adjoint polynomial, Δ, of grade μ, be taken to vanish to the second order at m, there is an adjoint polynomial, $\bar{\psi}$, of grade $(n-1)\sigma + n - 3 + \mu$, vanishing in the remaining $n\mu - 2$ zeros of Δ, whose other zeros consist of the places $m_1, \ldots, m_p$, each repeated. Take now any ϕ-polynomial, ϕ_0, vanishing to the first order at m, and let its other zeros be $A_1, \ldots, A_{2p-3}$; and take a polynomial $\Phi^{(3)}$ vanishing to the second order in each of $A_1, \ldots, A_{2p-3}$; then $\Phi^{(3)}$ will|| contain $5(p-1) - 2(2p-3), = p+1$, linearly independent terms, and will have $6(p-1) - 2(2p-3), = 2p$, further zeros. Let $X^{(1)}$ be any ϕ-polynomial of which all the zeros are of the second order. Consider the most general rational function, of order $2p$, whose poles consist of the place m, this being a pole of the second order, and of the zeros of $X^{(1)}$. This function will contain $2p - p + 1, = p + 1$, linearly independent terms and can be expressed in either of the forms $\Phi^{(3)}/\phi_0^2 X^{(1)}$, $\psi/\Delta X^{(1)}$, where ψ is any polynomial of grade $(n-1)\sigma + n - 3 + \mu$ which vanishes in the $n\mu - 2$ zeros of Δ other than m. Since now¶ ψ can be chosen, $= \bar{\psi}$, so that the zeros of this function are the places $m_1, \ldots, m_p$, each repeated, it follows that $\Phi^{(3)}$ can be equally chosen so that this is the case. So chosen it may be denoted by $X^{(3)}$. *Thus the places* $m_1, \ldots, m_p$ *arise as the remaining zeros of a form* $X^{(3)}$ *(with* $3(p-1), = p + 2p - 3$, *zeros, each of the second order), whose other* $2p-3$ *separate zeros are zeros of an arbitrary* ϕ*-polynomial,* ϕ_0*, which vanishes once at the place* m.

If now $n_1, \ldots, n_{p-1}$ be the places which, repeated, are the zeros of $X^{(1)}$, it follows, since $m, n_1, \ldots, n_{p-1}$, each repeated, are the poles, and $m_1, \ldots, m_p$, each repeated, are the zeros of a rational function, $X^{(3)}/\phi_0^2 X^{(1)}$, that, upon the dissected surface, we have

$$v_i^{m_p, m} - v_i^{n_1, m_1} - \ldots\ldots - v_i^{n_{p-1}, m_{p-1}} = -\tfrac{1}{2}(k_i + k_1'\tau_{i,1} + \ldots\ldots + k_p'\tau_{i,p}),$$

* Chap. VI. § 110 ff., and the references there given, and Klein, *Math. Annal.* XXXVI. p. 38.

† Chap. X. § 188, p. 281.

‡ Chap. X. § 179.

§ Chap. X. § 183, Chap. VI. § 92, Ex. ix.

|| Chap. VI. § 111.

¶ Chap. X. § 183.

where $k_1, \ldots, k_p, k_1', \ldots, k_p'$ are certain integers. Hence, as in § 241, it immediately follows that the rational function $X^{(3)}/\phi_0^2 X^{(1)}$, save for a constant factor, is the square of the function

$$e^{\Pi^{x,a}_{m_1,n_1}+\ldots\ldots+\Pi^{x,a}_{m_{p-1},n_{p-1}}+\Pi^{x,a}_{m_p,m}+\pi i(k_1'v_1^{x,a}+\ldots\ldots+k_p'v_p^{x,a})},$$

and therefore that the expression $\sqrt{X^{(3)}}/\phi_0\sqrt{X^{(1)}}$ may be regarded as a single-valued function on the dissected Riemann surface, whose values on the two sides of any period loop have a ratio constant along that loop. These constant ratios are equal to $e^{\pi i k_r'}$ and $e^{-\pi i k_r}$ for the rth loop of the first and second kind respectively. When the places $m_1, \ldots, m_p$ are regarded as given, these equations associate with the form $\sqrt{X^{(1)}}$ a definite characteristic

$$\tfrac{1}{2}k_1, \ldots, \tfrac{1}{2}k_p, \tfrac{1}{2}k_1', \ldots, \tfrac{1}{2}k_p'.$$

Also, if $Y^{(3)}$ be any polynomial which, beside vanishing to the second order in $A_1, \ldots, A_{2p-3}$, vanishes to the second order in places $m_1', \ldots, m_p'$, $Y^{(3)}/X^{(3)}$ is a rational function, and we have equations of the form

$$v_i^{m_1',m_1}+\ldots\ldots+v_i^{m_p',m_p}=\tfrac{1}{2}(\lambda_i+\lambda_1'\tau_{i,1}+\ldots\ldots+\lambda_p'\tau_{i,p}),$$

$$\sqrt{Y^{(3)}}/\sqrt{X^{(3)}}=Ae^{\Pi^{x,a}_{m_1',m_1}+\ldots\ldots+\Pi^{x,a}_{m_p',m_p}-\pi i(\lambda_1'\tau_{i,1}+\ldots\ldots+\lambda_p'\tau_{i,p})},$$

where $\lambda_1, \ldots, \lambda_p'$ are integers, A is a constant, and the paths of integration are limited to the dissected Riemann surface. These equations associate $\sqrt{Y^{(3)}}$ with the characteristic $\tfrac{1}{2}\lambda_1, \ldots, \tfrac{1}{2}\lambda_p, \tfrac{1}{2}\lambda_1', \ldots, \tfrac{1}{2}\lambda_p'$.

And, as in § 184, Chap. X., we infer that every odd characteristic is associated with a polynomial* $X^{(1)}$, and every even characteristic with a polynomial $Y^{(3)}$, which has $A_1, \ldots, A_{2p-3}$ for zeros of the second order; and it may happen that the polynomial $Y^{(3)}$ corresponding to an even characteristic has the form $\phi_0^2 Y^{(1)}$, in which case the places $m_1', \ldots, m_p'$ consist of the place m and the zeros of a form $Y^{(1)}$.

245. Let now $X^{(2\nu+1)}$ be any polynomial whose zeros consist of $(2\nu+1)(p-1)$ places, $z_1, z_2, \ldots$, each repeated; let ϕ_0 be as before, vanishing in $m, A_1, \ldots, A_{2p-3}$, and $X^{(3)}$ be as before, vanishing to the second order in $A_1, \ldots, A_{2p-3}, m_1, \ldots, m_p$. Then if $\Phi^{(\nu)}$ be any ϕ-polynomial whose zeros are $c_1, c_2, \ldots$, the function

$$\phi_0^2 X^{(2\nu+1)}/[\Phi^{(\nu)}]^2 X^{(3)}$$

* Or in particular cases with a lot of such polynomials, giving rise to coresidual sets of places.

is a rational function of order $2(2\nu+1)(p-1)+2$, whose zeros are $m, z_1, z_2, \ldots$, and whose poles consist of the places $m_1, \ldots, m_p$, and the zeros of $\Phi^{(\nu)}$, each repeated. Hence as before $\phi_0\sqrt{X^{(2\nu+1)}}/\Phi^{(\nu)}\sqrt{X^{(3)}}$ is a single-valued function on the dissected surface, and the form $\sqrt{X^{(2\nu+1)}}$ is associated with a characteristic $\frac{1}{2}q_1, \ldots, \frac{1}{2}q_p, \frac{1}{2}q_1', \ldots, \frac{1}{2}q_p'$, such that, on the dissected surface,

$$v_i^{z_1, m_1} + \ldots\ldots + v_i^{z_p, m_p} + v_i^{z_{p+1}, c_1} + \ldots\ldots = \tfrac{1}{2}(q_i + q_1'\tau_{i,1} + \ldots\ldots + q_p'\tau_{i,p}),$$
$$(i = 1, 2, \ldots, p);$$

and if, instead of $\Phi^{(\nu)}$, we had used any other polynomial $\Psi^{(\nu)}$, the characteristic could, by Abel's theorem, only be affected by the addition of integers.

Suppose now that $Y^{(2\mu+1)}$ is another polynomial, and take a polynomial $\Psi^{(\mu)}$; then if the characteristic of the function $\phi_0\sqrt{Y^{(2\mu+1)}}/\Psi^{(\mu)}\sqrt{X^{(3)}}$ differ from that of $\phi_0\sqrt{X^{(2\nu+1)}}/\Phi^{(\nu)}\sqrt{X^{(3)}}$ only by integers, we have when $x_1, x_2, \ldots$ denote the zeros of $\sqrt{Y^{(2\mu+1)}}$, and $d_1, d_2, \ldots$ denote the zeros of $\Psi^{(\mu)}$, the equation

$$v_i^{x_1, m_1} + \ldots\ldots + v_i^{x_p, m_p} + v_i^{x_{p+1}, d_1} + \ldots\ldots = \tfrac{1}{2}(q_i + q_1'\tau_{i,1} + \ldots\ldots + q_p'\tau_{i,p})$$
$$+ M_i + M_1'\tau_{i,1} + \ldots\ldots + M_p'\tau_{p,i},$$

where $M_1, \ldots, M_p, M_1', \ldots, M_p'$ denote integers; by adding this to the last equation we infer* that $\phi_0^2\sqrt{X^{(2\nu+1)}}\sqrt{Y^{(2\mu+1)}}/\Phi^{(\nu)}\Psi^{(\mu)}X^{(3)}$ is a rational function. Hence†, since there exists a rational function of the form $\phi_0^2X^{(1)}/X^{(3)}$, we infer, *when* $\sqrt{X^{(2\nu+1)}}$, $\sqrt{Y^{(2\mu+1)}}$ *have characteristics differing only by integers, there exists a form* $\Phi^{(\mu+\nu+1)}$ *whose zeros are the separate zeros of* $\sqrt{X^{(2\nu+1)}}$ *and* $\sqrt{Y^{(2\mu+1)}}$, *and we have* $\sqrt{X^{(2\nu+1)}}\sqrt{Y^{(2\mu+1)}} = \Phi^{(\nu+\mu+1)}$.

Hence, all possible forms $\sqrt{Y^{(2\mu+1)}}$, with the same value of μ, whose characteristics, save for integers, are the same, are expressible in the form $\Phi^{(\mu+\nu+1)}/\sqrt{X^{(2\nu+1)}}$, where $\Phi^{(\mu+\nu+1)}$ is a polynomial of the degree indicated, which vanishes once in the zeros of $\sqrt{X^{(2\nu+1)}}$. All such forms $\sqrt{Y^{(2\mu+1)}}$ are therefore expressible by such equations as

$$\sqrt{Y^{(2\mu+1)}} = \lambda_1\sqrt{Y_1^{(2\mu+1)}} + \ldots\ldots + \lambda_{2\mu(p-1)}\sqrt{Y_{2\mu(p-1)}^{(2\mu+1)}},$$

where $\sqrt{Y_1^{(2\mu+1)}}, \ldots, \sqrt{Y_{2\mu(p-1)}^{(2\mu+1)}}$ are special polynomials, and $\lambda_1, \ldots, \lambda_{2\mu(p-1)}$ are constants. The assignation of $2\mu(p-1)-1$, $=(2\mu+1)(p-1)-p$, zeros of $\sqrt{Y^{(2\mu+1)}}$ will determine the constants $\lambda_1, \ldots, \lambda_{2\mu(p-1)}$, and therefore determine the remaining p zeros. When $\mu = 0$ there may be a reduction in the number of zeros determined by the others.

It follows also that the zeros of any form $\sqrt{Y^{(2\mu+1)}}$ are the remaining zeros of a polynomial $\Phi^{(\mu+2)}$ which vanishes in the zeros of a form $\sqrt{X^{(3)}}$ having

* Chap. VIII. § 158. † Chap. VI. § 112.

the same characteristic as $\sqrt{Y^{(2\mu+1)}}$, or a characteristic differing from that of $\sqrt{Y^{(2\mu+1)}}$ only by integers. When the characteristic of $\sqrt{X^{(3)}}$ is odd, and $\sqrt{X^{(3)}} = \Phi^{(1)}\sqrt{X^{(1)}}$, we may take $\Phi^{(\mu+2)}$ to be of the form $\Phi^{(\mu+1)}\Phi^{(1)}$.

It can be similarly shewn that if $X^{(2\mu)}$ be a polynomial of even degree, 2μ, in the fundamental ϕ-polynomials, of which all the zeros are of the second order, and $\Phi^{(\mu)}$ be any polynomial of degree μ, the quotient $\sqrt{X^{(2\mu)}}/\Phi^{(\mu)}$ may be interpreted as a single-valued function on the dissected surface, and the form $\sqrt{X^{(2\mu)}}$ may be associated with a certain characteristic of half-integers. Further the zeros of $\sqrt{X^{(2\mu)}}$ are the remaining zeros of a form $\Phi^{(\mu+1)}$ which vanishes in the zeros of a form $\sqrt{X^{(2)}}$ of the same* characteristic as $\sqrt{X^{(2\mu)}}$. Also if $\sqrt{X^{(1)}}$, $\sqrt{Y^{(1)}}$ be two forms whose (odd) characteristics have a sum differing from the characteristic of $\sqrt{X^{(2)}}$ by integers, the ratio $\sqrt{X^{(2)}}/\sqrt{X^{(1)}Y^{(1)}}$ is a rational function; and if we determine $(p-1)$ pairs of odd characteristics, such that the sum of each pair is, save for integers, equal to the characteristic of $\sqrt{X^{(2)}}$, and $\sqrt{X_1^{(1)}}$, $\sqrt{Y_1^{(1)}}$, $\sqrt{X_2^{(1)}}$, $\sqrt{Y_2^{(1)}}$, ..., represent the corresponding forms, there exists an equation of the form

$$\sqrt{X^{(2)}} = \lambda_1\sqrt{X_1^{(1)}Y_1^{(1)}} + \lambda_2\sqrt{X_2^{(1)}Y_2^{(1)}} + \dots\dots + \lambda_{p-1}\sqrt{X_{p-1}^{(1)}\,Y_{p-1}^{(1)}}.$$

As a matter of fact every characteristic, except the zero characteristic, can, save for integers, be written as the sum of two odd characteristics in $2^{p-2}(2^{p-1}-1)$ ways.

246. In illustration of these principles we consider briefly the geometrical theory of a general plane quartic curve for which $p=3$. We may suppose the equation expressed homogeneously by the coordinates x_1, x_2, x_3 and take the fundamental ϕ-polynomials to be $\phi_1 = x_1$, $\phi_2 = x_2$, $\phi_3 = x_3$. There are then $2^{p-1}(2^p-1) = 28$ double tangents, $X^{(1)}$, of fixed position. There are 2^{2p}, $=64$, systems of cubic curves, $X^{(3)}$, each touching in six points. Of these six points of contact of a cubic, $X^{(3)}$, of prescribed characteristic, three may be arbitrarily taken; and we have in fact

$$\sqrt{X^{(3)}} = \lambda_1\sqrt{X_1^{(3)}} + \lambda_2\sqrt{X_2^{(3)}} + \lambda_3\sqrt{X_3^{(3)}} + \lambda_4\sqrt{X_4^{(3)}},$$

where λ_1, λ_2, λ_3, λ_4 are constants, and $\sqrt{X_1^{(3)}}$, $\sqrt{X_2^{(3)}}$, ..., are special forms of the assigned characteristic. The points of contact of all cubics $X^{(3)}$ of given odd characteristic are obtainable by drawing variable conics through the points of contact of the double tangent, D, associated with that odd characteristic. Let Ω_0 be a certain one of these conics and let X_0 denote the corresponding contact-cubic; then the rational function $X_0 D/\Omega_0^2$ has, clearly, no poles, and must be a constant, and therefore, absorbing the constant, we infer that the equation of the fundamental quartic can be written

$$4X_0 D - \Omega_0^2 = 0.$$

* Or a characteristic differing from that of $\sqrt{X^{(2\mu)}}$ by integers.

Three of the conics through the points of contact of D are $x_1 D=0$, $x_2 D=0$, $x_3 D=0$; the corresponding forms of $X^{(3)}$ are $x_1^2 D$, $x_2^2 D$, $x_3^2 D$. Hence all contact cubics of the same characteristic as $\sqrt{D}$ are included in the formula

$$\sqrt{X^{(3)}}=(\lambda_1 x_1+\lambda_2 x_2+\lambda_3 x_3)\sqrt{D}+\sqrt{X_0},$$

or

$$X^{(3)}=X_0+\Omega_0 P+DP^2,$$

where $P=\lambda_1 x_1+\lambda_2 x_2+\lambda_3 x_3$, λ_1, λ_2, λ_3 being constants; the conic through the points of contact of D which passes through the points of contact of $X^{(3)}$ is given by $\Omega=2\sqrt{DX^{(3)}}$, or $\Omega=2PD+\Omega_0$; and the fundamental quartic can equally be written

$$4X^{(3)}D-\Omega^2=4(X_0+\Omega_0 P+DP^2)D-(\Omega_0+2PD)^2=0.$$

If then we introduce space coordinates X, Y, Z, T given by

$$X=x_1,\ \ Y=x_2,\ \ Z=x_3,\ \ T=-\sqrt{X_0/D},$$

so that the general form of $\sqrt{X^{(3)}}$ with the same characteristic as $\sqrt{D}$ is given by

$$\sqrt{X^{(3)}}=\sqrt{D}\,(\lambda_1 X+\lambda_2 Y+\lambda_3 Z-T),$$

we have

$$4X_0(X,\ Y,\ Z)\,D(X,\ Y,\ Z)=\Omega_0^2(X,\ Y,\ Z),$$

$$2TD(X,\ Y,\ Z)+\Omega_0(X,\ Y,\ Z)=0,$$

where $X_0\ (X,\ Y,\ Z)$ is the result of substituting in X_0, for x_1, x_2, x_3, respectively X, Y, Z, etc.; by these equations the fundamental quartic is related to a curve of the sixth order in space of three dimensions, given by the intersection of the quadric surface

$$2TD(X,\ Y,\ Z)+\Omega_0(X,\ Y,\ Z)=0$$

and the quartic cone

$$4X_0(X,\ Y,\ Z)\,D(X,\ Y,\ Z)=\Omega_0^2(X,\ Y,\ Z);$$

the curve lies also on the cubic surface

$$T^2 D(X,\ Y,\ Z)+T\Omega_0(X,\ Y,\ Z)+X_0(X,\ Y,\ Z)=0,$$

which can also be written

$$(T-P)^2 D(X,\ Y,\ Z)+(T-P)\,\Omega(X,\ Y,\ Z)+X^{(3)}(X,\ Y,\ Z)=0,$$

where P denotes $\lambda_1 X+\lambda_2 Y+\lambda_3 Z$, $\Omega=2PD+\Omega_0$, and $X^{(3)}=DP^2+\Omega_0 P+X_0$, as above.

It can be immediately shewn (i) that the enveloping cone of the cubic surface just obtained, whose vertex is the point $X=0=Y=Z$, is the quartic cone whose intersection with the plane $T=0$ gives the fundamental quartic curve, (ii) that the tangent plane of the cubic surface at the point

$X=0=Y=Z$ is the plane $D(X, Y, Z)=0$, (iii) that the planes joining the point $X=0=Y=Z$ to the 27 straight lines of the cubic surface intersect the plane $T=0$ in the 27 double tangents of the fundamental quartic other than D, (iv) that the fundamental quartic curve may be considered as arising by the intersection of an arbitrary plane with the quartic cone of contact which can be drawn to an arbitrary cubic surface from an arbitrary point of the surface.

Thus the theory of the bitangents is reducible to the theory of the right lines lying on a cubic surface. Further development must be sought in geometrical treatises. Cf. Geiser, *Math. Annal.* Bd. I. p. 129, *Crelle* LXXII. (1870); also Frahm, *Math. Annal.* VII. and Toeplitz, *Math. Annal.* XI.; Salmon, *Higher Plane Curves* (1879), p. 231, note; Klein, *Math. Annal.* XXXVI. p. 51.

247. We have shewn that there are 28 double tangents each associated with one of the odd characteristics; the association depends upon the mode of dissection of the fundamental Riemann surface. We have stated moreover (§ 205, Chap. XI.), in anticipation of a result which is to be proved later, that there are $8.36=288$ ways in which all possible characteristics can be represented by combinations of one, two, or three of seven fundamental odd characteristics. These fundamental characteristics can be denoted by the numbers 1, 2, 3, 4, 5, 6, 7, and in what follows we shall, for the sake of definiteness, suppose them to be either the characteristics so denoted in the table given § 205, or one of the seven sets whose letter notation is given at the conclusion of § 205. Thus the sum of these seven characteristics is the characteristic, which, save for integers, has all its elements zero; or, as we may say, the sum of these characteristics is zero.

A double tangent whose characteristic is denoted by the number i will be represented by the equation $u_i=0$. A combination of two numbers also represents an odd characteristic (§ 205, Chap. XI.), so that there will also be 21 double tangents whose equations are of such forms as $u_{i,j}=0$. The three products $\sqrt{u_1u_{23}}$, $\sqrt{u_2u_{31}}$, $\sqrt{u_3u_{12}}$ will be radical forms, such as have been denoted by $\sqrt{X^{(2)}}$, each with the characteristic 123. Hence if suitable numerical multipliers be absorbed in u_1, u_3, we have (§ 245) an identity of the forms

$$\sqrt{u_1u_{23}}+\sqrt{u_2u_{31}}+\sqrt{u_3u_{12}}=0, \quad (u_2u_{31}+u_3u_{12}-u_1u_{23})^2=4u_2u_3u_{31}u_{12};$$

this must then be a form into which the equation of the fundamental quartic curve can be put. Further, each of the six forms

$$\sqrt{u_2u_{12}},\ \sqrt{u_3u_{13}},\ \sqrt{u_4u_{14}},\ \sqrt{u_5u_{15}},\ \sqrt{u_6u_{16}},\ \sqrt{u_7u_{17}}$$

has the same characteristic, denoted by the symbol 1. Thus, if suitable numerical multipliers be absorbed in u_2, u_4, the equation of the quartic can also be given in the form

$$(u_2u_{12}+u_4u_{14}-u_3u_{13})^2=4u_4u_2u_{12}u_{14}.$$

If therefore

$$f = u_2u_{31} + u_3u_{12} - u_1u_{23}, \quad \phi = u_2u_{12} + u_4u_{14} - u_3u_{13},$$

we have

$$(f-\phi)(f+\phi) = 4u_2u_{12}(u_3u_{13} - u_4u_{14}).$$

Now if $f-\phi$ were divisible by u_2, and $f+\phi$ divisible by u_{12}, the common point of the tangents $u_2 = 0$, $u_{12} = 0$ would make $f = 0$, and therefore be upon the fundamental quartic, $f^2 = 4u_2u_3u_{31}u_{12}$; this is impossible when the quartic is perfectly general. Hence, without loss of generality, we may take

$$f - \phi = 2\lambda u_2u_{12},$$

$$f + \phi = \frac{2}{\lambda}(u_3u_{13} - u_4u_{14}),$$

λ being a certain constant, and therefore

$$u_4u_{14} = u_3u_{13} - \lambda f + \lambda^2u_2u_{12}, \quad = u_3u_{13} - \lambda(u_2u_{31} + u_3u_{12} - u_1u_{23}) + \lambda^2u_2u_{12}.$$

Thérefore, when the six tangents u_1, u_2, u_3, u_{23}, u_{31}, u_{12} are given, the tangents u_4, u_{14} can be found by expressing the condition that the right-hand side should be a product of linear factors; as the right-hand is a quadric function of the coordinates this will lead to a sextic equation in λ, having the roots $\lambda = 0$, $\lambda = \infty$; if the other roots be substituted in turn on the right-hand, we shall obtain in turn four pairs of double tangents; these are in fact (u_4, u_{14}), (u_5, u_{15}), (u_6, u_{16}), (u_7, u_{17}). We use the equation obtained however in a different way; by a similar proof we clearly obtain the three equations

$$\begin{aligned} u_4u_{14} &= u_3u_{13} - \lambda_1(u_2u_{31} + u_3u_{12} - u_1u_{23}) + \lambda_1^2u_2u_{12}, \\ u_4u_{24} &= u_1u_{21} - \lambda_2(u_3u_{12} + u_1u_{23} - u_2u_{31}) + \lambda_2^2u_3u_{23}, \\ u_4u_{34} &= u_2u_{32} - \lambda_3(u_1u_{23} + u_2u_{31} - u_3u_{12}) + \lambda_3^2u_1u_{31}, \end{aligned} \tag{B}$$

and hence

$$u_4\left(\frac{u_{24}}{\lambda_2} + \frac{u_{34}}{\lambda_3}\right) = u_{23}\left(\lambda_2u_3 + \frac{u_2}{\lambda_3}\right) + u_1\left(\frac{u_{21}}{\lambda_2} + \lambda_3u_{31} - 2u_{23}\right);$$

from this we infer that the common point of the tangents u_1, u_4 either lies on u_{23} or on $\lambda_2u_3 + \frac{u_2}{\lambda_3} = 0$; as the fundamental quartic may be written in the form $\sqrt{Au_4u_{34}} + \sqrt{Bu_2u_{23}} + \sqrt{Cu_1u_{13}} = 0$, it follows that if u_1, u_4, u_{23} intersect, they intersect on the quartic, which is impossible. Hence u_4 must pass through the intersection of u_1 and $\lambda_2u_3 + \frac{u_2}{\lambda_3} = 0$; now we may assume that the tangents u_1, u_2, u_3 are not concurrent, since else, as follows from the equation $\sqrt{u_1u_{23}} + \sqrt{u_2u_{31}} + \sqrt{u_3u_{12}} = 0$, they would intersect upon the quartic; thus u_4 may be expressed linearly by u_1, u_2, u_3, and we may put

$$u_4 = a_1u_1 + a_2u_2 + a_3u_3 = a_1u_1 + \frac{1}{h_1}\left(\lambda_2u_3 + \frac{u_2}{\lambda_3}\right),$$

and so obtain $\lambda_2 = h_1 a_3$, $\lambda_3 = 1/h_1 a_2$, h_1 being a certain constant; then the equation under consideration becomes

$$u_4\left(\frac{u_{24}}{\lambda_2}+\frac{u_{34}}{\lambda_3}\right) = u_{23}h_1(u_4 - a_1u_1) + u_1\left(\frac{u_{21}}{\lambda_2} + \lambda_3 u_{31} - 2u_{23}\right),$$

or

$$u_4\left(\frac{u_{24}}{\lambda_2}+\frac{u_{34}}{\lambda_3} - h_1u_{23}\right) = u_1\left(\frac{u_{21}}{\lambda_2} + \lambda_3 u_{31} - 2u_{23} - a_1h_1u_{23}\right),$$

so that, if k_1 denote a proper constant,

$$\frac{u_{24}}{\lambda_2}+\frac{u_{34}}{\lambda_3} = h_1u_{23} - \frac{k_1}{h_1}u_1,$$

$$-k_1u_4 = \frac{u_{12}}{a_3} + \frac{u_{31}}{a_2} - h_1u_{23}(2 + a_1h_1).$$

We can similarly obtain the equations

$$-k_2u_4 = \frac{u_{12}}{a_3} + \frac{u_{23}}{a_1} - h_2u_{31}(2 + a_2h_2),$$

$$-k_3u_4 = \frac{u_{23}}{a_1} + \frac{u_{31}}{a_2} - h_3u_{12}(2 + a_3h_3),$$

where h_2, h_3, k_2, k_3 are proper constants; therefore, as u_{23}, u_{31}, u_{12} are not concurrent tangents, since else they would intersect on the fundamental quartic, we infer, by comparing the right-hand sides in these three equations,

$$-\frac{h_1}{k_1}(2 + a_1h_1) = \frac{1}{k_2a_1} = \frac{1}{k_3a_1}, \quad -\frac{h_2}{k_2}(2 + a_2h_2) = \frac{1}{k_3a_2} = \frac{1}{k_1a_2},$$

$$-\frac{h_3}{k_3}(2 + a_3h_3) = \frac{1}{k_1a_3} = \frac{1}{k_2a_3},$$

and hence, $k_1 = k_2 = k_3$, $= k$, say, and $1 + 2h_1a_1 + a_1^2h_1^2 = 0$ or $h_1 = -\dfrac{1}{a_1}$, $h_2 = -\dfrac{1}{a_2}$, $h_3 = -\dfrac{1}{a_3}$.

Thus

$$-ku_4 = \frac{u_{23}}{a_1} + \frac{u_{31}}{a_2} + \frac{u_{12}}{a_3},$$

or

$$\frac{u_{23}}{a_1} + \frac{u_{31}}{a_2} + \frac{u_{12}}{a_3} + k(a_1u_1 + a_2u_2 + a_3u_3) = 0. \qquad \text{(C)}$$

Further we obtained the equation

$$\frac{u_{24}}{\lambda_2}+\frac{u_{34}}{\lambda_3} = h_1u_{23} - \frac{k_1}{h_1}u_1;$$

thus we have

$$\frac{u_{24}}{\lambda_2}+\frac{u_{34}}{\lambda_3}+\frac{u_{23}}{a_1} = ka_1u_1, \quad \frac{u_{34}}{\lambda_3}+\frac{u_{14}}{\lambda_1}+\frac{u_{31}}{a_2} = ka_2u_2, \quad \frac{u_{14}}{\lambda_1}+\frac{u_{24}}{\lambda_2}+\frac{u_{12}}{a_3} = ka_3u_3,$$

and therefore, as $\lambda_2 = -\frac{a_3}{a_1}$, $\lambda_3 = -\frac{a_1}{a_2}$, and similarly $\lambda_1 = -\frac{a_2}{a_3}$, we have, by the equation (C),

$$-\frac{a_3}{a_2} u_{14} = \frac{u_{23}}{a_1} + k\,(a_2 u_2 + a_3 u_3),$$

$$-\frac{a_1}{a_3} u_{24} = \frac{u_{31}}{a_2} + k\,(a_3 u_3 + a_1 u_1),$$

$$-\frac{a_2}{a_1} u_{34} = \frac{u_{12}}{a_3} + k\,(a_1 u_1 + a_2 u_2).$$

But if we put

$$u_5 = b_1 u_1 + b_2 u_2 + b_3 u_3, \quad u_6 = c_1 u_1 + c_2 u_2 + c_3 u_3, \quad u_7 = d_1 u_1 + d_2 u_2 + d_3 u_3,$$

we have also three other equations such as (C), differing from (C) in the substitution respectively of the coefficients b_1, b_2, b_3 c_1, c_2, c_3 and d_1, d_2, d_3 in place of a_1, a_2, a_3, and of three constants, say l, m, n, in place of k. As the tangents u_5, u_6, u_7 are not concurrent (for the fundamental quartic can be written in a form $\sqrt{u_5 u_{15}} + \sqrt{u_6 u_{16}} + \sqrt{u_7 u_{17}} = 0$) we may use these three last equations to determine u_{23}, u_{31}, u_{12} in terms of u_1, u_2, u_3; the expressions obtained must satisfy the equation (C). Thus there exist, with suitable values of the multipliers A, B, C, D, the six equations

$$\frac{A}{a_1} + \frac{B}{b_1} + \frac{C}{c_1} + \frac{D}{d_1} = 0, \qquad Aka_1 + Blb_1 + Cmc_1 + Dnd_1 = 0,$$

$$\frac{A}{a_2} + \frac{B}{b_2} + \frac{C}{c_2} + \frac{D}{d_2} = 0, \qquad Aka_2 + Blb_2 + Cmc_2 + Dnd_2 = 0,$$

$$\frac{A}{a_3} + \frac{B}{b_3} + \frac{C}{c_3} + \frac{D}{d_3} = 0, \qquad Aka_3 + Blb_3 + Cmc_3 + Dnd_3 = 0.$$

From these equations the ratios of the constants k, l, m, n are determinable; suppose the values obtained to be written $\rho k', \rho l', \rho m', \rho n'$, where ρ is undetermined, and k', l', m', n' are definite; then, if we put α_i for $a_i\sqrt{k'}$, β_i for $b_i\sqrt{l'}$, γ_i for $c_i\sqrt{m'}$, δ_i for $d_i\sqrt{n'}$, v_{23} for u_{23}/ρ, v_{31} for u_{31}/ρ, and v_{12} for u_{12}/ρ, the equations obtained consist of

(i) four of the form

$$\frac{v_{23}}{\alpha_1} + \frac{v_{31}}{\alpha_2} + \frac{v_{12}}{\alpha_3} + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0 \qquad (C')$$

in which there occur in turn the sets of coefficients $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$, $(\gamma_1, \gamma_2, \gamma_3)$, $(\delta_1, \delta_2, \delta_3)$; from any three of these v_{23}, v_{31}, v_{12} may be expressed in terms of u_1, u_2, u_3;

(ii) four sets of the form

$$-\frac{\alpha_3}{\alpha_2} v_{14} = \frac{v_{23}}{\alpha_1} + \alpha_2 u_2 + \alpha_3 u_3, \quad -\frac{\alpha_1}{\alpha_3} v_{24} = \frac{v_{31}}{\alpha_2} + \alpha_3 u_3 + \alpha_1 u_1, \quad -\frac{\alpha_2}{\alpha_1} v_{34} = \frac{v_{12}}{\alpha_3} + \alpha_1 u_1 + \alpha_2 u_2,$$

where $v_{14} = u_{14}/\rho\sqrt{k'}$, $v_{24} = u_{24}/\rho\sqrt{k'}$, $v_{34} = u_{34}/\rho\sqrt{k'}$.

It will be recalled that in the course of the analysis the absolute values, and not merely the ratios of the coefficients in u_1, u_2, u_3, u_4, u_5, u_6, u_7, have been definitely fixed. Thus when these seven bitangents are given the values of a_1, a_2, a_3, b_1, b_2, b_3, etc. are definite; therefore the equations of the 15 bitangents v_{23}, v_{31}, v_{12}, v_{14}, v_{24}, v_{34}, ... are now determined from the seven given ones in an unique manner, and there is an unique quartic curve expressed by

$$\sqrt{u_1 v_{23}} + \sqrt{u_2 v_{31}} + \sqrt{u_3 v_{12}} = 0,$$

which has the seven given lines as bitangents.

It remains now to determine the remaining six double tangents whose characteristics are denoted by

$$45, \ 46, \ 47, \ 56, \ 57, \ 67.$$

If the characteristics 1, 2, 3, 4, 5, 6, 7 be taken in the order 1, 4, 5, 2, 3, 6, 7 it is clear that as we have determined the double tangents u_{23}, u_{31}, u_{12} in terms of u_1, u_2, u_3, so we can determine the tangents u_{45}, u_{51}, u_{14} in terms of u_1, u_4, u_5. Thus the tangent u_{45} can be found by substitutions in the foregoing work. For the actual deduction the reader is referred* to the original memoir, Riemann, *Ges. Werke* (Leipzig, 1876), p. 471, or Weber, *Theorie der Abel'schen Functionen vom Geschlecht* 3 (Berlin, 1876), pp. 98—100. Putting $\alpha_1 u_1 = x$, $\alpha_2 u_2 = y$, $\alpha_3 u_3 = z$, $v_{23}/\alpha_1 = \xi$, $v_{31}/\alpha_2 = \eta$, $v_{12}/\alpha_3 = \zeta$, $\beta_i/\alpha_i = A_i$, $\gamma_i/\alpha_i = B_i$, $\delta_i/\alpha_i = C_i$ $(i = 1, 2, 3)$, the quartic has the form

$$\sqrt{x\xi} + \sqrt{y\eta} + \sqrt{z\zeta} = 0,$$

and the 28 double tangents are given by the following scheme, where the number representing the characteristic is prefixed to each

(1) $x = 0$, (2) $y = 0$, (3) $z = 0$, (23) $\xi = 0$, (31) $\eta = 0$, (12) $\zeta = 0$,

(4) $x + y + z = 0$, (5) $A_1 x + A_2 y + A_3 z = 0$, (6) $B_1 x + B_2 y + B_3 z = 0$,

(7) $C_1 x + C_2 y + C_3 z = 0$,

(14) $\xi + y + z = 0$, (24) $\eta + z + x = 0$, (34) $\zeta + x + y = 0$,

(15) $\dfrac{\xi}{A_1} + A_2 y + A_3 z = 0$, (25) $\dfrac{\eta}{A_2} + A_3 z + A_1 x = 0$, (35) $\dfrac{\zeta}{A_3} + A_1 x + A_2 y = 0$,

(16) $\dfrac{\xi}{B_1} + B_2 y + B_3 z = 0$, (26) $\dfrac{\eta}{B_2} + B_3 z + B_1 x = 0$, (36) $\dfrac{\zeta}{B_3} + B_1 x + B_2 y = 0$,

(17) $\dfrac{\xi}{C_1} + C_2 y + C_3 z = 0$, (27) $\dfrac{\eta}{C_2} + C_3 z + C_1 x = 0$, (37) $\dfrac{\zeta}{C_3} + C_1 x + C_2 y = 0$,

* For the theory of the plane quartic curve reference may be made to geometrical treatises; developments in connection with the theta functions are given by Schottky, *Crelle*, cv. (1889), Frobenius, *Crelle*, xcix. (1885) and *ibid.* ciii. (1887); see also Cayley, *Crelle*, xciv. and Kohn, *Crelle*, cvii. (1890), where references to the geometrical literature will be found.

$$(67)\quad \frac{x}{1-A_2A_3}+\frac{y}{1-A_3A_1}+\frac{z}{1-A_1A_2}=0,$$

$$(45)\quad \frac{\xi}{A_1(1-A_2A_3)}+\frac{\eta}{A_2(1-A_3A_1)}+\frac{\zeta}{A_3(1-A_1A_2)}=0,$$

$$(75)\quad \frac{x}{1-B_2B_3}+\frac{y}{1-B_3B_1}+\frac{z}{1-B_1B_2}=0,$$

$$(46)\quad \frac{\xi}{B_1(1-B_2B_3)}+\frac{\eta}{B_2(1-B_3B_1)}+\frac{\zeta}{B_3(1-B_1B_2)}=0,$$

$$(56)\quad \frac{x}{1-C_2C_3}+\frac{y}{1-C_3C_1}+\frac{z}{1-C_1C_2}=0,$$

$$(47)\quad \frac{\xi}{C_1(1-C_2C_3)}+\frac{\eta}{C_2(1-C_3C_1)}+\frac{\zeta}{C_3(1-C_1C_2)}=0.$$

Here the six quantities $x, y, z, \xi, \eta, \zeta$ are connected by the equations

$$\begin{aligned}&\xi+\eta+\zeta+x+y+z=0,\\ &\frac{\xi}{A_1}+\frac{\eta}{A_2}+\frac{\zeta}{A_3}+A_1x+A_2y+A_3z=0,\\ &\frac{\xi}{B_1}+\frac{\eta}{B_2}+\frac{\zeta}{B_3}+B_1x+B_2y+B_3z=0,\\ &\frac{\xi}{C_1}+\frac{\eta}{C_2}+\frac{\zeta}{C_3}+C_1x+C_2y+C_3z=0.\end{aligned}\qquad (D)$$

Conversely, if we take arbitrary constants A_1, A_2, A_3, B_1, B_2, B_3, whose number, 6, is, when $p=3$, equal to $3p-3$, namely equal to the number of absolute constants upon which a Riemann surface depends when $p=3$, and, by the first three of the equations (D) determine ξ, η, ζ in terms of the arbitrary lines x, y, z, the last of the equations (D) will determine C_1, C_2, C_3 save for a sign which is the same for all; then it can be directly verified algebraically that the 28 lines here given are double tangents of the quartic curve $\sqrt{x\xi}+\sqrt{y\eta}+\sqrt{z\zeta}=0$.

248. Before leaving this matter we desire to point out further the connection between the two representations of the tangents which have been given. Comparing the two equations of the fundamental quartic curve expressed by the equations (§§ 246, 247)

$$\Omega_0^2=4X_0D,\quad (x\xi+y\eta-z\zeta)^2=4\xi\eta xy,$$

and putting, in accordance therewith,

$$D(x_1, x_2, x_3)=\xi,\quad \Omega_0(x_1, x_2, x_3)=z\zeta-x\xi-y\eta,\quad X_0(x_1, x_2, x_3)=xy\eta$$

and (cf. p. 382) replacing the fourth coordinate T by $T+u$, where

u is an arbitrary linear function of x, y, z or x_1, x_2, x_3, the equation of the cubic surface

$$(T+u)^2 D + (T+u)\,\Omega_0 + X_0 = 0,$$

becomes

$$T^2\xi + T(z\zeta - x\xi - y\eta + 2u\xi) + u^2\xi + u(z\zeta - y\eta - x\xi) + xy\eta = 0,$$

or

$$(T+u)^2\xi + (T+u)(z\zeta - x\xi - y\eta) + xy\eta = 0,$$

which will be found to be the same as

$$(T+u)(T+u-x-z)(T+u-x-\zeta) - (T+u-x)(T+u+y)(T+u+\eta) = 0.$$

Write now

$$v = u - x - z, \quad w = u - x - \zeta, \quad u' = u - x, \quad v' = u + y, \quad w' = u + \eta\,;$$

then we obtain the result, easy to verify, that if u, v, w, u', v', w' be arbitrary linear functions of the homogeneous space coordinates X, Y, Z, and T be the fourth coordinate, the tangent cone to the cubic surface*

$$(T+u)(T+v)(T+w) - (T+u')(T+v')(T+w') = 0 \qquad \text{(i)}$$

from the vertex $X = 0 = Y = Z$ can be written in the form

$$\sqrt{(P-P')(u-u')} + \sqrt{(u-v')(u-w')} + \sqrt{(u'-v)(u'-w)} = 0,$$

where $P - P' = u + v + w - u' - v' - w'$; we have in fact

$$x = u - u', \quad y = v' - u, \quad z = u' - v, \quad \eta = w' - u, \quad \zeta = u' - w,$$

$$\xi, = -(x + y + z + \eta + \zeta), = P - P'.$$

Now the 27 lines on the cubic surface (i) can be easily obtained†; and thence the forms obtained in § 247, for the bitangents of the quartic, can be otherwise established.

249. *Ex.* i. Prove that when the sum of the characteristics of three bitangents of the quartic is an even characteristic, their points of contact do not lie upon a conic.

By enumerating the constants we infer that it is possible to describe a plane quartic curve having seven arbitrary lines as double tangents. By the investigation of § 247 it follows that only one such quartic can be described when the condition is introduced that no three of the tangents shall have their points of contact upon a conic. By the theory here developed it follows that for a given quartic such a set of seven bitangents can be selected in $8 \,.\, 36 = 288$ ways.

Ex. ii. We have given an expression for the general radical form $\sqrt{X^{(3)}}$ of any given odd characteristic. Prove that a radical form $\sqrt{X^{(3)}}$ whose characteristic is even, denoted, suppose, by the index 123, can be written in the form

$$X^{(3)} = \lambda\sqrt{u_1u_2u_3} + \lambda_1\sqrt{u_1u_{12}u_{13}} + \lambda_2\sqrt{u_2u_{23}u_{21}} + \lambda_3\sqrt{u_3u_{31}u_{32}},$$

* Any cubic surface can be brought into this form, Salmon, *Solid Geometry* (1882), § 533.

† See Frost, *Solid Geometry* (1886), § 537. The three last equations (D) of § 247 are deducible from the equations occurring in Frost. The three equations correspond to the three roots of the cubic equation used by Frost.

where $\lambda, \lambda_1, \lambda_2, \lambda_3$ are constants, and u_i, u_{ij} denote double tangents of the characteristics denoted by the suffixes, as in § 247.

Ex. iii. If $(\frac{1}{2}q, \frac{1}{2}q')$, $(\frac{1}{2}r, \frac{1}{2}r')$ denote any two odd characteristics of half-integers, express the quotient

$$\vartheta(v^{x,z}; \tfrac{1}{2}q, \tfrac{1}{2}q')/\vartheta(v^{x,z}; \tfrac{1}{2}r, \tfrac{1}{2}r')$$

algebraically, when $p=3$.

Ex. iv. Obtain an expression of the quotient of any two radical forms $\sqrt{X^{(3)}}$, $\sqrt{Y^{(3)}}$, of assigned characteristics and known zeros, by means of theta functions, p being equal to 3.

250. Noether has given* an expression for the solution of the inversion problem in the general case in terms of radical forms, which is of importance as being capable of great generalization.

Using the places $m_1, \ldots, m_p$, associated as in Chap. X. with an arbitrary place m, and supposing them, each repeated, to be the remaining zeros of a form $X^{(3)}$, which vanishes to the second order in each of the places $A_1, \ldots, A_{2p-3}$ in which an arbitrary ϕ-polynomial, ϕ_0, which vanishes in m, further vanishes, as in § 244, let $\sqrt{Y^{(3)}}$ be any radical form, and $\Phi^{(1)}$ any ϕ-polynomial whose zeros are $a_1, \ldots, a_{2p-2}$. Then (§ 241) the consideration of the rational function $\phi_0^2 Y^{(3)}/[\Phi^{(1)}]^2 X^{(3)}$ leads to the equations

$$[v_i^{x_1, a_1} + v_i^{x_2, a_2} + \ldots\ldots + v_i^{x_{2p-3}, a_{2p-3}} + v_i^{z, a_{2p-2}}] - [v_i^{z, m} - v_i^{c_1, m_1} - \ldots\ldots - v_i^{c_p, m_p}]$$
$$= -\tfrac{1}{2}(\sigma_i + \sigma_1'\tau_{i,1} + \ldots\ldots + \sigma_p'\tau_{i,p}),$$

wherein the places

$$x_1, \ldots, x_{2p-3}, c_1, \ldots, c_p$$

are the zeros of $\sqrt{Y^{(3)}}$, all of $\sigma_1, \ldots, \sigma_p, \sigma_1', \ldots, \sigma_p'$ are integers, and z is an arbitrary place; and, as follows from these equations, the places $x_1, \ldots, x_{2p-3}$ may be arbitrarily assigned, the places $c_1, \ldots, c_p$ and the form $\sqrt{Y^{(3)}}$ being determinate, respectively, from these equations and the equation

$$\log \frac{\phi_0\sqrt{Y^{(3)}}}{\Phi^{(1)}\sqrt{X^{(3)}}} = \text{constant} + \Pi_{x_1, a_1}^{x, a} + \ldots\ldots + \Pi_{m, a_{2p-2}}^{x, a} + \Pi_{c_1, m_1}^{x, a} + \ldots\ldots + \Pi_{c_p, m_p}^{x, a}$$
$$+ \pi i[\sigma_1' v_1^{x, a} + \ldots\ldots + \sigma_p' v_p^{x, a}],$$

wherein the place a is arbitrary. Hence if we speak of

$$(\tfrac{1}{2}\sigma_1, \ldots, \tfrac{1}{2}\sigma_p, \tfrac{1}{2}\sigma_1', \ldots, \tfrac{1}{2}\sigma_p')$$

as the characteristic of $\sqrt{Y^{(3)}}$, it follows, if $\sqrt{Z^{(3)}}$ be another radical form with the characteristic

$$(\tfrac{1}{2}\rho_1, \ldots, \tfrac{1}{2}\rho_p, \tfrac{1}{2}\rho_1', \ldots, \tfrac{1}{2}\rho_p')$$

and the zeros

$$x_1, \ldots, x_{2p-3}, d_1, \ldots, d_p,$$

* *Math. Annal.* XXVIII. (1887), p. 354, "Zum Umkehrproblem in der Theorie der Abel'schen Functionen."

that the quotient $\sqrt{Y^{(3)}}/\sqrt{Z^{(3)}}$, which is equal to

$$Ae^{\Pi^{x,a}_{c_1,d_1}+\dots\dots+\Pi^{x,a}_{c_p,d_p}+\pi i[(\sigma_1'-\rho_1')v_1^{x,a}+\dots\dots+(\sigma_p'-\rho_p')v_p^{x,a}]},$$

wherein A is a quantity independent of x, is (§ 187, Chap. X.) also equal to

$$Ce^{\pi i[(\sigma_1'-\rho_1')v_1^{x,a}+\dots\dots+(\sigma_p'-\rho_p')v_p^{x,a}]}\frac{\Theta(v^{x,m}-v^{c_1,m_1}-\dots\dots-v^{c_p,m_p})}{\Theta(v^{x,m}-v^{d_1,m_1}-\dots\dots-v^{d_p,m_p})},$$

where C is a quantity independent of x; but by the equations here given this is the same as

$$Ce^{\pi i[(\sigma_1'-\rho_1')v_1^{x,a}+\dots\dots+(\sigma_p'-\rho_p')v_p^{x,a}]}\frac{\Theta(v^{x,a_{2p-2}}+v^{x_1,a_1}+\dots\dots+v^{x_{2p-3},a_{2p-3}}+\tfrac{1}{2}\Omega_\sigma)}{\Theta(v^{x,a_{2p-2}}+v^{x_1,a_1}+\dots\dots+v^{x_{2p-3},a_{2p-3}}+\tfrac{1}{2}\Omega_\rho)},$$

where $\tfrac{1}{2}\Omega_\sigma$ denotes p such quantities as $\tfrac{1}{2}(\sigma_i+\sigma_1'\tau_{i,1}+\dots\dots+\sigma_p'\tau_{i,p})$; thus, if we put

$$v=v^{x,a_{2p-2}}+v^{x_1,a_1}+\dots\dots+v^{x_{2p-3},a_{2p-3}}$$

and recall the formula (§ 175)

$$\Theta(v+\tfrac{1}{2}\Omega_\sigma)=e^{-\pi i\sigma'(v+\frac{1}{2}\sigma+\frac{1}{4}\tau\sigma')}\,\Theta(v;\ \tfrac{1}{2}\sigma,\tfrac{1}{2}\sigma'),$$

we infer that

$$\frac{\sqrt{Y^{(3)}}}{\sqrt{Z^{(3)}}}=E\frac{\Theta(v;\ \tfrac{1}{2}\sigma,\tfrac{1}{2}\sigma')}{\Theta(v;\ \tfrac{1}{2}\rho,\tfrac{1}{2}\rho')},$$

where E is a quantity independent of x.

Now in fact (§ 245) the general radical form $\sqrt{Y^{(3)}}$, of assigned characteristic $(\tfrac{1}{2}\sigma,\tfrac{1}{2}\sigma')$, is given by

$$\lambda_1\sqrt{Y_1^{(3)}}+\dots\dots+\lambda_{2p-2}\sqrt{Y_{2p-2}^{(3)}},$$

where $\sqrt{Y_1^{(3)}},\dots,\sqrt{Y_{2p-2}^{(3)}}$ are special forms of this characteristic, and $\lambda_1,\dots,\lambda_{2p-2}$ are constants. If we introduce the condition that $\sqrt{Y^{(3)}}$ vanishes at the places $x_1,\dots,x_{2p-3}$ we infer that $\sqrt{Y^{(3)}}$ is equal to $F\Delta_\sigma^{(3)}(x,x_1,\dots,x_{2p-3})$, where F is independent of x and $\Delta_\sigma^{(3)}(x,x_1,\dots,x_{2p-3})$ denotes the determinant

$$\begin{vmatrix}\sqrt{Y_1^{(3)}(x)}, & \dots\dots\dots\dots, & \sqrt{Y_{2p-2}^{(3)}(x)}\\ \dots & \dots & \dots\\ \sqrt{Y_1^{(3)}(x_i)}, & \dots\dots\dots\dots, & \sqrt{Y_{2p-2}^{(3)}(x_i)}\\ \dots & \dots & \dots\end{vmatrix}$$

in which i is to be taken in turn equal to $1, 2, \dots, 2p-3$. Hence we have

$$\frac{\Delta_\sigma^{(3)}(x,x_1,\dots,x_{2p-3})}{\Delta_\rho^{(3)}(x,x_1,\dots,x_{2p-3})}=G\frac{\Theta(v;\ \tfrac{1}{2}\sigma,\tfrac{1}{2}\sigma')}{\Theta(v;\ \tfrac{1}{2}\rho,\tfrac{1}{2}\rho')},$$

where, from the symmetry in regard to the places $x, x_1, \ldots, x_{2p-3}$, G is independent* of the position of any of these places, and v is given by

$$v = v^{x, a_{2p-2}} + v^{x_1, a_1} + \ldots\ldots + v^{x_{2p-3}, a_{2p-3}}.$$

To apply this equation to the solution of the inversion problem expressed by p such equations as

$$v^{x_1, \mu_1} + \ldots\ldots + v^{x_p, \mu_p} = u,$$

where $\mu_1, \ldots, \mu_p$ denote p arbitrary given places, we suppose the positions of the places $x_{p+1}, \ldots, x_{2p-3}$ to be given; then instead of $\Delta_\sigma(x, x_1, \ldots, x_{2p-3})$ we have an expression of the form

$$A_1 \sqrt{Y_1^{(3)}(x)} + \ldots\ldots + A_{p+1} \sqrt{Y_{p+1}^{(3)}(x)},$$

where $\sqrt{Y_1^{(3)}(x)}, \ldots, \sqrt{Y_{p+1}^{(3)}(x)}$ denote forms $\sqrt{Y^{(3)}(x)}$ vanishing in the given places $x_{p+1}, \ldots, x_{2p-3}$, and $A_1, \ldots, A_{p+1}$ are unknown constants. Since the arguments u are given, the arguments v are of the form $v^{x, a_{2p-2}} + w$, where w is known. If then in the equation

$$\frac{A_1 \sqrt{Y_1^{(3)}(x)} + \ldots\ldots + A_{p+1} \sqrt{Y_{p+1}^{(3)}(x)}}{B_1 \sqrt{Z_1^{(3)}(x)} + \ldots\ldots + B_{p+1} \sqrt{Z_{p+1}^{(3)}(x)}} = \frac{\Theta(v;\ \tfrac{1}{2}\sigma, \tfrac{1}{2}\sigma')}{\Theta(v;\ \tfrac{1}{2}\rho, \tfrac{1}{2}\rho')}$$

we determine the unknown ratios $A_1 : A_2 : \ldots\ldots : A_{p+1} : B_1 : \ldots\ldots : B_{p+1}$ by the substitution of $2p+1$ different positions for the place x, this equation itself will determine the places $x_1, \ldots, x_p$. They are, in fact, the zeros of either of the forms

$$A_1 \sqrt{Y_1^{(3)}(x)} + \ldots\ldots + A_{p+1} \sqrt{Y_{p+1}^{(3)}(x)},$$

$$B_1 \sqrt{Z_1^{(3)}(x)} + \ldots\ldots + B_{p+1} \sqrt{Z_{p+1}^{(3)}(x)}$$

other than the given zeros $x_{p+1}, \ldots, x_{2p-3}$. If the first of these forms be multiplied by an arbitrary form $\sqrt{Y^{(3)}(x)}$, of characteristic $(\tfrac{1}{2}\sigma, \tfrac{1}{2}\sigma')$, the places $x_1, \ldots, x_p$ are given as the zeros of a rational function of the form

$$A_1 \Phi_1^{(3)}(x) + \ldots\ldots + A_{p+1} \Phi_{p+1}^{(3)}(x),$$

of which $4p-6$ zeros are known, consisting, namely, of the places $x_{p+1}, \ldots, x_{2p-3}$ and the zeros of $\sqrt{Y^{(3)}(x)}$.

In regard to this result the reader may consult Weber, *Theorie der Abel'schen Functionen vom Geschlecht* 3 (Berlin, 1876), p. 157, the paper of Noether (*Math. Annal.* XXVIII.) already referred to, and, for a solution in which the radical forms are mth roots of rational functions, Stahl, *Crelle*, LXXXIX. (1880), p. 179, and *Crelle*, CXI. (1893), p. 104. It will be seen in the following chapter that the results may be deduced from another result of a simpler character (§ 274).

251. The theory of radical functions has far-reaching geometrical applications to problems of the contact of curves. See, for instance, Clebsch, *Crelle*, LXIII. (1864), p. 189. For the theory of the solution of the final algebraic equations see Clebsch and Gordan, *Abel'sche Functnen.* (Leipzig, 1866), Chap. X. Die Theilung; Jordan, *Traité des Substitutions* (Paris, 1870), p. 354, etc.; and now (Aug. 1896), for the bitangents in case $p=3$, Weber, *Lehrbuch der Algebra* (Braunschweig, 1896), II. p. 380.

* For the determination of G see Noether, *Math. Annal.* XXVIII. (1887), p. 368, and Klein, *Math. Annal.* XXXVI. (1890), pp. 73, 74.