CHAPTER III.

The Infinities of Rational Uniform Functions.

23. In this chapter and in general we shall use the term rational function to denote a uniform function of position on the surface of which all the infinities are of finite order, their number being finite. We deal first of all with the case in which these infinities are all of the first order.

If k places of the surface, say $a_1, a_2 \dots a_k$, be arbitrarily assigned we can always specify a function with p periods having these places as poles, of the first order, and otherwise continuous and uniform; namely, the function is of the form

$$\mu_0 + \mu_1 \Gamma^x_{a_1} + \dots + \mu_k \Gamma^x_{a_k},$$

where the coefficients $\mu_0, \mu_1 \dots \mu_k$ are constants, the zeros of the functions Γ being left undetermined. Conversely, as remarked in the previous chapter (§ 20), a rational function having $a_1, \dots, a_k$ as its poles must be of this form. In order that the expression may represent a rational function the periods must all be zero. Writing the periods of Γ^x_a in the form $\Omega_1(a), \dots, \Omega_p(a)$, this requires the equations

$$\mu_1 \Omega_i(a_1) + \mu_2 \Omega_i(a_2) + \dots + \mu_k \Omega_i(a_k) = 0,$$

for all the p values, $i = 1, 2, \dots, p$, of i. In what follows we shall for the sake of brevity say that a place c *depends upon* r places $c_1, c_2, \dots, c_r$ when for all values of i, the equations

$$\Omega_i(c) = f_1 \Omega_i(c_1) + \dots + f_r \Omega_i(c_r)$$

hold for finite values of the coefficients $f_1, \dots, f_r$, these coefficients being *independent of i.* Hence we may also say:

In order that a rational function should exist having k *assigned places as its poles, each simple, one at least of these places must depend upon the others.*

24. Taking the k places $a_1, a_2, \dots, a_k$ in the order of their suffixes, it may of course happen that several of them depend upon the others, say $a_{s+1}, \dots, a_k$

upon $a_1, \ldots, a_s$, the latter set $a_1, \ldots, a_s$ being independent: then we have equations of the form

$$\Omega_i(a_{s+1}) = n_{s+1,1}\,\Omega_i(a_1) + \ldots + n_{s+1,s}\,\Omega_i(a_s)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\Omega_i(a_k) \quad = n_{k,1}\,\Omega_i(a_1) \quad + \ldots + n_{k,s}\,\Omega_i(a_s),$$

the coefficients in any of the rows here being the same for all the p values of i. In particular, if s be as great as p and $a_1, \ldots, a_s$ be independent, equations of this form will hold for all positions of $a_{s+1}, \ldots, a_k$. For then we have enough disposeable coefficients to satisfy the necessary p equations.

When it does so happen, that $a_{s+1}, \ldots, a_k$ depend upon $a_1 \ldots a_s$, there exist rational functions, of the form

$$R_{s+1} = \sigma_{s+1} + \lambda_{s+1}\left[\Gamma^x_{a_{s+1}} - n_{s+1,1}\,\Gamma^x_{a_1} - \ldots\ldots - n_{s+1,s}\,\Gamma^x_{a_s}\right],$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$R_k \quad = \sigma_k \quad + \lambda_k \quad \left[\Gamma^x_{a_k} \quad - n_{k,1} \quad \Gamma^x_{a_1} - \ldots\ldots - n_{k,s} \quad \Gamma^x_{a_s}\right],$$

wherein $\sigma_{s+1} \ldots \sigma_k$, $\lambda_{s+1} \ldots \lambda_k$ are constants, which are all infinite once in $a_1 \ldots a_s$ and are, beside, infinite respectively at $a_{s+1}, \ldots, a_k$; and the most general function uniform on the dissected surface, which is infinite to the first order at $a_1, \ldots, a_k$, being, as remarked, of the form

$$\mu_0 + \mu_1\,\Gamma^x_{a_1} + \ldots\ldots + \mu_k\,\Gamma^x_{a_k},$$

can be written in the form

$$\mu_0 + \mu_1\,\Gamma^x_{a_1} + \ldots\ldots + \mu_s\,\Gamma^x_{a_s}$$
$$+\ \mu_{s+1}\left[\frac{1}{\lambda_{s+1}}\,R_{s+1} + n_{s+1,1}\,\Gamma^x_{a_1} + \ldots\ldots + n_{s+1,s}\,\Gamma^x_{a_s} - \frac{\sigma_{s+1}}{\lambda_{s+1}}\right]$$
$$+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+\ \mu_k\left[\frac{1}{\lambda_k} \quad R_k \ + n_{k,1} \quad \Gamma^x_{a_1} + \ldots\ldots + n_{k,s} \quad \Gamma^x_{a_s} - \frac{\sigma_k}{\lambda_k}\right],$$

namely, in the form

$$\nu_0 + \nu_1\,\Gamma^x_{a_1} + \ldots\ldots + \nu_s\,\Gamma^x_{a_s} + \nu_{s+1}R_{s+1} + \ldots\ldots + \nu_k R_k.$$

If this function is to have no periods, the equations

$$\nu_1\Omega_i(a_1) + \ldots\ldots + \nu_s\Omega_i(a_s) = 0, \quad (i = 1, 2, \ldots, p),$$

must hold. Since $a_1, \ldots, a_s$ are independent, such equations can only hold when $\nu_1 = 0 = \ldots = \nu_s$. Thus the *most general* rational function having k poles of the first order, at $a_1, \ldots, a_k$, is of the form

$$\nu_0 + \nu_{s+1}R_{s+1} + \ldots\ldots + \nu_k R_k,$$

and involves $k - s + 1$ linearly entering constants, s being the number of places among $a_1, \ldots, a_k$ which are independent. These constants will generally be called arbitrary: they are so only under the convention that a function which has all its poles *among* $a_1, \ldots, a_k$ be reckoned a particular case of a function having each of these as poles; for it is clear that, for instance, R_k is only infinite at $a_1, \ldots, a_s, a_k$. The proposition with a slightly altered enunciation, given below in § 27 and more particularly dealt with in § 37, is called the Riemann-Roch Theorem, having been first enunciated by Riemann*, and afterwards particularized by Roch†.

25. Take now other places $a_{k+1}, a_{k+2}, \ldots$ upon the surface *in a definite order*, and consider the possibility of forming a rational function, which beside simple infinities at $a_1, \ldots, a_k$ has other simple poles at, say, $a_{k+1}, a_{k+2}, \ldots, a_h$. By the first Article of the present chapter it follows that the least value of h for which this will be possible will be that for which a_h depends on $a_1 \ldots a_k a_{k+1} \ldots a_{h-1}$, that is, depends on $a_1 \ldots a_s a_{k+1} \ldots a_{h-1}$. This will certainly arise at latest when the number of these places $a_1 \ldots a_s a_{k+1} \ldots a_{h-1}$ is as great as p, namely $h - 1 = k + p - s$, and if none of the places $a_{k+1} \ldots a_{h-1}$ depend upon the preceding places $a_1 \ldots a_s$, it will not arise before; in that case there will be no rational function having for poles the places

$$a_1 \ldots\ldots a_k a_{k+1} \ldots\ldots a_{k+j}$$

for any value of j from 1 to $p - s$.

But in order to state the general case, suppose there is a value of j less than or equal to $p - s$, such that each of the places

$$a_{k+j+1} \ldots\ldots a_h$$

depends upon the places

$$a_1 \ldots\ldots a_s\ a_{k+1} \ldots\ldots a_{k+j},$$

the smallest value of j for which this occurs being taken, so that no one of $a_{k+1} \ldots a_{k+j}$ depends on the places which precede it in the series

$$a_1 \ldots\ldots a_s\ a_{k+1} \ldots\ldots a_{k+j}.$$

Then there exists no rational function with its poles at $a_1 \ldots a_k\ a_{k+1} \ldots a_{k+j}$, but there exist functions

$$R_{k+j+1} = \sigma_{k+j+1} + \lambda_{k+j+1} \Big[\Gamma^x_{a_{k+j+1}} - n_{k+j+1,1}\, \Gamma^x_{a_1} - \ldots\ldots$$
$$- n_{k+j+1,s}\, \Gamma^x_{a_s} - n_{k+j+1,k+1}\, \Gamma^x_{a_{k+1}} - \ldots\ldots - n_{k+j+1,k+j}\, \Gamma^x_{a_{k+j}}\Big],$$

....

$$R_{k+j+i} = \sigma_{k+j+i} + \lambda_{k+j+i} \Big[\Gamma^x_{a_{k+j+i}} - n_{k+j+i,1}\, \Gamma^x_{a_1} - \ldots\ldots$$
$$- n_{k+j+i,s}\, \Gamma^x_{a_s} - n_{k+j+i,k+1}\, \Gamma^x_{a_{k+1}} - \ldots\ldots - n_{k+j+i,k+j}\, \Gamma^x_{a_{k+j}}\Big],$$

whose poles are respectively at

$$a_1 \ldots\ldots a_s,\ a_{k+1} \ldots\ldots a_{k+j},\ a_{k+j+i}$$

for all values of i from 1 to $h - k - j$.

* Riemann, *Ges. Werke*, 1876, p. 101 (§ 5) and p. 118 (§ 14) and p. 120 (§ 16).

† Crelle, 64. Cf. also Forsyth, pp. 459, 464. The geometrical significance of the theorem has been much extended by Brill and Noether. (*Math. Ann.* vii.)

Then the most general rational function with poles at

$$a_1 \ldots\ldots a_s a_{s+1} \ldots\ldots a_k a_{k+1} \ldots\ldots a_{k+j} a_{k+j+1} \ldots\ldots a_{k+j+i}$$

is in fact

$$\nu_0 + \nu_{s+1} R_{s+1} + \ldots\ldots + \nu_k R_k + \nu_{k+j+1} R_{k+j+1} + \ldots\ldots + \nu_{k+j+i} R_{k+j+i}$$

and involves $k - s + i + 1$ arbitrary constants, namely the same number as that of the places of the set

$$a_1 \ldots\ldots a_s a_{s+1} \ldots\ldots a_k a_{k+1} \ldots\ldots a_{k+j} a_{k+j+1} \ldots\ldots a_{k+j+i}$$

which depend upon the places that precede them.

For such a function must have the form

$$\mu_0 + \mu_1 \Gamma^x_{a_1} + \ldots\ldots + \mu_s \Gamma^x_{a_s} + \mu_{s+1} \Gamma^x_{a_{s+1}} + \ldots\ldots + \mu_k \Gamma^x_{a_k} + \mu_{k+1} \Gamma^x_{a_{k+1}} + \ldots\ldots$$
$$+ \mu_{k+j} \Gamma^x_{a_{k+j}} + \mu_{k+j+1} \Gamma^x_{a_{k+j+1}} + \ldots\ldots + \mu_{k+j+i} \Gamma^x_{a_{k+j+i}},$$

namely,

$$\mu_0 + \mu_1 \Gamma^x_{a_1} + \ldots\ldots + \mu_s \Gamma^x_{a_s} + \mu_{k+1} \Gamma^x_{a_{k+1}} + \ldots\ldots + \mu_{k+j} \Gamma^x_{a_{k+j}}$$
$$+ \sum_{r=1}^{k-s} \mu_{s+r} \left[\frac{1}{\lambda_{s+r}} R_{s+r} + n_{s+r,1} \Gamma^x_{a_1} + \ldots\ldots + n_{s+r,s} \Gamma^x_{a_s} - \frac{\sigma_{s+r}}{\lambda_{s+r}}\right]$$
$$+ \sum_{t=1}^{t=i} \mu_{k+j+t} \left[\frac{1}{\lambda_{k+j+t}} R_{k+j+t} + n_{k+j+t,1} \Gamma^x_{a_1} + \ldots\ldots \right.$$
$$\left. + n_{k+j+t,s} \Gamma^x_{a_s} + n_{k+j+t,k+1} \Gamma^x_{a_{k+1}} + \ldots\ldots + n_{k+j+t,k+j} \Gamma^x_{a_{k+j}} - \frac{\sigma_{k+j+t}}{\lambda_{k+j+t}}\right],$$

which is of the form

$$\nu_0 + \nu_1 \Gamma^x_{a_1} + \ldots\ldots + \nu_s \Gamma^x_{a_s} + \nu_{s+1} R_{s+1} + \ldots\ldots + \nu_k R_k$$
$$+ \nu_{k+1} \Gamma^x_{a_{k+1}} + \ldots\ldots + \nu_{k+j} \Gamma^x_{a_{k+j}} + \nu_{k+j+1} R_{k+i+1} + \ldots\ldots + \nu_{k+j+i} R_{k+j+i};$$

and the p periods of this, each of the form

$$\nu_1 \Omega(a_1) + \ldots\ldots + \nu_s \Omega(a_s) + \nu_{k+1} \Omega(a_{k+1}) + \ldots\ldots + \nu_{k+j} \Omega(a_{k+j}),$$

cannot be zero unless each of $\nu_1 \ldots \nu_s \nu_{k+1} \ldots \nu_{k+j}$ be zero, for it is part of the hypothesis that none of $a_{k+1} \ldots a_{k+j}$ depend upon preceding places.

26. Proceeding in this way we shall clearly be able to state the following result—

Let there be taken upon the surface, in a definite order, an unlimited number of places $a_1, a_2, \ldots$. Suppose that each of $a_1 \ldots a_{Q_1-q_1}$ is independent of those preceding it, but each of $a_{Q_1-q_1+1} \ldots a_{Q_1}$ depends on $a_1 \ldots a_{Q_1-q_1}$. Suppose that each of $a_{Q_1+1} a_{Q_1+2} \ldots a_{Q_2-q_2}$ is independent of those that precede it in the series $a_1 \ldots a_{Q_1-q_1} a_{Q_1+1} \ldots a_{Q_2-q_2}$, but each of $a_{Q_2-q_2+1} \ldots a_{Q_2}$ depends upon $a_1 \ldots a_{Q_1-q_1} a_{Q_1+1} \ldots a_{Q_2-q_2}$. This requires that

$$Q_1 - q_1 + [Q_2 - q_2 - Q_1] \not> p.$$

Suppose that each of $a_{Q_2+1} \dots a_{Q_3-q_3}$ is independent of those that precede it in the series $a_1 \dots a_{Q_1-q_1} a_{Q_1+1} \dots a_{Q_2-q_2} a_{Q_2+1} \dots a_{Q_3-q_3}$, but each of $a_{Q_3-q_3+1} \dots a_{Q_3}$ depends upon the places of this series. This requires that

$$Q_1 - q_1 + [Q_2 - q_2 - Q_1] + [Q_3 - q_3 - Q_2] \ngtr p.$$

Let this enumeration be continued. We shall eventually come to places $a_{Q_{h-1}+1}, a_{Q_{h-1}+2}, \dots a_{Q_h-q_h}$, each independent of the places preceding, for which the total number of independent places included, that is, of places which do not depend upon those of our series which precede them, is p—so that the equation

$$\begin{aligned} p &= (Q_h - q_h - Q_{h-1}) + \dots\dots + (Q_2 - q_2 - Q_1) + (Q_1 - q_1) \\ &= Q_h - q_1 - q_2 - \dots\dots - q_h \end{aligned}$$

will hold. Then every additional place of our series, those, namely, chosen in order from $a_{Q_h-q_h+1}, a_{Q_h-q_h+2}, \dots$ will depend on the preceding places of the whole series.

This being the case, it follows, using R_f as a notation for a rational function having its poles among $a_1 \dots a_f$, that *rational functions*

$$R_1 \dots R_{Q_1-q_1};\ R_{Q_1+1} \dots R_{Q_2-q_2};\ R_{Q_2+1} \dots R_{Q_3-q_3};\ \dots\dots;\ R_{Q_{h-1}+1} \dots R_{Q_h-q_h}$$

do not exist.

The number of these non-existent functions is p.

For all other values of f, a rational function R_f *exists.*

To exhibit the general form of these existing rational functions in the present notation, let m be one of the numbers $1, 2, \dots, h$; i be one of the numbers $1, 2, \dots q_m$, and let the dependence of $a_{Q_m-q_m+i}$ upon the preceding places arise by p equations of the form

$$\begin{aligned} \Omega(a_{Q_m-q_m+i}) = [\rho_1 \Omega(a_1) + \dots + \rho_{Q_1-q_1} \Omega(a_{Q_1-q_1})] + \dots \\ + [\rho_{Q_{m-1}+1} \Omega(a_{Q_{m-1}+1}) + \dots + \rho_{Q_m-q_m} \Omega(a_{Q_m-q_m})]; \end{aligned}$$

then, denoting $\Gamma^x_{a_r}$ by Γ_r, there is a rational function

$$\begin{aligned} R_{Q_m-q_m+i} = A + B\{\Gamma_{Q_m-q_m+i} - [\rho_1\Gamma_1 + \dots + \rho_{Q_1-q_1}\Gamma_{Q_1-q_1}] - \dots \\ - [\rho_{Q_{m-1}+1}\Gamma_{Q_{m-1}+1} + \dots + \rho_{Q_m-q_m}\Gamma_{Q_m-q_m}]\}, \end{aligned}$$

which has its poles at

$$a_1 \dots a_{Q_1-q_1}, a_{Q_1+1} \dots a_{Q_2-q_2}, \dots, a_{Q_{m-1}+1} \dots a_{Q_m-q_m}, a_{Q_m-q_m+i},$$

and the general rational function having its poles at

$$a_1 \dots a_{Q_1} a_{Q_1+1} \dots a_{Q_2} a_{Q_2+1} \dots a_{Q_m-q_m+i}$$

is of the form

$$\nu_0 + \left[\nu_{Q-q_1+1} R_{Q_1-q_1+1} + \dots + \nu_{Q_1} R_{Q_1}\right] + \left[\nu_{Q_2-q_2+1} R_{Q_2-q_2+1} + \dots + \nu_{Q_2} R_{Q_2}\right]$$
$$+ \dots + \left[\nu_{Q_m-q_m+1} R_{Q_m-q_m+1} + \dots + \nu_{Q_m-q_m+i} R_{Q_m-q_m+i}\right],$$

and involves $q_1 + q_2 + \dots + q_{m-1} + i + 1$ arbitrary coefficients.

The result may be summarised by putting down the line of symbols

$$1, 2, \dots (Q_1 - q_1), \overline{(Q_1 - q_1 + 1), \dots, Q_1}, Q_1 + 1, \dots (Q_2 - q_2),$$
$$\overline{(Q_2 - q_2 + 1), \dots, Q_2}, Q_2 + 1, \dots, Q_{h-1} + 1, \dots, (Q_h - q_h), \overline{(Q_h - q_h + 1), \dots}$$

with a bar drawn above the indices corresponding to the places which depend upon those preceding them in the series. The bar beginning over $Q_h - q_h + 1$ is then continuous to any length. The total number of indices over which no bar is drawn is p. There exists a rational function R_f, in the notation above, for every index which is beneath a bar.

The proposition here obtained is of a very fundamental character. Suppose that for our initial algebraic equation or our initial surface, we were able only to shew, algebraically or otherwise, that for an arbitrary place a there exists a function K_a^x, discontinuous at a only and there infinite to the first order, this function being one valued save for additive multiples of k periods, and these periods finite and uniquely dependent upon a, then, taking arbitrary places $a_1, a_2, \dots$ upon the surface, in a definite order, and considering functions of the form

$$\lambda_1 K_{a_1}^x + \dots\dots + \lambda_N K_{a_N}^x,$$

that is, functions having simple poles at $a_1, \dots, a_N$, we could prove, just as above, that there are k values of N for which such functions cannot be one valued; and obtain the number of arbitrary coefficients in uniform functions of given poles. Namely, the proposition would furnish a definition of the characteristic number k—which is the deficiency, here denoted by p—based upon the properties of *the uniform rational functions.*

We shall sometimes refer to the proposition as *Weierstrass's gap theorem**.

27. When a place a is, in the sense here described, dependent upon places $b_1, b_2, \dots, b_r$, it is clear that of the equations

* "Lückensatz." The proposition has been used by Weierstrass, I believe primarily under the form considered below, in which the places $a_1, a_2, \dots$ are consecutive at one place of the surface, as the definition of p. Weierstrass's theory of algebraic functions, preliminary to a theory of Abelian functions, is not considered in the present volume. His lectures are in course of publication. The theorem here referred to is published by Schottky: Conforme Abbildung mehrfach zusammenhängender ebener Flächen, Crelle Bd. 83. A proof, with full reference to Schottky, is given by Noether, Crelle Bd. 97, p. 224.

$$A_1\Omega_1(b_1)+\ldots+A_p\Omega_p(b_1)=0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_1\Omega_1(b_r)+\ldots+A_p\Omega_p(b_r)=0$$
$$A_1\Omega_1(a)\;+\ldots+A_p\Omega_p(a)\;=0$$

the last is a consequence of those preceding—and conversely that when the last equation is a consequence of the preceding equations the place a depends upon the places $b_1, b_2, \ldots, b_r$.

Hence the conditions that the linear aggregate

$$\Omega(x)=A_1\Omega_1(x)+\ldots+A_p\Omega_p(x)$$

should vanish at the places

$$a_1\ldots a_{Q_1}a_{Q_1+1}\ldots a_{Q_2}\,a_{Q_2+1}\ldots a_{Q_m-q_m+i},$$

wherein $i \not> q_m$, are equivalent to only

$$(Q_1-q_1)+(Q_2-q_2-Q_1)+\ldots+(Q_m-q_m-Q_{m-1})$$

or

$$Q_m-q_1-\ldots-q_m$$

linearly independent equations.

If then $\tau+1$ be the number of linearly independent linear aggregates of the form $\Omega(x)$, which vanish in the Q_m-q_m+i specified places, we have

$$\tau+1=p-(Q_m-q_1-\ldots-q_m).$$

Denoting Q_m-q_m+i by Q, and the number of constants in the general rational function with poles at the Q specified places, of which constants one is merely additive, by $q+1$,

$$q+1=q_1+q_2+\ldots+q_{m-1}+i+1.$$

We therefore have

$$Q-q=p-(\tau+1).$$

Recalling the values of $\Omega_1(x)\ldots\Omega_p(x)$ and the fact (Chapter II. § 21) that every linear aggregate of them vanishes in just $2p-2$ places, we see that when Q is greater than $2p-2$, $\tau+1$ is necessarily zero.

In the case under consideration in the preceding article the number $\tau+1$ for the function $R_{Q_{h-1}}$, namely the number of linearly independent linear aggregates $\Omega(x)$ which vanish in the places

$$a_1a_2\ldots a_{Q_1}a_{Q_1+1}\ldots a_{Q_{h-1}},$$

is given, by taking $m=h-1$ and $i=q_{h-1}$ in the formula of the present article, by the equation

$$\begin{aligned}\tau+1&=p-(Q_{h-1}-q_1-\ldots-q_{h-1})\\&=Q_h-q_h-Q_{h-1}.\end{aligned}$$

Hence one such linear aggregate vanishes in the places

$$a_1 a_2 \dots a_{Q_{h-1}} a_{Q_{h-1}+1} \dots a_{Q_h - q_h - 1}$$

and therefore

$$Q_h - q_h - 1 \not> 2p - 2$$

or, *the index associated with the last place* $a_{Q_h - q_h}$ *of our series, corresponding to which a rational function* $R_{Q_h - q_h}$ *does not exist, is not greater than* $2p - 1$. A case in which this limit is reached, which also furnishes an example of the theory, is given below § 37, Ex. 2.

28. A limiting case of the problem just discussed is that in which the series of points a_1, a_2, ... are all consecutive at one place of the surface.

A rational function which becomes infinite only at a place, a, of the surface, and there like

$$\frac{C_1}{t} + \frac{C_2}{t^2} + \dots + \frac{C_r}{t^r},$$

where any of the constants C_1, C_2, ... C_{r-1}, but *not* C_r, may be zero, t being the infinitesimal, is said to be there infinite to the rth order. If $-\lambda_i = C_i/(i-1)!$, such a function can be expressed in a form

$$\lambda + \lambda_1 \Gamma_a^x + \lambda_2 D_a \Gamma_a^x + \dots + \lambda_r D_a^{r-1} \Gamma_a^x$$

where, in order that the function be one valued on the undissected surface, the p equations

$$\lambda_1 \Omega_i(a) + \lambda_2 D_a \Omega_i(a) + \dots + \lambda_r D_a^{r-1} \Omega_i(a) = 0$$

must be satisfied: and conversely these equations give sufficient conditions for the coefficients $\lambda_1, \lambda_2, \dots, \lambda_r$.

In other words, since λ_r cannot be zero because the function is infinite to the rth order, the p differential coefficients $D_a^{r-1}\Omega_i(a)$, each of the $\overline{r-1}$th order, must be expressible linearly in terms of those of lower order,

$$\Omega_i(a),\ D\Omega_i(a), \dots, D^{r-2}\Omega_i(a),$$

with coefficients which are independent of i. We imagine the p quantities $D_a^{r-1}\Omega_i(a)$, for $i = 1, 2, \dots, p$, written in a column, which we call the rth column; and for the moment we say that the necessary and sufficient condition for the existence of a rational function, infinite of the rth order at a, and not elsewhere infinite, is that the rth column be a linear function of the preceding columns.

Then as before, considering the columns in succession, they will divide themselves into two categories, those which are linear functions of the preceding ones and those which are not so expressible. And, since the number of elements in a column is p, the number of these latter independent columns will be just p. Let them be in succession the k_1th, k_2th, ..., k_pth. Then there exists no rational function infinite only at a, and there to these orders $k_1, k_2, \ldots, k_p$, though there are integrals of the second kind infinite to these orders. But if Q be a number different from $k_1, \ldots, k_p$, there does exist such a rational function of the Qth order, its most general expression being of the form

$$\lambda_Q D_a^{Q-1}\Gamma_a^x + \lambda_{Q-1}D_a^{Q-2}\Gamma_a^x + \ldots + \lambda_1\Gamma_a^x + \lambda,$$

namely, the integral of the second kind whose infinity is of order Q is expressible linearly by integrals of the second kind of lower order of infinity, with the addition of a rational function.

If $q+1$ be the number of linearly independent coefficients in this function, one being additive, we have an equation

$$Q - q = p - (\tau + 1),$$

where $p - (\tau + 1)$ is the number of the linearly independent equations of the form

$$\lambda_1\Omega_i(a) + \lambda_2 D\Omega_i(a) + \ldots + \lambda_Q D^{Q-1}\Omega_i(a) = 0, \quad (i = 1, 2, \ldots, p),$$

from which the others may be linearly derived. As before, $\tau + 1$ is the number of linearly independent linear aggregates of the form

$$A_1\Omega_1(x) + \ldots + A_p\Omega_p(x)$$

which satisfy the Q conditions

$$A_1D^r\Omega_1(a) + \ldots + A_pD^r\Omega_p(a) = 0$$

for $r = 0, 1, 2, \ldots, Q-1$.

29. In regard to the numbers $k_1 \ldots k_p$ we remark firstly that, unless $p = 0$, $k_1 = 1$—for if there existed a rational function with only one infinity of the first order, the positive integral powers of this function would furnish rational functions of all other orders with their infinity at this one place, and there would be no gaps (compare the argument Chapter II. § 21); and further that in general they are the numbers 1, 2, 3 ... p, that is to say, there is only a finite number of places on the surface for which a rational function can be formed infinite there to an order less than $p+1$ and not otherwise infinite. We shall prove this immediately by finding an upper and a lower limit to the number of such places (§ 31).

30. Some detailed algebraic consequences of this theory will be given in Chapter V. It may be* here remarked, what will be proved in Chapter VI. in considering the geometrical theory, that the zeros of the linear aggregate

$$A_1\Omega_1(x) + \ldots + A_p\Omega_p(x)$$

* It is possible that the reader may find it more convenient to postpone the complete discussion of § 30 until after reading Chapter vi.

can be interpreted in general as the intersections of a certain *curve*, of the form

$$\phi = A_1\phi_1(x) + \dots + A_p\phi_p(x) = 0,$$

wherein $\phi_1 \dots \phi_p$ are integral polynomials in x and y, with the curve represented by the fundamental equation of our Riemann surface. In such interpretation, the condition for the existence of a rational function of order Q with poles only at the place a, is that the fundamental curve be of such character at this place that every curve ϕ, obtained by giving different values to $A_1 \dots A_p$, which there cuts it in $Q-1$ consecutive points, necessarily cuts it in Q consecutive points. As an instance of such property, which seems likely also to make the general theory clearer, we may consider a Riemann surface associated with an equation of the form

$$f(x, y) = K + (x, y)_1 + (x, y)_2 + (x, y)_3 + (x, y)_4 = 0,$$

wherein $(x, y)_r$ is a homogeneous integral polynomial of the rth degree, with quite general coefficients, and K is a constant. Interpreted as a curve, this equation represents a general curve of the fourth degree; it will appear subsequently that the general integral of the first kind is

$$\int \frac{dx}{f'(y)}(A + Bx + Cy),$$

where $f'(y) = \partial f/\partial y$, and A, B, C are arbitrary constants; and thence, if we recall the fact that $\Omega_1(x), \dots, \Omega_p(x)$ are differential coefficients of integrals of the first kind, that the zeros of the aggregate

$$A_1\Omega_1(x) + \dots + A_p\Omega_p(x)$$

may be interpreted as the intersections of the quartic with a variable straight line.

Take now a point of inflexion of the quartic as the place a. Not every straight line there intersecting the curve in one point will intersect it in any other consecutive point; *but* every straight line there intersecting the curve in two consecutive points will necessarily intersect it there in three consecutive points. Hence it is possible to form a rational function of the third order whose only infinities are at the place of inflexion; in fact, if

$$A_0x + B_0y + 1 = 0$$

be the equation of the inflexional tangent, and

$$\lambda(A_0x + B_0y + 1) + \mu(Ax + By + 1) = 0$$

be the equation of any line through the fourth point of intersection of the inflexional tangent with the curve, the ratio of the expressions on the left hand side of these equations, namely

$$\lambda + \mu\frac{Ax + By + 1}{A_0x + B_0y + 1},$$

is a general rational function of the desired kind, as is immediately obvious on consideration of the places where it can possibly be infinite. Thus for the inflexional place the orders of two non-existent rational functions are 1, 2. It can be proved that in general there is no function of the fourth order—the gaps at the orders 1, 2, 4 are those indicated by Weierstrass' theorem.

In verification of a result previously enunciated we notice that since $Ax+By+1=0$ may be taken to be *any definite* line through the fourth intersection of the inflexional tangent with the curve, the function contains $q+1=2$ arbitrary constants. From the form of the integrals of the first kind which we have quoted, it follows that $p=3$; thus the formula

$$Q-q=p-(\tau+1),$$

wherein $Q=3$, requires $\tau+1=1$; now by § 28 $\tau+1$ should be the number of straight lines which can be drawn to have contact of the second order with the curve at the point: this is the case.

If the quartic possess also a point of osculation, a straight line passing through two consecutive points of the curve there will necessarily pass through three consecutive points and also necessarily through four. Hence, for such a place, we can form a rational function of the third order *and* one of the fourth. In fact, if $A_0x+B_0y+1=0$ be the tangent at the point of osculation and $A_1x+B_1y+1=0$ be any other line through this point, while $\lambda x+\mu y+\nu=0$ is any other line whatever, these functions are respectively, in their most general forms,

$$\lambda+\mu\frac{A_1x+B_1y+1}{A_0x+B_0y+1},\quad \frac{\lambda x+\mu y+\nu}{A_0x+B_0y+1},$$

wherein λ, μ, ν are arbitrary constants.

It can be shewn that in general we cannot form a rational function of the fifth order whose only infinity is at the place of osculation. Thus the gaps indicated by Weierstrass's theorem occur at the orders 1, 2, 5. (Cf. the concluding remark of § 34.)

In case, however, the place a be an ordinary point of the quartic, the lowest order of function, whose only infinity is there, is $p+1=4$: it will subsequently become clear that a general form of such a function in S'/S, where $S=0$ is *any* conic drawn to intersect the quartic in four consecutive points at a, and $S'=0$ is the most general conic drawn through the other four intersections of S with the quartic. S' will in fact be of the form $\lambda S+\mu T$, where T is any definite conic satisfying the conditions for S', and λ, μ are arbitrary constants; the equation $Q-q=p-(\tau+1)$ is clearly satisfied by $Q=4$, $q=1$, $p=3$, $\tau+1=0$.

The present article is intended only by way of illustration; the examples given appear to find their proper place here. The reader will possibly find it desirable to read them in connexion with the geometrical account given in Chapter VI.

31. Consider now what places of the surfaces are such that we can form a rational function infinite, only there, to an order as low as p.

For such a place, as follows from § 28, the determinant

$$\Delta = \begin{vmatrix} \Omega_1(x), & \Omega_2(x), & \ldots\ldots, & \Omega_p(x) \\ D\Omega_1(x), & D\Omega_2(x), & \ldots\ldots, & D\Omega_p(x) \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ D^{p-1}\Omega_1(x), & D^{p-1}\Omega_2(x), & \ldots\ldots, & D^{p-1}\Omega_p(x) \end{vmatrix}$$

must vanish. Assume for the present that none of the minors of Δ vanish at that place. It is clear by § 28 that Δ only vanishes at such places as we are considering.

Let v be any integral of the first kind. We can write

$$\Omega_i(x) = \frac{dv_i}{dt} \text{ in the form } \frac{dv}{dt}\frac{dv_i}{dv},$$

and similarly put

$$D\Omega_i(x) = \frac{d^2v}{dt^2}\frac{dv_i}{dv} + \left(\frac{dv}{dt}\right)^2 \frac{d^2v_i}{dv^2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$D^{p-1}\Omega_i(x) = \ldots\ldots\ldots + \left(\frac{dv}{dt}\right)^p \frac{d^pv_i}{dv^p},$$

and so write

$$\Delta = \left(\frac{dv}{dt}\right)^{\frac{1}{2}p(p+1)} D,$$

where D is the determinant whose rth row is formed with the quantities

$$\frac{d^rv_1}{dv^r}, \ \ldots\ldots, \ \frac{d^rv_p}{dv^r}.$$

Now $\dfrac{dv_i}{dv}$ is a rational function; and it is infinite only at the zeros of dv, whose aggregate number is $2p-2$; and $\dfrac{d^2v_i}{dv^2}$ is a rational function of the $(4p-4)$th order, its poles being also at the zeros of dv; and a similar statement can be made in regard to the other rows of D.

Hence D is a rational function whose infinities are of aggregate number

$$(2p-2)(1+2+\ldots+p) = (p-1)p(p+1),$$

and this is therefore the number of zeros of D.

Now Δ can vanish either by the vanishing of the factor D or by the vanishing of the factor $\left(\frac{dv}{dt}\right)^{\frac{1}{2}p(p+1)}$. The zeros of the last factor are, however, the poles of D. Hence *the aggregate number of zeros of* Δ *is* $(p-1)\,p\,(p+1)$. We shall see immediately that these zeros do not necessarily occur at as many as $(p-1)\,p\,(p+1)$ distinct places of the surface.

In order that a rational function should exist of order less than p, its infinity being entirely at one place, say of order $p-r$, it would be necessary that the r determinants formed from the matrix obtained by omitting the last r rows of Δ should all vanish at that place. We can, as in the case of Δ, shew that each of these minors will vanish only at a finite number of places. It is therefore to be expected that in general these minors will not have common zeros; that is, that the surface will need to be one whose $3p-3$ moduli are connected in some special way.

Moreover it is not in general true that a rational function of order $p+1$ exists for a place for which a function of order p exists, these functions not being elsewhere infinite. For then we could simultaneously satisfy the two sets of p equations

$$\lambda_1\Omega_i(a) + \lambda_2 D\Omega_i(a) + \ldots\ldots + \lambda_{p-1}D^{p-2}\Omega_i(a) + \lambda_p D^{p-1}\Omega_i(a) = 0,$$
$$\mu_1\Omega_i(a) + \mu_2 D\Omega_i(a) + \ldots\ldots + \mu_{p-1}D^{p-2}\Omega_i(a) + \mu_{p+1}D^{p}\Omega_i(a) = 0,$$

namely, Δ and $\frac{d\Delta}{dt}$ would both be zero at such a place. The condition that this be so would require that a certain function of the moduli of the surface—what we may call an absolute invariant—should be zero.

Therefore when of the p gaps required by Weierstrass's theorem, $p-1$ occur for the orders $1, 2, \ldots, p-1$, the other will in general occur for the order $p+1$. The reader will see that there is no such reason why, when a function of order p exists, a function of order $p+2$ or higher order should not exist.

32. The reader who has followed the example of § 30 will recall that the number of inflexions of a non-singular plane quartic* is 24 which is equal to the value of $(p-1)\,p\,(p+1)$ when $p=3$. The condition that the quartic possess a point of osculation is that a certain invariant should vanish†.

When the curve has a double point, there are only two integrals of the first kind‡, and p is equal to two. Thus in accordance with the theory above, there should be $(p-1)\,p\,(p+1)=6$ places for which we can form functions

* Salmon, *Higher Plane Curves* (1879), p. 213.

† The equation can be written so as to involve only $5=3p-3-1$ parametric constants (Chap. V. p. 98, Exs. 1, 2).

‡ Their forms are given Chapter II. § 17 β. Reasons are given in Chapter VI. The reader may compare Forsyth, p. 395.

of the second order infinite only at one of these places. In fact six tangents can be drawn to the curve from the double point: if $A_0x + B_0y = 0$ be the equation of one of these and $\lambda(Ax + By) + \mu(A_0x + B_0y) = 0$ be the equation of any line through the double point, the ratio

$$\xi = \lambda \frac{Ax + By}{A_0x + B_0y} + \mu$$

represents a function of second order infinite only at the point of contact of $A_0x + B_0y = 0$*.

For the point of contact of one of these tangents the p gaps occur for the orders 1 and 3.

The quartic with a double point can be birationally related to a surface expressed by an equation of the form

$$\eta^2 = (\xi, 1)_6,$$

ξ being the function above. The reader should compare the theory in Chapter I. and the section on the hyperelliptic case, Chapter V. below.

33. *Ex.* For the surface represented by the equation

$$f(x, y) = x^2y^2\{x, y\}_1 + xy\{x, y\}_2 + (x, y)_3 + (x, y)_2 + (x, y)_1 = 0$$

where the brackets indicate general integral polynomials of the order of the suffixes, p is equal to 4, and the general integral of the first kind is

$$\int dx\,(Axy + Bx + Cy + D)/f'(y)$$

where $f'(y) = \frac{\partial f}{\partial y}$. Prove that at the $(p-1)\,p\,(p+1) = 60$ places for which rational functions of the 4th order exist, infinite only at these places, the following equations are satisfied

$$2y'''/y' - 3\,(y''/y')^2 = 0,$$

$$2f_xf_y\left[\frac{\partial^3 f}{\partial x^3}f_y^3 - 3\frac{\partial^3 f}{\partial x^2\partial y}f_y^2f_x + 3\frac{\partial^3 f}{\partial x\partial y^2}f_yf_x^2 - \frac{\partial^3 f}{\partial y^3}f_x^3\right]$$
$$-3\left[\frac{\partial^2 f}{\partial x^2}f_y^2 - \frac{\partial^2 f}{\partial y^2}f_x^2\right]\left[\frac{\partial^2 f}{\partial x^2}f_y^2 - 2\frac{\partial^2 f}{\partial x\partial y}f_yf_x + \frac{\partial^2 f}{\partial x^2}f_x^2\right] = 0,$$

where $y' = \frac{dy}{dx}$, etc., $f_x = \frac{\partial f}{\partial x}$, etc.

Explain how to express these functions of the fourth order.

Enumerate all the zeros of the second differential expression here given.

Ex. 2. In general, the corresponding places are obtained by forming the differential equation of the pth order of all adjoint ϕ curves. In a certain sense Δ is a differential invariant, for all reversible rational transformations. (See Chapter VI.)

* Here the number of integrands of the integrals of the first kind, which are of the form $(Lx + My)/f'(y)$ (cf. Chapter III. § 28), which vanish in two consecutive points at the point of contact of $A_0x + B_0y = 0$, is clearly 1, or $\tau + 1 = 1$: hence the formula $Q - q = p - (\tau + 1)$ is verified by $Q = 2$, $q = 1$, $p = 2$, so that the form of function of the second order given in the text is the most general possible.

34. We pass now to consider whether the $(p-1)\,p\,(p+1)$ zeros of Δ will in general fall at separate places*.

Consider the determinant

$$\nabla = \begin{vmatrix} 0\ , \Omega_1(x) & , \dots\dots, \Omega_p(x) \\ \lambda_1, \Omega_1^{(k_1-1)}(\xi), & \dots\dots, \Omega_p^{(k_1-1)}(\xi) \\ \dots\dots\dots & \dots\dots\dots \\ \dots\dots\dots & \dots\dots\dots \\ \lambda_p, \Omega_1^{(k_p-1)}(\xi), & \dots\dots, \Omega_p^{(k_p-1)}(\xi) \end{vmatrix},$$

wherein $\Omega_i^{(\mu)}(\xi) = D_\xi^\mu \Omega_i(\xi)$, and $k_1, \dots, k_p$ are the orders of non-existent rational functions for a place ξ, in ascending order of magnitude, $(k_1 = 1)$; and let its value be denoted by

$$\lambda_1 \omega_1(x) + \dots + \lambda_p \omega_p(x),$$

so that $u_r = \int \omega_r(x)\, dt_x$ is an integral of the first kind.

Then $\omega_r(x)$ *vanishes at* ξ *to the* $(k_r - 1)$*th order.*

For $\omega_r(x)$ is the determinant

$$\nabla_r = (-)^r \begin{vmatrix} \Omega_1(x) & , \dots\dots, \Omega_p(x) \\ \Omega_1^{(k_1-1)}(\xi) & , \dots\dots, \Omega_p^{(k_1-1)}(\xi) \\ \dots\dots\dots & \dots\dots\dots \\ \Omega_1^{(k_{r-1}-1)}(\xi), & \dots\dots, \Omega_p^{(k_{r-1}-1)}(\xi) \\ \Omega_1^{(k_{r+1}-1)}(\xi), & \dots\dots, \Omega_p^{(k_{r+1}-1)}(\xi) \\ \dots\dots\dots & \dots\dots\dots \\ \Omega_1^{(k_p-1)}(\xi) & , \dots\dots, \Omega_p^{(k_p-1)}(\xi) \end{vmatrix};$$

now the $(k_r - 1)$th differential coefficient of this determinant (in regard to the infinitesimal at x) has at ξ a value which is in fact the minor of the element (1, 1) of ∇, save for sign. That this minor does not vanish is part of the definition of the numbers $k_1, k_2, \dots, k_p$. But all differential coefficients of ∇_r of lower than the $(k_r - 1)$th order do vanish at ξ: some, because for $x = \xi$ they are determinants having the first row identical with one of the following rows, this being the case for the differential coefficients of orders $k_1 - 1, k_2 - 1, \dots$; others, because when μ is not one of the numbers $k_1, k_2, \dots, k_p$, $D^{\mu-1}\Omega_i(\xi)$ is a linear function of those of $D^{k_1-1}\Omega_i(\xi)$, $D^{k_2-1}\Omega_i(\xi), \dots$ for which μ is greater than $k_1, k_2, \dots$, the coefficients of the linear functions being independent of i. This proves the proposition.

It is clear that the k_rth differential coefficient of ∇_r may also vanish at ξ. In particular $\omega_1(x)$ does not vanish at ξ: a result in accordance with a remark previously made (Chapter II. § 21), that there is no place at which the differentials of all the integrals of the first kind can vanish.

* The results in §§ 34, 35, 36 are given by Hurwitz, *Math. Annal.* 41, p. 409. They will be useful subsequently.

An important corollary is that *the highest order for which no rational function exists, infinite only at the place ξ, is less than $2p$.* For $\omega_p(x)$ vanishes only $2p-2$ times, namely, $k_p - 1 \overline{\overline{<}}\, 2p-2$.

35. We can now prove that *if $k_2 > 2$, the sum of the orders $k_1, k_2, \ldots, k_p$ is less than p^2.* For if there be a rational function of order m, infinite only at ξ, and r be one of the non-existent orders* $k_1 \ldots k_p$, $r-m$ is also one of these non-existent orders—otherwise the product of the existent rational function of order $r-m$ with the function of order m would be an existent function of order r. The powers of the function of order m are existent functions, hence none of $k_1 \ldots k_p$ are divisible by m.

Let r_i be the greatest of the non-existent orders $k_1 \ldots k_p$ which is congruent to $i\,(< m)$ for the modulus m: then, by the remark just made,

$$r_i,\ r_i - m,\ r_i - 2m,\ \ldots,\ m+i,\ i$$

are all non-existent orders—and all congruent to i for the modulus m. Since r_i occurs among $k_1 \ldots k_p$, all these also occur. Take i in turn equal to $1, 2, \ldots m-1$.

Then, the number of non-existent orders being p,

$$p = \left(1 + \frac{r_1 - 1}{m}\right) + \left(1 + \frac{r_2 - 2}{m}\right) + \ldots + \left(1 + \frac{r_{m-1} - (m-1)}{m}\right),$$

so that

$$\begin{aligned} r_1 + r_2 + \ldots + r_{m-1} &= mp - \tfrac{1}{2} m(m-1) \\ &= \tfrac{1}{2} m(2p - m + 1). \end{aligned}$$

Now the sum of the non-existent orders is

$$\sum_{i=1}^{m-1} \left[r_i + (r_i - m) + (r_i - 2m) + \ldots + i\right],$$

which is equal to

$$\frac{1}{2m} \sum_{i=1}^{m-1} (r_i + m - i)(r_i + i)$$

$$\begin{aligned} = \frac{1}{2m} \sum_i r_i\left[r_i - (2p-1)\right] &+ \frac{1}{2m} \sum_i r_i\left[2p + m - 1\right] \\ &+ \tfrac{1}{4} m(m-1) - \tfrac{1}{12}(m-1)(2m-1), \end{aligned}$$

and, since $\Sigma r_i = \frac{1}{2} m(2p - m + 1)$, this is equal to

$$\frac{1}{2m} \sum_i r_i\left[r_i - (2p-1)\right] + \tfrac{1}{4}\left[4p^2 - (m-1)^2\right] + \tfrac{1}{12}(m-1)(m+1),$$

or

$$p^2 - \frac{1}{2m} \sum_i r_i(2p - 1 - r_i) - \tfrac{1}{6}(m-1)(m-2).$$

* i.e. orders of rational functions, infinite only at ξ, which do not exist: and similarly in what follows.

Since, by the corollary of the preceding article, $2p-1$ is not less than r_i, this is less than p^2 unless m is 1 or 2. Now m cannot be equal to 1; and if it is 2 then also $k_2 > 2$. Hence the statement made at the beginning of the present Article is justified.

When there *is* a rational function of order 2, it is easy to prove that there are places for which $k_1 \dots k_p$ are the numbers 1, 3, 5, ..., $2p-1$, whose sum* is p^2. An example is furnished by § 32 above.

Ex. For the surface

$$y^3+y^2(x,\ 1)_1+y\,(x,\ 1)_2+(x,\ 1)_4=0,$$

for which $p=3$, there is, at $x=\infty$, only one place, and the non-existent orders are 1, 2, 5: whose sum is p^2-1.

36. We have in § 34 defined p integrals of the first kind

$$\int \omega_1(x)\,dt_x,\ \dots,\ \int \omega_p(x)\,dt_x$$

by means of a place ξ. Since the differential coefficients of these vanish at ξ to essentially *different* orders, these integrals cannot be connected by a homogeneous linear equation with constant coefficients. Hence a linear function of them with parametric constant coefficients is a general integral of the first kind. Therefore each of $\Omega_1(x) \dots \Omega_p(x)$ is expressible linearly in terms of $\omega_1(x) \dots \omega_p(x)$ in a form

$$\Omega_i(x) = c_{i1}w_1(x) + \dots + c_{ip}w_p(x),$$

where the coefficients are independent of x. Thus the determinant Δ (§ 31), which vanishes at places for which functions of order less than $p+1$ exist, is equal to

$$C \begin{vmatrix} \omega_1(x) & ,\dots\dots, & \omega_p(x) \\ D_x\omega_1(x) & ,\dots\dots, & D_x\omega_p(x) \\ \dots\dots & \dots\dots & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots \\ D_x^{p-1}\omega_1(x), & \dots\dots, & D_x^{p-1}\omega_p(x) \end{vmatrix},$$

where C is the determinant of the coefficients c_{ij}. It follows from the result of § 34 that the determinant here multiplied by C vanishes at ξ to the order

$$(k_1-1)+(k_2-2)+\dots+(k_p-p)=k_1+\dots+k_p-\tfrac{1}{2}p\,(p+1).$$

Thus, *the determinant Δ vanishes at any one of its zeros to an order equal to the sum of the non-existent orders for the place diminished by $\frac{1}{2}p\,(p+1)$.*

For example, it vanishes at a place where the non-existent orders are 1, 2, ..., $p-1$, $p+1$ to an order $\frac{1}{2}p\,(p-1)+p+1-\frac{1}{2}p\,(p+1)$ or to the first order. We have already remarked that such places are those which most usually occur.

* Cf. Burkhardt, *Math. Annal.* 32, p. 388, and the section in Chapter V., below, on the hyperelliptic case.

Hence, since $k_1 + \dots + k_p \leqq p^2$, Δ *vanishes at one of its zeros to an order* $\leqq \frac{1}{2}p(p-1)$.

Further, if r be the number of distinct places where Δ vanishes, and $m_1, m_2, \dots, m_r$ be the orders of multiplicity of zero at these places, it follows, from

$$m_1 + \dots + m_r = (p-1)p(p+1),$$

and

$$m_1 + \dots + m_r \leqq r \tfrac{1}{2} p (p-1),$$

that $r > 2p + 2$, or

there are at least $2p + 2$ *distinct places for which functions of less order than* $p + 1$, *infinite only thereat, exist*; this lower limit to the number of distinct places is only reached when there are places for which functions of the second order exist.

Ex. For the surface given by

$$x^4 + y^4 + (ax + by + c)^4 = 0,$$

p is equal to 3; there are $12 = 2p + 6$ distinct places where Δ vanishes.

37. We have called attention to the number of arbitrary constants contained in *the most general* rational function having simple poles in distinct places (§ 27) and to the number in the most general function infinite at a single place to prescribed order (§ 28): in this enumeration some of the constants may be multipliers of functions not actually becoming infinite in the most general way allowed them, that is, either of functions which are not really infinite at all the distinct places or of functions whose order of infinity is not so high as the prescribed order.

It will be convenient to state here the general result, the deduction of which follows immediately from the expression of the function in terms of integrals of the second kind:—

Let $a_1, a_2, \dots$ be any finite number of places on the surface, the infinitesimals at these places being denoted by $t_1, t_2, \dots$. The most general rational function whose expansion at the place a_i involves the terms

$$\frac{1}{t_i^{\lambda_i}},\ \frac{1}{t_i^{\mu_i}},\ \frac{1}{t_i^{\nu_i}},\ \dots$$

—whose number is finite, $= Q_i$ say,—and no other negative powers, involves $q + 1$ linearly entering arbitrary constants, of which one is additive, q being given by the formula

$$Q - q = p - (\tau + 1),$$

where Q is the sum of the numbers Q_i, and $\tau + 1$ is the number of linearly independent linear aggregates of the form

$$\Omega(x) = A_1\Omega_1(x) + \dots + A_p\Omega_p(x),$$

which satisfy the sets of Q_i relations, whose total number is Q, given by

$$A_1 D^{\lambda_i-1}\Omega_1(a_i) + A_2 D^{\lambda_i-1}\Omega_2(a_i) + \dots + A_p D^{\lambda_i-1}\Omega_p(a_i) = 0,$$
$$A_1 D^{\mu_i-1}\Omega_1(a_i) + A_2 D^{\mu_i-1}\Omega_2(a_i) + \dots + A_p D^{\mu_i-1}\Omega_p(a_i) = 0,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

As before, this general function will as a rule be an aggregate of functions of which not every one is as fully infinite as is allowed, *and it is clear from the present chapter that in the absence of further information in regard to the places $a_1, a_2, \dots$ it may quite well happen that not one of these functions is as fully infinite as desired, the conditions analogous to those stated in §§ 23, 28 not being satisfied. See Example 2 below.*

The equation $Q - q = p - (\tau + 1)$ will be referred to as the Riemann-Roch Theorem.

Ex. 1. For a rational function having only simple poles or, more generally, such that the numbers $\lambda_i, \mu_i, \nu_i, \dots$ for any pole are the numbers $1, 2, 3, \dots Q_i$,

if $Q > 2p - 2$, $\tau + 1$ is zero, since $\Omega(x)$ has only an aggregate number $2p - 2$ of zeros: the function involves $Q - p + 1$ constants,

if $Q = 2p - 2$, $\tau + 1$ cannot be greater than 1; for the ratio of two of the aggregates $\Omega(x)$ then vanishing at the poles, being expressible in a form $\dfrac{dV}{dW}$, where V, W are integrals of the first kind, would be a rational function without poles, namely a constant; then the linear aggregates $\Omega(x)$ would be identical: thus the function involves $Q - p + 1$ or $Q - p + 2$ constants, namely $p - 1$ or p constants,

if $Q = 2p - 3$, $\tau + 1$ cannot be greater than 1, since the ratio of two of the aggregates $\Omega(x)$ then vanishing at the poles would be a rational function of the first order and therefore p be equal to unity—in which case $2p - 3$ is negative: thus the function involves $p - 2$ or $p - 1$ constants,

if $Q = 2p - 4$, and $\tau + 1$ be greater than unity, the ratio of two of the vanishing aggregates $\Omega(x)$ would be a rational function of the second order: we have already several times referred to this possibility as indicative that the surface is of a special character—called hyperelliptic—and depends in fact only on $2p - 1$ independent moduli. In general such a function would involve $p - 3$ constants.

Ex. 2. Let V be an integral of the first kind and a be an arbitrary definite place which is not among the $2p - 2$ zeros of dV. We can form a rational function infinite to the first order at the $2p - 2$ zeros of dV and to the second order at a; the general form of such a function would contain $2p - 2 + 2 - p + 1 = p + 1$ arbitrary constants. *But there exists no rational function infinite to the first order at the zeros of dV and to the first order at* *the place a.* Such a function would indeed by the Riemann-Roch theorem here stated, contain $2p-2+1-p+1=p$ arbitrary constants: *but the coefficients of these constants are in fact infinite only at the zeros of dV.* For when the places $a_1, \dots, a_{2p-2}$ are all zeros of an aggregate of the form

$$A_1\Omega_1(x)+\dots+A_p\Omega_p(x),$$

the conditions that the periods of an expression

$$\lambda+\lambda_1\Gamma_{a_1}^{x}+\dots+\lambda_{2p-2}\Gamma_{a_{2p-2}}^{x}+\mu\Gamma_a^x$$

be all zero, namely the equations

$$\lambda_1\Omega_i(a_1)+\dots+\lambda_{2p-2}\Omega_i(a_{2p-2})+\mu\Omega_i(a)=0,\quad (i=1, 2, \dots, p),$$

lead to

$$\mu\left[A_1\Omega_1(a)+\dots+A_p\Omega_p(a)\right]=0,$$

and therefore to $$\mu=0.$$

Thus the function in question will be a linear aggregate of p functions whose poles are among the places $a_1, \dots, a_{2p-2}$. As a matter of fact, if W be a general integral of the first kind, expressible therefore in the form

$$\lambda V+\lambda_2 V_2+\dots+\lambda_p V_p,$$

wherein $V_2, \dots, V_p$ are integrals of the first kind, $\dfrac{dW}{dV}$ involves the right number of constants and is the function sought.

In this case the place a does not, in the sense of § 23, depend upon the places $a_1, \dots, a_{2p-2}$; the symbol suggested in § 26 for the places $a_1, \dots, a_{2p-2}$, $a, \dots$ is

$$1, 2, 3, \dots, p-1, \overline{p, p+1, \dots, 2p-2}, 2p-1, \overline{2p, 2p+1}, \dots.$$

It may be shewn quite similarly that there is no rational function having simple poles in $a_1, a_2, \dots, a_{2p-2}$ and infinite besides at a like the single term $\dfrac{1}{t^r}$, t being the infinitesimal at the place a.

Ex. 3. The most general rational function R which has the value c at each of Q given distinct places, $R-c$ being zero of the first order at each of these places, is obviously derivable by the remark that $1/(R-c)$ is infinite at these places.