PREFACE.

Mathematical Logic is a necessary preliminary to logical Mathematics. "Mathematical Logic" is the name given by PEANO to what is also known (after VENN) as "Symbolic Logic"; and Symbolic Logic is, in essentials, the Logic of Aristotle, given new life and power by being dressed up in the wonderful—almost magical—armour and accoutrements of Algebra. In less than seventy years, logic, to use an expression of DE MORGAN's, has so *thriven* upon symbols and, in consequence, so grown and altered that the ancient logicians would not recognize it, and many old-fashioned logicians will not recognize it. The metaphor is not quite correct: Logic has neither grown nor altered, but we now see more *of* it and more *into* it.

The primary significance of a symbolic calculus seems to lie in the economy of mental effort which it brings about, and to this is due the characteristic power and rapid development of mathematical knowledge. Attempts to treat the operations of formal logic in an analogous way had been made not infrequently by some of the more philosophical mathematicians, such as LEIBNIZ and LAMBERT; but their labors remained little known, and it was BOOLE and DE MORGAN, about the middle of the nineteenth century, to whom a mathematical—though of course non-quantitative—way of regarding logic was due. By this, not only was the traditional or Aristotelian doctrine of logic reformed and completed, but out of it has developed, in course of time, an instrument which deals in a sure manner with the task of investigating the fundamental concepts of mathematics—a task which philosophers have repeatedly taken in hand, and in which they have as repeatedly failed.

First of all, it is necessary to glance at the growth of symbolism in mathematics, where alone it first reached perfection. There have been three stages in the development of mathematical doctrines: first came propositions with particular numbers, like the one expressed, with signs subsequently invented, by "$2 + 3 = 5$"; then came more general laws holding for all numbers and expressed by letters, such as

$$``(a+b)\,c = ac + bc";$$

lastly came the knowledge of more general laws of functions and the formation of the conception and expression "function". The origin of the symbols for particular whole numbers is very ancient, while the symbols now in use for the operations and relations of arithmetic mostly date from the sixteenth and seventeenth centuries; and these "constant" symbols together with the letters first used systematically by VIÈTE (1540—1603) and DESCARTES (1596—1650), serve, by themselves, to express many propositions. It is not, then, surprising that DESCARTES, who was both a mathematician and a philosopher, should have had the idea of keeping the method of algebra while going beyond the material of traditional mathematics and embracing the general science of what thought finds, so that philosophy should become a kind of Universal Mathematics. This sort of generalization of the use of symbols for analogous theories is a characteristic of mathematics, and seems to be a reason lying deeper than the erroneous idea, arising from a simple confusion of thought, that algebraical symbols necessarily imply something quantitative, for the antagonism there used to be and is on the part of those logicians who were not and are not mathematicians, to symbolic logic. This idea of a universal mathematics was cultivated especially by GOTTFRIED WILHELM LEIBNIZ (1646—1716).

Though modern logic is really due to BOOLE and DE MORGAN, LEIBNIZ was the first to have a really distinct plan of a system of mathematical logic. That this is so appears from research—much of which is quite recent—into LEIBNIZ's unpublished work.

The principles of the logic of LEIBNIZ, and consequently of his whole philosophy, reduce to two[1]: (1) All our ideas are compounded of a very small number of simple ideas which form the "alphabet of human thoughts"; (2) Complex ideas proceed from these simple ideas by a uniform and symmetrical combination which is analogous to arithmetical multiplication. With regard to the first principle, the number of simple ideas is much greater than LEIBNIZ thought; and, with regard to the second principle, logic considers three operations—which we shall meet with in the following book under the names of logical multiplication, logical addition and negation—instead of only one.

"Characters" were, with LEIBNIZ, any written signs, and "real" characters were those which—as in the Chinese ideography—represent ideas directly, and not the words for them. Among real characters, some simply serve to represent ideas, and some serve for reasoning. Egyptian and Chinese hieroglyphics and the symbols of astronomers and chemists belong to the first category, but LEIBNIZ declared them to be imperfect, and desired the second category of characters for what he called his "universal characteristic".[2] It was not in the form of an algebra that LEIBNIZ first conceived his charateristic, probably because he was then a novice in mathematics, but in the form of a universal language or script.[3] It was in 1676 that he first dreamed of a kind of algebra of thought,[4] and it was the algebraic notation which then served as model for the characteristic.[5]

LEIBNIZ attached so much importance to the invention of proper symbols that he attributed to this alone the whole of his discoveries in mathematics.[6] And, in fact, his infinitesimal calculus affords a most brilliant example of the importance of, and LEIBNIZ's skill in devising, a suitable notation.[7]

Now, it must be remembered that what is usually understood by the name "symbolic logic", and which—though not its name—is chiefly due to BOOLE, is what LEIBNIZ called a *Calculus ratiocinator*, and is only a part of the Universal

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1 COUTURAT, *La Logique de Leibniz d'après des documents inédits*, Paris, 1901, pp. 431—432, 48.

2 *Ibid.*, p. 81. 3 *Ibid.*, pp. 51, 78. 4 *Ibid.*, p. 61.

5 *Ibid.*, p. 83. 6 *Ibid.*, p. 84. 7 *Ibid.*, p. 84—87.

Characteristic. In symbolic logic LEIBNIZ enunciated the principal properties of what we now call logical multiplication, addition, negation, identity, class-inclusion, and the null-class; but the aim of LEIBNIZ's researches was, as he said, to create "a kind of general system of notation in which all the truths of reason should be reduced to a calculus. This could be, at the same time, a kind of universal written language, very different from all those which have been projected hitherto; for the characters and even the words would direct the reason, and the errors—excepting those of fact—would only be errors of calculation. It would be very difficult to invent this language or characteristic, but very easy to learn it without any dictionaries". He fixed the time necessary to form it: "I think that some chosen men could finish the matter within five years"; and finally remarked: "And so I repeat, what I have often said, that a man who is neither a prophet nor a prince can never undertake any thing more conducive to the good of the human race and the glory of God".

In his last letters he remarked: "If I had been less busy, or if I were younger or helped by well-intentioned young people, I would have hoped to have evolved a characteristic of this kind"; and: "I have spoken of my general characteristic to the Marquis de l'Hôpital and others; but they paid no more attention than if I had been telling them a dream. It would be necessary to support it by some obvious use; but, for this purpose, it would be necessary to construct a part at least of my characteristic;—and this is not easy, above all to one situated as I am".

LEIBNIZ thus formed projects of both what he called a *characteristica universalis*, aud what he called a *calculus ratiocinator*; it is not hard to see that these projects are interconnected, since a perfect universal characteristic would comprise, it seems, a logical calculus. LEIBNIZ did not publish the incomplete results which he had obtained, and consequently his ideas had no continuators, with the exception of LAMBERT and some others, up to the time when BOOLE, DE MORGAN, SCHRÖDER, MACCOLL, and others rediscovered his theorems. But when the investigations of the principles of mathematics became the chief task of logical symbolism, the aspect of symbolic logic as a calculus ceased to be of such importance, as we see in the work of FREGE and RUSSELL. FREGE's symbolism, though far better for logical analysis than BOOLE's or the more modern PEANO's, for instance, is far inferior to PEANO's—a symbolism in which the merits of internationality and power of expressing mathematical theorems are very satisfactorily attained—in practical convenience. RUSSELL, especially in his later works, has used the ideas of FREGE, many of which he discovered subsequently to, but independently of, FREGE, and modified the symbolism of PEANO as little as possible. Still, the complications thus introduced take away that simple character which seems necessary to a calculus, and which BOOLE and others reached by passing over certain distinctions which a subtler logic has shown us must ultimately be made.

Let us dwell a little longer on the distinction pointed out by LEIBNIZ between a *calculus ratiocinator* and a *characteristica universalis* or *lingua characteristica.* The ambiguities of ordinary language are too well known for it to be necessary for us to give instances. The objects of a complete logical symbolism are: firstly, to avoid this disadvantage by providing an *ideography*, in which the signs represent ideas and the relations between them *directly* (without the intermediary of words), and secondly, so to manage that, from given premises, we can, in this ideography, draw all the logical conclusions which they imply by means of rules of transformation of formulas analogous to those of algebra,—in fact, in which we can replace reasoning by the almost mechanical process of calculation. This second requirement is the requirement of a *calculus ratiocinator.* It is essential that the ideography should be complete, that only symbols with a well-defined meaning should be used—to avoid the same sort of ambiguities that words have—and, consequently, that no suppositions should be introduced implicitly, as is commonly the case if the meaning of signs is not well defined. Whatever premises are necessary and sufficient for a conclusion should be stated explicitly.

Besides this, it is of practical importance,—though it is theoretically irrelevant,—that the ideography should be concise, so that it is a sort of stenography.

The merits of such an ideography are obvious: rigor of reasoning is ensured by the calculus character; we are sure of not introducing unintentionally any premise; and we can see exactly on what propositions any demonstration depends.

We can shortly, but very fairly accurately, characterize the dual development of the theory of symbolic logic during the last sixty years as follows: The *calculus ratiocinator* aspect of symbolic logic was developed by BOOLE, DE MORGAN, JEVONS, VENN, C. S. PEIRCE, SCHRÖDER, Mrs. LADD-FRANKLIN and others; the *lingua characteristica* aspect was developed by FREGE, PEANO and RUSSELL. Of course there is no hard and fast boundary-line between the domains of these two parties. Thus PEIRCE and SCHRÖDER early began to work at the foundations of arithmetic with the help of the calculus of relations; and thus they did not consider the logical calculus merely as an interesting branch of algebra. Then PEANO paid particular attention to the calculative aspect of his symbolism. FREGE has remarked that his own symbolism is meant to be a *calculus ratiocinator* as well as a *lingua characteristica*, but the using of FREGE's symbolism as a calculus would be rather like using a three-legged stand-camera for what is called "snap-shot" photography, and one of the outwardly most noticeable things about RUSSELL's work is his combination of the symbolisms of FREGE and PEANO in such a way as to preserve nearly all of the merits of each.

The present work is concerned with the *calculus ratiocinator* aspect, and shows, in an admirably succinct form, the beauty, symmetry and simplicity of the calculus of logic regarded as an algebra. In fact, it can hardly be doubted that some such form as the one in which SCHRÖDER left it is by far the best for exhibiting it from this point of view.[1] The content of the

[1] Cf. A. N. WHITEHEAD, *A Treatise on Universal Algebra with Applications,* Cambridge, 1898.

present volume corresponds to the two first volumes of SCHRÖDER's great but rather prolix treatise.[1] Principally owing to the influence of C. S. PEIRCE, SCHRÖDER departed from the custom of BOOLE, JEVONS, and himself (1877), which consisted in the making fundamental of the notion of *equality*, and adopted the notion of *subordination* or *inclusion* as a primitive notion. A more orthodox BOOLIAN exposition is that of VENN[2], which also contains many valuable historical notes.

We will finally make two remarks.

When BOOLE (cf. § 2 below) spoke of propositions determining a class of moments at which they are true, he really (as did MACCOLL) used the word "proposition" for what we now call a "propositional function". A "proposition" is a thing expressed by such a phrase as "twice two are four" or "twice two are five", and is always true or always false. But we might seem to be stating a proposition when we say: "Mr. WILLIAM JENNINGS BRYAN is Candidate for the Presidency of the United States", a statement which is sometimes true and sometimes false. But such a statement is like a mathematical *function* in so far as it depends on a *variable*—the time. Functions of this kind are conveniently distinguished from such entities as that expressed by the phrase "twice two are four" by calling the latter entities "propositions" and the former entities "propositional functions": when the variable in a propositional function is fixed, the function becomes a proposition. There is, of course, no sort of necessity why these special names should be used; the use of them is merely a question of convenience and convention.

In the second place, it must be carefully observed that, in § 13, 0 and 1 are not *defined* by expressions whose principal

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[1] *Vorlesungen über die Algebra der Logik*, Vol. I., Leipsic, 1890; Vol. II, 1891 and 1905. We may mention that a much shorter *Abriss* of the work has been prepared by EUGEN MÜLLER. Vol. III (1895) of SCHRÖDER's work is on the logic of relatives founded by DE MORGAN and C. S. PEIRCE,—a branch of Logic that is only mentioned in the concluding sentences of this volume.

[2] *Symbolic Logic*, London, 1881; 2nd ed., 1894.

copulas are relations of inclusion. A definition is simply the convention that, for the sake of brevity or some other convenience, a certain new sign is to be used instead of a group of signs whose meaning is already known. Thus, it is the sign of *equality* that forms the principal copula. The theory of definition has been most minutely studied, in modern times by FREGE and PEANO.

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