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A NUMERICAL STUDY OF THE RELATIVISTIC BURGERS AND EULER EQUATIONS ON A SCHWARZSCHILD BLACK HOLE EXTERIOR

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We study the dynamical behavior of compressible fluids evolving on the outer domain of communication of a Schwarzschild background. For both the relativistic Burgers equation and the relativistic Euler system, assuming spherical symmetry we introduce numerical methods that take the Schwarzschild geometry and, specifically, the steady state solutions into account. The schemes we propose preserve the family of steady state solutions and enable us to study the nonlinear stability of fluid equilibria and the behavior of solutions near the black hole horizon. We state and numerically demonstrate several properties about the late-time behavior of perturbed steady states.

1. Introduction

We are interested in compressible fluid flows on a Schwarzschild black hole background. Motivated by earlier works on relativistic fluid problems posed on curved spacetimes by LeFloch et al. [1; 4; 8; 18; 2; 19; 20] and on numerical methods by Glimm et al. [12; 13] and Russo et al. [28; 29; 30], who argue that steady state solutions should be included in the design of the scheme, as well as relying on the further analytical advances by LeFloch and Xiang [22], we design several

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numerical schemes for the approximation of shock wave solutions to the relativistic Burgers equation and to the compressible Euler system. We assume that the flows under consideration are spherically symmetric, and we design schemes that are asymptotic-preserving and allow us to investigate the late-time behavior of solutions. An important challenge we address here is taking the curved geometry into account at the level of the discretization and handling the behavior of solutions near the horizon of the black hole.

The *relativistic Burgers equation on a Schwarzschild background* reads as (see [23] for further details)

$$\partial_t\left(\frac{v}{(1-2M/r)^2}\right)+\partial_r\left(\frac{v^2-1}{2(1-2M/r)}\right)=0, \quad r>2M, \tag{1-1}$$

where we have normalized the light speed to unit and the unknown is the function $v=v(t,r)\in[-1,1]$. This equation can also be put in the nonconservative form

$$\partial_t v+\partial_r\left(\left(1-\frac{2M}{r}\right)\frac{v^2-1}{2}\right)=\frac{2M}{r^2}(v^2-1), \quad r>2M. \tag{1-2}$$

Here $M>0$ denotes the mass of the black hole and, clearly, we recover the standard Burgers equations when the mass vanishes.

For the relativistic Burgers model, we design here a finite volume method as well as a random choice method which both preserve steady state solutions. Then, we use these schemes and provide some support as well as some generalization to our theoretical results (briefly reviewed in Theorems 2.1–2.3 below). We treat the following issues:

- the global-in-time existence theory for the generalized Riemann problem and
- the late-time behavior of a steady state (and possibly discontinuous) solution under some initial perturbation.

In addition, our numerical study leads us to the following observations about general initial data.

Claim 1.1 (relativistic Burgers model). *Given any compactly perturbed steady shock taken as initial data, the corresponding solution to the relativistic Burgers model* (1-1) *converges* (*asymptotically in time*) *to a steady shock.*

Claim 1.2 (relativistic Burgers model). *Given initial data* $v_0=v_0(r)\in[-1,1]$ *defined on* $[2M,+\infty)$ *and prescribed at some time* t_0, *the corresponding solution* $v=v(t,r)$ *to the relativistic Burgers model* (1-1) *enjoys the following properties*:

- *If* $v_0(2M)=1$, *then there exists a finite time* $t_1>t_0$ *such that, for all* $t>t_1$, *the solution* $v=v(t,r)$ *is a single shock connecting the left-hand state* 1 *to the right-hand escape velocity profile* $-\sqrt{2M/r}$.

- *If* $v_0(2M) < 1$ *and* $\lim_{r\to+\infty} v_0(r) > 0$, *then there exists a finite time* $t_1 > t_0$ *such that, for all* $t > t_1$, *the solution globally coincides with the escape velocity profile* $v(t,r) = -\sqrt{2M/r}$.
- *If* $v_0(2M) < 1$ *and* $\lim_{r\to+\infty} v_0(r) \le 0$, *then there exists a finite time* $t_1 > t_0$ *such that, for all* $t > t_1$, *the solution coincides with*

$$v(t,r) = -\sqrt{1-(1-(v_0^\infty)^2)\Big(1-\frac{2M}{r}\Big)}, \qquad \lim_{r\to+\infty} v_0(r) =: v_0^\infty \le 0.$$

When the pressure is not assumed to vanish, we consider isothermal fluid flows with pressure law $p = k^2\rho$ where $k \in (0,1)$ represents the (constant) sound speed. Such an assumption guarantees the hyperbolicity and genuine nonlinearity of the Euler system which, on a Schwarzschild background, reads

$$\begin{aligned}
&\partial_t\Big(r^2\frac{1+k^2v^2}{1-v^2}\rho\Big) + \partial_r\Big(r(r-2M)\frac{1+k^2}{1-v^2}\rho v\Big) = 0,\\
&\partial_t\Big(r(r-2M)\frac{1+k^2}{1-v^2}\rho v\Big) + \partial_r\Big((r-2M)^2\frac{v^2+k^2}{1-v^2}\rho\Big)\\
&\qquad = 3M\Big(1-\frac{2M}{r}\Big)\frac{v^2+k^2}{1-v^2}\rho - M\frac{r-2M}{r}\frac{1+k^2v^2}{1-v^2}\rho + 2\frac{(r-2M)^2}{r}k^2\rho,
\end{aligned} \tag{1-3}$$

where the light speed is normalized to unit. By formally letting $k \to 0$, we recover the pressureless Euler system, from which in turn we derive the relativistic Burgers equation above. On the other hand, by letting the black hole mass $M \to 0$, we recover the relativistic Euler system.

We will also write the Euler equations in the alternative form

$$\begin{aligned}
&\partial_t\Big(\frac{1+k^2v^2}{1-v^2}\rho\Big) + \partial_r\Big((1-2M/r)\frac{1+k^2}{1-v^2}\rho v\Big) = -\frac{2}{r}(1-2M/r)\frac{1+k^2}{1-v^2}\rho v,\\
&\partial_t\Big(\frac{1+k^2}{1-v^2}\rho v\Big) + \partial_r\Big((1-2M/r)\frac{v^2+k^2}{1-v^2}\rho\Big)\\
&\qquad = \frac{-2r+5M}{r^2}\frac{v^2+k^2}{1-v^2}\rho - \frac{M}{r^2}\frac{1+k^2v^2}{1-v^2}\rho + 2\frac{r-2M}{r^2}k^2\rho.
\end{aligned} \tag{1-4}$$

We are going to design a finite volume method, with second-order accuracy, that preserves the family of steady state solutions to the Euler equations on a Schwarzschild background. Our numerical study suggests a global-in-time existence theory for the generalized Riemann problem, whose explicit form is not yet known theoretically. In particular, we will be able to exhibit solutions containing up to three steady state components, connected by a 1-wave and a 2-wave.

Claim 1.3 (relativistic Euler model). *Let* $(\rho_*, v_*) = (\rho_*, v_*)(r)$, $r > 2M$, *be a smooth steady state solution to the relativistic Euler equations on a Schwarzschild background* (1-3), *and consider the initial data* $(\rho_0, v_0) = (\rho_0, v_0)(r) = (\rho_*, v_*)(r) + (\delta_\rho, \delta_v)(r)$ *prescribed at some time* t_0, *where the perturbation* $(\delta_\rho, \delta_v) = (\delta_\rho, \delta_v)(r)$ *has compact support. Then, for sufficiently large times the corresponding solution* $(\rho, v) = (\rho, v)(t, r)$ *to* (1-3) *coincides with the given steady state solution*; *in other words, for some time* $t_1 > t_0$, *one has* $(\rho, v)(t, r) = (\rho_*, v_*)(r)$ *for all* $t > t_1$.

Using steady shocks (discussed in Section 7), we also have:

Claim 1.4 (relativistic Euler model). *Let* $(\rho_*, v_*) = (\rho_*, v_*)(r)$, $r > 2M$, *be a steady shock, and let* $(\rho_0, v_0) = (\rho_*, v_*)(r) + (\delta_\rho, \delta_v)(r)$ *where* $(\delta_\rho, \delta_v) = (\delta_\rho, \delta_v)(r)$ *is a compactly supported perturbation. Then there exists a finite time* $t > t_0$ *such that the solution is a steady shock for all later times.*

Our numerical random choice scheme is motivated by the methodology in Glimm, Marshall, and Plohr [13] for quasi-one-dimensional gas flows. We rely on static solutions and on the generalized Riemann problem, which we studied in [22] for the relativistic models under consideration here. The numerical analysis of hyperbolic problems posed on curved spacetimes was initiated in [1; 8; 18; 19; 20] using the finite volume methodology. For further background we also refer to [3; 5; 6; 7; 27; 10; 11; 14; 15; 16; 17; 21; 24; 25; 26; 31].

This paper is organized as follows. In Section 2, we briefly overview our theoretical results for the relativistic Burgers model. We include a full description of the family of steady state solutions, as well as some outline of the existence theory for the initial data problem and the nonlinear stability of piecewise steady solutions (see Figure 1). In Section 3, we introduce a finite volume method for the relativistic Burgers model (1-1), which is second-order accurate. In Section 4, we apply our scheme in order to study the generalized Riemann problem and to elucidate the late-time behavior of perturbations of steady solutions.

Building on our theoretical results, in Section 5 we implement a generalized Glimm scheme for the relativistic Burgers model (1-1). Our numerical method is based on an explicit generalized Riemann solver, and therefore, our method preserves all steady state solutions. Numerical experiments are presented in Section 6, in which we are able to validate and expand the theoretical results in Section 2. Our method avoids introducing numerical diffusion and provides an efficient approach for computing shock wave solutions. Furthermore, we apply both methods to the study of the initial problem for the relativistic Burgers equation when the initial velocity is rather arbitrary and we validate our Claims 1.1 and 1.2 and, along the way, clarify the behavior of the fluid flow near the black hole horizon.

Next, in Section 7, we turn our attention to the relativistic Euler model on a Schwarzschild background. We begin by reviewing some theoretical results,

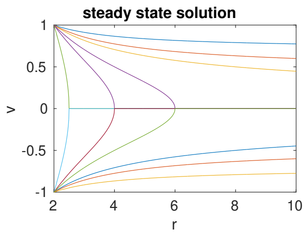

Figure 1. Burgers: steady state solutions.

including the existence theory for steady state solutions, the construction of a solver for the generalized Riemann problem, and the existence theory for the initial value problem. In Section 8 we design a finite volume method for the relativistic Euler model which is second-order accurate. With the proposed algorithm, in Section 9, we are able to tackle the generalized Riemann problem (whose solution is not known in a closed form) and we study the nonlinear stability of steady state solutions when the perturbation has compact support. This allows us to demonstrate numerically the validity of Claims 1.3 and 1.4 above.

2. Overview of the theory of the relativistic Burgers model

An important class of solutions to the relativistic Burgers model (1-1) is provided by the *steady state solutions*, that is, solutions depending on the space variable r only:

$$\partial_r \left(\frac{v^2 - 1}{2(1-2M/r)} \right) = 0. \tag{2-1}$$

The steady state solutions are given by

$$v(r) = \pm\sqrt{1 - K^2(1-2M/r)}, \tag{2-2}$$

in which $K > 0$ is an arbitrary constant and, clearly, the sign of a steady state cannot change. Clearly, each such solution is smooth in r and admits a finite limit $\lim_{r\to 2M} v(r) = \pm 1$ at the black hole horizon. Moreover, we have:

- When $0 < K < 1$, then the limit at space infinity is $\lim_{r\to+\infty} v(r) = \pm\sqrt{1-K^2}$.
- When $K = 1$, the solution is the critical steady state solution $v_*^\pm = \pm\sqrt{2M/r}$, which vanishes at infinity and also coincides with the *escape velocity profile.*
- When $K > 1$, the steady state solution is defined only on a bounded interval and stops being defined as the radius $r^\natural = 2MK^2/(1-K^2)$.

In addition to the smooth steady state solutions, we can also define the class of steady shock solutions to the relativistic Burgers equation

$$v(r) = \begin{cases} \sqrt{1-K^2(1-2M/r)}, & 2M < r < r_0, \\ -\sqrt{1-K^2(1-2M/r)}, & r > r_0, \end{cases} \tag{2-3}$$

where $K > 0$ is a constant and $r_0 > 2M$ is a given radius. The relevant solutions to the relativistic Burgers equation $v = v(t, r)$ have a range bounded by the light speed, that is, $v \in [-1, 1]$ for all $t > 0$ and $r > 2M$. An initial problem of particular importance is given by the generalized Riemann problem, associated with initial data made of two steady states separated by a jump discontinuity located at some given radius.

Theorem 2.1 (the generalized Riemann problem for the relativistic Burgers model). *There exists a unique solution to the generalized Riemann problem defined for all $t > 0$ realized either by a shock wave or a rarefaction wave. Moreover, the wave location*

- *tends to the black hole horizon if it initially converges towards the black hole,*
- *tends to the space infinity if it initially converges away from the black hole, and*
- *does not change if it is initially steady.*

In connection with the general existence theory for (1-1), we introduce the auxiliary variable $z := \mathrm{sgn}(v)\sqrt{(v^2 - 1)/(1 - 2M/r) + 1}$. It is obvious that z is a constant if v is a steady state solution. With this notation, we have the following result from [23].

Theorem 2.2 (existence theory for the relativistic Burgers model). *Consider the relativistic Burgers equation (1-1) posed on the outer domain of a Schwarzschild black hole with mass M. Then, for any initial velocity $v_0 = v_0(r) \in (-1, 1)$ such that $v_0 = v_0(r)$ has locally bounded total variation, there exists a corresponding weak solution to (1-1) $z = z(t, r)$ with locally finite total variation in space.*

We are going to design several numerical methods to study these solutions. In particular, we are interested in the behavior of solutions when the initial data $v_0 = v_0(r)$ is a piecewise smooth and steady state solution, to which we will add a compactly supported perturbation; i.e., we consider

$$v_0(r) = \begin{cases} v_L(r), & 2M < r < r_L, \\ \text{arbitrary values}, & r_L < r < r_R, \\ v_R(r), & r > r_R, \end{cases} \tag{2-4}$$

where $v_L = v_L(r)$ and $v_R = v_R(r)$ are two steady state solutions given by (2-2) and r_L, r_R are two fixed points.

Theorem 2.3 (time-asymptotic properties for the relativistic Burgers model). *Consider the asymptotic behavior of a relativistic Burgers solution $v = v(t, r)$ on a Schwarzschild background (1-1) whose initial data are composed of steady state solutions v_L, v_R with a compactly supported perturbation.*

- *If $v_L > v_R$, then the solution $v = v(t, r)$ converges asymptotically to a shock curve generated by a left-hand state v_L and a right-hand state v_R.*

- *If $v_L < v_R$, then a generalized N-wave $N = N(t,r)$ can be defined such that inside a rarefaction fan one has $|v(t,r) - N(t,r)| = O(t^{-1})$ while in a region supporting the evolution of the initial data one has $|v(t,r) - N(t,r)| = O(t^{-1/2})$. Otherwise, one has $v(t,r) = N(t,r)$.*
- *If $v_L = v_R$, then $\|v(t,r) - v_R(t,r)\|_{L^1(2M,+\infty)} = O(t^{-1/2})$.*

3. A finite volume scheme for the relativistic Burgers model

The first-order formulation. In this section, we propose a finite volume method for the relativistic Burgers equation (1-2) which takes the Schwarzschild geometry into consideration. In order to construct our approximations, we rely on the Riemann solver for the standard Burgers equation:

$$\partial_t v + \partial_x \frac{v^2}{2} = 0, \tag{3-1}$$

that is, an initial data problem with $v(t,r) = v_0(r)$ where $v_0 = v_0(r)$ is given as a piecewise constant function

$$v_0 = \begin{cases} v_L, & r < r_0, \\ v_R, & r > r_0, \end{cases}$$

for some fixed r_0 and two constants v_L, v_R. The solution to the standard Riemann problem reads

$$v(t,r) = \begin{cases} v_L, & r < s_L t + r_0, \\ (r - r_0)/t, & s_L t + r_0 < r < s_R t + r_0, \\ v_R, & r > s_R t + r_0, \end{cases} \tag{3-2}$$

with

$$s_L = \begin{cases} v_L, & v_L < v_R, \\ (v_L + v_R)/2, & v_L > v_R, \end{cases} \qquad s_R = \begin{cases} v_R, & v_L < v_R, \\ (v_L + v_R)/2, & v_L > v_R. \end{cases} \tag{3-3}$$

Denote by Δt, Δr the mesh lengths in time and in space, respectively, with ratio denoted by $\Lambda = \Delta t/\Delta r$. We also set $t_n = n\Delta t$ and $r_j = 2M + j\Delta r$. Introduce also the mesh point (t_n, r_j), $n \geq 0$ and $j \geq 0$, and the rectangle $R_{nj} = \{t_n \leq t < t_{n+1}, r_{j-1/2} \leq r < r_{j+1/2}\}$.

Integrate (1-2) from $r_{j-1/2}$ to $r_{j+1/2}$ in space and from t_n to t_{n+1} in time:

$$\int_{r_{j-1/2}}^{r_{j+1/2}} (v(t_{n+1}, r) - v(t_n, r))\, dr + \int_{t_n}^{t_{n+1}} \biggl((1 - 2M/r_{j+1/2}) \biggl(\frac{v(t, r_{j+1/2})^2 - 1}{2} \biggr) - (1 - 2M/r_{j-1/2}) \biggl(\frac{v^2(t, r_{j-1/2}) - 1}{2} \biggr) \biggr) dt - \int_{r_{j-1/2}}^{r_{j+1/2}} \int_{t_n}^{t_{n+1}} \frac{2M}{r^2} (v^2 - 1)\, dt\, dr = 0.$$

Denote by

$$V_j^n \simeq \frac{1}{\Delta r} \int_{r_{j-1/2}}^{r_{j+1/2}} v(t_n, r)\, dr$$

the approximate average of the solution in the space interval $(r_{j-1/2}, r_{j+1/2})$, and let us write a finite volume scheme for the relativistic Burgers equation on a Schwarzschild background in the form

$$V_j^{n+1} = V_j^n - \frac{\Delta t}{\Delta r}(F_{j+1/2} - F_{j-1/2}) - \Delta t \frac{2M}{r_j^2}({V_j^n}^2 - 1), \tag{3-4}$$

where $F_{j+1/2} = \mathcal{F}(r_{j+1/2}, V_j^n, V_{j-1}^n)$ with

$$\mathcal{F}(r, V_L, V_R) = \left(1 - \frac{2M}{r}\right)\tfrac{1}{2}(q(V_L, V_R)^2 - 1) \tag{3-5}$$

with $q(\cdot\,,\cdot)$ the standard solution to the Riemann problem centered at r given by (3-2). The CFL condition

$$\Lambda \max\left(1 - \frac{2M}{r}\right) v \leq 1$$

(the maximum being taken over all relevant values) guarantees that the solution to the Riemann problem does not leave the rectangle $R_{n,j}$ within one time step.

We now consider the boundary condition of our finite volume scheme. Let J be the number of the space mesh points, and we introduce ghost cells at the space boundaries: $R_{n,0} = \{t_n \leq t < t_{n+1},\ r_{-1/2} \leq r < r_{1/2}\}$ and $R_{n,J} = \{t_n \leq t < t_{n+1},\ r_{J-1/2} \leq r < r_{J+1/2}\}$. We solve the Riemann problem at the boundary of the interval $[r_1, r_2]$ with initial conditions

$$V_0(r) = \begin{cases} 1, & r < r_0, \\ V_0^n, & r > r_0, \end{cases} \qquad V_J(r) = \begin{cases} V_J^n, & r < r_J, \\ -1, & r > r_J. \end{cases}$$

A consistency property.

Claim 3.1. *The finite volume method for the relativistic Burgers model introduced in (3-4) satisfies the following properties*:

- *The scheme suitably preserves the steady state solutions to the Euler equations (7-1).*
- *The scheme is consistent*; *that is*, *if $v = v(t, r)$ is an exact solution to the relativistic Burgers model given by the ordinary differential equation (2-1), then for every fixed point $r > 2M$*

$$\mathcal{F}(r_R, V_L, V_R) - \mathcal{F}(r_L, V_L, V_R) = \frac{2M}{r^2}(v^2 - 1)(r_R - r_L) + O(r_R - r_L)^2 \tag{3-6}$$

holds as $V_L, V_R \to v$ and $r_L, r_R \to r$.

Proof. We write

$$\begin{aligned}
&F_{j+1/2} - F_{j-1/2} \\
&\quad = (1-2M/r_{j+1/2})\frac{q^2(V_j^n, V_{j+1}^n)-1}{2} - (1-2M/r_{j-1/2})\frac{q^2(V_{j-1}^n, V_j^n)-1}{2} \\
&\quad = \int_{j-1/2}^{j+1/2} \frac{2M}{r^2}(v^2-1)\,dr = \frac{2M}{r_j^2}(V_j^{n2}-1),
\end{aligned}$$

and therefore, $V_j^n = V_j^{n+1}$ holds. Next, recall that $\mathcal{F}(r, V_L, V_R) = (1-2M/r) \times (q^2(r, V_L, V_R)-1)/2$ is the numerical flux of the scheme determined by the standard Riemann solution. A Taylor expansion gives

$$\begin{aligned}
1-\frac{2M}{r'} &= 1-\frac{2M}{r} + \frac{2M}{r^2}(r-r') + O(r-r')^2, \\
\frac{q^2(r', V_L, V_R)-1}{2} &= \frac{v^2-1}{2} + v\partial_r v(r-r') + O(r-r')^2.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\mathcal{F}(r_R, V_L, V_R) - \mathcal{F}(r_L, V_L, V_R) \\
&\qquad = \frac{2M}{r^2}\frac{v^2-1}{2} + \left(1-\frac{2M}{r}\right) v\partial_r v(r_R-r_L) + O(r_R-r_L)^2 \\
&\qquad = \partial_r\left((1-2M/r)\frac{v^2-1}{2}\right) + O(r_R-r_L)^2 \\
&\qquad = \frac{2M}{r^2}(v^2-1)(r_R-r_L) + O(r_R-r_L)^2. \qquad \square
\end{aligned}$$

A second-order accurate formulation. We now extend the method to second-order accuracy. We follow the MUSCL methodology in order to achieve second-order accuracy in the space variable. Hence, the solution is now discretized as a piecewise linear function, and we define the min-mod expression

$$\Delta_j^n V = \begin{cases} \min(2|\Delta_{j-1/2}V^n|, 2|\Delta_{j+1/2}V^n|, |\Delta_j V^n|) \\ \qquad \text{if } \operatorname{sgn}\Delta_{j-1/2}V^n = \operatorname{sgn}\Delta_{j+1/2}V^n = \operatorname{sgn}\Delta_j V^n, \\ 0 \quad \text{otherwise,} \end{cases} \tag{3-7}$$

where

$$\Delta_j V^n = \tfrac{1}{2}(\Delta V_{j+1}^n - \Delta V_{j-1}^n), \qquad \Delta_{j+1/2}V^n = (\Delta V_{j+1}^n - \Delta V_j^n).$$

Then, our second-order scheme is stated as

$$\begin{aligned}
V_j^{n+1} = V_j^n - \frac{\Delta t}{\Delta r}\big(\mathcal{F}(r_{j+1/2}, V_j^{n+1/2,R}, V_{j+1}^{n+1/2,L}) - \mathcal{F}(r_{j-1/2}, V_{j-1}^{n+1/2,R}, V_j^{n+1/2,L})\big) \\
- \Delta t \frac{2M}{r_j^2}(V_j^2-1), \quad \text{(3-8)}
\end{aligned}$$

in which the numerical flux is still given by (3-5). Here, the two values $V_{j+1}^{n+1/2,L}$, $V_j^{n+1/2,R}$ are given by

$$
\begin{aligned}
V_j^{n+1/2,L} &:= V_j^{n,L} - \frac{\Delta t}{2}\left(\frac{(1-2M/r_j)V_j^n \Delta_j^n V}{\Delta r} - \frac{2M}{r_j^2}(V_j^{n2}-1)\right),\\
V_j^{n+1/2,R} &:= V_j^{n,R} - \frac{\Delta t}{2}\left(\frac{(1-2M/r_j)V_j^n \Delta_j^n V}{\Delta r} - \frac{2M}{r_j^2}(V_j^{n2}-1)\right),
\end{aligned}
\tag{3-9}
$$

where, with $\Delta_j^n V$ defined by (3-7), $V_j^{n,L} = V_j^n - \Delta_j^n V/2$ and $V_j^{n,R} = V_j^n + \Delta_j^n V/2$.

4. Numerical experiments for the relativistic Burgers model, I

Asymptotic-preserving property. We now present some numerical tests with the proposed finite volume method applied to the relativistic Burgers equation (1-2). As mentioned earlier, we work within the domain $r > 2M$, and the mass parameter M is taken to be $M = 1$ in all our tests. We work in the space interval $(r_{\min,}, r_{\max})$ with $r_{\min} = 2M = 2$ and $r_{\max} = 4$, and we take 256 points to discretize the space interval.

We begin by showing that the method at both first-order and second-order accuracy preserves the steady state solutions. For positive/negative steady state Burgers solutions $v = \pm\sqrt{3/4 + 1/(2r)}$, we see that the initial steady states are exactly conserved by the scheme. We also show that the following steady state shock is preserved by the scheme:

$$
v = \begin{cases} \sqrt{3/4+1/(2r)}, & 2 < r < 3,\\ -\sqrt{3/4+1/(2r)}, & r > 3. \end{cases}
$$

We obtain that our finite volume scheme preserves three typical forms for the static solutions, as is illustrated in Figures 2 and 3.

A moving shock separating two static solutions. In view of Theorem 2.1, whether the solution to the Riemann problem will move towards the black hole horizon depends only on the behavior of the initial velocity. We take again the space interval to be (2.0, 4.0) with 256 space mesh points. We take then two kinds of initial data

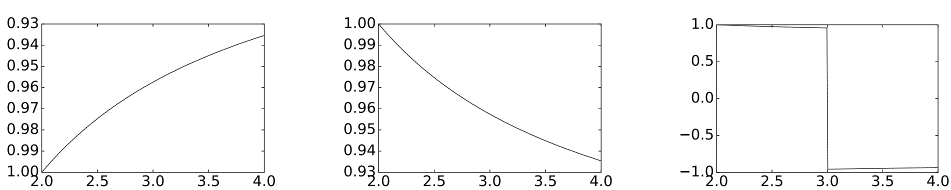

Figure 2. Burgers: three typical behaviors of steady states.

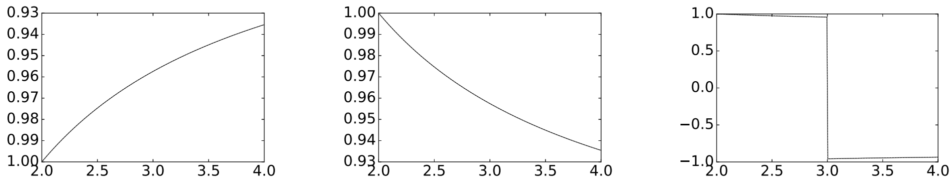

Figure 3. Burgers: solution at time $t = 20$ for three steady state solutions (second-order FVM).

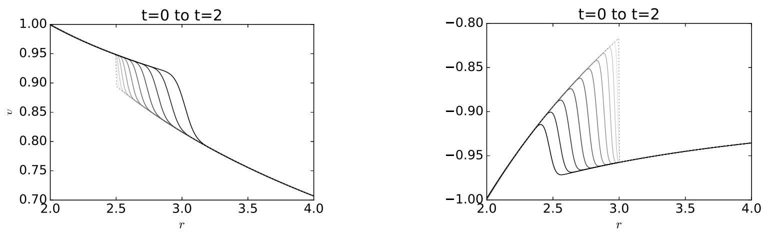

Figure 4. Burgers: right- and left-moving shocks (first-order FVM).

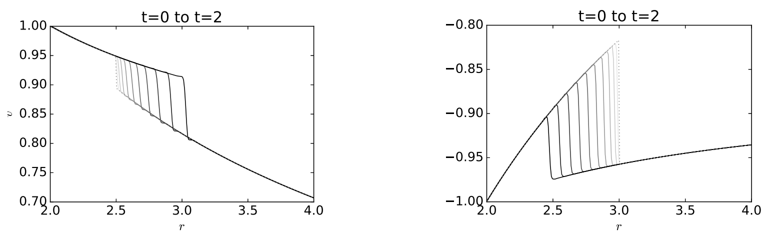

Figure 5. Burgers: right- and left-moving shocks (second-order FVM).

to be

$$v = \begin{cases} \sqrt{1/2 + 1/r}, & 2 < r < 2.5, \\ \sqrt{2/r}, & r > 2.5, \end{cases} \qquad v = \begin{cases} -\sqrt{2/r}, & 2 < r < 2.5, \\ -\sqrt{3/4 + 1/(4r)}, & r > 2.5. \end{cases}$$

The behavior of the two shock solutions obtained with the first-order and second-order accurate versions are shown in Figures 4 and 5.

Late-time behavior of solutions. We now study the late-time behavior of solutions whose initial data is given as (2-4), that is, steady state solution with a compactly supported perturbation. We treat the following two kinds of steady state solutions whose values at $r = 2M$ are ± 1, respectively:

$$v = \sqrt{1/2 + 1/r}, \qquad v = -\sqrt{1/2 + 1/r},$$

with compactly supported perturbations (see Figure 6).

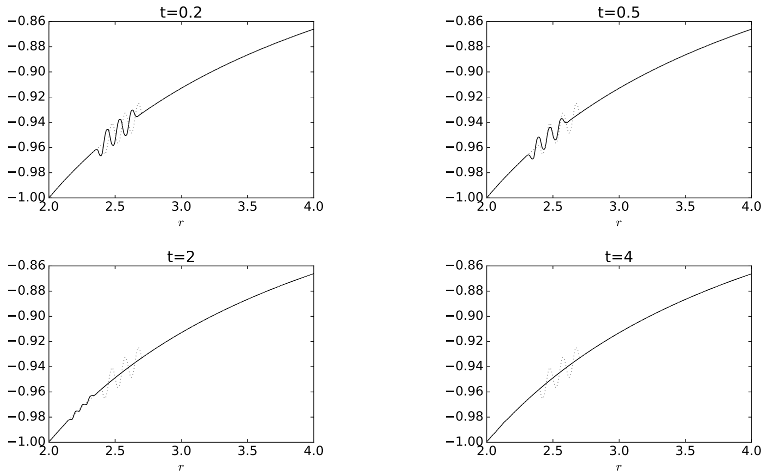

Figure 6. Burgers: evolution of a perturbed steady state (second order FVM).

5. A generalized random choice scheme for the relativistic Burgers model

Explicit solution to the generalized Riemann problem. In order to construct a Glimm method for the relativistic Burgers model, we need to first introduce the explicit form of the generalized Riemann problem of the relativistic Burgers equation (1-1), which is an initial problem whose initial data $v_0 = v_0(r)$ is given as

$$v_0(r) = \begin{cases} v_L(r), & 2M < r < r_0, \\ v_R(r), & r > r_0, \end{cases} \tag{5-1}$$

where r_0 is a fixed point in space and $v_L = v_L(r)$ and $v_R = v_R(r)$ are two steady state solutions of Burgers equation with explicit form

$$v_L(r) = \operatorname{sgn}(v_L^0)\sqrt{1 - K_L^2\left(1 - \frac{2M}{r}\right)}, \qquad v_R(r) = \operatorname{sgn}(v_R^0)\sqrt{1 - K_R^2\left(1 - \frac{2M}{r}\right)}, \tag{5-2}$$

where $K_L, K_R > 0$ are two constants and we denote $v_L^0 = v_L(r_0)$ and $v_R(r_0) = v_R^0$. The existence of the generalized Riemann problem is stated in Theorem 2.1. More precisely, the solution to the Riemann problem $v = v(t, r)$ can be realized by either a shock wave or a rarefaction wave which is given explicitly by the form

$$v(t,r) = \begin{cases} v_L(r), & r < r_L(t), \\ \tilde{v}(t,r), & r_L(t) < r < r_R(t), \\ v_R(r), & r > r_R(t). \end{cases} \tag{5-3}$$

Here, $r_L(t)$ and $r_R(t)$ are bounds of rarefaction regions satisfying

$$R_j(r_j(t)) - R_j(r_0) = t, \tag{5-4}$$

where $R_j = R_j(r)$ is given by

$$R_j(r) := \frac{R^{v_j}(r)}{2} + \chi_{[v_j^0<v_k^0]}(r)\frac{R^{v_j}(r)}{2} + \chi_{[v_j^0<v_k^0]}(r)\frac{R^{v_k}(r)}{2} \tag{5-5}$$

with $j = L, R$ and $k = R, L$,

$$\chi_{[v_j^0 \gtrless v_k^0]}(r) = \begin{cases} 1, & v_j^0 \gtrless v_k^0, \\ 0, & \text{otherwise}, \end{cases}$$

and the function $R_j^v = R_j^v(r)$ given by

$$\begin{aligned} R^{v_j}(r) &:= \operatorname{sgn}(v_j)\frac{1}{(1-K_j^2)^{3/2}}\Bigg(2M(1-K_j^2)^{3/2}\ln(r-2M) \\ &- 2M(1-K_j^2)^{3/2}\ln\left(2r\sqrt{1-K_j^2\left(1-\frac{2M}{r}\right)}+(2M-r)K_j^2\right) \\ &+ 1\left(r\sqrt{1-K_j^2}\sqrt{1-{K_j}^2\left(1-\frac{2M}{r}\right)}\right. \\ &\left.\left.+ M(2-3K_j^2)\ln\left(r\sqrt{1-K_j^2}\sqrt{1-K_j^2\left(1-\frac{2M}{r}\right)}+(M-r)K_j^2+r\right)\right)\right). \end{aligned} \tag{5-6}$$

The function $\tilde{v} = \tilde{v}(t, r)$ denotes the generalized rarefaction wave

$$\tilde{v}(t, r) = \operatorname{sgn}(r - r_0)\sqrt{1 - K^2(t, r)\left(1 - \frac{2M}{r}\right)}, \tag{5-7}$$

where $K = K(t, r)$ is characterized by the condition

$$\operatorname{sgn}(r - r_0) = \frac{\widetilde{R}(r, K) - \widetilde{R}(r_0, K)}{t}, \tag{5-8}$$

where

$$\begin{aligned} \widetilde{R}(r, K) &:= \frac{1}{(1-K^2)^{3/2}}\Bigg(2M(1-K^2)^{3/2}\ln(r-2M) \\ &- 2M(1-K^2)^{3/2}\ln\left(2r\sqrt{1-K^2\left(1-\frac{2M}{r}\right)}+(2M-r)K^2\right) \\ &+ \left(r\sqrt{1-K^2}\sqrt{1-K^2\left(1-\frac{2M}{r}\right)}\right. \\ &\left.\left.+ M(2-3K^2)\ln\left(r\sqrt{1-K^2}\sqrt{1-K^2\left(1-\frac{2M}{r}\right)}+(M-r)K^2+r\right)\right)\right). \end{aligned} \tag{5-9}$$

One can check that (5-3) satisfies the Rankine–Hugoniot jump conditions and the entropy inequalities. Importantly, the solution to the generalized Riemann problem is globally defined in time and space.

A generalized random choice method. The random choice method is a scheme based on generalized Riemann problems. We use again the time-space grid where the mesh lengths in time and in space are Δt, Δr with $t_n = n\Delta t$ and $r_j = 2M + j\Delta r$ where we recall $2M$ is the black hole horizon. Denote by V_j^n the numerical solution $V(n\Delta t, 2M + j\Delta r)$. Let (w_n) be a sequence equidistributed in $(-\frac{1}{2}, \frac{1}{2})$, and write $r_{n,j} = 2M + (j + w_n)\Delta r$. We define our Glimm-type approximations as

$$V_j^{n+1} = V_{\mathcal{R}}^{j,n}(t_{n+1}, r_{n,j}), \tag{5-10}$$

where $V_{\mathcal{R}}^{j,n} = V_{\mathcal{R}}^{j,n}(t, r)$ is the solution to the Riemann problem with the initial data

$$V_0^{j,n} = \begin{cases} V_L^{j,n}(r), & r < r_{j+\mathrm{sgn}(w_n)/2}, \\ V_R^{j,n}(r), & r > r_{j+\mathrm{sgn}(w_n)/2}, \end{cases} \tag{5-11}$$

where the left-hand state $V_L^{j,n} = V_L^{j,n}(r)$ and the right-hand state $V_R^{j,n} = V_R^{j,n}(r)$ are steady state solutions to (2-1) with initial conditions

$$\begin{cases} V_L^{j,n}(r_j) = V_j^n, & w_n \geq 0, \\ V_L^{j,n}(r_{j-1}) = V_{j-1}^n, & w_n < 0, \end{cases} \qquad \begin{cases} V_R^{j,n}(r_j) = V_j^n, & w_n > 0, \\ V_R^{j,n}(r_{j+1}) = V_{j+1}^n, & w_n \geq 0. \end{cases}$$

We choose a random number only once at each time level $t = t_n$ rather than in every mesh cell as was done in the original Glimm method.

In order to have an equidistributed sequence, the random values (w_n) are defined by following Chorin [9]: we give two large prime numbers $p_1 < p_2$ and define a sequence of integers (q_n) by

$$q_0 \quad \text{given } q_0 < p_2, \qquad q_n := (p_1 + q_{n-1}) \bmod p_2, \quad n \geq 1. \tag{5-12}$$

Then we define the sequence $w_n' = (q_n + w_n + \frac{1}{2})/p_2 - \frac{1}{2}$, which is to be used in our Glimm method instead of (w_n). It is direct to see that $w_n' \in (-\frac{1}{2}, \frac{1}{2})$.

6. Numerical experiments for the relativistic Burgers model, II

Consistency property. We now present numerical experiments with the proposed Glimm method for the Burgers equation on a Schwarzschild background (1-1). Recall that $r > 2M$, and we choose again $M = 1$ for the black hole mass. The space interval in consideration is $(r_{\min,}, r_{\max})$ with $r_{\min} = 2M = 2$ and $r_{\max} = 4$. To introduce the random sequence, we fix two prime integers, specifically $p_1 = 937$ and $p_2 = 997$ and $q_0 = 800$. Since the solution to every local generalized Riemann problem (1-1) with (5-1) is exact, the following observation is immediate.

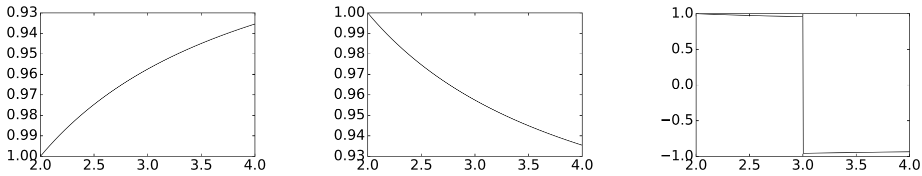

Figure 7. Burgers: evolution at time $t = 20$ from a steady state initial data (Glimm scheme).

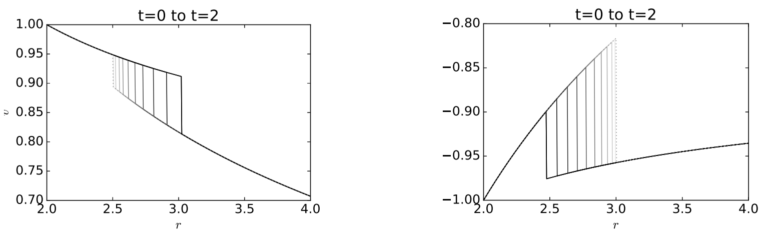

Figure 8. Burgers: right- and left-moving shocks (Glimm scheme).

Claim 6.1. *Consider a given initial velocity $v_0 = v_0(r)$ as a steady state solution such that the static Burgers model (2-1) holds. Then the approximate solution to the relativistic Burgers equation (1-1) constructed by the Glimm method (5-10) is exact for such data.*

We will still observe the evolution of those three types of solutions shown in Figure 2, that is, the two steady state solutions $v = \pm\sqrt{3/4 + 1/(2r)}$ and the steady shock

$$v = \begin{cases} \sqrt{3/4 + 1/(2r)}, & 2 < r < 3, \\ -\sqrt{3/4 + 1/(2r)}, & r > 3. \end{cases}$$

Different types of shocks. We consider two different shocks whose initial speeds are positive and negative. As was observed by the finite volume method, whether the position of the shock will go toward the black hole horizon is determined uniquely by their initial behavior. We can recover the same conclusion with the Glimm method. Again, we take two kinds of initial data:

$$v = \begin{cases} \sqrt{1/2 + 1/r}, & 2 < r < 2.5, \\ \sqrt{2/r}, & r > 2.5, \end{cases} \qquad v = \begin{cases} -\sqrt{2/r}, & 2 < r < 2.5, \\ \sqrt{3/4 + 1/(4r)}, & r > 2.5. \end{cases}$$

Since our Riemann solver is exact, the numerical solutions contain no numerical diffusion (see Figure 8).

Asymptotic behavior. We are now interested in the evolution of solutions whose initial data are given as piecewise steady state solutions satisfying (2-1). As was

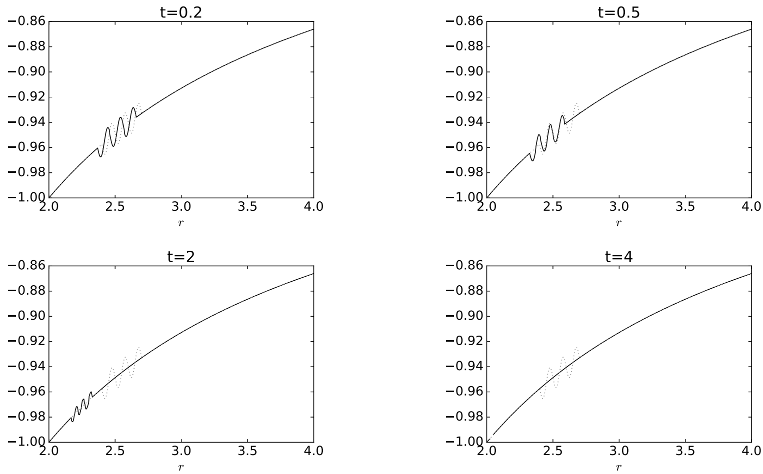

Figure 9. Burgers: evolution from an initially perturbed steady state (Glimm method).

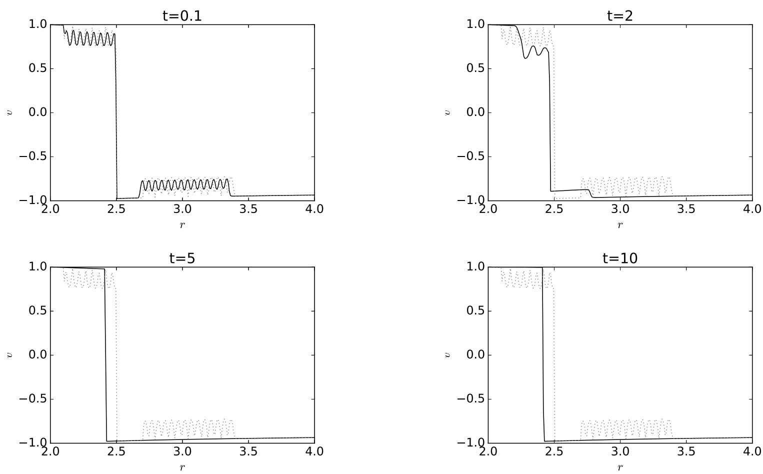

Figure 10. Burgers: evolution of a perturbed steady shock (Glimm method).

done earlier, we take into account two kinds of initial data:

$$v = \sqrt{1/2 + 1/r}, \qquad v = \begin{cases} \sqrt{1/2 + 1/r}, & 2 < r < 2.5, \\ \sqrt{2/r}, & r > 2.5, \end{cases}$$

perturbed by compactly supported functions (see Figure 9).

Steady shock with perturbation. The behavior of a smooth steady state solution to the relativistic Burgers model (1-1) perturbed by a function on a compactly supported function is understood both numerically and theoretically: the solution converges to the same initial steady state solution. The steady shock (2-3) is a

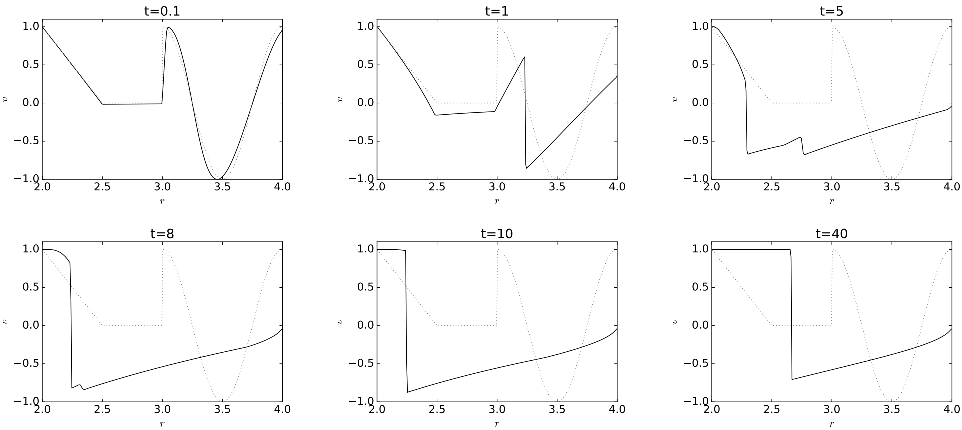

Figure 11. Burgers: evolution with prescribed velocity 1 at $r = 2M$ and at $r = +\infty$ (Glimm scheme).

solution to the static equation (2-1) in the distribution sense. We are interested in the asymptotic behavior, and our numerical results in Figure 10 lead us to the following:

Conclusion 6.1. *Consider a perturbed steady shock given as* (2-3):

$$v_0 = \begin{cases} \sqrt{1 - K^2(1 - 2M/r)}, & 2M < r < r_0, \\ -\sqrt{1 - K^2(1 - 2M/r)}, & r > r_0, \end{cases}$$

where K is a given constant and $r_0 > 2M$ is a fixed radius out of the Schwarzschild black hole region. The solution to the relativistic Burgers model (1-1) *converges at some finite time to a solution of the form* (*with possibly $r_1 \neq r_0$*)

$$v = \begin{cases} \sqrt{1 - K^2(1 - 2M/r)}, & 2M < r < r_1, \\ -\sqrt{1 - K^2(1 - 2M/r)}, & r > r_1. \end{cases}$$

Late-time behavior of general solutions. It is obvious that the steady state solution satisfying (2-1) serves as a solution to the relativistic Burgers equation on a Schwarzschild background. Notice that, on the black hole horizon $r = 2M$, the steady state solution equals the light speed, that is, either 1 or -1, which equals exactly the light speed and obviously their boundary values will not change as time evolves. The value of a steady state solution at infinity is also given explicitly. Observations on the numerical method shows that the asymptotic behavior of the Burgers model (1-1) is mainly determined by the values of the initial data at the black hole horizon $r = 2M$ and the space infinity $r = +\infty$. More precisely, suppose that a given velocity $v_0 = v_0(r)$ does not satisfy the static Burgers equation (2-1); then we have the following conclusion (see Figures 11, 12, and 13).

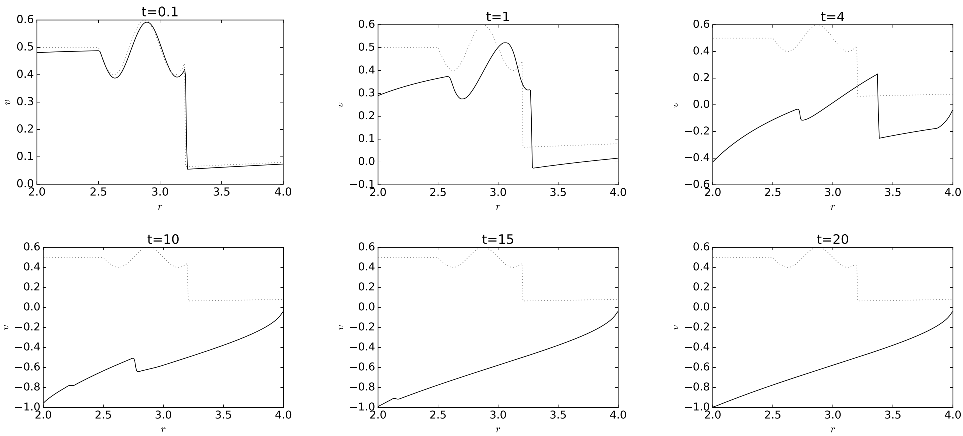

Figure 12. Burgers: evolution with given velocity less than 1 at $r = 2M$ and positive at $r = +\infty$ (Glimm scheme).

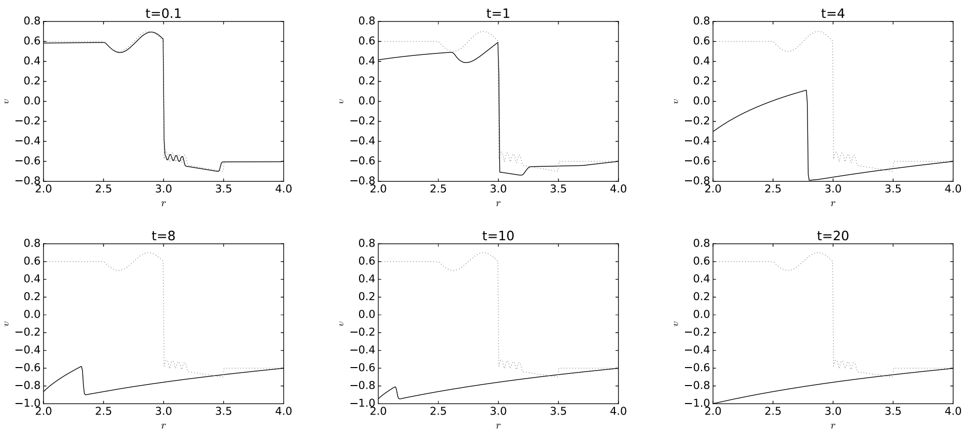

Figure 13. Burgers: evolution with velocity less than 1 at $r = 2M$ and negative at $r = +\infty$ (Glimm scheme).

Conclusion 6.2. (1) *If the initial velocity* $\lim_{r\to 2M} v_0(r) = 1$, *then the solution to the Burgers equation (1-1) satisfies that there exists a time* $t > t_0$ *such that for all* $t > t_0$ *the solution* $v = v(t, r)$ *is a shock with left-hand state* 1 *and right-hand state* v_*^- *with* $v_*^-(r) = -\sqrt{2M/r}$ *the negative critical steady solution.*

(2) *If the initial velocity* $\lim_{r\to 2M} v_0(r) < 1$ *and* $\lim_{r\to +\infty} v_0(r) > 0$, *there exists a time* $t_0 > 0$ *such that the solution to the Burgers equation* $v(t, r) = v_*^-(r)$ *for all* $t > t_0$ *where* $v_*^-(r) = -\sqrt{2M/r}$ *is the negative critical steady state solution to the relativistic Burgers model.*

(3) *If the initial velocity* $\lim_{r\to 2M} v_0(r) < 1$ *and* $\lim_{r\to+\infty} v_0(r) \le 0$, *then the solution to the relativistic Burgers model satisfies that*

$$v(t,r) = -\sqrt{1-(1-v_0^{\infty 2})\left(1-\frac{2M}{r}\right)}$$

for $t > t_0$ *for a time* $t_0 > 0$ *where* $0 \ge v_0^\infty = \lim_{r\to+\infty} v_0(r)$.

7. Overview of the theory of the relativistic Euler model

Continuous and discontinuous steady state solutions. Steady solutions to the relativistic Euler model on a Schwarzschild background (1-3) are given by the differential system

$$\begin{aligned} &\partial_r\left(r(r-2M)\frac{1}{1-v^2}\rho v\right) = 0,\\ &\partial_r\left((r-2M)^2\frac{v^2+k^2}{1-v^2}\rho\right)\\ &\qquad = \frac{M}{r}\frac{(r-2M)}{1-v^2}(3\rho v^2+3k^2\rho-\rho-k^2\rho v^2)+\frac{2k^2}{r}(r-2M)^2\rho. \end{aligned} \tag{7-1}$$

Smooth steady states associated with a radius $r_0 > 2M$, a density $\rho_0 > 0$, and a velocity $|v_0| < 1$ are given by solving the algebraic system

$$\begin{aligned} &\operatorname{sgn}(v)(1-v^2)|v|^{2k^2/(1-k^2)}\frac{r^{4k^2/(1-k^2)}}{(1-2M/r)}\\ &\qquad = \operatorname{sgn}(v_0)(1-v_0^2)|v_0|^{2k^2/(1-k^2)}\frac{r_0^{4k^2/(1-k^2)}}{(1-2M/r_0)},\\ &r(r-2M)\rho\frac{v}{1-v^2} = r_0(r_0-2M)\rho_0\frac{v_0}{1-v_0^2}. \end{aligned} \tag{7-2}$$

We have also the expressions of the first-order derivatives

$$\begin{aligned} \frac{d\rho}{dr} &= -\frac{2(r-M)}{r(r-2M)}\rho - \frac{(1+v^2)(1-k^2)}{r(r-2M)}\rho\left(\frac{2k^2}{1-k^2}(r-2M)-M\right)/(v^2-k^2),\\ \frac{dv}{dr} &= v\frac{(1-v^2)(1-k^2)}{r(r-2M)}\left(\frac{2k^2}{1-k^2}(r-2M)-M\right)/(v^2-k^2). \end{aligned} \tag{7-3}$$

We denote the *critical steady state solution to the relativistic Euler model* (1-3) (ρ, v) with its velocity $v = v(r)$ satisfying

$$\begin{aligned} &\frac{1-\epsilon^2v^2}{1-2M/r}(r^2|v|)^{2\epsilon^2k^2/(1-\epsilon^2k^2)}\\ &\qquad = (1+3\epsilon^2k^2)k^{2\epsilon^2k^2/(1-\epsilon^2k^2)}\left(\frac{1+3\epsilon^2k^2}{2\epsilon^2k^2}M\right)^{4\epsilon^2k^2/(1-\epsilon^2k^2)}. \end{aligned} \tag{7-4}$$

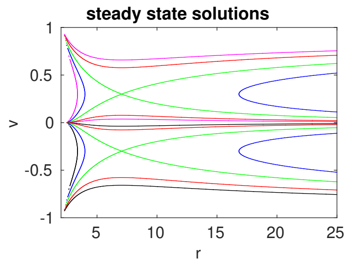

Figure 14. Euler: steady state solutions.

Unlike the static Burgers model (2-1), steady state solutions to the relativistic Euler model do not have an explicit form. We recall the following result established in [22]. See Figure 14 for an illustration.

Theorem 7.1 (smooth steady flows on a Schwarzschild background). *Denoting by* $k \in [0, 1]$ *the sound speed and by* $M > 0$ *the mass of the black hole*, *let us consider the relativistic Euler model on a Schwarzschild background* (1-3). *For any given radius* $r_0 > 2M$, *density* $\rho_0 > 0$, *and velocity* $|v_0| < 1$, *there exists a unique smooth steady state solution* $\rho = \rho(r)$ *and* $v = v(r)$ *satisfying* (7-2) *and the initial conditions* $\rho(r_0) = \rho_0$ *and* $v(r_0) = v_0$. *Moreover*, *the velocity component is such that the signs of* $v(r)$ *and* $|v(r)| - k$ *do not change within the domain of definition of this solution. Two different families of solutions can be distinguished.*

- *If there exists no sonic point at which*, *by definition*, *the fluid velocity equals the sound speed*, *the* (*smooth*) *steady state solution is defined globally on the whole interval outside of the black hole horizon* $(2M, +\infty)$.
- *Otherwise*, *the steady state solution cannot be extended as a smooth solution once it reaches the sonic point.*

It is natural to then consider steady shock waves to (1-3), that is, two steady state solutions connected by a standing shock:

$$(\rho, v) = \begin{cases} (\rho_L, v_L)(r), & 2M < r < r_0, \\ (\rho_R, v_R)(r), & r > r_0, \end{cases} \tag{7-5}$$

where $r_0 > 2M$ is a given radius and (ρ_L, v_L) and (ρ_R, v_R) are steady state solutions satisfying (7-2) and

$$v_R(r_0) = \frac{k^2}{v_L(r_0)}, \quad \rho_R(r_0) = \frac{v_L(r_0)^2 - k^4}{k^2(1 - v_L(r_0)^2)} \rho_L(r_0), \quad v_L(r_0) \in (-k, -k^2) \cup (k, 1). \tag{7-6}$$

We refer to such a solution as a *steady shock of the relativistic Euler model*, that is, a function of the form (7-5) and (7-6) satisfying (7-1) in the distributional sense, satisfying the Lax entropy inequality and the Rankine–Hugoniot jump conditions.

Observe that, for a fixed radius $r_1 \neq r_0$ and (ρ_L, v_L), (ρ_R, v_R) satisfying (7-5), the following function is *not* a steady shock of the Euler model (1-3):

$$(\rho, v) = \begin{cases} (\rho_L, v_L)(r), & 2M < r < r_1, \\ (\rho_R, v_R)(r), & r > r_1. \end{cases}$$

Generalized Riemann problem and Cauchy problem. The generalized Riemann problem for the relativistic Euler system (1-3) is the Cauchy problem with initial data

$$(\rho_0, v_0)(r) = \begin{cases} (\rho_L, v_L)(r), & 2M < r < r_0, \\ (\rho_R, v_R)(r), & r > r_0, \end{cases} \tag{7-7}$$

where $r = r_0$ is a fixed radius and $\rho_L = \rho_L(r)$, $v_L = v_L(r)$, $\rho_R = \rho_R(r)$, and $v_R = v_R(r)$ are two smooth steady state solutions satisfying the static Euler equations (7-1). Referring to [22], we can construct an approximate solver $\widetilde{U} = (\tilde{\rho}, \tilde{v}) = (\tilde{\rho}, \tilde{v})(t, r)$ of the generalized Riemann problem of the relativistic Euler model (1-3) whose initial data is (7-7) such that:

- $\|\widetilde{U}(t, \cdot) - U(t, \cdot)\|_{L^1} = O(\Delta t^2)$ for any fixed $t > 0$ where $U = (\rho, v) = (\rho, v)(t, r)$ satisfies (1-3) and (7-7) and Δt is the time step in the construction.
- $\widetilde{U} = (\tilde{\rho}, \tilde{v})$ is exact outside the rarefaction fan regions.
- $\widetilde{U} = (\tilde{\rho}, \tilde{v})$ (and the exact solution U) contains at most three steady states: the two states given in the initial data (ρ_L, v_L), (ρ_R, ρ_R) and the uniquely defined intermediate (ρ_M, v_M) connected by a 1-family wave (either 1-shock or 1-rarefaction) and a 2-family wave (either 2-shock or 2-rarefaction).

Theorem 7.2 (existence theory of the relativistic Euler model). *Consider the Euler system describing fluid flows on a Schwarzschild geometry* (1-3)*. For any initial density $\rho_0 = \rho_0(r) > 0$ and velocity $|v_0| = |v_0(r)| < 1$ satisfying*

$$TV_{[2M+\delta,+\infty)}(\ln \rho_0) + TV_{[2M+\delta,+\infty)}\left(\ln \frac{1-v_0}{1+v_0}\right) < +\infty,$$

where $\delta > 0$ is a constant, there exists a weak solution $(\rho, v) = (\rho, v)(t, r)$ defined on $(0, T)$ for any given $T > 0$ and satisfying the prescribed initial data at the initial time and, with a constant C independent of time,

$$\sup_{t\in[0,T]}\left(TV_{[2M+\delta,+\infty)}(\ln \rho(t,\cdot)) + TV_{[2M+\delta,+\infty)}\left(\ln \frac{1-v(t,\cdot)}{1+v(t,\cdot)}\right)\right) \\ \leq TV_{[2M+\delta,+\infty)}(\ln \rho_0) + TV_{[2M+\delta,+\infty)}\left(\ln \frac{1-v_0}{1+v_0}\right)e^{CT}.$$

8. A finite volume method for the relativistic Euler model

A semidiscretized numerical scheme. We write the relativistic equations on a Schwarzschild background (1-4) as

$$\partial_t U + \partial_r\left(\left(1 - \frac{2M}{r}\right)F(U)\right) = S(r, U), \tag{8-1}$$

$$U = \begin{pmatrix} U^0 \\ U^1 \end{pmatrix} = \begin{pmatrix} \dfrac{1+k^2v^2}{1-v^2}\rho \\ \dfrac{1+k^2}{1-v^2}\rho v \end{pmatrix}, \qquad F(U) = \begin{pmatrix} \dfrac{1+k^2}{1-v^2}\rho v \\ \dfrac{v^2+k^2}{1-v^2}\rho \end{pmatrix}, \tag{8-2}$$

with source term

$$S(r, U) = \begin{pmatrix} -\dfrac{2}{r}(1-2M/r)\dfrac{1+k^2}{1-v^2}\rho v \\ \dfrac{-2r+5M}{r^2}\dfrac{v^2+k^2}{1-v^2}\rho - \dfrac{M}{r^2}\dfrac{1+k^2v^2}{1-v^2}\rho + 2\dfrac{r-2M}{r^2}k^2\rho \end{pmatrix}.$$

The Jacobian matrix

$$D_U F(U) = \begin{pmatrix} 0 & 1 \\ (-v^2+k^2)/(1-k^2v^2) & 2(1-k^2)v/(1-k^2v^2) \end{pmatrix} \tag{8-3}$$

admits two real and distinct eigenvalues, denoted $\mu_\mp = (1-2M/r)(v\mp k)/(1\mp k^2v)$. We also have

$$v = \frac{1+k^2 - \sqrt{(1+k^2)^2 - 4k^2(U^1/U^0)^2}}{2k^2U^1/U^0} \in (-1, 1)$$

and $\rho = U^1(1-v^2)/(v(1+k^2))$.

Denote by Δt and Δr the mesh lengths in time and in space, respectively, and assume the CFL condition

$$\frac{\Delta t}{\Delta x}\max(|\mu_-|, |\mu_+|) \le \tfrac{1}{2}. \tag{8-4}$$

We write $t_n = n\Delta t$ and $r_j = 2M + j\Delta r$, and we consider the corresponding mesh points (t_n, r_j) for all integers $n \ge 0$ and $j \ge 0$. We also set $\rho(t_n, r_j) = \rho_j^n$, $v(t_n, r_j) = v_j^n$, and $U(t_n, r_j) \simeq U_n^j$ where $U = U(t, r)$ is a solution to (8-1).

We search for our approximations $U_j^n = (1/\Delta r)\int_{r_{j-1/2}}^{r_{j+1/2}} U(t_n, r)\,dr$ in the finite volume form

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta r}(F_{j+1/2}^n - F_{j-1/2}^n) + \Delta t S_j^n, \tag{8-5}$$

where the numerical flux is expressed in the form

$$F^n_{j-1/2} = \mathcal{F}_l(r_{j-1/2}, U^n_{j-1}, U^n_j) = \left(1 - \frac{2M}{r_{j-1/2}}\right)\mathcal{F}(U^n_{j-1/2-}, U^n_{j-1/2+}), \quad (8\text{-}6)$$

and $U_{j+1/2\pm}$ and $U_{j-1/2\pm}$ as well as the source term $S^n_j = (1/\Delta r)\int_{r_{j-1/2}}^{r_{j+1/2}} S(t,_n r)\,dr$ must still be determined. For definiteness, we choose the Lax–Friedrichs flux

$$\mathcal{F}(U_L, U_R) = \frac{F(U_L) + F(U_R)}{2} - \frac{1}{\lambda}\frac{U_R - U_L}{2} \quad (8\text{-}7)$$

with $\lambda = \Delta r/\Delta t$, where F is the exact flux (8-1).

Taking the curved geometry into account. It remains to determine the states $U_{j+1/2\pm}$ and $U_{j-1/2\pm}$ as well as the discretized source S^n_j, which must take into account the Schwarzschild geometry. For a steady state solution $U = U(r)$, the equation $\partial_r((1 - 2M/r)F(U)) = S(r, U)$ holds, where U, F, and the source term S are given by (8-1). (Equivalently, the solution (ρ, v) satisfies the static Euler equations (7-1).) We propose to represent the numerical solution in each cell $(r_{j-1/2}, r_{j+1/2})$ as a steady state solution, whenever such a solution is available. Hence, we require the algebraic relations

$$\begin{aligned}
&(1 - {v^n_{j+1/2-}}^2){v^n_{j+1/2-}}^{2k^2/(1-k^2)} r_{j+1/2}^{4k^2/(1-k^2)}/(1 - 2M/r_{j+1/2}) \\
&\qquad\qquad = (1 - v^{n2}_j){v^n_j}^{2k^2/(1-k^2)} r_j^{4k^2/(1-k^2)}/(1 - 2M/r_j), \\
&r_{j+1/2}(r_{j+1/2} - 2M)\rho^n_{j+1/2-}\frac{v^n_{j+1/2-}}{1 - {v^n_{j+1/2-}}^2} \\
&\qquad\qquad = r_j(r_j - 2M)\rho^n_j\frac{v^n_j}{1 - v^{n2}_j}, \\
&(1 - {v^n_{j+1/2+}}^2){v^n_{j+1/2+}}^{2k^2/(1-k^2)} r_{j+1/2}^{4k^2/(1-k^2)}/(1 - 2M/r_{j+1/2}) \\
&\qquad\qquad = (1 - {v^n_{j+1}}^2){v^n_{j+1}}^{2k^2/(1-k^2)} r_{j+1}^{4k^2/(1-k^2)}/(1 - 2M/r_{j+1}), \\
&r_{j+1/2}(r_{j+1/2} - 2M)\rho^n_{j+1/2+}\frac{v^n_{j+1/2+}}{1 - {v^n_{j+1/2+}}^2} \\
&\qquad\qquad = r_j + 1(r_{j+1} - 2M)\rho^n_{j+1}\frac{v^n_{j+1}}{1 - {v^n_{j+1}}^2}.
\end{aligned} \quad (8\text{-}8)$$

A difficulty arises here from the fact that a steady state solution need not be defined globally on the whole interval $(2M, +\infty)$, and it is possible that (8-8) does not admit a solution. However, this difficulty can be solved as follows: we simply set $(\rho^n_{j+1/2-}, v^n_{j+1/2-}) = (\rho^n_j, v^n_j)$ when the first two equations in (8-8) do not have a solution, while we set $(\rho^n_{j+1/2-}, v^n_{j+1/2-}) = (\rho^n_{j+1}, v^n_{j+1})$ when the last two equations in (8-8) do not admit a solution.

Next, integrating (8-5) by parts, we obtain an expression for the source terms, i.e.,

$$\begin{aligned} S^n_j &= \frac{1}{\Delta r}\int_{r_{j-1/2}}^{r_{j+1/2}} S(t_n, r)\,dr = \frac{1}{\Delta r}\int_{r_{j-1/2}}^{r_{j+1/2}} \partial_r((1-2M/r)F(U(t_n, r)))\,dr \\ &= \frac{1}{\Delta r}\big((1-2M/r_{j+1/2})F(U^n_{j+1/2-}) - (1-2M/r_{j-1/2+})F(U^n_{j-1/2+})\big), \end{aligned} \tag{8-9}$$

where the states $U^n_{j+1/2-}$ and $U^n_{j-1/2+}$ are determined by (8-8) and $F(\cdot)$ denotes the flux of the Euler system (8-1). Finally, second-order accuracy in time is achieved in a standard manner via the MUSCL methodology.

Theorem 8.1. *The finite volume scheme proposed for the relativistic Euler equations on a Schwarzschild background (1-4) satisfies*:

- *The scheme preserves the steady state solution to the Euler equations (7-1).*
- *The scheme is consistent*; *that is, for an exact solution $U = U(t, r)$ and the states $U_L, U_R \to U$ and $r_L, r_R \to r$, we have*

$$\mathcal{F}_r(r_R, U_L, U_R) - \mathcal{F}_l(r_L, U_L, U_R) = S(r, U)(r_R - r_L) + O((r_R - r_L)^2), \tag{8-10}$$

  where $\mathcal{F}_l, \mathcal{F}_r$ are numerical fluxes given by (8-6) and $S(r, U)$ is the source term given by (8-1).
- *The scheme has second-order accuracy in space and first-order accuracy in time.*

Proof. For a steady state given by (7-1), we have $U_{j+1/2+} = U_{j+1/2-}$. Hence, the flux of the finite volume method (8-6) satisfies $F_{j+1/2} = (1-2M/r_{j+1/2})F(U_{j+1/2+}) = (1-2M/r_{j+1/2})F(U_{j+1/2-})$, which gives

$$\begin{aligned} &\frac{1}{\Delta r}(F^n_{j+1/2} - F^n_{j-1/2}) \\ &\qquad = (1-2M/r_{j+1/2})F(U_{j+1/2-}) - (1-2M/r_{j-1/2})F(U_{j-1/2+}) = S^n_j. \end{aligned}$$

Therefore, the scheme preserves the steady state solutions. Next, according to (8-8) and (8-9), there exist four states U_L^l, U_R^l, U_L^r, U_R^r such that

$$\begin{aligned}
&\mathscr{F}_r(r_R, U_L, U_R) - \mathscr{F}_l(r_L, U_L, U_R)\\
&\qquad = (1-2M/r_R)\mathscr{F}(U_L^r, U_R^r) - (1-2M/r_L)\mathscr{F}(U_L^l, U_R^l)\\
&\qquad = (1-2M/r + 2M/r^2(r_R - r) + O(r_R - r))\\
&\qquad\qquad\qquad \times (\mathscr{F}(U,U) + \partial_1\mathscr{F}(U,U)(U_R - U) + o(U_R - U))\\
&\qquad\qquad - (1-2M/r + 2M/r^2(r_L - r) + O(r_L - r))\\
&\qquad\qquad\qquad \times (\mathscr{F}(U,U) + \partial_2\mathscr{F}(U,U)(U_L - U) + o(U_L - U)).
\end{aligned}$$

By (8-8), $U_R - U_L = O(r_R - r_L)S(r, U)$. Moreover, since $U = U(t, r)$ is exact, we have $\mathscr{F}(U, U) = F(U)$ and $\partial_1\mathscr{F}(U, U) = \partial_2\mathscr{F}(U, U) = \partial_U F(U)$. Therefore,

$$\begin{aligned}
&\mathscr{F}_r(r_R, U_L, U_R) - \mathscr{F}_l(r_L, U_L, U_R)\\
&\quad = \frac{2M}{r^2}(r_R - r_L)F(U) + (1-2M/r)\partial_U F(U)(U_R - U_L) + O((r_R - r_L)^2)\\
&\quad = S(r, U)(r_R - r_L) + O((r_R - r_L)^2).
\end{aligned}$$

Next, a Taylor expansion with respect to time yields us

$$U_j^{n+1} = U_j^n + \partial_t U_j^n \Delta t + \partial_{tt}^2 U_j^n \Delta t^2 + O(\Delta t^3).$$

Recall that our scheme gives

$$\begin{aligned}
U_j^{n+1} &= U_j^n - \frac{\Delta t}{\Delta r}\big((1-2M/r_{j+1/2})F_{j+1/2}^n - (1-2M/r_{j-1/2})F_{j-1/2}^n - \Delta r S_j^n\big).\\
&= U_j^n - \frac{1}{\lambda}\bigg((1-2M/r_{j+1/2})\bigg(\frac{F(U_{j+1/2+}) - F(U_{j+1/2-})}{2} - \frac{1}{\lambda}\frac{U_{j+1/2+} - U_{j+1/2-}}{2}\bigg)\\
&\qquad + (1-2M/r_{j-1/2})\bigg(\frac{F(U_{j-1/2+}) - F(U_{j-1/2-})}{2} + \frac{1}{\lambda}\frac{U_{j-1/2+} - U_{j-1/2-}}{2}\bigg)\bigg).
\end{aligned}$$

According our construction, we have

$$\begin{aligned}
&\left(1 - \frac{2M}{r_{j+1/2}}\right)(F(U_{j+1/2+}) - F(U_{j+1/2-}))\\
&\qquad = \left(1 - \frac{2M}{r_{j+1}}\right)F(U_{j+1}^n) - \left(1 - \frac{2M}{r_j}\right)F(U_j^n) - \int_{r_j}^{r_{j+1}} S(r, U(t_n, r))\,dr.
\end{aligned}$$

A Taylor expansion to Δr gives us $U_{j+1/2+} - U_{j+1/2-} = O(\Delta r^3)$ and

$$\left(1 - \frac{2M}{r_{j\pm1}}\right) = 1 - \frac{2M}{r_j} \pm \frac{2M}{r_j^2}\Delta r - \frac{2M}{r_j^3}\Delta r^2 + O(\Delta r^3),$$

$$F(U^n_{j\pm1}) = F(U^n_j) + \partial_U F(U^n_j)(\pm\partial_r U^n_j \Delta r + \tfrac{1}{2}\partial^2_{rr} U^n_j \Delta r^2) \\ + \tfrac{1}{2}(\partial_r U^n_j)^T \partial^2_{UU} F(U^n_j)\partial_r U^n_j \Delta r^2 + O(\Delta r^3),$$

$$\int_{r_j}^{r_{j+1}} S(r, U(t_n, r))\, dr = S(r_j, U^n_j)\Delta r + \partial_r S(r_j, U^n_j)\Delta r^2 + O(\Delta r^3).$$

Hence, we conclude $\partial_t U^n_j + \partial_r((1-2M/r_j)F(U^n_j)) - S(r_j, U^n_j) + O(\Delta t + \Delta r^2) = 0$. □

Numerical steady state solution. Recall that the steady state solution to the relativistic Euler model is given by a static Euler system (7-1). Hence, if $U = U(t, r)$ is a steady state solution, it trivially satisfies $\int |\partial_r F((1-2M/r)U) - S(r, U)|\, dr = 0$, where $F = (F^0, F^1)^T$ is the flux and $S = (S^0, S^1)^T$ the source term given by (8-1). In order to describe the steady state solution numerically, we define the total variation in time

$$E^n := E(t_n) = \sum_j \sum_{i=0,1} \big|(1 - 2M/r_{j+1/2})(F^i(U^n_{j+1/2+}) - F^i(U^n_{j-1/2-})) \\ - (1 - 2M/r_{j-1/2})(F^i(U^n_{j-1/2+}) - F^i(U^n_{j-1/2-}))\big|. \quad (8\text{-}11)$$

Clearly, we have the following property.

Claim 8.2. *If $U = U(t, r)$ is a numerical solution to the relativistic Euler model constructed by (8-5)–(8-9), then U is a steady state solution (smooth or with a shock) for $t \geq T$ where $T > 0$ is a finite time if and only if there exists an integer N such that, for all $n > N$, the total variation $E^n \equiv 0$.*

9. Numerical experiments for the relativistic Euler model

Nonlinear stability of steady state solutions. Before studying the stability of steady state solutions, we check that our scheme preserves smooth steady state solutions to the relativistic Euler model (1-4). Recall that $r > 2M$ with $M = 1$ being the black hole mass. We work on the space interval $(r_{\min,}, r_{\max})$ with $r_{\min} = 2M = 2$ and $r_{\max} = 10$, and we take 500 points to discretize this interval. We consider the evolution of two steady state solutions satisfying the algebraic relation (7-2) of the Euler model with the density $\rho(10) = 1.0$ and velocity $v(10) = 0.6$ and the density $\rho(10) = 1.0$ and velocity $v(10) = -0.8$, respectively. We also provides the evolution of a steady state shock (see Figures 15 and 16).

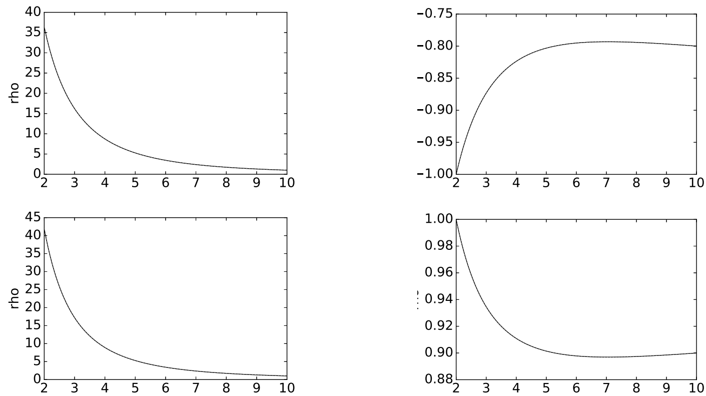

Figure 15. Euler: evolution of steady state solutions plotted at time $t = 50$.

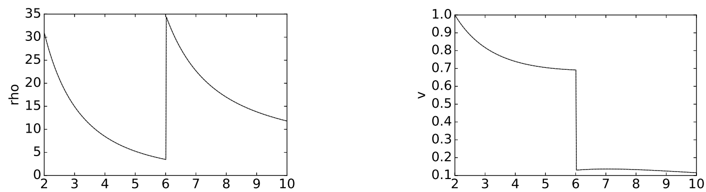

Figure 16. Euler: evolution of a steady shock plotted at time $t = 50$.

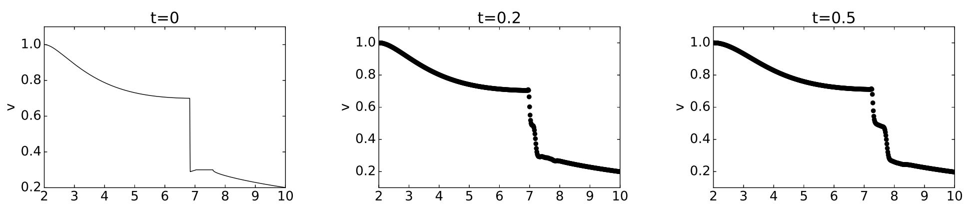

Figure 17. Euler: solution to the Riemann problem (1-shock and 2-shock).

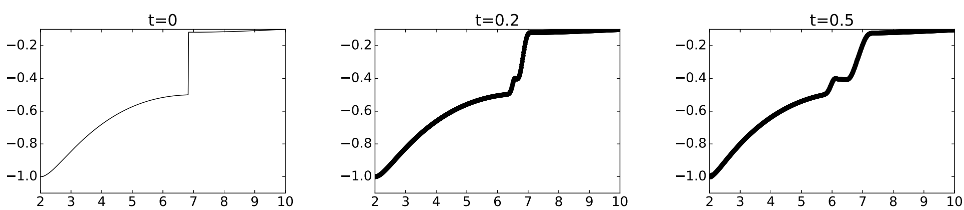

Figure 18. Euler: solution to the Riemann problem (1-rarefaction and 2-rarefaction).

Propagation of discontinuities. Referring to [22], we recall that there exists a solution to the generalized Riemann problem (1-3) with (7-7) consisting of at most three steady state solutions. Figures 17 and 18 show the evolution of two

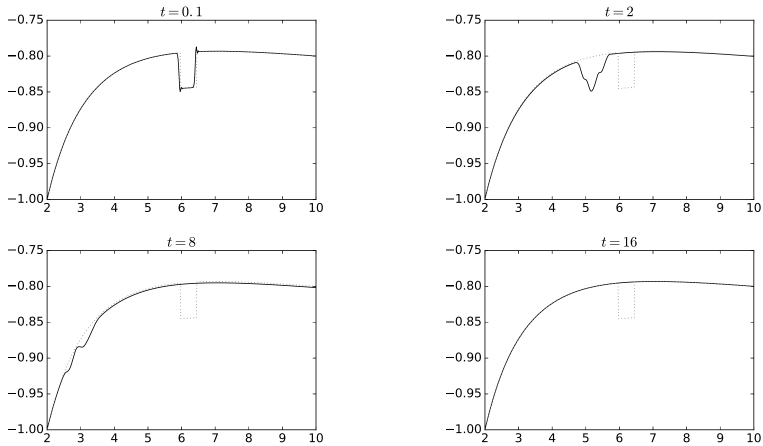

Figure 19. Euler: evolution of a perturbed steady state — convergence to the same steady state.

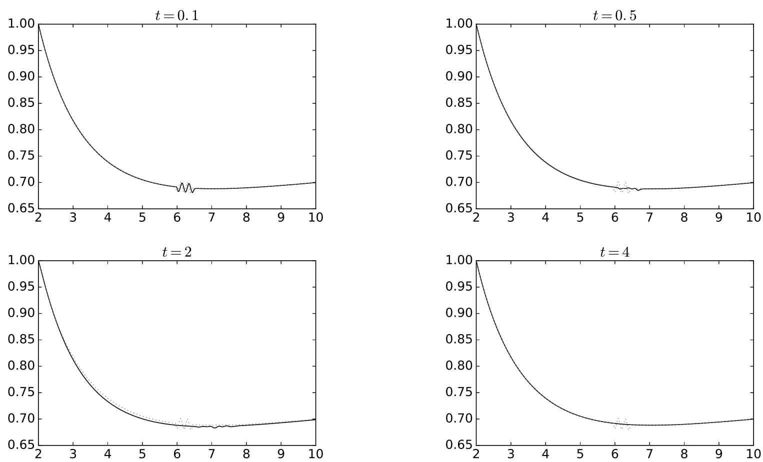

Figure 20. Euler: evolution of a perturbed steady state — convergence to the same steady state.

generalized Riemann problems with an initial discontinuity. Furthermore, we are now interested in the late-time behavior of solutions whose initial data is a steady state solution perturbed by a compactly supported solution. Numerical tests lead us to the following result.

Conclusion 9.1 (stability of smooth steady state solutions to the Euler model). *Let* $(\rho_*, v_*) = (\rho_*, v_*)(r)$, $r > 2M$, *be a smooth steady state solution to the Euler equations* (7-1) *and* $(\rho_0, v_0) = (\rho_0, v_0)(r) = (\rho_*, v_*)(r) + (\delta_\rho, \delta_v)(r)$ *where* $(\delta_\rho, \delta_v) = (\delta_\rho, \delta_v)(r)$ *is a function with compact support*; *then the solution to the relativistic Euler equations on a Schwarzschild background* (1-4) *denoted by* $(\rho, v) = (\rho, v)(t, r)$ *satisfies that* $(\rho, v)(t, \cdot) = (\rho_*, v_*)$ *for all* $t > t_0$ *where* $t_0 > 0$

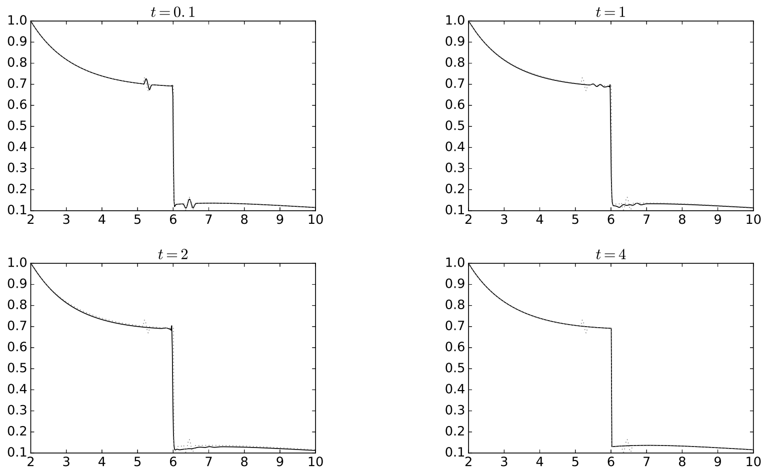

Figure 21. Euler: evolution of a perturbed steady state shock — convergence to the same steady state.

is a finite time. Numerical experiments show that there exists a finite time $t_0 > 0$ such that $(\rho, v)(t, r) = (\rho_*, v_*)(r)$ for all $t > t_0$.

We observe the phenomenon described in Claim 1.3 in Figures 19 and 20, where we have plotted the evolution of different steady state solutions to the Euler model with an initial perturbation. The steady shock given by (7-5) and (7-6) is a weak solution satisfying the static Euler equations (7-1). As is done in the Burgers model, we are also interested in the behavior of steady shocks with perturbations. We summarize our results as follows; see Figure 21.

Conclusion 9.2. *Consider a steady shock $(\rho_*, v_*) = (\rho_*, v_*)(r)$, $r > 2M$, given by (7-5) and (7-6) whose point of discontinuity is at $r = r_*$, and we give the initial data $(\rho_0, v_0) = (\rho_0, v_0)(r) = (\rho_*, v_*)(r) + (\delta_\rho, \delta_v)(r)$ with $(\delta_\rho, \delta_v) = (\delta_\rho, \delta_v)(r)$ a compactly supported function; then there exists a finite time $t > t_0$ such that, for all $t > t_0$, the solution is a steady state shock.*

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