REGENERATIVE PHENOMENA AND THE CHARACTERIZATION OF MARKOV TRANSITION PROBABILITIES

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1. Markov chains in continuous time

It is the object of this paper to draw together certain lines of research which during the last decade have grown out of the problem of characterizing the functions which can arise as transition probabilities of continuous time Markov chains. This problem is now solved (see Sections 8 and 9), although as usual its solution has thrown up further problems which demand attention.

The evolution of a Markov chain X_t in continuous time, with stationary transition probabilities, on a countable state space S, is as usual [2] described by the functions

$$p_{i,j}(t) = \mathbf{P}(X_{s+t} = j \mid X_s = i) \tag{1.1}$$

for $i, j \in S$, $t > 0$. These necessarily satisfy the conditions

$$p_{i,j}(t) \geqq 0, \qquad \sum_{j \in S} p_{i,j}(t) = 1, \tag{1.2}$$

and

$$p_{i,j}(t + u) = \sum_{k \in S} p_{i,k}(t) p_{k,j}(u), \tag{1.3}$$

to which it is usual to add the continuity condition

$$\lim_{t \to 0} p_{i,j}(t) = p_{i,j}(0) = \delta_{i,j}. \tag{1.4}$$

Conversely, given any array $(p_{i,j};\, i, j \in S)$ of functions satisfying (1.2) and (1.3), a Markov chain X_t can be constructed so as to satisfy (1.1).

It is therefore not surprising that a substantial part of the theory of Markov chains should be concerned with the consequences of (1.2), (1.3), and (1.4) for the functions $p_{i,j}$. It is possible to regard this as a problem in pure analysis, but those methods that have proved most powerful have had strong probabilistic motivation. The following list of typical results, taken from [2], will illustrate the achievements of this part of the theory (they are arranged in roughly increasing order of difficulty):

(I) $p_{i,i}(t) > 0$ *for all* $t > 0$;

(II) $p_{i,j}$ *is uniformly continuous on* $[0, \infty)$;

(III) (Doob) *the limit* $q_i = \lim_{t\to 0} t^{-1}\{1 - p_{i,i}(t)\}$ *exists in* $0 \leqq q_i \leqq \infty$, *and* $p_{i,i}(t) \geqq e^{-q_i t}$.

(IV) (Kolmogorov) *the limit* $\pi_{i,j} = \lim_{t\to\infty} p_{i,j}(t)$ *exists* (and a good deal is known about its properties);

(V) (Kolmogorov) *the finite limit* $q_{i,j} = \lim_{t\to 0} t^{-1}p_{i,j}(t)$ *exists when* $i \neq j$;

(VI) (Austin-Ornstein) *the function* $p_{i,j}$ *is either identically zero or always positive on* $(0, \infty)$;

(VII) (Austin-Ornstein-Chung) *the function* $p_{i,j}$ *is continuously differentiable in* $(0, \infty)$ *for* $i = j$ *and in* $[0, \infty)$ *for* $i \neq j$.

This last result cannot be substantially improved; Yuskevitch showed that $p_{i,j}$ need not have a second derivative, and Smith showed that $p'_{i,i}$ need not be continuous at the origin.

In the light of results such as these, the question at once arises [13] of characterizing the functions $p_{i,j}$. It is clear from (1.4) that there are two distinct cases to be considered, according as $i = j$ (the *diagonal* case) or $i \neq j$ (the *nondiagonal* case).

The natural first step in approaching such a problem is to ask whether it can be solved in the usually simpler situation of discrete time, when the variables t, u in (1.2) and (1.3) take only integer values. (Condition (1.4) then has no force.) The answer was given long ago by Feller and Chung in terms of the notion of a *renewal sequence*. A sequence $(u_n: n \geqq 0)$ is called a renewal sequence if there exists a sequence $(f_n; n \geqq 1)$ satisfying

$$f_n \geqq 0, \qquad \sum_{n=1}^{\infty} f_n \leqq 1, \tag{1.5}$$

and such that (u_n) is determined recursively by the equations

$$u_0 = 1, \qquad u_n = \sum_{r=1}^{n} f_r u_{n-r}, \qquad n \geqq 1. \tag{1.6}$$

Then the results of Feller and Chung may be summarized as follows.

THEOREM 1.1 [6]. *A sequence* (a_n) *can be expressed in the form*

$$a_n = p_{i,i}(n), \tag{1.7}$$

for some discrete Markov chain if and only if (a_n) *is a renewal sequence. A sequence* (b_n) *can be expressed in the form*

$$b_n = p_{i,j}(n), \qquad i \neq j, \tag{1.8}$$

if and only if there exists a renewal sequence (u_n) *and a sequence* (f'_n) *satisfying* (1.5) *such that*

$$b_n = \sum_{r=1}^{n} f'_r u_{n-r}. \tag{1.9}$$

It should be noted that the solution for the diagonal case is effective in the sense that, given a sequence (a_n), we can test whether it is a renewal sequence by using (1.6) to compute the corresponding f_n. This is not however true of the nondiagonal case; the representation (1.9) is far from unique, and I know of no sure way of deciding of a sequence (b_n) whether it can be expressed in this form. This feature will persist in the much more difficult solution of the continuous time problem.

2. Regenerative phenomena

It is a notable feature of some (but not all) of the arguments used in Markov chain theory that they really only concern one or two states of the set S. Thus if interest centers on one particular state i, it is often possible to lump all the remaining states together in a single state "not i." More precisely, it may suffice to consider, not X_t, but the process

$$Z_t = \varphi(X_t), \tag{2.1}$$

where $\varphi(i) = 1$, $\varphi(j) = 0, j \neq i$.

The process Z_t is in general non-Markovian, but it has a simple structure governed by the function $p_{i,i}$. If $\mathbf{P}_i$ denotes probability conditional upon $\{X_0 = i\}$, then for $0 = t_0 < t_1 < t_2 < \cdots < t_n$,

$$\begin{aligned} \mathbf{P}_i(Z_{t_r} = 1; r = 1, 2, \cdots, n) \\ = \mathbf{P}_i(X_{t_r} = i; r = 1, 2, \cdots, n) \\ = \prod_{r=1}^{n} p_{i,i}(t_r - t_{r-1}). \end{aligned} \tag{2.2}$$

This suggests the following definition.

A *regenerative phenomenon* with *p-function p* is a stochastic process $(Z_t; t > 0)$ taking values 0 and 1, such that for $0 = t_0 < t_1 < t_2 < \cdots < t_n$,

$$\mathbf{P}(Z_{t_r} = 1; r = 1, 2, \cdots, n) = \prod_{r=1}^{n} p(t_r - t_{r-1}). \tag{2.3}$$

A function p: $(0, \infty) \rightarrow [0, 1]$ is called a *p-function* if there is a regenerative phenomenon having p as p-function.

The left side of (2.3) is of course equal to

$$\mathbf{E}(Z_{t_1} Z_{t_2} \cdots Z_{t_n}), \tag{2.4}$$

and thus the p-function p determines the expectation of any linear combination of products of values of the process Z. In particular, p determines

$$\mathbf{P}(Z_{t_r} = \alpha_r; r = 1, 2, \cdots, n) = (-1)^{\Sigma\alpha_r}\mathbf{E}\left\{\sum_{r=1}^{n} (1 - \alpha_r - Z_{t_r})\right\}, \tag{2.5}$$

whenever $\alpha_r = 0$ or 1, so that the finite dimensional distributions of Z are known once p has been specified.

It is of course necessary that when (2.5) is calculated the result should be nonnegative. This requirement imposes, for each n, 2^n inequalities on the function p, though it turns out that all but 2 of these are consequences of those for smaller values of n. Thus p satisfies, for each $n \geqq 1$, a pair of inequalities, which may be written, for $n = 1$,

$$0 \leqq p(t) \leqq 1, \tag{2.6}$$

for $n = 2$,

$$p(t)p(u) \leqq p(t + u) \leqq 1 - p(t) + p(t)p(u), \tag{2.7}$$

for $n = 3$,

$$\begin{aligned} p(t)p(u + v) + p(t + u)p(v) - p(t)p(u)p(v) \\ \leqq p(t + u + v) \\ \leqq 1 - p(t) - p(t + u) + p(t)p(u) + p(t)p(u + v) \\ + p(t + u)p(v) - p(t)p(u)p(v), \end{aligned} \tag{2.8}$$

and so on. Conversely, the Daniell-Kolmogorov theorem establishes the existence of the process Z whenever this infinite family of functional inequalities is satisfied, showing that these inequalities are both necessary and sufficient for p to be a p-function.

For any $h > 0$, the events $E_n = \{Z_{nh} = 1\}$ form a recurrent event in the sense of Feller [6], since for $0 = r_0 < r_1 < \cdots < r_k$,

$$\mathbf{P}\left(\bigcap_{k=1}^{n} E_{r_k}\right) = \prod_{k=1}^{n} p\{(r_k - r_{k-1})h\}. \tag{2.9}$$

Thus, the sequence $(p(nh))$ is a renewal sequence. This simple remark is one of the most powerful tools in the theory of p-functions.

In view of (2.2), we can now say that (2.1) defines a regenerative phenomenon with p-function $p_{i,i}$. Thus, *any diagonal Markov transition function* $p_{i,i}$ *is a* p-*function.* Theorem 1.1 might suggest that the converse ought to be true; regrettably it is not.

3. Standard p-functions

A p-function is called *standard* (by analogy with Chung's terminology for chains satisfying (1.4)) if

$$\lim_{t \to 0} p(t) = 1, \tag{3.1}$$

and the class of standard p-functions is denoted by $\mathscr{P}$. Then (1.4) shows that the p-function $p_{i,i}$ is standard, and if $\mathscr{P}\mathscr{M}$ denotes the class of all diagonal Markov transition functions then

$$\mathscr{P}\mathscr{M} \subseteqq \mathscr{P}. \tag{3.2}$$

If we combine (3.1) and (2.7) with the fact that $(p(nh))$ is a renewal sequence, a number of the simpler results known for $\mathscr{P}\mathscr{M}$ can be proved in the wider class $\mathscr{P}$. Thus, the following theorem is proved by quite elementary arguments.

THEOREM 3.1 [13]. *If p is any standard p-function, then*

(i) *p is uniformly continuous and strictly positive on* $(0, \infty)$,

(ii) *the limit*

$$p(\infty) = \lim_{t \to \infty} p(t) \tag{3.3}$$

exists, and

(iii) *the limit*

$$q = \lim_{t \to 0} t^{-1} \{1 - p(t)\} \tag{3.4}$$

exists in $0 \leqq q \leqq \infty$ *and*

$$p(t) \geqq e^{-qt}. \tag{3.5}$$

Because $(p(nh))$ is a renewal sequence, there must exist a sequence $(f_n(h))$ with

$$f_n(h) \geqq 0, \qquad \sum_{n=1}^{\infty} f_n(h) \leqq 1, \tag{3.6}$$

such that (1.6) holds with $f_n = f_n(h)$, $u_n = p(nh)$. It follows easily that, for $|z| < 1$,

$$\sum_{n=0}^{\infty} p(nh)z^n = \left\{1 - \sum_{n=1}^{\infty} f_n(h)z^n\right\}^{-1}. \tag{3.7}$$

Now fix $\theta > 0$, and set $z = e^{-\theta h}$ in (3.6). If the left side is multiplied by h it converges, as $h \to 0$; to the Laplace transform

$$r(\theta) = \int_0^{\infty} p(t)e^{-\theta t}\, dt \tag{3.8}$$

of p. The limiting behavior of the right side may be examined using the Helly compactness theorem, and a rather technical argument then leads to the following basic characterization of the class $\mathscr{P}$.

THEOREM 3.2 [13]. *If p is any standard p-function, there is a unique measure* μ *on the Borel subsets of* $(0, \infty]$ *with*

$$\int_{(0,\infty]} (1 - e^{-x})\mu(dx) < \infty, \tag{3.9}$$

such that, for all $\theta > 0$,

$$\int_0^{\infty} p(t)e^{-\theta t}\, dt = \left\{\theta + \int_{(0,\infty]} (1 - e^{-\theta x})\mu(dx)\right\}^{-1}. \tag{3.10}$$

Conversely, if μ *is any Borel measure on* $(0, \infty]$ *satisfying* (3.9), *there exists a unique continuous function* p *satisfying* (3.10), *and* p *is a standard* p*-function.*

Thus, (3.10) sets up a one to one correspondence between $\mathscr{P}$ and the class of measures μ satisfying (3.9). The limits whose existence is asserted in Theorem 3.1 are simply expressed in terms of μ;

$$p(\infty) = \left\{1 + \int x\mu(dx)\right\}^{-1}, \tag{3.11}$$

$$q = \mu(0, \infty]. \tag{3.12}$$

A regenerative phenomenon with $q < \infty$ is called *stable*, and one with $q = \infty$ *instantaneous*.

As an example, suppose that μ concentrates all its mass q at a single point a, $0 < a < \infty$. Then

$$r(\theta) = (\theta + q - qe^{-\theta a})^{-1}, \tag{3.13}$$

which inverts to give

$$p(t) = \sum_{n=0}^{[t/a]} \pi_n\{q(t - na)\}, \tag{3.14}$$

where π_n denotes the Poisson probability

$$\pi_n(\lambda) = \frac{\lambda^n e^{-\lambda}}{n!}. \tag{3.15}$$

This is an oscillating p-function, converging to the limit $p(\infty) = (1 + qa)^{-1}$. It is differentiable everywhere except at the point $t = a$, where it has left and right derivatives

$$D_-p(a) = -qe^{-qa}, \qquad D_+p(a) = q - qe^{-qa}. \tag{3.16}$$

In view of the theorem cited as (VII) in Section 1, the p-function (3.14) cannot therefore come from a Markov chain. Thus, *the inclusion* (3.2) *is strict.*

The differentiability behavior of (3.14) is entirely typical of that of p-functions for which the corresponding measures μ have atoms in $(0, \infty)$. In fact, let $m(t) = \mu(t, \infty]$, so that m is finite, nonincreasing and right continuous in $(0, \infty)$, and integrable on $(0, 1)$ (because of (3.9)). If m_n denotes the n-fold convolution of m with itself, we have the following result.

THEOREM 3.3 [15]. *The series*

$$b(t) = \sum_{n=1}^{\infty} (-1)^{n-1} m_n(t) \tag{3.17}$$

is uniformly absolutely convergent on compact intervals in $(0, \infty)$, *and all terms except possibly the first are continuous. The equation*

$$p(t) = 1 - \int_0^t b(u)\, du \tag{3.18}$$

holds, and shows that p has finite right and left derivatives at all points t in $0 < t < \infty$, *and that*

$$D_+p(t) - D_-p(t) = \mu\{t\}. \tag{3.19}$$

In particular, p is continuously differentiable in $(0, \infty)$ *if and only if* μ *has no atoms in* $(0, \infty)$.

Thus, the (diagonal case of the) Austin-Ornstein differentiability theorem is equivalent to the statement that, for p in $\mathscr{P}\mathscr{M}$, the measure μ has no atoms, except perhaps at ∞. This result will be considerably strengthened in Section 7.

The fundamental formula (3.10) has a number of other important uses. It can for example be used to examine the rate of convergence of $p(t)$ to $p(\infty)$, to establish the Volterra equation

$$p(t) = 1 - \int_0^t p(t-u)\mu(u, \infty]\, du, \tag{3.20}$$

(of which (3.17) and (3.18) describe an iterative solution), and to generalize a theorem of Kendall [8] by showing that every function in $\mathscr{P}$ admits a Fourier representation of the form

$$p(t) = p(\infty) + \int_0^\infty \varphi(\omega) \cos \omega t \, d\omega, \tag{3.21}$$

where $\varphi \geqq 0$. For these results and others, the reader is referred to [13] and [15].

4. Additive processes

The right side of (3.10) strongly suggests a connection with the theory of additive processes (processes with stationary independent increments), a connection which in the Markov case was exploited by Lévy [25]. If Z is a regenerative phenomenon with standard p-function p, it is easy to check that

$$\lim_{h \to 0} \mathbf{P}(Z_{t+h} \neq Z_t) = 0, \tag{4.1}$$

so that Z has a measurable version, and it makes sense to consider the process

$$\tau_t = \int_0^t Z_u \, du. \tag{4.2}$$

Since the sample functions of τ are continuous and nondecreasing, there exists an inverse process T defined by

$$T_t = \inf\{s > 0; \tau_s \geqq t\}. \tag{4.3}$$

Then [13] T is an additive process, with Lévy-Khinchin representation

$$\log \mathbf{E}(e^{-\theta T_t}) = -t\left\{\theta + \int_{(0,\infty]} (1 - e^{-\theta x})\mu(dx)\right\}^{-1}. \tag{4.4}$$

The term $-t\theta$ signifies a constant drift, so that

$$T_t = t + \xi_t, \tag{4.5}$$

where the additive process ξ increases in jumps.

If $q = \mu(0, \infty] < \infty$, the jumps of ξ occur at the points of a Poisson process of rate q, the height of each jump having distribution function

$$F(x) = q^{-1}\mu(0, x]. \tag{4.6}$$

Translating this back into a description of Z, it shows that the sample functions of (a separable version of) Z are step functions. The lengths of the intervals of constancy are independent random variables, those with $Z_t = 1$ having distribution function $1 - e^{-qx}$, and those with $Z_t = 0$ having distribution function F.

When $q = \infty$, the jumps of ξ are dense, and the sample function behavior of Z becomes much more complex. A version can be chosen in which the set $\{t; Z_t = 0\}$ is a countable union of intervals, but the complement $\{t; Z_t = 1\}$ can never be so, and instead resembles a Cantor set (though of positive measure). The measure μ determines the lengths of the component intervals of the former set, in the sense that, for $c > 0$, the intervals of length greater than c are well ordered, with distribution function

$$F_c(x) = \frac{\mu(c, x]}{\mu(c, \infty]}. \tag{4.7}$$

The problem of choosing a suitable version of Z in the instantaneous case has been considered (in unpublished lectures) by Kendall, who remarks that if T is a right continuous, strong Markov, additive process satisfying (4.4), then

$$\{t; Z_t = 1\} = \{T_u; u \geqq 0\} \tag{4.8}$$

defines a convenient version of the regenerative phenomenon whose p-function is given by (3.10). This construction permits the calculation of some useful distributions. For instance, the backward recurrence time

$$\beta_t = \inf\{u > 0; Z_{t-u} = 1\} \tag{4.9}$$

has distribution

$$\begin{aligned} \mathbf{P}\{\beta_t = 0\} &= p(t), \\ \mathbf{P}\{\beta_t \in (u, u + du)\} &= p(t - u)\mu(u, \infty]\cdot du. \end{aligned} \tag{4.10}$$

In fact, β is a Markov process, and

$$\{t; \tilde{Z}_t = 1\} = \{t; \beta_t = 0\} \tag{4.11}$$

defines a version $\tilde{Z}$ of Z. For related work, raising the possibility of a "strong regenerative property," see [7].

5. Markov measures

Returning now to the problem of describing the class $\mathscr{PM}$ of diagonal Markov transition functions $p_{i,i}$, we first exhibit a large subclass of $\mathscr{PM}$. Let (u_n) be any renewal sequence; then by Theorem 1.1 there exists a discrete time Markov chain with $u_n = p_{a,a}(n)$ for one state a. If c is any positive constant, consider the functions

$$\tilde{p}_{i,j}(t) = \sum_{n=0}^{\infty} p_{i,j}(n)\pi_n(ct), \tag{5.1}$$

where π_n is the Poisson probability (3.15). An elementary computation shows that these satisfy (1.2), (1.3), and (1.4), so that the function $\tilde{p}_{a,a}$ belongs to $\mathscr{PM}$. Thus, for any renewal sequence (u_n) and any $c > 0$, $\mathscr{PM}$ contains the function

$$p(t) = \sum_{n=0}^{\infty} u_n\pi_n(ct). \tag{5.2}$$

The class of functions of the form (5.2) is denoted by $\mathscr{Q}$. Since $\mathscr{Q}$ contains only stable p-functions, it cannot exhaust $\mathscr{PM}$, so that

$$\mathscr{Q} \subset \mathscr{PM} \subset \mathscr{P}. \tag{5.3}$$

For any p in $\mathscr{P}$, and any positive integer k, the sequence $(p(nk^{-1}))$ is a renewal sequence, so that

$$p_k(t) = \sum_{n=0}^{\infty} p(nk^{-1})\pi_n(kt) \tag{5.4}$$

belongs to $\mathscr{Q}$. It is a simple consequence of the weak law of large numbers that, for all t,

$$p(t) = \lim_{k\to\infty} p_k(t), \tag{5.5}$$

so that every function in $\mathscr{P}$ is the pointwise limit of a sequence of functions in $\mathscr{Q}$.

It is useful to express this fact in more formal topological language. Every p-function is a function from $(0, \infty)$ into $[0, 1]$, and may therefore be regarded as an element of the product space

$$\Pi = [0, 1]^{(0,\infty)}, \tag{5.6}$$

whose product topology is compact by Tychonov's theorem. The set of p-functions (standard or not), being defined by the inequalities (2.6), (2.7), (2.8), $\cdots$, is clearly closed in Π, and thus inherits a compact Hausdorff topology. In this topology $\mathscr{P}$ is not closed (consider the sequence $p_n(t) = e^{-nt}$), and the subspace topology on $\mathscr{P}$, though Hausdorff and indeed metrizable [5], is not compact. Equation (5.5) shows that $\mathscr{Q}$, and *a fortiori* $\mathscr{PM}$, is dense in $\mathscr{P}$. It is this fact that makes the identification of $\mathscr{PM}$ as a subset of $\mathscr{P}$ a somewhat delicate matter.

In the one to one correspondence set up by Theorem 3.2 between $\mathscr{P}$ and the class of measures satisfying (3.9), the latter class inherits a topology from that of $\mathscr{P}$. This topology has been identified by Kendall [10] and Davidson [5]; a sequence (p_n) in $\mathscr{P}$ converges to p in $\mathscr{P}$ if and only if the corresponding measures μ_n, μ satisfy

$$\lim_{n\to\infty} \int_{(0,\infty]} (1 - e^{-x})\varphi(x)\mu_n(dx) = \int_{(0,\infty]} (1 - e^{-x})\varphi(x)\mu(dx) \tag{5.7}$$

for every bounded continuous function φ on $(0, \infty]$.

In the correspondence (3.10), the subset $\mathscr{P}\mathscr{M}$ of $\mathscr{P}$ corresponds to a proper (albeit dense in the sense of (5.7)) subset of the class of measures satisfying (3.9). It will be convenient to describe the members of this subset simply as *Markov measures*, so that $\mathscr{P}\mathscr{M}$ is determined once the Markov measures have been characterized. In view of the discussion in Section 4, this amounts to the characterization of the possible distributions of lengths of excursions from a particular state in a Markov chain.

Examples of Markov measures can be obtained by computing the measure μ corresponding to p-functions of the form (5.2). It is not difficult to calculate that, if (f_n) is the sequence related to (u_n) by (1.6), then

$$\begin{aligned} \mu(dt) &= c^2\, dt \sum_{n=2}^{\infty} f_n \pi_{n-2}(ct), \\ \mu\{\infty\} &= c\left\{1 - \sum_{n=1}^{\infty} f_r\right\}. \end{aligned} \tag{5.8}$$

The point to note about this measure is that, apart from a possible atom at ∞, it has a density with respect to Lebesgue measure of the form $e^{-ct}P(t)$, where P is a power series with nonnegative coefficients.

More general examples can be constructed by a technique due to Yuskevitch ([26], see also [9]). Consider first a Markov chain of the special type constructed by Lévy [25], on the nonnegative integers, in which any excursion from 0 only visits one other state. The states $i \geqq 1$ must be stable, but 0 may be instantaneous. Fix positive integers b_i, m_i, $i = 1, 2, \cdots$, and construct a new chain on the state space,

$$\{0\} \cup \{(i, \alpha); i = 1, 2, \cdots, \alpha = 0, 1, 2, \cdots, m_i\}, \tag{5.9}$$

by replacing a sojourn in i by a progress through the states $(i, 0)$, $(i, 1)$, $\cdots$, (i, m_i) and return to 0, the time spent in (i, α) having probability density $b_i e^{-b_i t}$. It is not difficult to show that, in this new chain, the measure μ corresponding to the p-function $p_{0,0}$ has density in $(0, \infty)$ of the form

$$h(t) = \sum_{i=1}^{\infty} c_i t^{m_i} e^{-b_i t}, \tag{5.10}$$

where c_i is a nonnegative constant depending on the parameters of the original chain.

By a suitable choice of these parameters [17], one can obtain any measure satisfying (3.9) whose density is of the form

$$h(t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{m,n} t^m e^{-nt}, \qquad a_{m,n} \geqq 0. \tag{5.11}$$

Any such measure is therefore a Markov measure, regardless of the value of $\mu\{\infty\}$. It thus becomes urgent to know which functions of a positive variable t can be expressed in the form (5.11); the answer to a slightly more general question is given by the following theorem.

THEOREM 5.1 [23]. *A function h: $(0, \infty) \rightarrow [0, \infty]$, not identically zero, is expressible in the form*

$$h(t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n} t^m e^{-nt} \tag{5.12}$$

with $a_{m,n} \geqq 0$ if and only if it is lower semicontinuous and satisfies

$$h(t) \geqq a t^m e^{-nt} \tag{5.13}$$

for some $a > 0$ and some nonnegative integers m, n.

Thus, the Yuskevitch construction shows that μ is a Markov measure whenever it has a density h in $(0, \infty)$ which is lower semicontinuous and satisfies (5.13). We shall see that these sufficient conditions come very close to being necessary.

6. Quasi-Markov chains

The notion of regenerative phenomenon is an abstraction of the process obtained by lumping together all states of a Markov chain except one. For some purposes, however, this is too drastic; if for instance one is interested in the transition probability $p_{i,j}$, $i \neq j$, then both states i and j should retain their identities. This suggests the following more general definition ([14], [16]).

A *quasi-Markov chain* of order N is a stochastic process Z_t, $t \geqq 0$, taking values $0, 1, 2, \cdots, N$, in such a way that, for $0 = t_0 < t_1 < t_2 < \cdots < t_n$ and $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n \in \{1, 2, \cdots, N\}$,

$$\begin{aligned} \mathbf{P}\{Z_{t_k} = \alpha_k (k = 1, 2, \cdots, n) \,|\, Z_0 = \alpha_0\} \\ = \prod_{k=1}^{n} p_{\alpha_{k-1},\alpha_k}(t_k - t_{k-1}). \end{aligned} \tag{6.1}$$

Here the functions $p_{\alpha,\beta}$, which determine the finite dimensional distributions of Z so long as $Z_0 \neq 0$, are the elements of a matrix

$$\mathbf{p}(t) = (p_{\alpha,\beta}(t);\, \alpha, \beta = 1, 2, \cdots, N), \tag{6.2}$$

called the p-matrix of the chain. It is important to note that (6.1) is *not* required to hold if $\alpha_k = 0$ for any k; the state 0 is anomalous. A quasi-Markov chain of order 1 is essentially a regenerative phenomenon. A quasi-Markov chain is said to be standard if

$$\lim_{t \to 0} \mathbf{p}(t) = \mathbf{I}. \tag{6.3}$$

the identity matrix of order N.

If X_t is a Markov chain, and $i_1, i_2, \cdots, i_N$ are any N distinct states, then it is immediate that the process

$$Z_t = \psi(X_t), \tag{6.4}$$

where

$$\psi(i_\alpha) = \alpha, \qquad \psi(j) = 0, \qquad j \notin \{i_1, i_2, \cdots, i_N\}, \tag{6.5}$$

is a standard quasi-Markov chain with p-matrix

$$\big(p_{i_\alpha, i_\beta}(t);\ \alpha, \beta = 1, 2, \cdots, N\big). \tag{6.6}$$

The analysis described in Sections 2 and 3 can be extended to the more general situation of a quasi-Markov chain; for the details see [14]. For present purposes it suffices to quote the main characterization theorem which generalizes Theorem 3.2.

THEOREM 6.1 [14]. *In order that a continuous* $(N \times N)$ *matrix valued function* $\mathbf{p}(t)$, $t > 0$, *be the p-matrix of a standard quasi-Markov chain, it is necessary and sufficient that its Laplace transform*

$$\mathbf{r}(\theta) = \int_0^\infty \mathbf{p}(t) e^{-\theta t}\, dt \tag{6.7}$$

should have, for all $\theta > 0$, *an inverse*

$$\mathbf{r}(\theta)^{-1} = \big(r^{\alpha,\beta}(\theta)\big), \tag{6.8}$$

with

$$r^{\alpha,\alpha}(\theta) = \theta + a_\alpha + \int_{(0,\infty)} (1 - e^{-\theta x}) \mu_\alpha(dx), \tag{6.9}$$

and (for $\alpha \neq \beta$*)*

$$r^{\alpha,\beta}(\theta) = -\int_{[0,\infty)} e^{-\theta x} \mu_{\alpha,\beta}(dx), \tag{6.10}$$

where μ_α *is a Borel measure on* $(0, \infty)$ *satisfying* (3.9), $\mu_{\alpha,\beta}$ *a totally finite Borel measure on* $[0, \infty)$, *and, for all* α,

$$\sum_{\beta \neq \alpha} \mu_{\alpha,\beta}[0, \infty) \leqq a_\alpha. \tag{6.11}$$

It is possible to give probabilistic interpretations of the quantities occurring in these formulae. Comparing (6.9) with (3.10) it appears that $1/r^{\alpha,\alpha}(\theta)$ is the Laplace transform of a p-function p_α; this turns out to be a taboo function in the sense of Chung [2],

$$p_\alpha(t) = {}_H p_{\alpha,\alpha}(t), \quad H = \{\beta;\, 1 \leqq \beta \leqq N, \beta \neq \alpha\}. \tag{6.12}$$

The measure $\mu_{\alpha,\beta}$ has the following interpretation, for a suitably regular version of Z. Scan a sample function of Z for the first times $\sigma \leqq \tau$ with

$$Z_{\sigma-} = \alpha, \qquad Z_{\tau+} = \beta, \qquad Z_t = 0, \qquad \sigma < t < \tau. \tag{6.13}$$

Then $\mu_{\alpha,\beta}$ is a multiple of the distribution of $(\tau - \sigma)$.

The formulae of Theorem 6.1 take on particularly useful forms when $N = 2$. Inverting the (2×2) matrix $\mathbf{r}(\theta)^{-1}$ directly, we have

$$\mathbf{r}(\theta) = \det[\mathbf{r}(\theta)] \begin{pmatrix} r^{2,2}(\theta) & -r^{1,2}(\theta) \\ -r^{2,1}(\theta) & r^{1,1}(\theta) \end{pmatrix}, \tag{6.14}$$

so that

$$\frac{r_{1,2}(\theta)}{r_{2,2}(\theta)} = -\frac{r^{1,2}(\theta)}{r^{1,1}(\theta)} = r_1(\theta) \int e^{-\theta x} \mu_{1,2}(dx). \tag{6.15}$$

Using (6.12), this implies that

$$p_{1,2}(t) = \int_0^t f_{1,2}(s) p_{2,2}(t-s)\, ds, \tag{6.16}$$

where

$$f_{1,2}(t) = \int_0^t {}_2p_{1,1}(t-u) \mu_{1,2}(du). \tag{6.17}$$

Applying this in particular to the quasi-Markov chain obtained from the Markov chain by lumping together all states except i and j, we obtain the equation

$$p_{i,j}(t) = \int_0^t f_{i,j}(s) p_{j,j}(t-s)\, ds, \tag{6.18}$$

where

$$f_{i,j}(t) = \int_0^t {}_jp_{i,i}(t-u) \mu_{i,j}(du). \tag{6.19}$$

Equation (6.18) is the celebrated *first passage decomposition* (usually proved by quite different methods [2]); the fact that $f_{i,j}$ itself has the decomposition (6.19) is crucial. Condition (6.11) is in this context equivalent to the inequality

$$\int_0^\infty f_{i,j}(t)\, dt = \int_0^\infty {}_jp_{i,i}(t)\, dt \cdot \mu_{i,j}[0, \infty) \leqq 1. \tag{6.20}$$

If in the above argument we had started with $r_{1,2}(\theta)/r_{1,1}(\theta)$ instead of

$r_{1,2}(\theta)/r_{2,2}(\theta)$, we should have obtained the *last exit decomposition*,

$$p_{i,j}(t) = \int_0^t p_{i,i}(t - s) g_{i,j}(s)\, ds, \tag{6.21}$$

where

$$g_{i,j}(t) = \int_0^t {}_ip_{j,j}(t - u) \mu_{i,j}(du). \tag{6.22}$$

Such identities are naturally written in convolution notation:

$$\begin{aligned} f_{i,j} &= {}_jp_{i,i} * d\mu_{i,j}, & p_{i,j} &= f_{i,j} * p_{j,j} = {}_jp_{i,i} * d\mu_{i,j} * p_{j,j}, \\ g_{i,j} &= {}_ip_{j,j} * d\mu_{i,j}, & p_{i,j} &= g_{i,j} * p_{i,i} = {}_ip_{j,j} * d\mu_{i,j} * p_{i,i}. \end{aligned} \tag{6.23}$$

7. Properties of Markov measures

The first passage and last exit decompositions can be combined, by an argument shown to me by Professor Reuter, to give important positive information about the measure μ associated by Theorem 3.2 with the p-function $p_{i,i}$ in a Markov chain. By (1.3),

$$p_{i,i}(t + u) = p_{i,i}(t) p_{i,i}(u) + \sum_{j \neq i} p_{i,j}(t) p_{j,i}(u), \tag{7.1}$$

so that, using (6.18) and (6.21),

$$\begin{aligned} &p_{i,i}(t + u) - p_{i,i}(t) p_{i,i}(u) \\ &\qquad = \sum_{j \neq i} \int_0^t p_{i,i}(t - s) g_{i,j}(s)\, ds \int_0^u f_{j,i}(v) p_{i,i}(u - v)\, dv. \end{aligned} \tag{7.2}$$

Multiplying by $e^{-\alpha t - \beta u}$ and integrating over $t > 0$, $u > 0$, we have

$$\frac{r_{i,i}(\alpha) - r_{i,i}(\beta)}{\beta - \alpha} - r_{i,i}(\alpha) r_{i,i}(\beta) = \sum_{j \neq i} r_{i,i}(\alpha) \hat{g}_{i,j}(\alpha) \hat{f}_{j,i}(\beta) r_{i,i}(\beta), \tag{7.3}$$

where $\alpha, \beta > 0$, $\alpha \neq \beta$, and $\hat{\varphi}$ denotes the Laplace transform of the function φ. Using (3.10) to express $r_{i,i}$ in terms of the measure μ, and simplifying, we have

$$\int_{(0,\infty)} \frac{e^{-\alpha x} - e^{-\beta x}}{\beta - \alpha} \mu(dx) = \sum_{j \neq i} \hat{g}_{i,j}(\alpha) \hat{f}_{j,i}(\beta). \tag{7.4}$$

It is not difficult to see that this implies that μ has a density h in $(0, \infty)$, and that

$$h(t + u) = \sum_{j \neq i} g_{i,j}(t) f_{j,i}(u), \tag{7.5}$$

for almost all (t, u). That μ is absolutely continuous was suggested without proof by Lévy [25]; in view of the remarks following Theorem 3.3, it implies at once the continuous differentiability of $p_{i,i}$.

If in (7.5) we replace t by $(t - u)$ and integrate with respect to $u \in (0, t)$, we obtain the formula

$$th(t) = \sum_{j \neq i} \int_0^t g_{i,j}(t - u) f_{j,i}(u)\, du, \tag{7.6}$$

for almost all t. Thus, one version of the density of μ is given by

$$h(t) = t^{-1} \sum_{j \neq i} k_j(t); \tag{7.7}$$

where

$$k_j = g_{i,j} * f_{j,i} = {}_i p_{j,j} * d\mu_{i,j} * d\mu_{j,i} * {}_i p_{j,j}. \tag{7.8}$$

Now for any p-function p, $p^{(2)} = p * p$ is a continuous function on $(0, \infty)$ with $\lim_{t \to 0} p^{(2)}(t) = 0$. It follows easily that $k_j = {}_i p_{j,j}{}^{(2)} * d\mu_{i,j} * d\mu_{j,i}$ is continuous, and hence that the density h in (7.7) is lower semicontinuous in $(0, \infty)$.

Now suppose that h is not identically zero. Then there must exist $j \neq i$ for which k_j is not identically zero, and for this j (7.8) shows that neither $\mu_{i,j}$ nor $\mu_{j,i}$ can be identically zero. The p-function ${}_i p_{j,j}$ satisfies, as a simple consequence of the left inequality of (2.7), the inequality

$${}_i p_{j,j}(t) \geqq e^{-\alpha t}, \tag{7.9}$$

for some α and all sufficiently large t. If (7.9) is substituted into (7.8) and (7.7), it follows that, for some β, and all sufficiently large t, $h(t) \geqq e^{-\beta t}$.

A further result follows from the Austin-Ornstein positivity theorem cited as (VI) in Section 1. With the particular value of j chosen above, (6.18) and (6.19) show that neither $p_{i,j}$ nor $p_{j,i}$ vanishes identically. The theorem then asserts that, for all $t > 0$, $p_{i,j}(t) > 0$, $p_{j,i}(t) > 0$. Using (6.18) and (6.19) again, it follows that, for all $\varepsilon > 0$,

$$\mu_{i,j}[0, \varepsilon) > 0, \qquad \mu_{j,i}[0, \varepsilon) > 0. \tag{7.10}$$

Substituting (7.10) into (7.8) and (7.7), we have $h(t) > 0$, $t > 0$.

Combining all these results, we have a set of necessary conditions for a measure μ to correspond (in (3.10)) to a function in $\mathscr{PM}$.

THEOREM 7.1. *Every Markov measure μ has (as well as a possible atom at infinity) a lower semicontinuous density h on $(0, \infty)$ which is either identically zero or satisfies (for some β) the inequalities*

$$\begin{aligned} h(t) &> 0 && \textit{for all } t > 0, \\ h(t) &\geqq e^{-\beta t} && \textit{for sufficiently large } t. \end{aligned} \tag{7.11}$$

8. The solution of the Markov characterization problem

If Theorems (5.1) and (7.1) are compared, it will be seen that the gap between the sufficient and the necessary conditions, for μ to be a Markov measure, is just the gap between the inequalities (5.13) and (7.11). In fact, the gap is narrower than might at first appear, since a positive lower semicontinuous function is bounded away from zero on every compact interval. It follows easily that *a lower semicontinuous function h satisfying* (7.11) *satisfies* (5.12) *if and only if, for some integer m,*

$$h(t) \geqq t^m \tag{8.1}$$

for all sufficiently small t.

All that remains therefore to complete the characterization of $\mathscr{P}\mathscr{M}$ is to determine to what extent (8.1) is necessary. In fact, it is not necessary at all, as was shown in [23] by calculating the measure μ for an *escalator* [2], an infinite string of states through which the process moves in a finite time (that is, a divergent pure birth process with instantaneous return). For such a chain, it is easy [21] to see that $h(t) \to 0$ as $t \to 0$ faster than any monomial. Moreover, it is possible (and essential) to go much further than this, and to prove the following result.

THEOREM 8.1 [23]. *Let ω be any positive continuous function on* $(0, 1]$. *Then there exists an escalator with the property that the probability density h of the time spent in traversing it satisfies*

$$h(t) \leqq \omega(t), \qquad 0 < t \leqq 1. \tag{8.2}$$

With this as a tool, the solution now proceeds with only technical difficulties. In the Yuskevitch construction, each finite string of states is replaced by a suitably chosen escalator, and it is then shown to be possible to realize any positive lower semicontinuous function h satisfying (7.11). In other words, the converse of Theorem 7.1 is true. Collecting the various results together, we therefore have the following fundamental theorem.

THEOREM 8.2 [23]. *A continuous function $p(t) (t > 0)$ can be expressed in the form $p(t) = p_{i,i}(t)$ in some Markov chain if and only if its Laplace transform can be expressed in the*

$$\int_0^\infty p(t)e^{-\theta t}\,dt = \left\{\theta + a + \int_0^\infty (1 - e^{-\theta t})h(t)\,dt\right\}^{-1}, \tag{8.3}$$

where $a \geqq 0$ and h is a lower semicontinuous function which is either identically zero or satisfies the inequalities (7.11).

Because of the way the theorem has been proved, one can in fact say rather more, for it shows that any function in $\mathscr{P}\mathscr{M}$ can be realized in a special sort of chain, a *bouquet of escalators*. If a person moves through the state space according to such a chain, his wanderings can be described as follows. Starting at the distinguished state i, he is presented with a choice of escalators. He chooses one, ascends it, and on reaching the top after an infinite number of jumps returns at once to i, where he again has the opportunity to choose. Notice that in this chain all the states, except perhaps i, are stable.

Now suppose the chain is modified by providing each state except i with a cinema showing a film of a totally instantaneous chain (independent for each state, and for each visit to a state) which the person is required to watch while waiting in the state. If the specification is enlarged to include the state of the film, the result is another chain in which every state, except perhaps i, is instantaneous.

It follows that the function $p_{i,i}$ contains no information about the stability or otherwise of the other states of the chain. This is in line with the experience of authors who have constructed pathological Markov chains (for example, [11]), who have found that extreme sample function irregularity is consistent with simple functional forms for the individual transition probabilities.

9. The nondiagonal problem

Theorem 8.2 is the complete continuous time analogue of the first half of Theorem 1.1, and it therefore remains to find the analogue of the second, nondiagonal, part. The key to this problem lies in equations (6.18) and ((6.19), which combine to give

$$p_{i,j} = {}_jp_{i,i} * d\mu_{i,j} * p_{j,j}. \tag{9.1}$$

In this decomposition, the first and third members clearly belong to $\mathscr{PM}$. The totally finite measure $\mu_{i,j}$ has, as noted in Section 6, a probabilistic interpretation as a multiple of the distribution of $(\tau - \sigma)$, where (if $X_0 = i$),

$$\begin{aligned} \tau &= \inf\{t;\, X_t = j\}, \\ \sigma &= \sup\{t < \tau;\, X_t = i\}. \end{aligned} \tag{9.2}$$

From this, it is not difficult to show [17] that, apart from a possible atom $b = \mu_{i,j}\{0\}$, μ is a Markov measure.

It is shown in [17] that these conditions are sufficient as well as necessary, in the sense of the following theorem.

THEOREM 9.1 [17]. *A function $q(t)$, $t > 0$, can be expressed in the form $q(t) = p_{i,j}(t)$, $i \neq j$, in some Markov chain if and only if it can be written*

$$q = bp_1 * p_2 + p_1 * h * p_2, \tag{9.3}$$

where b is a nonnegative constant, p_1 and p_2 belong to $\mathscr{PM}$, h is a lower semicontinuous function, either identically zero or satisfying (7.11), *and*

$$\left\{b + \int_0^\infty h(t)\,dt\right\} \int_0^\infty p_1(t)\,dt \leqq 1. \tag{9.4}$$

As in the discrete time case, there appears to be no canonical form of the decomposition (9.3), and no effective way of deciding, of a given function q, whether it is expressible in the form (9.3). The effective description of the nondiagonal Markov transition probabilities therefore remains an open problem, presumably more difficult than the corresponding discrete time problem.

If this problem can be solved, it will probably be easy to characterize functions of the form

$$p_{i,A}(t) = \sum_{j\in A} p_{i,j}(t), \qquad A \subseqq S, \tag{9.5}$$

and to solve the corresponding problem [19] for purely discontinuous Markov processes on uncountable state spaces.

10. Multiplicative properties of p-functions

If p_1 and p_2 are p-functions, we can construct corresponding regenerative phenomena Z^1 and Z^2 on distinct probability spaces Ω_1, Ω_2. Then the process Z defined on the product space $\Omega_1 \times \Omega_2$, with the product probability measure, by

$$Z_t(\omega_1, \omega_2) = Z_t^1(\omega_1)Z_t^2(\omega_2), \tag{10.1}$$

is a regenerative phenomenon with p-function

$$p(t) = p_1(t)p_2(t). \tag{10.2}$$

Thus, the product of two p-functions is itself a p-function.

If p_1 and p_2 are standard, then so is $p = p_1p_2$, so that $\mathscr{P}$ is a commutative Hausdorff topological semigroup, with identity e given by $e(t) \equiv 1$. The arithmetical properties of this semigroup have been extensively studied by Kendall [10] and Davidson [5]. In particular, Kendall observed that there is a strong resemblance between $\mathscr{P}$ and the convolution semigroup $\mathscr{W}$ of probability measures on the line, an observation which is the starting point of the theory of *delphic semigroups*. This theory, like the classical theory of $\mathscr{W}$, leans heavily on the concept of an *infinitely divisible* element of the semigroup, one which can be expressed in the form

$$p = (p_n)^n, \tag{10.3}$$

for every integer $n \geqq 2$, where p_n belongs to the semigroup.

It was observed in [16] that $\mathscr{P}$ contains every continuous function p with $0 < p(t) \leqq p(0) = 1$ such that

$$\varphi(t) = -\log p(t) \tag{10.4}$$

is concave. Such p-functions are clearly infinitely divisible, for we may take

$$p_n(t) = \exp\{-n^{-1}\varphi(t)\}. \tag{10.5}$$

Kendall showed that they are the only infinitely divisible elements of $\mathscr{P}$.

It is very natural to ask which of the infinitely divisible p-functions belong to $\mathscr{P}\mathscr{M}$. In principle this question is answered by Theorem 8.2, but this is difficult to apply since the measure μ depends in a very complicated way on φ (see [22]).

A more useful answer is given by the following theorem (of which the regrettably complex proof will be published elsewhere).

THEOREM 10.1. *Let p be an infinitely divisible element of $\mathscr{P}$, and write the concave function φ in the form*

$$\varphi(t) = \int_{(0,\infty]} \min(t, x)\lambda(dx), \tag{10.6}$$

λ being a measure on $(0, \infty]$ with

$$\int_{(0,\infty]} \min(1, x)\lambda(dx) < \infty. \tag{10.7}$$

Then p belongs to $\mathscr{PM}$ if λ has a lower semicontinuous density in $(0, \infty)$, and $\lambda(0, \varepsilon) > 0$ for all $\varepsilon > 0$ (unless $\lambda(0, \infty) = 0$).

A special class of infinitely divisible p-functions is the class of completely monotonic functions, that is, those expressible in the form

$$p(t) = \int_{[0,\infty)} e^{-tx}\nu(dx) \tag{10.8}$$

for probability measures ν on $[0, \infty)$. These all belong to $\mathscr{PM}$; indeed, it was proved in [18] that they are exactly the diagonal transition probabilities of *reversible* chains. The problem of characterizing the nondiagonal transition functions of reversible chains remains open.

There are a number of other open problems in the multiplicative theory of $\mathscr{P}$, often motivated by the corresponding problems for the classical semigroup $\mathscr{W}$. To take just one example, Davidson has conjectured that, if p is any infinitely divisible element of $\mathscr{P}$ which is not of the form $p(t) = e^{-qt}$, then p has a factor which is not infinitely divisible.

11. Inequalities for p-functions

Freedman has asked the following question. If in a Markov chain, for some $i \in S$, $t > 0$, $c > 0$, it is known that

$$p_{i,i}(t) \geqq c, \tag{11.1}$$

can one give a lower bound for $p_{i,i}(s)$ $(s < t)$? He has given a partial answer to this question (in joint work with Blackwell [1]; the result was independently discovered by Davidson [5]) in the form of the inequality

$$p_{i,i}(s) \geqq \tfrac{1}{2} + (c - \tfrac{3}{4})^{1/2}, \tag{11.2}$$

as long as $c > \frac{3}{4}$.

The elegant proof of this inequality uses only the right inequality of (2.7); it is therefore true of all p in $\mathscr{P}$ that, if $c > \frac{3}{4}$, then

$$p(t) \geqq c \text{ implies } p(s) \geqq \tfrac{1}{2} + (c - \tfrac{3}{4})^{1/2} \tag{11.3}$$

for $s < t$. The result can indeed be extended, since $\mathscr{P}$ is not closed in the product space Π, to all functions in its closure $\bar{\mathscr{P}}$. Now the elements of $\bar{\mathscr{P}}$ are p-functions (since the set of p-functions is closed in Π) and those in $\bar{\mathscr{P}} - \mathscr{P}$ might be called *semistandard*. The p-functions which are not standard have been studied in [20], where the following rather deep theorem is proved.

THEOREM 11.1 [20]. *Every p-function satisfies one and only one of the following four conditions*:

(i) *p is standard;*

(ii) *there exists a constant $a \in (0, 1)$ and a standard p-function $\bar{p}$ such that $p(t) = a\bar{p}(t)$;*

(iii) *$p(t) = 0$ for almost all t;*

(iv) *p is not Lebesgue measurable.*

The functions of type (ii) are easy to handle, and are all semistandard since

$$a\bar{p}(t) = \lim_{n \to \infty} \exp\{-\min[nt, -\log a]\}\bar{p}(t). \tag{11.4}$$

Those of type (iv) are sufficiently pathological to ignore, though it is conceivable that some may perhaps be semistandard. The functions of type (iii) require however more attention; they include for instance the functions of the form

$$p(t) = \begin{cases} u_t & t \text{ integral,} \\ 0 & \text{otherwise,} \end{cases} \tag{11.5}$$

where (u_n) is any renewal sequence.

If p is any semistandard p-function of type (iii), then $p(t) \leqq \frac{3}{4}$ for all $t > 0$, since otherwise (11.3) would imply that $p(s) > 0$ for all $s \in (0, t)$. Hence,

$$\gamma = \sup\{p(t);\, t > 0,\, p \text{ semistandard of type (iii)}\} \tag{11.6}$$

satisfies

$$\gamma \leqq \tfrac{3}{4}. \tag{11.7}$$

To obtain an inequality in the other direction, consider the standard p-function (3.14), set $a = 1 - b/q$, and let $q \to \infty$ for fixed b. The result is the semistandard p-function of the form (11.5), with

$$u_n = \pi_n(nb). \tag{11.8}$$

This is greatest when $n = b = 1$, when it has the value e^{-1}, so that

$$\gamma \geqq e^{-1}. \tag{11.9}$$

All the evidence suggests that in fact $\gamma = e^{-1}$, but no proof is known. Any proof would almost certainly imply a substantial improvement on the Davidson-

Blackwell-Freedman inequality (11.2), and might even describe the subsets of the plane defined by

$$\begin{aligned} \Gamma_1 &= \{(p(s), p(t)); p \in \mathscr{P}\mathscr{M}\}, \\ \Gamma_2 &= \{(p(s), p(t)); p \in \mathscr{P}\}, \\ \Gamma_3 &= \{(p(s), p(t)); p \in \bar{\mathscr{P}}\} = \bar{\Gamma}_1. \end{aligned} \tag{11.10}$$

12. Approximate regenerative phenomena

It is appropriate to end on a tentative note. The idea of a regenerative phenomenon is intended to describe the situation in which a Markov process returns to its starting point for a set of time points of positive measure. This is natural when handling countable state spaces, and is sometimes relevant [19] in more general situations. But many Markov processes on continuous state spaces do not return to their starting point, or do so only on a set of time instants of zero measure. For these it is more natural to think in terms of return to small neighborhoods of the starting point. Thus, instead of a single process Z_t one has a family of processes Z_t^N corresponding to neighborhoods N, and a corresponding family of "approximate p-functions" p^N. Whether there is too little structure here for a valuable theory only time will tell, but any such development would, in effect, be an abstract version of the theory of local time for diffusion processes.

A very similar situation occurs in the boundary theory of Markov chains ([3], [4]). Here one is led to an equation very like (3.10), but missing the crucial "drift" term:

$$\int e^{-\theta t} p(dt) = \left\{\int (1 - e^{-\theta x})\mu(dx)\right\}^{-1}. \tag{12.1}$$

It is no longer possible to assert that P is absolutely continuous, and the complexity of the resulting theory shows very clearly how much reliance is placed on the drift term θ in (3.10). For example, even the proper formulation of the Volterra equation which generalizes (3.20) requires analysis of formidable depth [12]. The problem is nevertheless mentioned here, in the hope that an approximate regenerative theory, of comparable scope to the exact one related in this paper, might one day be found (see the remarks of Professor Chung and Dr. Williams in the discussion of [16]).

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