SUPPORTS OF GAUSSIAN MEASURES

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1. Introduction

The present paper is a continuation of the work, carried out in [4] and [5] of investigating the relationship between a Gaussian process and its reproducing kernel Hilbert subspace. Our main result gives a characterization of the topological support of a Gaussian measure defined on a linear topological space of functions on an arbitrary set. As special cases we consider Gaussian processes on Banach spaces and on duals of Fréchet spaces.

2. Preliminary results

In this section we state, and in some cases prove, several results concerning the equivalence of Gaussian measures under translation. Many of these are modifications of well known results. They are included here since the modified versions (for the most part having to do with the removal of separability requirements) do not seem to be available in the literature. The most important result is Theorem 2.2, which provides us with the basic technique for the proofs of our support theorems. It was obtained as Lemma 6 of [4] and used there to derive certain zero-one laws for Gaussian processes.

Let T be a nonempty set, X a linear space of real valued functions on T and $\mathscr{A} = \mathscr{A}(X{:}\, T)$ the smallest σ-field of subsets of X under which all the evaluation maps $x \to x(t)$, $t \in T$, are measurable. Let a Gaussian probability measure P_0 be given on $\mathscr{A}$ such that its mean function $Ex(t) = 0$ for all t in T and

$$R(t, s) = \int_X x(t)x(s)P_0(dx) \tag{2.1}$$

is its covariance kernel. The symbol $\mathscr{A}_0 = \mathscr{A}_0(X;\, T)$ will denote the completion of $\mathscr{A}$ with respect to P_0. Let $H(R)$ be the reproducing kernel Hilbert space (RKHS) determined by R. For the definition of a RKHS see [4] where further references are given. We shall assume that $H(R)$ is a space of functions on T and that the basic space X is rich enough to contain $H(R)$.

If S is any countable subset of T, write $\mathscr{A}_S = \mathscr{A}(X;\, S)$ for the smallest σ-field of subsets of X with respect to which the maps $x \to x(t)$, $t \in S$, are measurable,

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and write $\mathscr{A}_0(X; S)$ for the completion of $\mathscr{A}_S$ with respect to P_0. For any countable set $S \subset T$ we write H_S or $H_S(R)$ for the closed linear subspace of $H(R)$ spanned by $\{R(\cdot, t), t \in S\}$. Define

$$K_S = \{x \in X: x(\cdot) = x'(\cdot) \text{ on } S \text{ for some } x' \in H_S\}. \tag{2.2}$$

For each m in X the transformation $\sigma_m: X \to X$ defined by $\sigma_m(x) = x + m$ clearly sends $\mathscr{A}$ sets into $\mathscr{A}$ sets and $\mathscr{A}_S$ sets into $\mathscr{A}_S$ sets. Furthermore, if f is any real, $\mathscr{A}$ measurable function on X, then $f(x + m)$ is jointly $\mathscr{A} \times \mathscr{A}$ measurable in (x, m). The measure P_m given by $P_m(A) = P_0(\sigma_m^{-1}A)$, $A \in \mathscr{A}$, is Gaussian with mean function m and the same covariance kernel as P_0.

LEMMA 2.1. *Let T be countable. Then*

$$P_m \equiv P_0[\mathscr{A}] \tag{2.3}$$

(that is, mutually absolutely continuous relative to $\mathscr{A}$) if and only if

$$m \in H(R). \tag{2.4}$$

Also

$$H(R) \in \mathscr{A}. \tag{2.5}$$

PROOF. The first part of the lemma is well known. To prove (2.5) proceed as follows: since $\chi_A(x - m)$, where χ_A is the characteristic function of A, is jointly measurable in (x, m), $P_m(A) = \int \chi_A(x - m)P_0(dx)$ is measurable in m for each $A \in \mathscr{A}$ and, of course, a measure in A for each $m \in X$. Moreover, $\mathscr{A}$ is countably generated since T is countable. Hence, by a result of Doob (*Stochastic Processes*, p. 616, example 2.7) we can write

$$P_m(A) = \int_A f(x, m)P_0(dx) + Q(m, A) \tag{2.6}$$

where $Q(m, \cdot)$ is singular with respect to P_0 and $f(\cdot, \cdot)$ is jointly measurable. Now from (2.3) and (2.4) we have

$$H(R) = \{m \in X : P_m \equiv P_0\} = \left\{m \in X : \int_X f(x, m)P_0(dx) = 1\right\}. \tag{2.7}$$

Hence $H(R)$ is $\mathscr{A}$ measurable.

For an arbitrary (that is, not necessarily countable) T we have the following result.

LEMMA 2.2. *Let $S \subset T$ be countable and let P'_m and P'_0 be the restrictions of P_m and P_0 to $\mathscr{A}_0(X; S)$. Then*

$$P'_m \equiv P'_0 \tag{2.8}$$

if and only if

$$m \in K_S. \tag{2.9}$$

If $m \notin K_S$, P'_m and P'_0 are mutually singular. Furthermore

$$K_S \in \mathscr{A}(X; S). \tag{2.10}$$

PROOF. Denoting by $x|_S$ the restriction of the function x to S, write $Y = \{x|_S : x \in X\}$. Let $\varphi : X \to Y$ be defined by $\varphi(x) = x|_S$. The following facts are easily verified: $\tilde{P}_0 = P_0\varphi^{-1}$ is a Gaussian measure on Y with zero mean function and covariance kernel R' = the restriction of R to $S \times S$, Y is a linear space of real functions on S, $H(R') \subset Y$ and

$$\varphi^{-1}(H(R')) = K_S. \tag{2.11}$$

Since $A \in \mathscr{A}(Y)$ if and only if $\varphi^{-1}(A) \in \mathscr{A}(X; S)$, for any $B \in \mathscr{A}(X; S)$ we have $P_0'(B) = P_0(B) = \tilde{P}_0(\varphi(B))$. Now let $m \in X$ and $B \in \mathscr{A}(X; S)$ such that $P_0'(B) > 0$. Then

$$P_0'(B + m) > 0 \Leftrightarrow \tilde{P}_0(\varphi(B + m)) > 0. \tag{2.12}$$

Since φ is a linear map, this is also equivalent to $\tilde{P}_0(\varphi(B) + \varphi(m)) > 0$, which by Lemma 2.1 is equivalent to $\varphi(m) \in H(R')$.

This gives

$$\begin{aligned} \varphi(m) \in H(R') &\Leftrightarrow m \in \varphi^{-1}(H(R')) \\ &\Leftrightarrow m \in K_S \end{aligned} \tag{2.13}$$

according to (2.11). Thus, $K_S \in \mathscr{A}(X; S)$ since $H(R') \in \mathscr{A}(Y)$.

We now deduce

THEOREM 2.1. *Let T be an arbitrary nonempty set. Then*

$$P_m \equiv P_0[\mathscr{A}] \tag{2.14}$$

if and only if

$$m \in \bigcap_S K_S, \tag{2.15}$$

where S runs over all countable subsets of T.

PROOF. If $P_m \equiv P_0$ then $P_m' \equiv P_0'$ relative to $\mathscr{A}(X; S)$ and, hence, $m \in K_S$. Since S is an arbitrary countable subset of T, it follows that $m \in \bigcap_S K_S$. Conversely, if $m \in \bigcap_S K_S$ then $P_m' \equiv P_0'$ $[\mathscr{A}(X; S)]$ for any countable $S \subset T$ so that $P_m \equiv P_0[\mathscr{A}]$.

Before proceeding further we observe that from (2.2)

$$H(R) \subset K_S. \tag{2.16}$$

Let $m \in H(R)$. Then $m \in \bigcap_S K_S$ and we have $P_m \equiv P_0$. Denote the Radon-Nikodym derivative dP_m/dP_0 by ρ_m. Note that $\rho_0 = 1$ a.s. P_0. In what follows $(,)$, $\|\cdot\|$, and $(,)_H$, $\|\cdot\|_H$ will denote the inner product and norm in $L^2(X, \mathscr{A}_0(X; S), P_0)$ and H_S, respectively. The expression $L_0(X; S)$ denotes the closed linear subspace of constant random variables and $L_1(X; S)$ the closed linear manifold in $L^2(X, \mathscr{A}_0(X; S), P_0)$ generated by the finite linear combinations $\Sigma\, c_i x(t_i)$, $t_i \in S$. The following lemmas needed in the proof of Theorem 2.2 are stated below without proof. (See [4].) We also observe that if $A \in \mathscr{A}(X; S)$ and

$m_1, m_2 \in X$ such that $m_1|_S = m_2|_S$, then $A + m_1 = A + m_2$. This observation is useful in deducing part (b) of Lemma 2.3 from (a).

LEMMA 2.3. (a) *For every* $m \in H_S$, $\rho_m \in L^2(X, \mathscr{A}_0(X; S), P_0)$ *and* $(\rho_{m_1}, \rho_{m_2}) = \exp\{(m_1, m_2)_H\}$, $m_1, m_2 \in H_S$.

(b) *If* $m \in K_S$, *then* $\rho_m \in L^2(X, \mathscr{A}_0(X, S), P_0)$, $\rho_m = \rho_{m'}$ *where* $m' \in H_S$ *such that* $m' = m$ *on* S, *and* $(\rho_{m_1}, \rho_{m_2}) = \exp\{(m_1', m_2')_H\}$.

LEMMA 2.4. *The density* ρ_m *is a continuous function from* $H(R)$ *to* $L^2(X, \mathscr{A}_0(X, S), P_0)$.

LEMMA 2.5. *The family* $\{\rho_m : m \in H_S\}$ *spans* $L^2(X, \mathscr{A}_0(X; S), P_0)$.

LEMMA 2.6. *Let* $g \in L^2(X, \mathscr{A}_0(X, S), P_0)$. *Then*

$$g \in L_0(X; S) \tag{2.17}$$

if and only if

$$(g, \rho_m) = (g, \rho_0) \text{ for all } m \in H_S. \tag{2.18}$$

THEOREM 2.2. *Let* $\{e_j\}_{j=1}^{\infty}$ *be a complete orthonormal system in* H_S *and let* g *be an* $\mathscr{A}_0(X; S)$ *measurable real function such that for each* $x \in X$ *and every rational* r

$$g(x + re_j) = g(x), \qquad j = 1, 2, 3, \cdots. \tag{2.19}$$

Then

$$g(x) = \text{constant a.s. } (P_0). \tag{2.20}$$

Since the proof is short and the theorem is of independent interest we repeat it here (see [4]).

PROOF. We shall first assume that $g \in L^2(X, \mathscr{A}_0(X; S), P_0)$. Let m be an arbitrary element of H_S. Then, using the separability of H_S, it is easy to find a sequence $\{m^{(p)}\}$, $m^{(p)} \in H_S$, such that

$$m^{(p)} = \sum_{j=1}^{n_p} c_j^{(p)} e_j \tag{2.21}$$

where $c_j^{(p)}$ are rationals, the sum (2.21) is finite and

$$\|m^{(p)} - m\|_H \to 0 \text{ as } p \to \infty. \tag{2.22}$$

By repeated application of (2.19) we obtain

$$g(x + m^{(p)}) = g(x), \tag{2.23}$$

for every $x \in X$ and $p = 1, 2, 3, \cdots$. Hence, from the standard formula for change of variable under a measurable transformation, we find that for every positive integer p

$$\begin{aligned} \int_X g(x) P_0(dx) &= \int_X g(x + m^{(p)}) P_0(dx) \\ &= \int_X g(x) \rho_{m^{(p)}}(x) P_0(dx). \end{aligned} \tag{2.24}$$

As $p \to \infty$ the right side of (2.24) converges to $\int_X g(x)\rho_m(x)P_0(dx)$ because of Lemma 2.4 and (2.22). The resulting relations from (2.24) can then be written as

$$(2.25) \qquad (g, \rho_0) = (g, \rho_m).$$

Since (2.25) holds for every $m \in H_S$, it follows from Lemma 2.6 that $g \in L_0(X; S)$. In other words,

$$(2.26) \qquad g = \text{constant a.e. } P_0.$$

Next suppose that g is $\mathscr{A}_0(X; S)$ measurable. If N is any positive integer, define $g_N(x) = g(x)$ if $|g(x)| \leqq N$ and 0 if $|g(x)| > N$. Then, since $g_N \in L^2(X, \mathscr{A}_0(X; S), P_0)$ and satisfies (2.19), it follows that $g_N(x) =$ constant a.e. P_0. Since N is arbitrary we have $g =$ constant a.e. P_0.

Since $H_S \subset H(R)$ we deduce the following.

COROLLARY 2.1. *Let g be an $\mathscr{A}$ measurable, real valued function such that for every $h \in H(R)$, and each $x \in X$*

$$(2.27) \qquad g(x + h) = g(x).$$

Then

$$(2.28) \qquad g(x) = \text{constant a.s. } P_0.$$

PROOF. Since g is $\mathscr{A}$ measurable, g is $\mathscr{A}(X; S)$ measurable for an $\mathscr{A}(X; S)$ constructed with respect to some countable set S. Further, for $m \in H_S$, $g(x + m) = g(x)$. Hence, by Theorem 2.2, $g =$ constant a.e. P_0.

COROLLARY 2.2. *Let g be an $\mathscr{A}(X; S)$ measurable, real valued function such that for all $m \in K_S$ and $x \in X$,*

$$(2.29) \qquad g(x + m) = g(x).$$

Then

$$(2.30) \qquad g = \text{constant a.s. } P_0.$$

PROOF. This is obvious since $H_S \subset K_S$.

COROLLARY 2.3. *Let A be an $\mathscr{A}(X; S)$ measurable set such that $A + m = A$ for all $m \in H_S$. Then*

$$(2.31) \qquad P_0(A) = 0 \quad or \quad P_0(A) = 1.$$

3. General theorems on the support of a Gaussian measure

In this section we equip the linear space X with a topology and describe the support of the measure P_0. As in the previous section, P_0 is a Gaussian measure on $(X, \mathscr{A})$ with zero mean function and covariance kernel R. We shall also assume that

$$(3.1) \qquad H(R) \subset X.$$

Condition (3.1) presupposes that $H(R)$ consists of functions on T (rather than equivalence classes of functions). This is ensured by assuming that R is proper, that is, that for every finite subset $t_1, \cdots, t_n$ of T, and real numbers $c_1, \cdots, c_n$

$$\sum R(t_i, t_j)c_i c_j = 0 \quad \text{implies} \quad \sum c_i R(\cdot, t_i) = 0. \tag{3.2}$$

Later (at the end of Section 5) we shall relax this restriction.

Let τ be a topology on X under which (X, τ) has the following properties.

(i) The space (X, τ) is a linear topological space such that for each t in T the evaluation map $x \to x(t)$ is continuous.

(ii) The relation $A \in \mathscr{A}$ implies that the τ closure of $A \in \mathscr{A}$. (We denote the τ closure of A by $\bar{A}$ or $\bar{A}^{\tau}$.)

(iii) Given a τ closed linear subspace $L \in \mathscr{A}$ there exists a decreasing sequence $\{U_n\}$ of τ open neighborhoods of 0 in X such that $L + U_n \in \mathscr{A}$, $U_n \in \mathscr{A}$ and

$$\bigcap_{n=1}^{\infty} (L + U_n) = L. \tag{3.3}$$

REMARKS. (a) Property (i) implies that $\tau \cap \mathscr{A}$ generates $\mathscr{A}$. We shall call the members of $\tau \cap \mathscr{A}$ the open sets in $\mathscr{A}$.

(b) *Let* $\hat{\mathscr{B}}_\tau$ = the σ-field generated by all the τ open sets of X (that is, the τ topological Borel sets in X). Also let τ_s denote the weakest topology on X (under which (X, τ_s) is a linear topological space) for which the maps $x \to x(t)$, $t \in T$, are continuous. Then

$$\mathscr{A} \subset \hat{\mathscr{B}}_{\tau_s} \subset \hat{\mathscr{B}}_\tau. \tag{3.4}$$

We shall write $\mathscr{B}_\tau$ for the σ-field of Baire subsets of X in the τ topology.

DEFINITION 3.1. *By the topological support of* P_0, *denoted by* Supp (P_0) *or* $\text{Supp}_\tau (P_0)$ *we mean the set of all points x in X such that every τ open set in $\mathscr{A}$ containing x has positive P_0 measure.*

Supp (P_0) is obviously a τ closed set but may not belong to $\mathscr{A}$.

THEOREM 3.1. *Let equations* (3.1) *and* (3.2) *and properties* (i), (ii), *and* (iii) *hold. Then*

$$\text{Supp}\,(P_0) \subset \bigcap_{S \in \Delta} \bar{K}_S, \tag{3.5}$$

where Δ is the set of all countable subsets of T and $\bar{K}_S$ is the τ closure of K_S.

PROOF. From (2.10) and (ii), $\bar{K}_S \in \mathscr{A}$. Since $\bar{K}_S$ is a τ closed linear subspace of X, by (iii) there exists a decreasing sequence $\{U_n\}$ of open neighborhoods of 0 such that $\bigcap_{n=1}^{\infty} (\bar{K}_S + U_n) = \bar{K}_S$. Let $x_0 \in$ Supp (P_0) and let $A_n = x_0 + \bar{K}_S + U_n$. Then $\bigcap_{n=1}^{\infty} A_n = x_0 + \bar{K}_S$. Since $A_n \in \mathscr{A}$, there exists some $T_0 \in \Delta$ such that $A_n \in \mathscr{A}(X; T_0)$. It has been shown in Section 2 that $H(R) \subset K_S$. Hence, if $h \in H_{T_0}$, $h \in K_S$ (since $H_{T_0} \subset H(R)$) so that $A_n + h = A_n$. By Corollary 2.3 to the main theorem of the preceding section we conclude that $P_0(A_n) = 0$ or 1. But $x_0 \in$ Supp (P_0) implies that $P_0(A_n) > 0$. Thus $P(A_n) = 1$ for all n and so

$$P_0(x_0 + \bar{K}_S) = 1. \tag{3.6}$$

Now, P_0, being Gaussian with zero mean function, is symmetric. Hence,

$$P_0\{-(x_0 + \bar{K}_S)\} = P_0(x_0 + \bar{K}_S) = 1. \tag{3.7}$$

From (3.6) and (3.7) it follows that there exist elements y and z in $\bar{K}_S$ such that

$$x_0 + y = -x_0 + z, \tag{3.8}$$

that is, $2x_0 = z - y$, which implies that $x_0 \in \bar{K}_S$ since $\bar{K}_S$ is a linear space. Thus Supp $(P_0) \subset \bar{K}_S$ and (3.5) follows since $S \in \Delta$ is arbitrary.

The conclusion of the above theorem can be strengthened if we make one more assumption of Lindelöf type concerning $\mathscr{A}$ measurability and the τ topology.

THEOREM 3.2. *Assume, in addition to equations* (3.1) *and* (3.2) *and properties* (i), (ii), *and* (iii), *that given any τ closed set A in $\mathscr{A}$ and an open cover $\{U_\alpha\}$ of A by elements U_α of $\mathscr{A}$, we can extract a countable subcover $\{U_{\alpha_j}\}_{j=1}^{\infty}$ of A. Then we have*

$$\text{Supp}\,(P_0) = \bigcap_{S\in\Delta} \bar{K}_S. \tag{3.9}$$

PROOF. First of all it is clear from the proof of Theorem 3.1 that for each $S \in \Delta$

$$P_0(\bar{K}_S) = 1. \tag{3.10}$$

Let $x_0 \in \bigcap_{S\in\Delta} \bar{K}_S$ and let V be an open set in $\mathscr{A}$ containing x_0. Since $V \in \mathscr{A}$, $V \in \mathscr{A}(X; S)$ for some countable S. From Lemma 2.2 we see that if P_0' and P_m' denote, respectively, the restriction of the measures P_0 and P_m to the σ-field $\mathscr{A}(X; S)$, where $m \in K_S$, then $P_0' \equiv P_m'$ relative to $\mathscr{A}(X; S)$. Hence, if $P_0(V) = 0$ it follows that

$$P_0(V + m) = 0 \qquad \text{for every } m \in K_S. \tag{3.11}$$

Now the family $\{V + m\}_{m\in K_S}$ is an $\mathscr{A}$ measurable open covering of the τ closed set $\bar{K}_S$. Hence, by the assumptions in Theorem 3.2, there is a countable subcover of $\bar{K}_S$ which, together with (3.11) implies $P_0(\bar{K}_S) = 0$. But this contradicts (3.10). Thus, we have shown that $P_0(V)$ must be positive for any open V in $\mathscr{A}$ containing x_0, that is, $x_0 \in$ Supp (P_0) and we have

$$\bigcap_{\Delta} \bar{K}_S \subset \text{Supp}\,(P_0). \tag{3.12}$$

Since the opposite inclusion has been shown in Theorem 3.1, the proof of (3.9) is complete.

COROLLARY 3.1. *Under equations* (3.1) *and* (3.2) *and properties* (i), (ii), *and* (iii) *we always have*

$$\overline{H(R)} \subset \text{Supp}\,(P_0). \tag{3.13}$$

This follows from Theorem 3.2 and the fact that $H(R) \subset K_S$ for every $S \in \Delta$.

THEOREM 3.3. *Suppose one of the following additional conditions holds:* (a) *there exists a countable set S such that $\bar{K}_S = \overline{H(R)}$*; (b) *$\tau = \tau_s$. Then*

$$\text{Supp}\,(P_0) = \overline{H(R)}. \tag{3.14}$$

PROOF. Part (a) is obvious. For (b) suppose $x_0 \in \text{Supp}(P_0)$ and V is a τ open neighborhood of x_0 (belonging to $\mathscr{A}$) which contains no points of $H(R)$. Since $\tau = \tau_s$, without loss of generality we may assume V to be of the form

$$V = \{x \in X : |x(t_i) - x_o(t_i)| < 1, \quad i = 1, \cdots, k\}. \tag{3.15}$$

From (3.9) taking $S = \{t_1, \cdots, t_k\}$ we have $x_0 \in \bar{K}_S$. Hence there exists $y \in K_S$ such that $y \in V$, that is, such that

$$|y(t_i) - x_0(t_i)| < 1, \qquad i = 1, \cdots, k. \tag{3.16}$$

But, by the definition of K_S, there is a point x' in H_S and, hence, belonging to $H(R)$ such that $y = x'$ on S. Hence $x' \in V$ and $x_0 \in \overline{H(R)}$. Thus we have shown that

$$\text{Supp}(P_0) \subset \overline{H(R)}. \tag{3.17}$$

Equation (3.14) now follows in view of Corollary 3.1 to Theorem 3.2.

It is natural to expect that Supp (P_0) is the intersection of all τ closed sets in $\mathscr{A}$ of probability one. Under an additional hypothesis on τ this is indeed the case.

THEOREM 3.4. *Assume, in addition to the hypotheses of Theorem* 3.2, *that for every τ closed set A in $\mathscr{A}$ and a point $x \notin A$ there exists an open set U in $\mathscr{A}$ such that $x \in U$ and $U \cap A = \varnothing$. Then*

$$\text{Supp}(P_0) = \bigcap_{\mathscr{F}} B \tag{3.18}$$

where $\mathscr{F}$ is the family of all τ closed sets in $\mathscr{A}$ of probability one.

PROOF. Let C denote the set on the right side of (3.18). Then clearly $C \subset \text{Supp}(P_0)$. If $C \neq \text{Supp}(P_0)$ there is a τ closed set $B \in \mathscr{F}$ such that for every countable subset S of T we have

$$\bar{K}_S \not\subset B. \tag{3.19}$$

Let S be a countable set such that B is in the σ-field $\mathscr{A}(X; S)$. From (3.19) there exists $x \in K_S$ such that $x \notin B$. By the assumption in Theorem 3.4 there is an open set U in $\mathscr{A}$ such that $x \in U$ and U and B are disjoint. This implies that $P_0(B) < 1$ since $P_0(U)$ is positive. We, thus, have a contradiction. Hence, $C = \text{Supp}(P_0)$ and (3.18) is proved.

Equation (3.18) always holds if (X, τ) is a locally convex linear topological space and $\mathscr{A} = \hat{\mathscr{B}}_\tau$ since, in this case, the assumption of Theorem 3.4 is satisfied.

4. Gaussian measures in separable Banach spaces

We shall not go into details in this section as this problem has been studied in three recent papers ([5], [6], [8]) motivated largely by the introduction of the notion of abstract Wiener space by L. Gross [2].

THEOREM 4.1. *Let X be a separable, infinite dimensional Banach space and τ, the norm topology in X. Suppose that P_0 is a Gaussian probability measure on $(X, \mathscr{A})$ with zero mean and proper covariance functional R. Then there exists a* *separable Hilbert space H which, if R is proper, is a dense subspace of X such that if $\bar{H}$ is its closure in X with respect to τ,*

$$\bar{H} = \operatorname{Supp}(P_0). \tag{4.1}$$

In this result H is the generating Hilbert subspace of P_0, a Hilbert subspace congruent to the RKHS $H(R)$. For the proof of its existence we refer the reader to the cited papers. In [5] equation (4.1) is proved under the restriction that R is continuous. The condition $R(y, y) = 0$ implies $y = 0$ ensures that R is proper. Since it is well known that in a separable Banach space $\mathscr{A} = \hat{\mathscr{B}}_\tau$ and since the Lindelöf theorem holds in X, among the conditions assumed for Theorem 3.3 it is necessary only to comment on (iii). The neighborhoods U_n may be taken to be the spheres $\{x \in X : \|x\| < \frac{1}{n}\}$, since these belong to $\mathscr{A}$. It is easily seen that for any closed set A in X, $\bigcap_{n=1}^{\infty} (A + U_n) = A$. Observe that since $\bar{H} \in \mathscr{A}$, the proof of Theorem 3.1 with $\bar{K}_S$ replaced by $\bar{H}$ yields $\operatorname{Supp}(P_0) \subset \bar{H}$, while the opposite inclusion follows from (3.18). Thus (4.1) is proved.

5. Gaussian measures on duals of Fréchet spaces

Let F be a separable Fréchet space and $E = F'$, the topological dual of F. Take $(X, \tau) = (E, \tau_s)$ where τ_s is the $\sigma(E, F)$ topology in E. It is known that (see [7])

$$\mathscr{A} = \mathscr{B}_\tau. \tag{5.1}$$

We shall assume that P_0 is a Gaussian measure on $(E, \mathscr{A})$ with zero mean and covariance functional R. For convenience, we first make the assumption that R is proper.

LEMMA 5.1. *Under the above assumptions,*

$$H(R) \subset E, \tag{5.2}$$

and

$$H(R) \tag{5.3}$$

is a separable Hilbert space.

PROOF. Write $\sigma^2(y) = R(y, y), y \in F$. Let $\langle , \rangle$ be the natural linear form in E. First, we show that if y_0 is any point in F and $\{y_n\}$ in F converges to y_0, then

$$\sigma^2(y_n - y_0) \to 0. \tag{5.4}$$

Following the procedure in [3] the sequence of random variables $\{\langle x, y_n \rangle\}$ on $(E, \mathscr{A}, P_0)$ converges in probability to $\langle x, y_0 \rangle$ since they converge a.s. (indeed for every x) to it. Hence, for every $\varepsilon > 0$,

$$P_0(\{x \in E : |\langle x, y_n \rangle - \langle x, y_0 \rangle| > \varepsilon\}) < \varepsilon, \tag{5.5}$$

for all sufficiently large n. The probability on the left side equals

$$\left(\frac{2}{\pi}\right)^{1/2} \int_{\varepsilon/\sigma(y_n - y_0)}^{\infty} e^{-t^2/2}\, dt. \tag{5.6}$$

If (5.4) is not true, for some $\varepsilon' > 0$ there is a subsequence $\{y_{n'}\}$ such that $\sigma(y_{n'} - y_0) \geqq \varepsilon' > 0$. Then the integral in (5.6), with $n = n'$, is bounded below by

$$\left(\frac{2}{\pi}\right)^{1/2} \int_{\varepsilon/\varepsilon'}^{\infty} e^{-t^2/2}\, dt. \tag{5.7}$$

This implies a contradiction of (5.5) if ε is chosen small enough, and (5.4) is established. The inequality

$$\begin{aligned} |R(y_1, y_2) - R(y_1^0, y_2^0)| \leqq \sigma(y_1 - y_1^0)\sigma(y_2 - y_2^0) &+ \sigma(y_1^0)\sigma(y_2 - y_2^0) \\ &+ \sigma(y_2^0)\sigma(y_1 - y_1^0), \end{aligned} \tag{5.8}$$

where (y_1, y_2), (y_1^0, y_2^0) belong to $F \times F$ shows that R is continuous on $F \times F$. Since F is separable we have (5.3). Furthermore, by the reproducing property, for every f in $H(R)$ and y in F

$$f(y) = (f, R(\cdot, y))_{H(R)} \tag{5.9}$$

which leads to

$$|f(y)| \leqq \|f\|_{H(R)}\sigma(y, y). \tag{5.10}$$

Clearly, f is linear on F and by (5.4) and (5.10) it is, moreover, continuous. Hence, $f \in F'$ and (5.2) is proved.

LEMMA 5.2. *Let A be any closed subset of E. For every open cover $\{U_\alpha\}$ of A there exists a countable subcover of A.*

PROOF. Choose a decreasing sequence V_n of neighborhoods of 0 in F. Then the polars V_n^0 are compact and metrizable and

$$E = \bigcup_{n=1}^{\infty} V_n^0. \tag{5.11}$$

Since the relative topology of V_n^0 has a countable base, by Lindelöf's theorem the open cover $\{U_\alpha \cap V_n^0\}$ of $A \cap V_n^0$ contains a countable subcover. Since this is true for each n, the assertion of the lemma follows from (5.11).

The assumption of Theorem 3.2 is an immediate consequence. We also deduce condition (iii) as follows. If A is any τ closed linear subspace of E, then

$$A = \bigcap_{y \in A^0} \{x \in E : \langle x, y\rangle = 0\}. \tag{5.12}$$

(See [9], [10]). Since A is a closed linear subspace of $(E, \sigma(E, F))$, A^0 is a closed linear subspace of the topological space $(F, \sigma(F, E))$. Since A^0 is convex, A^0 is closed in F as a Fréchet space. (If $M \subset F$ is convex, M is closed if and only if it is weakly closed in F). Since F is a separable Fréchet space, there exists a countable subset $D = \{y_j\}$ of A^0 which is dense in the relative topology of A^0.

Then

$$A = \bigcap_{j=1}^{\infty} \{x \in E : \langle x, y_j \rangle = 0\}. \tag{5.13}$$

Let x_0 belong to the set on the right side in (5.13). If y is any point in A^0, then there exists a subsequence $\{y'_j\}$ in D such that $y'_j \to y$ (in the initial topology of F). Since $x_0 \in E = F'$, $\langle x_0, y'_j \rangle \to \langle x_0, y \rangle$. Hence $\langle x_0, y \rangle = 0$, that is, $x_0 \in A$ by (5.12), and (5.13) is shown. For each $n = 1, 2, \cdots$ define

$$U_n = \bigcap_{j=1}^{n} \{x \in E : |\langle x, y_j \rangle| < 2^{-n}\}. \tag{5.14}$$

$\{U_n\}$ is a decreasing sequence of τ open neighborhoods of 0 in E. It will now be shown that

$$\bigcap_{n=1}^{\infty} (A + U_n) = A. \tag{5.15}$$

Suppose $x \in A + U_n$ for each n. Then

$$\begin{aligned} x &= a_n + u_n, \qquad a_n \in A, u_n \in U_n, \\ \langle x, y_j \rangle &= \langle a_n, y_j \rangle + \langle u_n, y_j \rangle = \langle u_n, y_j \rangle \end{aligned} \tag{5.16}$$

from (5.13). For fixed j and $n \geqq j$,

$$|\langle x, y_j \rangle| = |\langle u_n, y_j \rangle| < 2^{-n} \to 0 \qquad \text{as } n \to \infty. \tag{5.17}$$

Hence $\langle x, y_j \rangle = 0$ for every j and $x \in A$, that is, $\bigcap_{n=1}^{\infty} (A + U_n) \subset A$. Equation (5.15) follows since the reverse inclusion is obvious. Condition (iii) of Theorem 3.1 has thus been verified. Since all the conditions of Theorem 3.3 are satisfied for the Gaussian measure space $(E, \mathscr{A}, P_0)$ with the topology τ_s for τ in E (condition (ii) is guaranteed by (5.1)) we are in a position to prove the main result of this section.

THEOREM 5.1. *If P_0 is a Gaussian probability measure with zero mean and covariance R on $(E, \mathscr{A})$ with $\tau = \tau_s$, then*

$$\operatorname{Supp}(P_0) = \overline{H(R)} = E. \tag{5.18}$$

It remains only to prove the second assertion in (5.18), namely, that $H(R)$ is dense in E. If $\overline{H(R)} \neq E$, by the Hahn–Banach theorem there exists an element $y \neq 0$ in the dual of $(E, \sigma(E, F))$ such that y vanishes identically on $\overline{H(R)}$. Since the dual is $(F, \sigma(F, E))$ we have $y \in F$. Next, $\overline{H(R)}$ belongs to $\mathscr{A}$ from (5.1) and that $P_0(\overline{H(R)}) = 1$. Finally, from

$$R(y, y) = \int_E \langle x, y \rangle^2 P_0(dx) = \int_{\overline{H(R)}} \langle x, y \rangle^2 P_0(dx) = 0 \tag{5.19}$$

and the assumption that R is proper, it follows that $y = 0$, a contradiction. The proof of (5.18) is complete.

REMARK. Let $\tau_1 (\geqq \tau_s)$ be any locally convex topology in E compatible with the duality of E and F. Then the τ_1 and τ_s closures of $H(R)$ are the same. If, further, the assumption of Theorem 3.2 holds for (E, τ_1) (for example, if $\tau_1 = \tau_c$), then it is easy to show that

$$\mathrm{Supp}_{\tau_1}(P_0) = \mathrm{Supp}_{\tau_s}(P_0) = \overline{H(R)}. \tag{5.20}$$

To complete the results of this section we abandon the restriction that R is proper. Define

$$\Gamma = \{y \in F : R(y, y) = 0\}. \tag{5.21}$$

Since R is continuous (Lemma 5.1), Γ is a closed linear subspace of F. It is well known that the quotient topology on F/Γ is metrizable and complete. Furthermore, since F is separable, it follows that $\tilde{F} = F/\Gamma$ is a separable Fréchet space. Let $\tilde{E} = \tilde{F}'$, the dual of $\tilde{F}$ ($\tilde{F}$ being regarded as a separable Fréchet space). Then it is easily seen that $\tilde{E}$ is the linear subspace of E given by Γ^0, the polar of Γ, that is,

$$\tilde{E} = \bigcap_{y \in \Gamma} \{x \in E : \langle x, y\rangle = 0\}. \tag{5.22}$$

Clearly, $\tilde{E}$ is a closed linear subspace of E if we take in E, $\tau = \sigma(E, F)$. From (5.1) $\mathscr{A} = \hat{\mathscr{B}}_\tau = \mathscr{B}_\tau$, $\mathscr{B}_\tau$ being the σ-field of Baire subsets of E in the $\sigma(E, F)$ topology. Also, since from Lemma 5.2 it follows that (E, τ) is a Lindelöf space, P_0 given on $\mathscr{A}$ is a τ-smooth probability measure (see [1], Corollary 1.9.2). Direct the set of all finite subsets $\{\gamma\}$ of Γ by: $\gamma_1 \geqq \gamma_2$ if and only if $\gamma_1 \subset \gamma_2$ and write $C_\gamma = \bigcap_{y \in \gamma} \{x \in E : \langle x, y\rangle = 0\}$. If $\gamma_1 \geqq \gamma_2$ then $C_{\gamma_1} \supset C_{\gamma_2}$. Therefore, $\{C_\gamma\}$ is a decreasing net such that ((5.22))

$$\tilde{E} = \bigcap_{\gamma} C_\gamma. \tag{5.23}$$

The τ smoothness of P_0 implies that

$$P_0(\tilde{E}) = \lim_{\gamma} P_0(C_\gamma). \tag{5.24}$$

But for each γ, $P_0(C_\gamma) = 1$ since $P_0(\{x \in E : \langle x, y\rangle = 0\}) = 1$ for every $y \in \Gamma$. Hence,

$$P_0(\tilde{E}) = 1. \tag{5.25}$$

Denote the σ-field $\tilde{E} \cap \mathscr{A}$ by $\tilde{\mathscr{A}}$. Then $\tilde{\mathscr{A}}$ is the σ-field $\mathscr{A}_{\tilde{F}}(\tilde{E})$ and by (5.25), P_0 is a Gaussian probability measure on $(\tilde{E}, \tilde{\mathscr{A}})$. Furthermore it is well known that the topology $\sigma(\tilde{E}, \tilde{F})$ is the same as the topology induced on $\tilde{E}$ by $\sigma(E, F)$. This fact implies two conclusions: first, writing $\tilde{\tau} = \sigma(\tilde{E}, \tilde{F})$, it is easy to see that

$$\mathrm{Supp}_{\tilde{\tau}}(P_0) = \mathrm{Supp}_{\tau}(P_0); \tag{5.26}$$

secondly, $\tilde{E}$ being the dual of the separable Fréchet space $\tilde{F}$, recalling that $\tilde{\tau} = \sigma(\tilde{E}, \tilde{F})$, we have as in (5.1)

$$\tilde{\mathscr{A}} = \hat{\mathscr{B}}_{\tilde{\tau}}. \tag{5.27}$$

The function

$$\tilde{R}(\eta_1, \eta_2) = R(y_1, y_2), \tag{5.28}$$

where $\eta_i \in \tilde{F}$ and y_i is any element from η_i is the covariance functional of P_0 on $(\tilde{E}, \tilde{\mathscr{A}})$. Furthermore, it is obvious that $\tilde{R}$ is proper. Hence Theorem 5.1 can now be applied to $(\tilde{E}, \tilde{\mathscr{A}}, P_0)$. Finally, observing that the RKHS $H(\tilde{R})$ is contained in $\tilde{E}$ and that the $\tilde{\tau}$ closure of $H(\tilde{R})$ = τ closure of $H(\tilde{R})$, we obtain the following result.

THEOREM 5.2. *Let P_0 be a Gaussian measure on $(E, \mathscr{A})$ with zero mean and covariance R. If $\tau = \sigma(E, F)$ and $\overline{H(\tilde{R})}$ denotes the τ closure of $H(\tilde{R})$ then $\overline{H(\tilde{R})}$ is a separable subspace of E, and*

$$\overline{H(\tilde{R})} = \mathrm{Supp}_\tau (P_0) = \tilde{E}. \tag{5.29}$$

In Theorems 5.1 and 5.2 we have considered Gaussian measures on the topological dual E of a separable Fréchet space and the topology τ in E is taken to be either τ_s (that is, $\sigma(E, F)$) or τ_c, the topology of uniform convergence on compact subsets of F. In checking the applicability of Theorem 3.2 the main point has been to verify property (iii) and the assumption of Theorem 3.2. We learn from a conversation with J. Hoffman-Jorgensen that these conditions are verifiable in the more general situation when P_0 is given on a space E which is the topological dual of an analytic space and τ is either τ_s or τ_c. This is so since (E, τ) has the hereditary Lindelöf, as well as the hereditary separable, property. Furthermore, it is also true that in this case $\mathscr{A} = \hat{\mathscr{B}}_\tau$ so that condition (ii) is automatically satisfied.

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