HARNESSES

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1. Long-range misorientation in the crystalline structure of metals

Sometime in 1955 or 1956 at the Atomic Energy Research Establishment at Harwell, Professor A. H. Cottrell set me a problem on long-range misorientation in the crystalline structure of metals. This made me think about multidimensional martingales, which I propose to call harnesses. The whole subject seems largely unexplored, and I have only some sketchy and tentative remarks to make; but I resurrect this material in the hope that others will bring it to fruition.

When we analyze the micro-structure of a large lump of metal, we find two major features; first, the metal consists of an assemblage of imperfect metal crystals, called grains; and second, the grains consist of small contiguous domains (called subgrains) of irregular size and shape, in each of which the atoms are packed on a perfect crystalline lattice. Each subgrain has a unit vector specifying the spatial orientation of its lattice. The vectors of adjacent subgrains subtend small irregular angles, one with the next, say random angles with some specified distribution. It is these angles between subgrains that are the manifestation of the imperfection of the crystal structure in the grains. Except in special circumstances (which do not concern us here) the orientations of neighboring grains are quite independent; but those of neighboring subgrains are not independent because the change of orientation from one to the next is small. A grain contains a large number of subgrains. One might suppose that the small irregularities between neighboring subgrains would cumulate, so that the long-range misorientation (namely the angle between the vectors of two subgrains within the same grain but many subgrains apart) would increase with increasing distance between the subgrains and so ultimately become quite large. Yet observation shows this is not to be the case: the long-range misorientation seems to be of the same order of magnitude as the short-range misorientation between adjacent subgrains. Indeed, it is this smallness of the long-range misorientation which is the manifestation of the (albeit imperfect) overall crystalline coherency of the grain. The general orientation of the structure can only change by large amounts at the so-called "large-angle grain boundaries" between one grain and the next. Much the same sort of effect seems to occur if one tries to distort at random a sheet of paper by crumpling it up. The normal to the surface of the paper has a more or less constant direction (or else varies quite smoothly and not at all randomly) except at the creases in the crumpled paper, where it undergoes a sudden drastic change. The creases in the paper are analogous to the large-angle grain boundaries in the metal.

(It may entertain mathematical statisticians to investigate the mathematics of crumpled paper: at least this will afford them a theoretical study which of itself generates its own abundant supply of experimental material.)

Both for paper and metal, and for a number of other physical situations, the general question confronts us of whether d-dimensional Euclidian space can suffer random distortion; and, if so, to what extent; and how does that extent depend upon d. We shall see that the question is in some rather ill-understood way bound up with the topological properties of d-dimensional space. Nearly all stochastic processes deal with the case $d = 1$, and we know quite a lot about that case; but we know virtually nothing about the cases $d > 1$. Since the problem is difficult, I shall make several simplifications, some quite drastic. However, it is worth making one remark at once about the function $\mathbf{d}(\mathbf{r})$ defined in the next paragraph. This is a vector function $\mathbf{d}$ of a vector variable $\mathbf{r}$; and we shall (effectively) reduce it to a scalar function y of a vector variable $\mathbf{r}$. This reduction is largely a matter of convenience, and not a bypassing of the intrinsic multidimensionality of the problem. The intrinsic multidimensionality lies in the vectorial character of the parameter $\mathbf{r}$; and it is this parameter $\mathbf{r}$ which embodies the topological peculiarities of the situation. By contrast, the vectorial character of $\mathbf{d}$ is trivial: it could have been reduced to a scalar function in many other ways than the one adopted here. For instance, we might have taken a linear functional of $\mathbf{d}$ or we might have considered functionals on the orthogonal group of rotations; and to have done so might have given an illusion of mathematical elegance and paid a courtship to fashionable "modern" mathematics. But in the presence of real mathematical difficulties it is as well to argue as simply as possible.

Let $\mathbf{r}$ denote the position vector of a typical point in the metal, and let $\mathbf{d}(\mathbf{r})$ be the unit vector associated with the subgrain containing $\mathbf{r}$. We have to consider fluctuations of $\mathbf{d}(\mathbf{r})$, which is a vector function of a vector variable. The first simplification will reduce this to a problem on the fluctuations of a scalar function of a vector variable. Let P and Q be two distant points in the metal, and take a sequence of points $P = P_0, P_1, \cdots, P_n = Q$ such that the P_i are in the successive separate subgrains visited in travelling from P to Q along some reasonably direct path. Let $\mathbf{d}_i = \mathbf{d}(\mathbf{r}_i) = (x_i, y_i, z_i)$ where $\mathbf{r}_i$ is the position vector of P_i, and suppose that the coordinate system has been chosen so that $\mathbf{d}_0 = (1, 0, 0)$. If θ_n is the angle between the unit vector at P and Q, we have

$$\sin^2 \theta_n = 1 - \cos^2 \theta_n = 1 - x_n^2 = y_n^2 + z_n^2. \tag{1.1}$$

The symmetry of the problem makes us presume that

$$Ey_n = Ez_n = 0; \; Ey_n^2 = Ez_n^2. \tag{1.2}$$

Hence,

$$E(\sin^2 \theta_n) = 2 \operatorname{var} y_n. \tag{1.3}$$

We can therefore get some information on the magnitude of the long-range misorientation θ_n by studying the fluctuations of y_n, which is a scalar function of the vector variable $\mathbf{r}_n - \mathbf{r}_0$.

If θ_n is reasonably small, even though n is large, the vectors $\mathbf{d}_0, \mathbf{d}_1, \cdots, \mathbf{d}_n$ will be approximately parallel, and the differences $y_i - y_{i-1}$ will have a common distribution. We shall later see that the assumption that the differences $y_i - y_{i-1}$ are identically (but not independently) distributed with a small variance σ^2, does eventually lead to the conclusion that θ_n is also small, even for large n; so this assumption is (a posteriori) a reasonable one, and we shall make it.

So far we have said nothing about the irregular (random) shapes and sizes of the subgrain, apart from the tacit presumption, arising from some sort of homogeneity, that n will be roughly proportional to the distance between P and Q. The subgrains fit together to form a continuous solid, so the problem of specifying their joint shape-size distribution is a really formidable question in statistical topology, a subject that merits study in its own right. But we are up against enough difficulties without this one, and I propose to ignore it by assuming that all the subgrains are cubes of equal size. Then we may represent each subgrain by integer coordinates (r, s, t); and each subgrain will carry a scalar function $y_{r,s,t}$ whose fluctuations interest us. Hereafter, when we speak about a lattice or lattice-points, we shall mean this lattice of integers (r, s, t) and later more general lattices, but never the atomic crystalline lattices of the metallic subgrains. It is not easy to decide how drastic is this assumption of cubic subgrains; but perhaps it is not too far out.

Sometimes we shall write the scalars in the form y_{rst} when we wish to emphasize the lattice points to which they belong. At other times we shall write them in the form y_i when we wish to regard them as values at the i-th point of some path on the lattice (r, s, t).

Since $y_0 = 0$, we have

$$\text{var } y_n = \text{var}\left[\sum_{i=1}^{n} (y_i - y_{i-1})\right]; \tag{1.4}$$

and, if the differences $y_i - y_{i-1}$ were independent, we should have

$$\text{var} \sum_{i=1}^{n} (y_i - y_{i-1}) = \sum_{i=1}^{n} \text{var } (y_i - y_{i-1}) = n\sigma^2, \tag{1.5}$$

and the long-range misorientation θ_n would behave like a multiple of $n^{1/2}$. But we cannot reasonably assume independence for the differences $y_i - y_{i-1}$, for these differences are associated with steps between neighboring lattice points, and the summation $\sum$ in (1.4) is a path-sum over successive steps of a path. If we choose two different paths joining a pair of specified end-points, the two path-sums must be equal. Indeed, a path-sum around any closed path must be zero. Hence, the differences $y_i - y_{i-1}$ are dependent in a rather complicated topological fashion, and it is this that vitiates the familiar relation var $\sum =$ $\sum$ var. Equally, we wish to discover what happens to this familiar relation when $\sum$ is path-summation in several dimensions.

From the point of view of mathematical simplicity, it is very desirable to express the physical situation in terms of a system of independent random variables. Quantities like $y_i - y_{i-1}$ are associated with the arcs of the lattice, that is to say, line segments joining nearest-neighbor lattice-points; and we have seen that it is unreasonable to assign independent random variables to these arcs of the lattice. There are, as it were, too many arcs. In fact, there are three times as many arcs as lattice-points, and it is this excess of arcs over lattice-points which distinguishes the multidimensional case from the one-dimensional case. Let us, therefore, try to associate independent random variables with the lattice-points instead of the arcs.

2. Martingales and harnesses

Since the differences $y_i - y_{i-1}$ have zero expectations by hypothesis, we have the conditional expectation

$$E(y_i | y_{i-1}, y_{i-2}, \cdots) = E(y_i | y_{i-1}) = y_{i-1}. \tag{2.1}$$

This is a martingale in its simplest form. In terms of lattice coordinates, we might write this in the form

$$E(y_{r,s,t} | y_{r-1,s,t}) = y_{r-1,s,t}; \tag{2.2}$$

but, equally, we might write

$$E(y_{r,s,t} | y_{r,s-1,t}) = y_{r,s-1,t}. \tag{2.3}$$

This suggests the relation

$$E(y_{r,s,t} | y_{r-1,s,t}, y_{r,s-1,t}, y_{r,s,t-1}) = \tfrac{1}{3}(y_{r-1,s,t} + y_{r,s-1,t} + y_{r,s,t-1}), \tag{2.4}$$

which in turn suggests

$$y_{r,s,t} = \tfrac{1}{3}(y_{r-1,s,t} + y_{r,s-1,t} + y_{r,s,t-1}) + \epsilon_{r,s,t}, \tag{2.5}$$

where $\epsilon_{r,s,t}$ is a zero-mean random variable associated with the lattice-point (r, s, t). Alternatively, we might consider a system given by

$$\begin{aligned} y_{r,s,t} = \tfrac{1}{6}(&y_{r-1,s,t} + y_{r,s-1,t} + y_{r,s,t-1} \\ &+ y_{r+1,s,t} + y_{r,s+1,t} + y_{r,s,t+1}) + \epsilon^*_{r,s,t}, \end{aligned} \tag{2.6}$$

where $\epsilon^*_{r,s,t}$ is a zero-mean random variable associated with the lattice point (r, s, t). I shall call systems, like (2.4), (2.5), (2.6), harnesses. I deliberately leave this definition of harness in a vague form because I am very uncertain what form of definition is most profitable.

The idea behind the terminology is the following. In gaming, a martingale is a fair gambling system, and this is probably the immediate source of the stochastic sense of "martingale." But in turn, the gaming term seems to have its origin in the equestrian sense of the word "martingale." In that sense, a martingale is a strap that prevents a horse from throwing up his head. If the horse is proceeding in the positive sense of the parameter i, and his mouth and breast are at heights y_i and z_i, respectively, above the ground at time i, then he will be moving in a steady horizontal fashion when his breast is now at the same height as his mouth was at the previous moment; thus, $y_i = z_i = y_{i-1}$ in conformity with (2.1). Since the strap checks upward but not downward movements of the head, it comes closer to what a mathematician would call a submartingale. If there are constraints from several different directions, as in (2.4), we may imagine them caused by several different straps, or by a harness.

The history of martingales goes back a long way, and there are elaborate reliefs in the British Museum depicting martingales in the reigns of Tiglath-Pileser III (745–727 B.C.), Sennacherib (705–681 B.C.), and Assurbanipal (668–626 B.C.). Anderson [1] writes of the Assyrians: "They manage their horses with bit and bridle, and later reliefs show a remarkable anticipation of the modern martingale (not used as far as I know by any other ancient people). The reins are attached to a large tassel hanging below the horse's neck, which continues to provide a certain check on the horse's mouth. The rider is thus enabled to use both hands for his weapons, and can shoot the bow at full gallop." Müseler [8] and Hitchcock [4] give information about the various types of modern martingale (the standing, running, and Irish martingales), and the latter author has a colorful passage in which he says: "The standing martingale, which is used as a check to prevent the horse from throwing up his head and hitting the rider in the face, or carrying it too high, is a good remedy for stargazing, or for horses which have ewe-necks. . . . This type of martingale is used universally on the polo ground."

3. One-sided harnesses and their generating functions

We shall call (2.5) a one-sided harness, because the constraints on $y_{r,s,t}$ all come from one side of the plane $r + s + t =$ constant. Since (2.5) is a difference equation, it needs some boundary conditions. For simplicity we suppose that these are

$$y_{0,s,t} = y_{r,0,t} = y_{r,s,0} = 0. \tag{3.1}$$

With these boundary conditions, any system of $\epsilon_{r,s,t}$ (including, of course, the identically distributed independent ones we propose to use) will uniquely determine all $y_{r,s,t}$ for all nonnegative r, s, t. To see this, we note that the $y_{r,s,t}$ (with $r \geq 0$, $s \geq 0$, $t \geq 0$) are defined on the planes $r + s + t = u$ recursively for $u = 0, 1, 2, \cdots$ by (2.5) and (3.1).

We wish to study the fluctuations of $y_{r,s,t}$, but at the same time we do not want the boundary conditions (3.1) to influence the result unduly. We shall therefore examine

$$V(\ell, m, n) = \lim_{r,s,t \to \infty} E(y_{r+\ell,s+m,t+n} - y_{r,s,t})^2. \tag{3.2}$$

Consider the ν-dimensional version of (2.5), (3.1), and (3.2). We have

$$y_{r_1,r_2,\cdots,r_\nu} = \frac{1}{\nu} \{y_{r_1-1,r_2,r_3,\cdots,r_\nu} + \cdots + y_{r_1,r_2,\cdots,r_{\nu-1},r_\nu-1}\} + \epsilon_{r_1,r_2,\cdots,r_\nu}, \tag{3.3}$$

$$y_{0,r_2,r_3,\cdots,r_\nu} = \cdots = y_{r_1,r_2,\cdots,r_{\nu-1},0} = 0, \tag{3.4}$$

$$V(\ell_1, \ell_2, \cdots, \ell_\nu) = \lim_{r_1,\cdots,r_\nu\to\infty} E(y_{r_1+\ell_1,\cdots,r_\nu+\ell_\nu} - y_{r_1,r_2,\cdots,r_\nu})^2. \tag{3.5}$$

To evaluate V, we introduce the generating functions

$$Y(u_1, u_2, \cdots, u_\nu) = \sum_{r_1\geq 0,\cdots,r_\nu\geq 0} y_{r_1,r_2,\cdots,r_\nu} u_1^{r_1} u_2^{r_2} \cdots u_\nu^{r_\nu}, \tag{3.6}$$

$$X(u_1, u_2, \cdots, u_\nu) = \sum_{r_1\geq 1,\cdots,r_\nu\geq 1} \epsilon_{r_1,r_2,\cdots,r_\nu} u_1^{r_1} u_2^{r_2} \cdots u_\nu^{r_\nu}. \tag{3.7}$$

Notice that the lower limits of summation are different in (3.6) and (3.7). On multiplying (3.3) by $u_1^{r_1}u_2^{r_2} \cdots u_\nu^{r_\nu}$ and summing over $r_1 \geq 1, \cdots, r_\nu \geq 1$ we get, by virtue of (3.4),

$$Y = \nu^{-1}(u_1 + u_2 + \cdots + u_\nu)Y + X. \tag{3.8}$$

Next write

$$z_{r_1,r_2,\cdots,r_\nu} = y_{r_1+\ell_1,\cdots,r_\nu+\ell_\nu} - y_{r_1,r_2,\cdots,r_\nu} \tag{3.9}$$

and

$$Z(u_1, u_2, \cdots, u_\nu) = \sum_{r_1\geq 1,\cdots,r_\nu\geq 1} z_{r_1,r_2,\cdots,r_\nu} u_1^{r_1} u_2^{r_2} \cdots u_\nu^{r_\nu}. \tag{3.10}$$

Then

$$Z = (u_1^{-\ell_1}u_2^{-\ell_2} \cdots u_\nu^{-\ell_\nu} - 1)Y = \frac{(u_1^{-\ell_1}u_2^{-\ell_2}, \cdots, u_\nu^{-\ell_\nu} - 1)}{1 - (u_1 + u_2 + \cdots + u_\nu)/\nu} X, \tag{3.11}$$

by (3.9) and (3.8). Hence, $Ez^2_{r_1,r_2,\cdots,r_\nu}$ is the coefficient of $(u_1u_2 \cdots u_\nu)^0 v_1^{r_1}v_2^{r_2} \cdots v_\nu^{r_\nu}$ in the expansion of

$$\begin{aligned} &E\{Z(u_1v_1, \cdots, u_\nu v_\nu)Z(u_1^{-1}, u_2^{-1}, \cdots, u_\nu^{-1})\} \\ &\quad = \frac{\{(u_1v_1)^{-\ell_1} \cdots (u_\nu v_\nu)^{-\ell_\nu} - 1\}\{u_1^{\ell_1} \cdots u_\nu^{\ell_\nu} - 1\}}{\left\{1 - \frac{1}{\nu}\sum_{i=1}^{\nu} u_i v_i\right\}\left\{1 - \frac{1}{\nu}\sum_{i=1}^{\nu} u_i\right\}} E_X \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} E_X &= E\{X(u_1v_1, \cdots, u_\nu v_\nu)X(u_1^{-1}, u_2^{-1}, \cdots, u_\nu^{-1})\} \\ &= \sum_{r_1\geq 1,\cdots,r_\nu\geq 1} v_1^{r_1}v_2^{r_2} \cdots v_\nu^{r_\nu}\sigma^2 = \left(\prod_{j=1}^{\nu} \frac{v_j}{1 - v_j}\right)\sigma^2 \end{aligned} \tag{3.13}$$

because, by hypothesis, the $\epsilon_{r_1,r_2,\cdots,r_\nu}$ are independently distributed, each with zero mean and variance σ^2.

It is perhaps worth mentioning that questions of convergence do not arise in any of the foregoing work. We are only concerned with picking out specified coefficients in the power series and these series can all be truncated at a suitable point if necessary. From (3.12) and (3.13) we have

$$(3.14)\quad \sum_{r_1\ge 1,\cdots,r_\nu\ge 1} v_1^{r_1}v_2^{r_2}\cdots v_\nu^{r_\nu}Ez^2_{r_1,r_2,\cdots,r_\nu}$$
$$= \sigma^2\prod_{j=1}^{\nu}\left\{\frac{v_j\sigma^2}{2\pi(1-v_j)}\int_{-\pi}^{\pi}d\theta_j\right\}$$
$$\cdot\frac{\{(e^{i\theta_1}v_1)^{-\ell_1}\cdots(e^{i\theta_\nu}v_\nu)^{-\ell_2}-1\}\{e^{i\ell_1\theta_1}\cdots e^{i\ell_\nu\theta_\nu}-1\}}{\left\{1-\frac{1}{\nu}\sum_{j=1}^{\nu}e^{i\theta_j}v_j\right\}\left\{1-\frac{1}{\nu}\sum_{j=1}^{\nu}e^{-i\theta_j}\right\}}.$$

Here we have used the ordinary properties of Fourier series to pick out the coefficient of $(u_1u_2\cdots u_\nu)^0$ in (3.12). If $|v_j| < 1$, the expressions on the right of (3.14) can be expanded and the coefficients identified with those in the formal expressions (3.12) and (3.13). Hence, (3.14) holds for all $|v_j| < 1$, and the series on the left of (3.14) converges under these conditions.

Writing Alim for the Abelian limit, which by regularity is equal to the ordinary limit whenever the latter exists, we get

$$(3.15)\quad V(\ell_1,\ell_2,\cdots,\ell_\nu) = \operatorname*{Alim}_{r_1\to\infty,\cdots,r_\nu\to\infty} Ez^2_{r_1,r_2,\cdots,r_\nu}$$
$$= \lim_{v_1\to 1,\cdots,v_\nu\to 1}\sigma^2(1-v_1)\cdots(1-v_\nu)\sum_{r_1\ge 1,\cdots,r_\nu\ge 1} v_1^{r_1}v_2^{r_2}\cdots v_\nu^{r_\nu}Ez^2_{r_1,r_2,\cdots,r_\nu}$$
$$= \sigma^2\prod_{j=1}^{\nu}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta_j\right\}\frac{\{e^{-i(\ell_1\theta_1+\cdots+\ell_\nu\theta_\nu)}-1\}\{e^{i(\ell_1\theta_1+\cdots+\ell_\nu\theta_\nu)}-1\}}{\left\{1-\frac{1}{\nu}\sum_{j=1}^{\nu}e^{i\theta_j}\right\}\left\{1-\frac{1}{\nu}\sum_{j=1}^{\nu}e^{-i\theta_j}\right\}}.$$

There is no difficulty in justifying the operation of taking limits under the integral signs above. However, we have not verified that the limit in (3.2) actually exists. This is a minor point, which I shall not pursue: if the limit should fail to exist (which seems unlikely), then we can redefine V in (3.2) using an Abelian limit.

Our next task is to study the behavior of $V(\ell_1,\ell_2,\cdots,\ell_\nu)$ for large values of

$$(3.16)\quad n = (\ell_1^2+\ell_2^2+\cdots+\ell_\nu^2)^{1/2}.$$

Since (3.15) is obviously symmetrical in $\ell_1,\ell_2,\cdots,\ell_\nu$ and remains unaltered if we reverse the signs of all ℓ_j simultaneously, we may suppose without loss of generality that

$$(3.17)\quad |\ell_1|\le|\ell_2|\le\cdots\le|\ell_{\nu-1}|\le\ell_\nu.$$

The next section consists entirely of computations. Those interested in results should proceed to section 5.

4. Approximation of integrals

Define

$$(4.1)\quad V = V(\ell_1,\ell_2,\cdots,\ell_\nu),$$

$$(4.2)\quad A = e^{i(\ell_1\theta_1+\cdots+\ell_{\nu-1}\theta_{\nu-1})},$$

$$B = \nu - \sum_{j=1}^{\nu-1} e^{i\theta_j}, \tag{4.3}$$

$$C = \nu - \sum_{j=1}^{\nu-1} e^{-i\theta_j}, \tag{4.4}$$

$$\ell = \ell_\nu, \qquad \theta = \theta_\nu. \tag{4.5}$$

Then

$$\begin{aligned} V/\nu^2 &= \sigma^2 \left\{ \prod_{j=1}^{\nu-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta_j \right\} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \frac{(A^{-1}e^{-\ell\theta} - 1)(Ae^{\ell\theta} - 1)}{(B - e^{i\theta})(C - e^{-i\theta})} \\ &= \sigma^2 \left\{ \prod_{j=1}^{\nu-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta_j \right\} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \frac{(2 - Ae^{\ell\theta} - A^{-1}e^{-\ell\theta})}{BC(1 - B^{-1}e^{i\theta})(1 - C^{-1}e^{-i\theta})} \\ &= \sigma^2 \left\{ \prod_{j=1}^{\nu-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta_j \right\} \Big\{ 2 \cdot \text{coefficient of } e^{i0\theta} \\ &\quad - A \cdot \text{coefficient of } e^{-i\ell\theta} - A^{-1} \cdot \text{coefficient of } e^{i\ell\theta} \Big\} \\ &\quad \text{in } \frac{(1 - B^{-1}e^{i\theta})^{-1}(1 - C^{-1}e^{-i\theta})^{-1}}{BC}. \end{aligned} \tag{4.6}$$

Now

$$\frac{(1 - B^{-1}e^{i\theta})^{-1}(1 - C^{-1}e^{-i\theta})^{-1}}{BC} = \frac{1}{BC} \sum_{r,s \geq 0} B^{-r}C^{-s}e^{i(r-s)\theta}. \tag{4.7}$$

The coefficient of $e^{i0\theta}$ in (4.7) is

$$\frac{1}{BC} \sum_{r \geq 0} (BC)^{-r} = \frac{1}{BC - 1}. \tag{4.8}$$

The coefficient of $e^{-i\ell\theta}$ in (4.7) is

$$\frac{1}{BC} \sum_{r \geq 0} B^{-r}C^{-(r+\ell)} = \frac{C^{-\ell}}{BC - 1}. \tag{4.9}$$

The coefficient of $e^{i\ell\theta}$ in (4.7) is

$$\frac{1}{BC} \sum_{s \geq 0} B^{-(s+\ell)}C^{-s} = \frac{B^{-\ell}}{BC - 1}. \tag{4.10}$$

Hence (4.6) yields

$$\frac{V}{\nu^2} = \sigma^2 \left\{ \prod_{j=1}^{\nu-1} \int_{-\pi}^{\pi} \frac{d\theta_j}{2\pi} \right\} \frac{1}{BC - 1} \left\{ 2 - \frac{A}{C^\ell} - \frac{1}{AB^\ell} \right\}. \tag{4.11}$$

We shall prove first that V is bounded if $\nu \geq 4$. From (4.2), (4.3), and (4.4), we have $|A| = 1$, $|B| \geq 1$, $|C| \geq 1$. So

$$\left| \frac{V}{\nu^2} \right| \leq \sigma^2 \left\{ \prod_{j=1}^{\nu-1} \int_{-\pi}^{\pi} \frac{d\theta_j}{2\pi} \right\} \frac{4}{BC - 1}. \tag{4.12}$$

Moreover, $BC - 1$ is a continuous function of $\theta_1, \theta_2, \cdots, \theta_{\nu-1}$ which only vanishes (within the region of integration) when $B = C = 1$, that is when $\theta_1 = \theta_2 = \cdots = \theta_{\nu-1} = 0$. For small θ_j, writing

$$S_k = \sum_{j=1}^{\nu-1} \theta_j^k, \qquad (k = 1, 2, \cdots), \tag{4.13}$$

we have

$$\begin{aligned} BC - 1 &= (1 - iS_1 + \tfrac{1}{2}S_2 - \cdots)\ (1 + iS_1 + \tfrac{1}{2}S_2 + \cdots) - 1 \\ &= S_1^2 + S_2 + 0(\theta^4). \end{aligned} \tag{4.14}$$

Hence,

$$\begin{aligned} V &= O\left[\left\{\prod_{j=1}^{\nu-1} \int_{-\pi}^{\pi} d\theta_j\right\} \frac{1}{S_2}\right] \\ &= O\left[\int_0^1 (S_2^{1/2})^{\nu-4}\, d(S_2^{1/2})\right] = O(1), \qquad (\nu \geq 4), \end{aligned} \tag{4.15}$$

where the last line in (4.15) is the result of transforming to polar coordinates in the $\boldsymbol{\theta}$-space.

The integral V is not bounded for smaller values of ν, and a more delicate analysis is required. We begin with the case $\nu = 3$. In the following we shall write ϑ for a real number lying in the closed interval $[-1, 1]$. The value of ϑ may vary from occurrence to occurrence: thus, we have relations like $\vartheta^2 = \vartheta$, $\vartheta - \vartheta = 2\vartheta$. We also write

$$\lambda = (\theta_1 + \theta_2)/2, \qquad \mu = (\theta_1 - \theta_2)/2, \qquad \rho = 3\lambda^2 + \mu^2. \tag{4.16}$$

By Taylor's expansion, we have

$$\cos\lambda = 1 - \tfrac{1}{2}\lambda^2 + \tfrac{1}{24}\lambda^4 \cos\vartheta\lambda = 1 - \tfrac{1}{2}\lambda^2 + \tfrac{1}{24}\lambda^4\vartheta. \tag{4.17}$$

We shall make extensive use of expressions and manipulations like (4.17) without further explicit mention.

We have, from (4.4) with $\nu = 3$,

$$\begin{aligned} Ce^{-2i\lambda-\rho} &= e^{-2i\lambda-\rho}(3 - e^{-i\theta_1} - e^{-i\theta_2}) \\ &= e^{-2i\lambda-\rho}(3 - e^{-i(\lambda+\mu)} - e^{-i(\lambda-\mu)}) \\ &= e^{-\rho}(3e^{-2i\lambda} - 2e^{-3i\lambda}\cos\mu) \\ &= e^{-\rho}(3\cos 2\lambda - 2\cos 3\lambda\cos\mu) \\ &\qquad - ie^{-\rho}(3\sin 2\lambda - 2\sin 3\lambda\cos\mu) \\ &= e^{-\rho}\,[3 - 6\lambda^2 + 2\lambda^4\vartheta - 2\,(1 - \tfrac{9}{2}\lambda^2 + \tfrac{27}{8}\lambda^4\vartheta)\cos\mu] \\ &\qquad - ie^{-\rho}\,[6\lambda - 4\lambda^3\vartheta - 2\,(3\lambda - \tfrac{9}{2}\lambda^3\vartheta)\cos\mu] \\ &= e^{-\rho}\,[3 - 6\lambda^2 + 2\lambda^4\vartheta - 2\,(1 - \tfrac{1}{2}\mu^2 + \tfrac{1}{24}\mu^4\vartheta) \\ &\qquad + 9\lambda^2\,(1 - \tfrac{1}{2}\mu^2\vartheta) - \tfrac{27}{4}\lambda^4\vartheta] \\ &\qquad - ie^{-\rho}\,[6\lambda - 4\lambda^3\vartheta - 6\lambda\,(1 - \tfrac{1}{2}\mu^2\vartheta) + 9\lambda^3\vartheta] \end{aligned} \tag{4.18}$$

$$= e^{-\rho}\left[1 + 3\lambda^2 + \mu^2 + \tfrac{35}{4}\lambda^4\vartheta + \tfrac{9}{2}\lambda^2\mu^2\vartheta + \tfrac{1}{12}\mu^4\vartheta\right] - i\lambda e^{-\rho}(13\lambda^2\vartheta + 3\mu^2\vartheta)$$

$$= e^{-\rho}\left[1 + 3\lambda^2 + \mu^2 + (\tfrac{36}{4}\lambda^4 + \tfrac{12}{2}\lambda^2\mu^2 + \mu^4)\,\vartheta\right] - i\lambda e^{-\rho}\,(13\lambda^2\vartheta + \tfrac{13}{3}\mu^2\vartheta)$$

$$= e^{-\rho}(1 + \rho) + \rho^2 e^{-\rho}\vartheta - \tfrac{13}{3}i\lambda e^{-\rho}\rho\vartheta.$$

$$= e^{-\rho}(1 + \rho) + \rho^2\vartheta - \frac{13i}{3\sqrt{3}}\,\rho^{3/2}\vartheta.$$

Now define

$$f(\rho) = e^{-\rho}(1 + \rho). \tag{4.19}$$

We have

$$f'(0) = 0;\; f''(\rho) = -(1 - \rho)e^{-\rho}. \tag{4.20}$$

It is easy to prove that

$$-1 \le f''(\rho) \le e^{-2} < 1, \qquad (\rho \ge 0). \tag{4.21}$$

Hence,

$$e^{-\rho}(1 + \rho) = 1 + \tfrac{1}{2}\rho^2\vartheta. \tag{4.22}$$

Since $-\pi \le \theta_1 \le \theta_2 \le \pi$, we have

$$\rho = 3\left(\frac{\theta_1 + \theta_2}{2}\right)^2 + \left(\frac{\theta_1 - \theta_2}{2}\right)^2 \le 4\pi^2. \tag{4.23}$$

Hence,

$$Ce^{-2i\lambda-\rho} = 1 + \rho^{3/2}\left(\tfrac{3}{2}\,\vartheta\rho^{1/2} - \frac{13i}{3\sqrt{3}}\,\vartheta\right) \tag{4.24}$$

$$= 1 + \rho^{3/2}\left(3\pi\vartheta - \frac{13i}{3\sqrt{3}}\,\vartheta\right)$$

$$= 1 + 10\rho^{3/2}\vartheta^*,$$

when ϑ^* is a complex number such that $|\vartheta^*| < 1$.

Next let $\mathfrak{R}$ denote the real part of a number, and consider

$$\left|\mathfrak{R}\left\{\frac{A}{C^\ell} - \frac{A}{(e^{\rho+2i\lambda})^\ell}\right\}\right| \le \left|\frac{A}{C^\ell} - \frac{A}{e^{(\rho+2i\lambda)\ell}}\right| \tag{4.25}$$

$$= e^{-\rho\ell}|(Ce^{-2i\lambda-\rho})^{-\ell} - 1| = e^{-\rho\ell}|(1 + 10\rho^{3/2}\vartheta^*)^{-\ell} - 1|.$$

If z is any complex number, we have

$$e^z - 1 = \int_0^z e^u\,du, \tag{4.26}$$

where the integral is taken along the straight line from 0 to z. Hence,

$$|e^z - 1| \le \int_0^z |e^u|\,|du| \le |z|e^{|z|}. \tag{4.27}$$

Similarly, if $|z| < 1$, and we take the integral along a straight line, we have

$$(4.28)\qquad |\log(1+z)| = \left|\int_0^z \frac{du}{1+u}\right| \le \int_0^z \frac{|du|}{|1+u|}$$
$$\le \frac{|z|}{\inf\limits_{0\le t\le 1}|1+tz|} \le \frac{|z|}{\inf\limits_{0\le t\le 1}(1-t|z|)} = \frac{|z|}{1-|z|}.$$

Now choose an absolute constant $\rho_0 > 0$ such that

$$(4.29)\qquad 10\rho^{1/2}/(1-10\rho^{3/2}) \le \tfrac{1}{2}, \qquad (0 \le \rho \le \rho_0).$$

If $0 \le \rho \le \rho_0$, then $|10\rho^{3/2}\vartheta^*| < 1$, and we have

$$(4.30)\qquad |(1+10\rho^{3/2}\vartheta^*)^{-\ell} - 1| = |e^{-\ell\log(1+10\rho^{3/2}\vartheta^*)} - 1|$$
$$\le |-\ell\log(1+10\rho^{3/2}\vartheta^*)|e^{|-\ell\log(1+10\rho^{3/2}\vartheta^*|}$$
$$\le \ell|\log(1+10\rho^{3/2}\vartheta^*)|e^{\ell|\log(1+10\rho^{3/2}\vartheta^*|}$$
$$\le \frac{10\ell\rho^{3/2}}{1-10\ell\rho^{3/2}}\exp\left\{\frac{10\ell\rho^{3/2}}{1-10\ell\rho^{3/2}}\right\} \le \tfrac{1}{2}\,\ell\rho^{\ell\rho/2}.$$

Inserting this result into (4.26), we obtain

$$(4.31)\qquad \mathcal{R}(A/C^\ell) = e^{-\rho\ell}\mathcal{R}(Ae^{-2i\lambda\ell}) + \tfrac{1}{2}\vartheta\ell\rho e^{-\rho\ell/2}, \qquad (0 \le \rho \le \rho_0).$$

Since B and C are complex conjugates, by (4.3) and (4.4), we have

$$(4.32)\qquad BC - 1 = |C|^2 - 1 = |3 - 2e^{-i\lambda}\cos\mu|^2 - 1$$
$$= (3 - 2\cos\lambda\cos\mu)^2 + 4\sin^2\lambda\cos^2\mu - 1$$
$$= 8 - 12\cos\lambda\cos\mu + 4\cos^2\mu$$
$$= 4\,[2 - 3\,(1 - \tfrac{1}{2}\lambda^2)\cos\mu + \cos^2\mu] + \tfrac{1}{2}\lambda^4\vartheta$$
$$= 4(2-\cos\mu)(1-\cos\mu) + 6\lambda^2\cos\mu + \tfrac{1}{2}\lambda^4\vartheta$$
$$= 4(2-\cos\mu)\,(\tfrac{1}{2}\mu^2 + \tfrac{1}{24}\mu^4\vartheta) + 6\lambda^2 + 3\lambda^2\mu^2\vartheta + \tfrac{1}{2}\lambda^4\vartheta$$
$$= 2\mu^2(2-\cos\mu) + 6\lambda^2 + \tfrac{1}{2}\lambda^4\vartheta + 3\lambda^2\mu^2\vartheta + \tfrac{1}{2}\mu^4\vartheta$$
$$= 2\mu^2\,(1 + \tfrac{1}{2}\mu^2\vartheta) + 6\lambda^2 + \tfrac{1}{2}\lambda^4\vartheta + 3\lambda^2\mu^2 + \tfrac{1}{2}\mu^4\vartheta$$
$$= 2\mu^2 + 6\lambda^2 + (9\lambda^4 + 6\lambda^2\mu^2 + \mu^4)\vartheta = 2\rho + \rho^2\vartheta.$$

As (4.12) shows, the integrand in the integral for V is bounded except in the neighborhood of $\theta_1 = \theta_2 = 0$; so we have

$$(4.33)\qquad V = O(1) + \frac{9\sigma^2}{4\pi^2}\iint\limits_{0\le\rho\le\rho_0} d\theta_1\,d\theta_2\,\frac{1}{BC-1}\left\{2 - \frac{A}{C^\ell} - \frac{1}{AB^\ell}\right\}$$
$$= O(1) + \frac{9\sigma^2}{8\pi^2}\iint\limits_{0\le\rho\le\rho_0} \frac{d\theta_1\,d\theta_2}{\rho\,(1+\frac{1}{2}\,\rho\vartheta)}\left\{2 - \frac{A}{C^\ell} - \frac{1}{AB^\ell}\right\}$$
$$= O(1) + \frac{9\sigma^2}{8\pi^2}\iint\limits_{0\le\rho\le\rho_0} \frac{d\theta_1\,d\theta_2}{\rho}\left\{2 - \frac{A}{C^\ell} - \frac{1}{AB^\ell}\right\}$$
$$- \frac{9\sigma^2}{8\pi^2}\iint\limits_{0\le\rho\le\rho_0} \frac{d\theta_1\,d\theta_2}{1+\frac{1}{2}\,\vartheta\rho}\,\frac{1}{2}\,\vartheta 4$$

on resolving $1/\rho(1 + (\frac{1}{2})\rho\vartheta)$ into partial fractions and noting that the expression in braces does not exceed 4 in magnitude. Since $\rho_0 < 1$, the final integrand is bounded, and we get

$$V = O(1) + \frac{9\sigma^2}{8\pi^2} \iint\limits_{0 \le \rho \le \rho_0} \frac{d\theta_1\, d\theta_2}{\rho} 2\{1 - \Re(A/C^{\ell})\} \tag{4.34}$$

where we have used the fact that A/C^{ℓ} and $1/AB^{\ell}$ are complex conjugates.

We can satisfy (4.16) by writing

$$\frac{\theta_1 + \theta_2}{2} = \lambda = \left(\frac{\rho}{3}\right)^{1/2} \sin \omega, \qquad \frac{\theta_1 - \theta_2}{2} = \mu = \rho^{1/2} \cos \omega. \tag{4.35}$$

The Jacobian of this transformation is

$$\frac{\partial(\theta_1, \theta_2)}{\partial(\rho, \omega)} = \frac{1}{\sqrt{3}}. \tag{4.36}$$

Hence,

$$V = O(1) + \frac{3\sqrt{3}\cdot\sigma^2}{4\pi^2} \int_0^{\rho_0} \frac{d\rho}{\rho} \int_0^{2\pi} d\omega \{1 - \Re(A/C^{\ell})\}. \tag{4.37}$$

According to (4.31), the error introduced by replacing $\Re(A/C^{\ell})$ by $e^{-\rho\ell}\Re(Ae^{-2i\lambda\ell})$ in (4.37) will be

$$\frac{3\sqrt{3}\cdot\sigma^2}{4\pi} \int_0^{\rho_0} \frac{d\rho}{\rho} \int_0^{2\pi} d\omega\, \tfrac{1}{2}\rho\ell e^{-\rho\ell}\vartheta = \frac{3\sqrt{3}\cdot\sigma^2}{4\pi} \vartheta \int_0^{\infty} d(\rho\ell) e^{-\rho\ell} = O(1). \tag{4.38}$$

Hence,

$$V = O(1) + \frac{3\sqrt{3}\cdot\sigma^2}{4\pi^2} \int_0^{\rho_0} \frac{d\rho}{\rho} \int_0^{2\pi} d\omega \{1 - e^{-\rho\ell}\Re(Ae^{-2i\lambda\ell})\}. \tag{4.39}$$

Now

$$\begin{aligned} \Re(Ae^{-2i\lambda\ell}) &= \Re \exp \{i[\ell_1(\lambda + \mu) + \ell_2(\lambda - \mu) - 2\lambda\ell_3]\} \\ &= \cos [(\ell_1 + \ell_2 - 2\ell_3)\lambda + (\ell_1 - \ell_2)\mu] \\ &= \cos \left[\rho^{1/2}\left\{\frac{\ell_1 + \ell_2 - 2\ell_3}{3} \sin \omega + (\ell_1 - \ell_2) \cos \omega\right\}\right] \\ &= \cos [N\rho^{1/2} \cos (\omega + \omega_0)], \end{aligned} \tag{4.40}$$

where ω_0 is a constant depending only on ℓ_1, ℓ_2, ℓ_3, and

$$\begin{aligned} N^2 &= \tfrac{1}{3}(\ell_1 + \ell_2 - 2\ell_3)^2 + (\ell_1 - \ell_2)^2 \\ &= \tfrac{4}{3}(\ell_1^2 + \ell_2^2 + \ell_3^2 - \ell_2\ell_3 - \ell_3\ell_1 - \ell_1\ell_2). \end{aligned} \tag{4.41}$$

Therefore,

$$(4.42)\qquad V = O(1) + \frac{3\sqrt{3}\cdot\sigma^2}{4\pi^2}\int_0^{\rho_0}\frac{d\rho}{\rho}\int_0^{2\pi}d\omega\,\{1 - e^{-\rho\ell}\cos[N\rho^{1/2}\cos(\omega+\omega_0)]\},$$

$$= O(1) + \frac{3\sqrt{3}\cdot\sigma^2}{4\pi^2}\int_0^{\rho_0}\frac{d\rho}{\rho}\int_0^{2\pi}d\omega\,\{1 - e^{-\rho\ell}\cos(N\rho^{1/2}\cos\omega)\},$$

$$= O(1) + \frac{3\sqrt{3}\cdot\sigma^2}{2\pi}\int_0^{\rho_0}\frac{d\rho}{\rho}\{1 - e^{-\rho\ell}J_0(N\rho^{1/2})\},$$

where $J_0(x)$ is the Bessel function of the first kind and zero-th order.

Now write $k = N/\sqrt{\ell}$ and consider

$$(4.43)\qquad I = \int_0^{\rho_0}\frac{d\rho}{\rho}\{1 - e^{-\rho\ell}J_0(N\rho^{1/2})\} = \int_0^{\rho_0\ell}\frac{dx}{x}\{1 - e^{-x}J_0(kx^{1/2})\}$$

$$= \int_0^1\frac{dx}{x}\{1 - J_0(kx^{1/2})\} + \int_1^{\rho_0\ell}\frac{dx}{x} - \int_1^{\rho_0\ell}\frac{dx}{x}e^{-x}J_0(kx^{1/2}) + \int_0^1\frac{dx}{x}(1-e^{-x})J_0(kx^{1/2})$$

$$= \int_0^1\frac{dx}{x}\{1 - J_0(kx^{1/2})\} + \log(\rho_0\ell) + O(1),$$

since $|J_0(kx^{1/2})| \le 1$ and $(1 - e^{-x})/x \le 1$ for $x \ge 0$. If $k = 0$, the final integral vanishes, since $J_0(0) = 1$. If $k > 0$, we have

$$(4.44)\qquad \int_0^1\frac{dx}{x}\{1 - J_0(kx^{1/2})\} = \int_0^{k^2}\frac{dy}{y}\{1 - J_0(\sqrt{y})\}.$$

Now $\{1 - J_0(\sqrt{y})\}/y$ is a bounded function of y; and $J_0(\sqrt{y}) = 0(y^{-1/4})$ as $y \to \infty$. Hence,

$$(4.45)\qquad \int_0^{k^2}\frac{dy}{y}\{1 - J_0(\sqrt{y})\} = \begin{cases} O(1), & \text{if } k^2 \le 1 \\ \displaystyle\int_1^{k^2}\frac{dy}{y} + O(1) = \log k^2 + O(1), & \text{if } k^2 \ge 1. \end{cases}$$

Collecting results from (4.43), (4.44), and (4.45), we get

$$(4.46)\qquad I = \begin{cases} \log\ell + O(1) & \text{if } k^2 \le 1 \\ \log k^2 + \log\ell + O(1) = \log N^2 + O(1) & \text{if } k^2 \ge 1. \end{cases}$$

If $k^2 \ge 1$, then $N^2 \ge \ell$. On the other hand, $N^2 \le 8\ell^2$ by (4.41). So $\log N^2 = \log\ell + O(1)$ when $k^2 \ge 1$. Hence in every case we get

$$(4.47)\qquad I = \log\ell + O(1) = \log n + O(1),$$

by (3.16), (3.17), and (4.5). Finally (4.42) and (4.47) give

$$V(\ell_1, \ell_2, \ell_3) = \frac{3\sqrt{3}}{2\pi}\sigma^2 \log n + O(1) \quad \text{as} \quad n = (\ell_1^2 + \ell_2^2 + \ell_3^2)^{1/2} \to \infty. \tag{4.48}$$

Next we consider the one-dimensional case of (3.15). The answer for this case is, of course, known; but it is comforting to be able to check the result, and in any case we shall need one of the integrals (4.49) in dealing with the two-dimensional case in a moment. For $\nu = 1$, (3.16) gives $\ell_1 = n$ and (3.15) gives

$$V(\ell_1) = \frac{\sigma^2}{2\pi}\int_{-\pi}^{\pi} \frac{1 - \cos n\theta}{1 - \cos\theta}\, d\theta = \sigma^2 n \tag{4.49}$$

since

$$\frac{1 - \cos n\theta}{1 - \cos\theta} = \frac{1 - \cos (n-1)\theta}{1 - \cos\theta} + 1 + 2\sum_{r=1}^{n-1} \cos r\theta. \tag{4.50}$$

For two dimensions, we start from (4.11). We have

$$BC - 1 = 4(1 - \cos\theta_1); \tag{4.51}$$

and hence

$$\begin{aligned} V &= \frac{\sigma^2}{2\pi}\int_{-\pi}^{\pi} \frac{d\theta_1}{(1 - \cos\theta_1)}\, 2\mathcal{R}\left[1 - \frac{e^{i\ell_1\theta_1}}{(2 - e^{-i\theta_1})^{\ell_2}}\right] \\ &= \frac{\sigma^2}{\pi}\int_{-\pi}^{\pi} \frac{d\theta}{1 - \cos\theta}\frac{1}{2^{\ell_2}}\mathcal{R}\left[\left(1 - \frac{1}{2}\right)^{-\ell_2} - e^{i\ell_1\theta}\left(1 - \frac{1}{2}e^{-i\theta}\right)^{-\ell_2}\right] \\ &= \frac{\sigma^2}{\pi}\int_{-\pi}^{\pi} \frac{d\theta}{1 - \cos\theta}\frac{1}{2^{\ell_2}}\sum_{r=0}^{\infty}\mathcal{R}\binom{-\ell_2}{r}\left\{\left(-\frac{1}{2}\right)^r - e^{i\ell_1\theta}\left(-\frac{1}{2}e^{-i\theta}\right)^r\right\} \\ &= \frac{\sigma^2}{\pi}\frac{1}{2^{\ell_2}}\sum_{r=0}^{\infty}\binom{-\ell_2}{r}\left(-\frac{1}{2}\right)^r\int_{-\pi}^{\pi}\frac{1 - \cos(\ell_1 - r)\theta}{1 - \cos\theta}\, d\theta \\ &= \frac{2}{2^{\ell_2}}\sum_{r=0}^{\infty}\binom{-\ell_2}{r}\left(-\frac{1}{2}\right)^r |\ell_1 - r|, \end{aligned} \tag{4.52}$$

by (4.49). This, however, shows that V is twice the expected value of $|\ell_1 - r|$ when r is distributed as a negative binomial variate with parameters $-\frac{1}{2}$ and $-\ell_2$. When n is large, ℓ_2 is large, and we can approximate by means of the central limit theorem. In fact, V will approach twice the expected value of $|\ell_1 - \ell_2 - y(2\ell_2)^{1/2}|$, where y is normally distributed with zero mean and unit variance. Thus,

$$\begin{aligned} V &\sim \frac{2\sigma^2}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} |\ell_1 - \ell_2 - y(2\ell_2)^{1/2}| e^{-y^2/2}\, dy, \\ &\sim \frac{4\sigma^2}{\sqrt{\pi}}\ell_2^{1/2}e^{-\alpha^2/2} + 2\sigma^2|\ell_1 - \ell_2|\Phi(\alpha), \end{aligned} \tag{4.53}$$

where

$$\alpha = |\ell_1 - \ell_2|(2\ell_2)^{1/2}, \qquad \Phi(\alpha) = (2\pi)^{-1/2}\int_{-\alpha}^{\alpha} e^{-t^2/2}\, dt. \tag{4.54}$$

Here we have been assuming (3.17). For general ℓ_1, ℓ_2, we obtain

$$V \sim \frac{2^{7/4}\sigma^2}{\sqrt{\pi}} [n^2 + |\ell_1^2 - \ell_2^2|]^{1/4} e^{-\alpha^2/2} + 2\sigma^2|\ell_1 - \ell_2|\Phi(\alpha), \tag{4.55}$$

where

$$\alpha = |\ell_1 - \ell_2|[2n^2 + 2|\ell_1^2 - \ell_2^2|]^{-1/4}, \qquad n^2 = \ell_1^2 + \ell_2^2. \tag{4.56}$$

5. Summary of results for one-sided harnesses

We summarize the foregoing results in theorem 1.

THEOREM 1. *Let $V(\ell_1, \ell_2, \cdots, \ell_\nu)$ denote the Abelian limit of*

$$E(y_{r_1+\ell_1, r_2+\ell_2, \cdots, r_\nu+\ell_\nu} - y_{r_1, r_2, \cdots, r_\nu})^2 \tag{5.1}$$

as $r_1, r_2, \cdots, r_\nu \to \infty$, where $y_{r_1, r_2, \cdots, r_\nu}$ is the one-sided harness defined by (3.3) *subject to the boundary conditions* (3.4). *Then*

$$V(\ell_1) = \sigma^2\ell_1; \tag{5.2}$$

and, when $\ell_1, \ell_2, \cdots, \ell_\nu \to \infty$,

$$V(\ell_1, \ell_2) \sim \frac{4\sigma^2}{\sqrt{(2\pi)}} [2(\ell_1^2 + \ell_2^2 + |\ell_1^2 - \ell_2^2|)]^{1/4} \left\{e^{-\alpha^2/2} + \alpha \int_0^\alpha e^{-t^2/2}\, dt\right\} \tag{5.3}$$

where

$$\alpha = |\ell_1 - \ell_2|[2(\ell_1^2 + \ell_2^2 + |\ell_1^2 - \ell_2^2|)]^{-1/4}; \tag{5.4}$$

$$V(\ell_3, \ell_2, \ell_3) = \frac{3\sqrt{3}}{\pi} \sigma^2 \log (\ell_1^2 + \ell_2^2 + \ell_3^2) + O(1); \tag{5.5}$$

and

$$V(\ell_1, \ell_2, \cdots, \ell_\nu) = O(1) \qquad \text{for} \quad \nu \geq 4. \tag{5.6}$$

This theorem shows how long-range fluctuations depend upon distance and dimensionality, assuming, of course, that a one-sided harness is the right mathematical model. These fluctuations will be proportional to the square root of V. Thus, in one dimension, we get the familiar result that the fluctuations are proportional to the square root of the distance $n = (\ell_1^2 + \cdots + \ell_\nu^2)^{1/2}$. In two dimensions there is a complicated situation. The magnitude of the fluctuations depends upon the parameter α, which characterizes the direction of the distance in question. If α is not large, then the fluctuations are proportional to $n^{1/4}$; but, if α is large, the fluctuations are proportional to $n^{1/2}$, as in the one-dimensional case. Large values of α correspond to the case when the distance is measured perpendicular to the direction in which the one-sided harness is recursively constructed; while small values of α represent distances parallel to this direction. For intermediate directions, the fluctuations will be intermediately proportional to $n^{1/4}$ and $n^{1/2}$. In three dimensions, the fluctuations are always proportional to $(\log n)^{1/2}$; and are therefore virtually constant, unless n is enormously larger than anything one is likely to encounter in a physical situation. In four or more dimensions, the fluctuations are always bounded. Professor A. H. Cottrell has picturesquely stated this result as follows.

THEOREM 2. *In four or more dimensions, a harness is a strait jacket.*

Presumably this theorem is well-known to four-dimensional horses.

6. Central harnesses

The sensitivity to direction, already noted in (5.3), is a most undesirable feature of one-sided harnesses, which suggests that they do not provide a satisfactory mathematical model of long-range misorientation in crystalline structure. After all, the physical problem is isotropic and our treatment should be independent of our choice of coordinate axes. We therefore turn to central harnesses, that is to say, systems of the type (2.6). This definition shows much more symmetry, but unfortunately it leads to a new type of difficulty, which would appear to make it even less suitable than the one-sided harness as a mathematical model.

This difficulty already appears in the central one-dimensional harness; and for simplicity we shall look merely at this case. We have

$$y_r = \tfrac{1}{2}y_{r+1} + \tfrac{1}{2}y_{r+1} + \epsilon_r, \tag{6.1}$$

where the ϵ's are supposed to be independently distributed with zero mean and common variance σ^2. The difference equation (6.1) has the general solution

$$y_r = A + Br + 2\sum_{s=0}^{r}(r - s)\epsilon_s, \tag{6.2}$$

where A and B are constants to be determined by boundary conditions. If we choose the boundary conditions to be

$$y_0 = y_k = 0 \tag{6.3}$$

for some fixed k (and these seem to typify the most general boundary conditions that would be physically acceptable), then

$$y_r = 2\sum_{s=0}^{r}(r - s)\epsilon_s - \frac{2r}{k}\sum_{s=0}^{k}(k - s)\epsilon_s. \tag{6.4}$$

This leads to

$$V(r) = \operatorname{var} y_r = \begin{cases} 2\sigma^2 r(k - r)[1 + 2r(k - r)]/3k, & 0 \le r \le k, \\ 2\sigma^2 r(r - k)[1 + 2k(r - k)]/3k, & k \ge r. \end{cases} \tag{6.5}$$

There is no way of choosing k so that $V(r) = O(r)$ for all r, or even for only those r which satisfy $0 \le r \le k$.

It is disappointing that central harnesses will not therefore serve our purpose, but perhaps hardly surprising. One-sided harnesses obey first-order difference equations, whereas central harnesses obey second-order equations; and it is familiar that the injection of randomness into an undamped second-order system can provoke unacceptably large fluctuations.

7. Serial harnesses

The following line of attack was suggested to me by Mr. D. G. Champernowne [2] in a private communication (undated, but around about 1957). So far we have only considered hypercubical lattices, but now we extend the treatment to a more general class of lattices.

Let L denote a countable lattice which forms an Abelian group (with at least one infinite cyclic subgroup) under vector addition: that is to say, if $\mathbf{v}$ and $\mathbf{w}$ are vectors representing points of L, then $\mathbf{v} \pm \mathbf{w}$ also belong to L. Let $\mathbf{p}(\mathbf{w})$ denote a given discrete *symmetric* probability measure on L: that is to say,

$$0 \leq p(\mathbf{v}) = p(-\mathbf{v}); \sum_{\mathbf{v}\in L} p(\mathbf{v}) = 1. \tag{7.1}$$

We also suppose that $p(\mathbf{v}_0) > 0$ for some vector $\mathbf{v}_0$ which generates an infinite cyclic subgroup of L. *Inter alia,* this prevents $p(\mathbf{v})$ from being the trivial distribution

$$p_0(\mathbf{0}) = 1, \qquad p_0(\mathbf{v}) = 0, \qquad (\mathbf{v} \neq \mathbf{0}). \tag{7.2}$$

We shall consider Markovian walks W on L, with one-step transition probabilities

$$P(\mathbf{v} \rightarrow \mathbf{w}) = p(\mathbf{v} - \mathbf{w}). \tag{7.3}$$

The $(n + 1)$-step transition probabilities of W are then (in the usual fashion)

$$p_{n+1}(\mathbf{v}) = \sum_{\mathbf{w}\in L} p_n(\mathbf{w})p(\mathbf{v} - \mathbf{w}), \qquad (n \geq 0), \tag{7.4}$$

when p_0 is defined in (7.2).

For each point of L, we suppose that there is a sequence of random variables $\epsilon_i(\mathbf{v})$, $(i = 0, 1, 2, \cdots)$, where all the ϵ's are mutually independent identically distributed random variables with zero mean and common variance σ^2. We define the serial harnesses $Y_t(\mathbf{v})$, $(t = 0, 1, 2, \cdots)$ by means of

$$Y_0(\mathbf{v}) = \epsilon_0(\mathbf{v}); \quad Y_{t+1}(\mathbf{v}) = \epsilon_{t+1}(\mathbf{v}) + \sum_{\mathbf{w}\in L} p(\mathbf{w})Y_t(\mathbf{v} - \mathbf{w}). \tag{7.5}$$

We shall be interested in the limiting behavior of the fluctuations of $Y_t(\mathbf{v})$ as $t \rightarrow \infty$. It is obvious that $EY_t(\mathbf{v}) = 0$; and we define

$$V_t(\mathbf{v}, u) = \mathbf{0}[Y_{t+u}(\mathbf{v}) - Y_t(\mathrm{var})] = E[Y_{t+u}(\mathbf{v}) - Y_t(\mathbf{0})]^2, \tag{7.6}$$

and consider

$$V(\mathbf{v}, u) = \lim_{t\rightarrow\infty} V_t(\mathbf{v}, u). \tag{7.7}$$

We have from (7.5) and (7.4),

$$
\begin{aligned}
(7.8)\qquad & Y_t(\mathbf{v}) \\
&= \epsilon_t(\mathbf{v}) + \sum_{\mathbf{w}\in L} p(\mathbf{w})\{\epsilon_t(\mathbf{v}-\mathbf{w}) + \sum_{\mathbf{x}\in L} p(\mathbf{x})\{\epsilon_{t-1}(\mathbf{v}-\mathbf{w}-\mathbf{x}) + \cdots \\
&= \sum_{r=0}^{t} \sum_{\mathbf{w}\in L} \epsilon_{t-r}(\mathbf{v}-\mathbf{w})p_r(\mathbf{w}) \\
&= \sum_{r=0}^{t} \sum_{\mathbf{w}\in L} p_r(\mathbf{v}-\mathbf{w})\epsilon_{t-r}(\mathbf{w}) \\
&= \sum_{r=0}^{t} \sum_{\mathbf{w}\in L} p_{t-r}(\mathbf{v}-\mathbf{w})\epsilon_r(\mathbf{w}).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(7.9)\qquad Y_{t+u}(\mathbf{v}) - Y_t(\mathbf{0}) &= \sum_{r=0}^{t} \sum_{\mathbf{w}\in L} [p_{t+u-r}(\mathbf{v}-\mathbf{w}) - p_{t-r}(-\mathbf{w})]\epsilon_r(\mathbf{w}) \\
&\quad + \sum_{r=t+1}^{t+u} \sum_{\mathbf{w}\in L} p_{t+u-r}(\mathbf{v}-\mathbf{w})\epsilon_r(\mathbf{w}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(7.10)\qquad V_t(\mathbf{v}, u) &= \sum_{r=0}^{t} \sum_{\mathbf{w}\in L} [p_{t+u-r}(\mathbf{v}-\mathbf{w}) - p_{t-r}(-\mathbf{w})]^2\sigma^2 \\
&\quad + \sum_{r=t+1}^{t+u} \sum_{\mathbf{w}\in L} [p_{t+u-r}(\mathbf{v}-\mathbf{w})]^2\sigma^2 \\
&= \sigma^2\Big\{\sum_{r=0}^{t} \sum_{\mathbf{w}\in L} [p_{r+u}(\mathbf{w}-\mathbf{v}) - p_r(\mathbf{w})]^2 \\
&\quad + \sum_{r=0}^{u-1} \sum_{\mathbf{w}\in L} [p_r(\mathbf{w}-\mathbf{v})]^2\Big\} \\
&= \sigma^2\Big\{\sum_{r=0}^{t} \sum_{\mathbf{w}\in L} [p_{r+u}(\mathbf{w}-\mathbf{v})p_{r+u}(\mathbf{v}-\mathbf{w}) + p_r(\mathbf{w})p_r(-\mathbf{w}) \\
&\quad - 2p_{r+u}(\mathbf{v}-\mathbf{w})p_r(\mathbf{w})] + \sum_{r=0}^{u-1} \sum_{\mathbf{w}\in L} p_r(\mathbf{w}-\mathbf{v})p_t(\mathbf{v}-\mathbf{w})\Big\} \\
&= \sigma^2\Big\{\sum_{r=0}^{t} [p_{2r+2u}(\mathbf{0}) + p_{2r}(\mathbf{0}) - 2p_{2r+u}(\mathbf{v})] + \sum_{r=0}^{u-1} p_{2r}(\mathbf{0})\Big\} \\
&= 2\sigma^2 \sum_{r=0}^{t} [p_{2r}(\mathbf{0}) - p_{2r+u}(\mathbf{v})] + \sigma^2 \sum_{r=t+1}^{t+u} p_{2r}(\mathbf{0}).
\end{aligned}
$$

Now, by Schwarz's inequality, we have

$$
\begin{aligned}
(7.11)\qquad [p_{2r}(\mathbf{v})]^2 &= [\sum_{\mathbf{w}\in L} p_r(\mathbf{v}-\mathbf{w})p_r(\mathbf{w})]^2 \\
&\le [\sum_{\mathbf{w}\in L} p_r^2(\mathbf{v}-\mathbf{w})][\sum_{\mathbf{w}\in L} p_r^2(\mathbf{w})] = [\sum_{\mathbf{w}\in L} p_r(\mathbf{w})p_r(-\mathbf{w})]^2 = [p_{2r}(\mathbf{0})]^2.
\end{aligned}
$$

Hence,

$$p_{2r+u}(\mathbf{v}) = \sum_{\mathbf{w}\in L} p_{2r}(\mathbf{w})p_u(\mathbf{v}-\mathbf{w}) \leq p_{2r}(\mathbf{0}) \sum_{\mathbf{w}\in L} p_u(\mathbf{v}-\mathbf{w}) = p_{2r}(\mathbf{0}); \tag{7.12}$$

and consequently,

$$p_{2r}(\mathbf{0}) - p_{2r+u}(\mathbf{v}) \geq 0. \tag{7.13}$$

Thus each term in the first series of the last line of (7.10) is nonnegative; so, when $t \to \infty$, this series either converges or diverges to $+\infty$. The second series in the last line of (7.10) has a fixed number of terms, namely u; and each of these terms tends to zero when $t \to \infty$, because $p(\mathbf{v}_0) > 0$ for some $\mathbf{v}_0$ which generates an infinite cyclic subgroup of L. The final result is theorem 3.

THEOREM 3. *The limit $V(\mathbf{v}, u)$ is equal to*

$$V(\mathbf{v}, u) = 2\sigma^2 \sum_{r=0}^{\infty} [p_{2r}(\mathbf{0}) - p_{2r+u}(\mathbf{v})], \tag{7.14}$$

where each term in this series is nonnegative (and so the series either converges or has the formal value $+\infty$).

This theorem is a straightforward generalization of a result originally proved by Champernowne.

The precise behavior of the function $V(\mathbf{v}, u)$ for large $\mathbf{v}$, u depends to some extent on the basic transition function $p(\cdot)$. Rather than get involved in too detailed an exact discussion, it seems more profitable to proceed heuristically in the hope of revealing the general overall features to be expected of $V(\mathbf{v}, u)$, without worrying too much if these results should prove false in certain special cases.

Let us suppose that the one-step transitions have a finite variance, that is to say that the matrix

$$\sum_{\mathbf{w}\in L} \mathbf{w}\,\mathbf{w}'\, p(\mathbf{w}) = \mathbf{A} \tag{7.15}$$

is finite, where $\mathbf{w}$ is a column vector and, $\mathbf{w}'$ is its transpose. Then the walks W will obey the central limit theorem, and if some Tauberian version of the central limit theorem (such as [3]) applies, then we may expect that

$$p_n(\mathbf{v}) \sim \|2\pi\, n\mathbf{A}\|^{-1/2}\lambda^{-1} \exp\{-\mathbf{v}'\mathbf{A}^{-1}\mathbf{v}/2n\} \tag{7.16}$$

where $\|\mathbf{A}\|$ is the determinant of $\mathbf{A}$, and λ is the average number of lattice points per unit volume. In particular, we shall have

$$p_n(\mathbf{0}) \sim cn^{-k/2}/2\sigma^2, \tag{7.17}$$

where c is some constant and k is the rank of $\mathbf{A}$, that is, the effective dimensionality of the walks W on L.

Hence, in the first place,

$$V(\mathbf{v}, u) = O(1) \qquad \text{if} \quad k \geq 3. \tag{7.18}$$

When $k = 1$ or 2, we shall look at two separate situations, one when $\mathbf{v} = \mathbf{0}$ and u is large, the other when $u = 0$ and $\mathbf{v}$ is large.

Consider then the case when $k = 1$, $\mathbf{v} = \mathbf{0}$, and u is large. We have

$$(7.19)\qquad \begin{aligned} V(\mathbf{0}, u) &= O(1) + \lim_{t\to\infty} \sum_{r=0}^{t} \frac{c}{(2r)^{1/2}} - \frac{c}{(2r+u)^{1/2}} \\ &= O(1) + \lim_{t\to\infty} \frac{c}{\sqrt{2}} \int_0^t [x^{-1/2} - (x + \tfrac{1}{2}u)^{-1/2}]\, dx \\ &= O(1) + \lim_{t\to\infty} c\sqrt{2}\, [t^{1/2} - (t + \tfrac{1}{2}u)^{1/2} + (\tfrac{1}{2}u)^{1/2}] \\ &= c\sqrt{u} + O(1) \qquad \text{as} \quad u \to \infty. \end{aligned}$$

When $k = 2$, $\mathbf{v} = \mathbf{0}$, and u is large,

$$(7.20)\qquad \begin{aligned} V(\mathbf{0}, u) &= O(1) + \lim_{t\to\infty} \tfrac{1}{2}c \int_0^t [x^{-1} - (x + \tfrac{1}{2}u)^{-1}]\, dx \\ &= O(1) + \tfrac{1}{2}c \log u \qquad \text{as} \quad u \to \infty. \end{aligned}$$

On the other hand, if $u = 0$ and $\mathbf{v}$ is large, we have

$$(7.21)\qquad V(\mathbf{v}, 0) \sim c \sum_{r=0}^{\infty} [p_{2r}(\mathbf{0}) - p_{2r}(\mathbf{v})].$$

Now the sum

$$(7.22)\qquad \sum_{r=0}^{t} [p_{2r}(\mathbf{0}) - p_{2r}(\mathbf{v})]$$

represents, for a random walk W starting at $\mathbf{0}$, the expected excess of even-step visits to the origin over even-step visits to $\mathbf{v}$, during the first t steps of the walk. Suppose t is very large, much larger than the number of steps needed to first visit the hyperplane in which $\mathbf{0}$ and $\mathbf{v}$ are mirror images. When the walk first reaches this hyperplane, then by symmetry, in the ensuing part of the walk the expected excess of visits to $\mathbf{0}$ over visits to $\mathbf{v}$ will be zero. Hence

$$(7.23)\qquad \sum_{r=0}^{\infty} [p_{2r}(\mathbf{0}) - p_{2r}(\mathbf{v})]$$

is, for a random walk W starting from $\mathbf{0}$, the expected number of visits to $\mathbf{0}$ before first reaching this hyperplane. Thus

$$(7.24)\qquad \sum_{r=0}^{\infty} [p_{2r}(\mathbf{0}) - p_{2r}(\mathbf{v})] = \sum_{r=0}^{N} p_{2r}(\mathbf{0}),$$

where N is the expected number of steps before W first crosses the hyperplane. However, N will be nearly equal to some multiple of $\mathbf{v}^2$, the square of the length of $\mathbf{v}$. Thus $V(\mathbf{v}, 0)$ will be asymptotically proportional to the length of $\mathbf{v}$ when $k = 1$, and asymptotically proportional to the logarithm of the length of $\mathbf{v}$ when $k = 2$.

The foregoing argument is, of course, very rough and ready, and is, moreover, wrong in certain special respects. For example, suppose L is the hypercubical lattice in d dimensions, and let

$$(7.25)\qquad p(\mathbf{x}) = 2^{-d}$$

whenever $\mathbf{x}$ is one of the 2^d vectors with coordinates all ± 1, $p(\mathbf{x})$ being zero for all other $\mathbf{x}$. Then $p_{2r+u}(\mathbf{v})$ will be zero if u has opposite parity to the sum of the coordinates of $\mathbf{v}$; and accordingly, $V(\mathbf{v}, u)$ will be infinite when u and $\mathbf{v}$ are so related and $d = 1$ or 2. In particular, $V(\mathbf{v}, 0)$ will be finite if and only if either the sum of the coordinates of $\mathbf{v}$ is even or $d > 2$.

Nevertheless, the general picture yielded by the heuristic argument is probably fair. For example, subject to the reservations on parity quoted above, it yields the correct result when applied to the particular case (7.25), as Champernowne showed by an exact evaluation of (7.14).

Let us now compare and contrast the general conclusions just reached with those reached in theorem 1 for one-sided harnesses. We shall suppose that L lies in d-dimensional Euclidean space, and that the walks W are also honestly d-dimensional. We write L_u for the $(d + 1)$-dimensional space $(\mathbf{v}, u)$. When L_u is at least 4-dimensional $(d + 1 \geq 4)$, then $V(\mathbf{v}, u)$ is bounded, as in (5.6). When L_u is 3-dimensional $(d = 2)$, then $V(\mathbf{v}, u)$ increases logarithmically with $\mathbf{v}$ or u, as in (5.5). When L_u is 2-dimensional, then $V(\mathbf{v}, u)$ behaves like $\sqrt{u}$ when $\mathbf{v} = \mathbf{0}$ and like $|\mathbf{v}|$ when $u = 0$. This corresponds to the behavior of (5.3), since the coordinate axis of u in L_u corresponds to the direction in which the serial harness is recursively constructed. When Champernowne first produced his results, we both felt that this similarity with one-sided harnesses showed that the formulation (7.5) (or specifically, its formulation with respect to the special case (7.25), that being the actual formulation then employed) did not lead to any improvement upon the anisotropic shortcomings of one-sided harnesses. We had been hoping to find in two or more dimensions that the fluctuations would be smaller than in one dimension, and were disappointed to find that, when L_u was 2-dimensional, the fluctuations perpendicular to the line of recurrent advance were still the same order of magnitude as for ordinary martingales. Strangely enough, it is just this feature which now appears to me to save the situation.

I now believe that the right way of looking at the matter is to look at L and not at L_u. When L is one-dimensional, we have in L itself, that is, taking $u = 0$, the ordinary one-dimensional fluctuations. When L is two-dimensional, the fluctuations increase logarithmically with distance; and when L is three-dimensional or more, the fluctuations are bounded.

We therefore take $u = 0$ in what follows, and look at $V(\mathbf{v}, 0)$, which we shall simply write as $V(\mathbf{v})$. We have as a restatement of theorem 3 for this specialization the following theorem. The expression (7.26) is very similar to, though not identical with, the potential kernel [9].

THEOREM 4. *The following equality holds:*

$$V(\mathbf{v}) = 2\sigma^2 \sum_{r=0}^{\infty} [p_{2r}(0) - p_{2r}(\mathbf{v})]. \tag{7.26}$$

Next we shall prove

THEOREM 5. *For all* $\mathbf{v}_1, \mathbf{v}_2 \in L$,

$$V(\mathbf{v}_1 + \mathbf{v}_2) + V(\mathbf{v}_1 - \mathbf{v}_2) \leq 2V(\mathbf{v}_1) + 2V(\mathbf{v}_2). \tag{7.27}$$

Moreover, the set of all $\mathbf{v}$ *at which* $V(\mathbf{v})$ *is finite forms a subgroup* L^* *of* L.

Consider

$$(7.28)\qquad \begin{aligned} &2p_{2r}(\mathbf{v}_1) - p_{2r}(\mathbf{v}_1 + \mathbf{v}_2) - p_{2r}(\mathbf{v}_1 - \mathbf{v}_2) \\ &\quad = \sum_{\mathbf{w}\in L} [p_r(\mathbf{w})p_r(\mathbf{v}_1 - \mathbf{w}) + p_r(\mathbf{v}_2 + \mathbf{w})p_r(\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{w}) \\ &\qquad\qquad - p_r(\mathbf{v}_1 - \mathbf{w})p_r(\mathbf{v}_2 + \mathbf{w}) - p_r(\mathbf{w})p_r(\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{w})] \\ &\quad = \sum_{\mathbf{w}\in L} [p_r(\mathbf{w}) - p_r(\mathbf{v}_2 + \mathbf{w})][p_r(\mathbf{v}_1 - \mathbf{w}) - p_r(\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{w})] \\ &\quad = \sum_{\mathbf{w}\in L} [p_r(\mathbf{w}) - p_r(\mathbf{v}_2 + \mathbf{w})][p_r(\mathbf{w} - \mathbf{v}_1) - p_r(\mathbf{v}_2 + \mathbf{w} - \mathbf{v}_1)] \\ &\quad \le \sum_{\mathbf{w}\in L} [p_r(\mathbf{w}) - p_r(\mathbf{v}_2 + \mathbf{w})]^2, \end{aligned}$$

since, by Schwarz's inequality, $\sum a_i a_i^* \le \sum a_i^2$ whenever the a_i^* are a permutation of the a_i. Further,

$$(7.29)\qquad \begin{aligned} &\sum_{\mathbf{w}\in L} [p_r(\mathbf{w}) - p_r(\mathbf{v}_2 + \mathbf{w})]^2 \\ &\quad = \sum_{\mathbf{w}\in L} [p_r(\mathbf{w}) - p_r(\mathbf{v}_2 + \mathbf{w})][p_r(-\mathbf{w}) - p_r(-\mathbf{v}_2 - \mathbf{w})] \\ &\quad = \sum_{\mathbf{w}\in L} [p_r(\mathbf{w})p_r(-\mathbf{w}) + p_r(\mathbf{v}_2 + \mathbf{w})p_r(-\mathbf{v}_2 - \mathbf{w}) \\ &\qquad\qquad - p_r(\mathbf{w})p_r(-\mathbf{v}_2 - \mathbf{w}) - p_r(\mathbf{v}_2 + \mathbf{w})p_r(-\mathbf{w})] \\ &\quad = 2p_{2r}(\mathbf{0}) - p_{2r}(-\mathbf{v}_2) - p_{2r}(\mathbf{v}_2) = 2p_{2r}(\mathbf{0}) - 2p_{2r}(\mathbf{v}_2). \end{aligned}$$

On combining (7.28) and (7.29), we have

$$(7.30)\qquad \begin{aligned} &2[p_{2r}(\mathbf{0}) - p_{2r}(\mathbf{v}_1)] + 2[p_{2r}(\mathbf{0}) - p_{2r}(\mathbf{v}_2)] \\ &\qquad \ge [p_{2r}(\mathbf{0}) - p_{2r}(\mathbf{v}_1 + \mathbf{v}_2)] + [p_{2r}(\mathbf{0}) - p_{2r}(\mathbf{v}_1 - \mathbf{v}_2)]; \end{aligned}$$

and hence, by summing (7.30) over r, we get (7.27). Finally, if $V(\mathbf{v}_1)$ and $V(\mathbf{v}_2)$ are both finite, then (7.27) shows that both the nonnegative quantities $V(\mathbf{v}_1 + \mathbf{v}_2)$ and $V(\mathbf{v}_1 - \mathbf{v}_2)$ must be finite. This proves that L^* is a subgroup of L.

It also follows from (7.27) that, when $\mathbf{v}_1$ and $\mathbf{v}_2$ are given, then at least one of the two inequalities $V(\mathbf{v}_1 + \mathbf{v}_2) \le V(\mathbf{v}_1) + V(\mathbf{v}_2)$ and $V(\mathbf{v}_1 - \mathbf{v}_2) \le V(\mathbf{v}_1) + V(\mathbf{v}_2)$ must be true. It seems a reasonable conjecture that both are actually true. Since $V(\mathbf{v}) = V(-\mathbf{v})$ by symmetry, this conjecture may be put in the following form.

CONJECTURE 1. *For all* $\mathbf{v}_1, \mathbf{v}_2 \in L$,

$$(7.31)\qquad V(\mathbf{v}_1 + \mathbf{v}_2) \le V(\mathbf{v}_1) + V(\mathbf{v}_2).$$

THEOREM 6. *If* $\mathbf{v}$ *is a point of* L *such that* $p_2(\mathbf{v}) > 0$, *then* $\mathbf{v} \in L^*$.

Suppose $\mathbf{v} \ne 0$. Let $s_0, s_1, \cdots, s_k$ be a sequence of numbers, each equal to ± 1, such that for a random walk W of t steps $s_i = +1$ if the i-th even-step visit to either $\mathbf{0}$ or $\mathbf{v}$ is to $\mathbf{0}$, and $s_i = -1$ if the i-th even-step visit to either $\mathbf{0}$ or $\mathbf{v}$ is to $\mathbf{v}$. We take $s_0 = +1$ to correspond to the zero-th visit to $\mathbf{0}$, namely to the start of W from $\mathbf{0}$. Then

$$(7.32)\qquad \sum_{r=0}^{t} [p_{2r}(\mathbf{0}) - p_{2r}(\mathbf{v})] = E\sum_{i=0}^{k} s_i.$$

Let S denote the sum of the sequence $s_0, s_1, \cdots, s_k$ omitting any consecutive terms at the end of the sequence which are all -1. Then

$$E \sum_{i=0}^{k} s_i \leq ES. \tag{7.33}$$

Now S is the sum of a sequence which consists of consecutive blocks, such that all s_i in a block are alike while the s_i in adjacent blocks are of opposite sign. The first and last blocks have s_i all equal to $+1$. By the symmetry of the situation, that is to say the symmetry $p(v) = p(-v)$, the expected number of terms in each block (except perhaps the last, which may be truncated) is independent of the block. Hence ES equals the expected number of symbols in the last block. The probability that successive s_i are of opposite signs is at least $p_2(\mathbf{v})$. Hence,

$$ES \leq \sum_{n=1}^{\infty} np_2(\mathbf{v})[1 - p_2(\mathbf{v})]^{n-1} = 1 + [p_2(\mathbf{v})]^{-1}. \tag{7.34}$$

Consequently, $V_t(\mathbf{v})$ is bounded (independently of t) and $V(\mathbf{v}) < \infty$. Thus $\mathbf{v} \in L^*$ if $\mathbf{v} \neq \mathbf{0}$. Obviously $\mathbf{0} \in L$. This completes the proof of theorem 6.

THEOREM 7. *If* $\sum_{r=0}^{\infty} p_{2r}(\mathbf{0}) < \infty$, *then* $L^* = L$. *If* $\sum_{r=0}^{\infty} p_{2r}(\mathbf{0}) = \infty$, *then* L^* *consists of all points of* L *which can be reached from* $\mathbf{0}$ *in an even number of steps with positive probability.*

Theorem 6, together with the fact that L^* is a subgroup of L, shows that L^* contains all points of L which can be reached from $\mathbf{0}$ in an even number of steps with positive probability. If $\mathbf{v}$ is any point which cannot be reached from $\mathbf{0}$ in an even number of steps with positive probability, then $p_{2r}(\mathbf{v}) = 0$ and

$$V(\mathbf{v}) = 2\sigma^2 \sum_{r=0}^{\infty} p_{2r}(\mathbf{0}); \tag{7.35}$$

and theorem 7 follows at once.

Theorem 7 shows that L^* is nontrivial (that is, consists of more than $\mathbf{0}$) when $p(\mathbf{0}) < 1$; and that, if we wish, we can make $L^* = L$ by choosing a system of random walks W such that $p(\mathbf{0}) > 0$.

8. Distributional properties of serial harnesses

Consider a random walk $W(t, \mathbf{v})$ of t steps starting at $\mathbf{v}$. We define the walk-sum $S[W(t, \mathbf{v})]$ by means of

$$S[W(t, \mathbf{v})] = \sum_{r=0}^{t} \epsilon_r(\mathbf{w}_r) \tag{8.1}$$

where $\mathbf{w}_0 = \mathbf{v}$, and $\mathbf{w}_r$ is the point of L which $W(t, \mathbf{v})$ visits at its r-th step. Then define

$$Y_t^*(\mathbf{v}) = E_{W(t,\mathbf{v})} S[W(t, \mathbf{v})], \tag{8.2}$$

when $E_{W(t,\mathbf{v})}$ denotes the expectation over all t-stepped walks $W(t, \mathbf{v})$ starting at $\mathbf{v}$. Comparing (8.2) with (7.5) we see that the only difference between $Y_t(\mathbf{v})$

and $Y_t^*(\mathbf{v})$ is that we have written $\epsilon_r(\mathbf{w})$ in place of $\epsilon_{t-r}(\mathbf{w})$. Thus the distributional properties of Y_t and Y_t^* are the same. Thus

$$H_t(\mathbf{v}) = Y_t^*(\mathbf{v}) - Y_t^*(0) \tag{8.3}$$

is a weighted sum of the random variables $\epsilon_r(w)$, say

$$H_t(\mathbf{v}) = \sum_{r=0}^{t} \sum_{\mathbf{w}\in L} c(r, \mathbf{v}, \mathbf{w})\epsilon_r(\mathbf{w}). \tag{8.4}$$

However, when $\mathbf{v} \in L^*$,

$$\sigma^2 \sum_{r=0}^{t} \sum_{\mathbf{w}\in L} [c(r, \mathbf{v}, \mathbf{w})]^2 = V_t(\mathbf{v}) \leq V(\mathbf{v}) < \infty; \tag{8.5}$$

and hence,

$$\sum_{r=0}^{\infty} \sum_{\mathbf{w}\in L} [c(r, \mathbf{v}, \mathbf{w})]^2 < \infty, \qquad \mathbf{v} \in L^*. \tag{8.6}$$

The random variables $\epsilon_r(\mathbf{w})$ have zero mean and are independent. So, applying Kolmogorov's three series theorem to (8.6), we deduce that

$$H(\mathbf{v}) = \lim_{t\to\infty} H_t^*(\mathbf{v}), \qquad \mathbf{v} \in L^*, \tag{8.7}$$

exists almost everywhere in the product probability space of the ϵ's. The limit in (7.36) exists in the same sense. We can, if we prefer, take (8.7) as a definition of a serial harness.

The problem of specifying what possible distributions $H(\mathbf{v})$ may have is an interesting and probably rather difficult question. I formulate it below but do not answer it. Let $K_\epsilon(x) = \log Ee^{ix\epsilon}$ denote the cumulant generating function of the common distribution of the ϵ's, and let $K_{H(\mathbf{v})}(x)$ be the corresponding cumulant generation function of $H(\mathbf{v})$. Then since

$$H(\mathbf{v}) = \sum_{r=0}^{\infty} \sum_{\mathbf{w}\in L} [p_r(\mathbf{w}) - p_r(\mathbf{w} - \mathbf{v})]\epsilon_r(\mathbf{w}), \tag{8.8}$$

we have

$$K_{H(\mathbf{v})}(x) = \sum_{r=0}^{\infty} \sum_{\mathbf{w}\in L} K_\epsilon\{[p_r(\mathbf{w}) - p_r(\mathbf{w} - \mathbf{v})]x\}. \tag{8.9}$$

Except in special cases, this will be an awkward expression to evaluate. Even harder is the problem of determining whether functions $K_\epsilon(x)$ and $p(\mathbf{w})$ exist such that $K_{H(\mathbf{v})}(x)$ has some prescribed form.

One particular case may, however, be answered completely. If each of the ϵ's is normally distributed, then $H(\mathbf{v})$ is normally distributed, whatever the transition function $p(\mathbf{v})$ may be. The converse is also true. If $H^*(\mathbf{v}) = P + Q$ is normally distributed with P and Q independent, then P and Q must be normally distributed; and we can take, for any prescribed r_0 and $\mathbf{w}_0$,

$$P = [p_{r_0}(\mathbf{w}_0) - p_{r_0}(\mathbf{v} - \mathbf{w}_0)]\epsilon_{r_0}(\mathbf{w}_0), \tag{8.10}$$

$$Q = H^*(\mathbf{v}) - P, \tag{8.11}$$

which shows that $e_{r_0}(\mathbf{w}_0)$ is normally distributed.

There is, moreover, a property of the normal distribution which may be relevant to harness theory, particularly in connection with the remarks on extension of the Markovian concept in the next section. This property is exhibited in the following theorem, which Professor E. Lukacs communicated to me privately in a letter [6] in answer to a question about harnesses, and which he has kindly allowed me to reproduce here.

THEOREM 8. *Let z_1 and z_2 be two independently and identically distributed random variables with zero mean and finite variance σ^2. If the conditional mean and the conditional variance of $z_1 - z_2$, given $z_1 + z_2$, are both constant, then the common distribution of z_1 and z_2 is normal.*

We have the conditional expectations

$$E(z_1 - z_2|z_1 + z_2) = E(z_1 - z_2) = 0; \tag{8.12}$$

and hence,

$$\operatorname{var}(z_1 - z_2|z_1 + z_2) = E[(z_1 - z_2)^2|z_1 + z_2] = E(z_1 - z_2)^2 = 2\sigma^2. \tag{8.13}$$

Lukacs [7] has proved that if x and y are random variables with Ey finite, and if $E(y|x) = E(y)$ almost everywhere, then

$$E(ye^{itx}) = E(y)E(e^{itx}) \tag{8.14}$$

for all real t. Applying (8.14) to (8.13) we get

$$E[(z_1 - z_2)^2e^{it(z_1+z_2)}] = 2\sigma^2E[e^{it(z_1+z_2)}] = 2\sigma^2[f(t)]^2, \tag{8.15}$$

where

$$f(t) = Ee^{itz_1} = Ee^{itz_2} \tag{8.16}$$

is the characteristic function of the common distribution of z_1 and z_2. Expanding (8.15), we have

$$2\sigma^2[f(t)]^2 = 2E[z_1^2e^{itz_1}] - 2E^2[z_1e^{itz_1}], \tag{8.17}$$

since z_1 and z_2 have the same distribution and are independent.

Since

$$f'(t) = iE(z_1e^{itz_1}), \qquad f''(t) = -E(z_1^2e^{itz_1}), \tag{8.18}$$

we get

$$f''(t) - [f'(t)]^2 = -\sigma^2[f(t)]^2. \tag{8.19}$$

Since $f(t) \neq 0$ in some neighborhood of $t = 0$, we may there divide by $[f(t)]^2$ to obtain

$$\frac{d}{dt}\left[\frac{f'(t)}{f(t)}\right] = -\sigma^2. \tag{8.20}$$

Hence,

$$f'(t)/f(t) = -\sigma^2t \tag{8.21}$$

because $E(z) = 0$; and finally

$$\log f(t) = -\sigma^2t^2/2. \tag{8.22}$$

According to a theorem on analytic characteristic functions, due to R. P. Boas, this solution can be continued so that it is valid for all real t, and accordingly, z_1 and z_2 are normally distributed.

9. Extensions of the Markovian concept

The problem of formulating an extension of the familiar Markov property to a multidimensional situation has exercised mathematical statisticians for many years, although very little progress has so far been made. I wish to make some extremely tentative and speculative remarks about this. In the simplest one-dimensional situation we have a sequence of random variables $z_1, z_2, \cdots$, such that the distributional properties of $z_n, z_{n+1}, \cdots$, given $z_1, z_2, \cdots, z_m$ where $m < n$, are the same as the distributional properties of $z_n, z_{n+1}, \cdots$, given z_m. This is the ordinary Markov property. (It would not matter if the z's were vectors: what is important is that the indexing variables m and n are one-dimensional.) The problem is how to extend this concept to the situation when the indexing variables m and n are multidimensional.

Consider, for simplicity, the case when the indexing variable is a vector $\mathbf{v}$ belonging to a lattice L. The random variables are then of the form $z_{\mathbf{v}}$. Let A be any arbitrary subset of L, and (in a sense to be defined presently) let B be the boundary of A. We write A' for the complement of A with respect to L and suppose that $A' \supset B$. Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ be any subset of lattice points all belonging to A. Then we want the joint conditional distribution of $z_{\mathbf{v}_1}, z_{\mathbf{v}_2}, \cdots, z_{\mathbf{v}_k}$, given $z_{\mathbf{w}}$ for all $\mathbf{w} \in A'$, to be the same as their joint conditional distribution, given $z_{\mathbf{w}}$ for all $\mathbf{w} \in B$. This is a possible and natural extension of the Markovian concept to a multidimensional index $\mathbf{v}$.

The system of harnesses introduced via (7.5) seems to satisfy this extension, provided that we define B suitably. Given A, let us define B as the set of all points $\mathbf{w} \in A'$ such that $\mathbf{w}$ can be reached in one step from some point of A with positive probability for walks governed by the transition law $p(\mathbf{v})$. That is to say, $\mathbf{w} \in B$ if $p(\mathbf{w} - \mathbf{v}) > 0$ with some $\mathbf{v} \in A$ and $\mathbf{w} \in A'$: the $\mathbf{v}$ in question may depend upon $\mathbf{w}$. For example, if L is the square lattice and $p(\mathbf{v}) = 0$ except for transitions between nearest neighbors, then the boundary of A is the set of points of A' which immediately surround A in the ordinary geometric sense.

Suppose now that we wish to define a random variable $Y^*(\mathbf{v})$ such that $Y^*(\mathbf{v})$ has prescribed values on A', say $Y^*(\mathbf{v}) = y(\mathbf{v})$ when $\mathbf{v} \in A'$. We proceed as in (8.1), but with a slight modification. If $\mathbf{v} \in A$, we write $W^B(\mathbf{v})$ for a random walk which starts at $\mathbf{v}$ and which terminates immediately after first leaving A, namely, as soon as it reaches B. We then write

$$S[W^B(\mathbf{v})] = [\sum_r \epsilon_r(\mathbf{w}_r)] + y(\mathbf{w}_B) \tag{9.1}$$

where $\mathbf{w}_0 = v$, and $\mathbf{w}_r$ are the successive points of A which $W^B(\mathbf{v})$ visits, and $\mathbf{w}_B$ is the terminal points of the walk in B. Then we take

$$Y^B(\mathbf{v}) = E_{W^B(\mathbf{v})} S[W^B(\mathbf{v})] \tag{9.2}$$

when the expectation is over all walks $W^B(\mathbf{v})$. We shall have

$$Y^B(\mathbf{v}) = \sum_r \left[\sum_{\mathbf{w}\in A} p_r^B(\mathbf{w}, \mathbf{v})\epsilon_r(\mathbf{w}) + \sum_{\mathbf{w}\in B} f_r^B(\mathbf{w}, \mathbf{v})\epsilon_r(\mathbf{w}) \right] \tag{9.3}$$

where $p_r^B(\mathbf{w}, \mathbf{v})$ is the probability that a walk starting at $\mathbf{v}$ visits $\mathbf{w}$ at the r-th step without having previously left A, and $f_r^B(\mathbf{w}, \mathbf{v})$ is the probability that a walk starting at $\mathbf{v}$ first reaches B at the r-th step with $\mathbf{w} \in B$ as the point of B thus reached.

I have not yet had time to study properly systems such as (9.2) or (9.3), but it seems plausible at first sight that they may provide some sort of mathematical model for an extended Markovian concept. Lévy [5] has some results.

In particular, suppose we are considering the simplest possible case, that of the one-dimensional integer lattice and that A consists of a single point, say the integer v. Then if $p(1) = p(-1) = \frac{1}{2}$, the boundary of A will be the two points $v - 1$ and $v + 1$. For instance, we are given $y(v - 1) = y_{-1}$ and $y(v + 1) = y_{+1}$, and we have to consider distributions for $Y(v) = Y_0$, say. Write $z_1 = Y_0 - y_{-1}$ and $z_2 = y_{+1} - Y$. Then $z_1 + z_2$ is given and the distribution of Y_0 is entirely determined by that of $z_1 - z_2$. If the physical situation requires that the conditional mean and variance of Y_0 is constant, then, according to theorem 8, the distribution of Y_0 must be normal. The relevance of this remark to the general question of an extended Markovian concept is not at all clear to me at this stage, but I nevertheless hope it worth putting on record to provoke discussion.

10. Harnessed processes

We shall extend the treatment of section 8 to a more general specification which allows us to discern certain relations between central and serial harnesses. We proceed formally in order to fix attention on the underlying ideas: rigor would be out of place at this early stage of the investigation.

We suppose, as before, that we are given a lattice L and a probability transition function $p(\mathbf{v})$ defined for $\mathbf{v} \in L$. Let (Ω, A, μ) be some probability space with points $\omega \in \Omega$. Let $T: \omega \to T\omega$ denote some measure-preserving transformation on Ω. With each point $\mathbf{v} \in L$ we associate a sequence of random variables

$$f_0(\mathbf{v}, \omega) = f(\mathbf{v}, \omega), \qquad f_r(\mathbf{v}, \omega) = f(\mathbf{v}, T^r\omega), \qquad r = 1, 2, \cdots. \tag{10.1}$$

We do *not* suppose that the $f_r(\mathbf{v}, \omega)$ are independent. The complete set of all $f_r(\mathbf{v}, \omega)$ constitutes a stochastic process which we denote by $F(L, \omega)$. Next we define

$$g(\mathbf{v}, \omega) = f_0(\mathbf{v}, \omega) + \sum_{r=1}^{\infty} \sum_{\mathbf{w}\in L} p_r(\mathbf{w} - \mathbf{v}) f_r(\mathbf{w}, \omega). \tag{10.2}$$

Then $g(v, \mathbf{w})$ satisfies

$$g(\mathbf{v}, \omega) = f(\mathbf{v}, \omega) + \sum_{\mathbf{w}\in L} p(\mathbf{v} - \mathbf{w}) g(\mathbf{w}, T\omega). \tag{10.3}$$

We may define the difference operators ∇_p^2 and Δ_T by means of

$$\nabla_p^2 h(\mathbf{v}, \omega) = -h(\mathbf{v}, \omega) + \sum_{\mathbf{w}\in L} p(\mathbf{v} - \mathbf{w}) h(\mathbf{w}, \omega) \tag{10.4}$$

and

$$\Delta_T h(\mathbf{v}, \omega) = -h(\mathbf{v}, \omega) + h(\mathbf{v}, T\omega). \tag{10.5}$$

(The operator ∇_p^2 has sometimes been called the *pedestrian operator* because it arises from a random walk. When L is hypercubical and the walk is the ordinary Pólya walk, ∇_p^2 reduces to the finite-difference analogue of the Laplacian operator.) In terms of these operators we can write

$$f(\mathbf{v}, \omega) + \Delta_T g(\mathbf{v}, \omega) + \nabla_p^2 g(\mathbf{v}, T\omega) = 0. \tag{10.6}$$

Still more briefly we may write

$$f + (\Delta_T + T\,\nabla_p^2)g = 0, \tag{10.7}$$

or

$$f + \Box_{T,p} g = 0. \tag{10.8}$$

This equation may well have many solutions; but the solution (10.2), which is, in a sense, its Neumann solution, is of special interest, and we denote it by

$$\Box_{T,p}^{-1} f(\mathbf{v}, \omega) = \sum_{r=0}^{\infty} \sum_{\mathbf{w} \in L} p_r(\mathbf{w} - \mathbf{v}) f(\mathbf{w}, T^r\omega). \tag{10.9}$$

Thus, ignoring all questions of convergence and existence, we have produced a process

$$\Box_{T,p}^{-1} F(L, \omega) = \{\Box_{T,p}^{-1} f(\mathbf{v}, \omega)\}. \tag{10.10}$$

We may call $\Box_{T,p}^{-1} F(L, \omega)$ the harnessed process derived from the process $F(L, \omega)$, and consider $\Box_{T,p}^{-1}$ as a harnessing operator.

If we choose $f(\mathbf{v}, \omega) = \epsilon_0(\mathbf{v}, \omega)$ all independent, and choose T so that all $f(\mathbf{v}, T^r\omega)$ are independent both over $\mathbf{v}$ and r, then we shall obtain the serial harnesses

$$H(\mathbf{v}, \omega) = \Box_{T,p}^{-1}[f(\mathbf{v}, \omega) - f(\mathbf{0}, \omega)]. \tag{10.11}$$

If, on the other hand, we choose $f(\mathbf{v}, \omega) = \epsilon_0(\mathbf{v}, \omega)$ and take T to be the identity transformation, (10.7) will reduce to

$$f + \nabla_p^2 g = 0, \tag{10.12}$$

and we shall be led to a central harness. We see, in effect, that the first-order operator Δ_T in (10.7) is producing some sort of damping of the kind sought in section 6.

Clearly, the operator $\Box_{T,p}^{-1}$ offers plenty of scope for deriving a variety of new processes. For example, we could consider harnessed harnesses $\Box_{T,p}^{-1} H(\mathbf{v}, \omega)$, and so on.

11. Conclusions

For the original metallurgical problem, the conclusions seem to be fairly satisfactory. If our model is a three-dimensional serial harness, then the long-range misorientation is bounded. With a three-dimensional one-sided harness, the long-range misorientation will only increase as the square root of the logarithm of the distance, and, with the numbers involved in the physical situation, this function increases so slowly as to be virtually bounded. This finding agrees with physical expectations, according to which there can only be long-range misorientation if some phenomenon (such as a large angle grain boundary) occurs which is much more drastic than small local variations between neighboring domains.

From the point of view of statistical theory, this paper is exploratory, tentative, and doubtless suffers from prolixity and muddled thinking. But the general problems considered, rather than the methods here used to approach them, are of considerable importance, and I hope that this paper will encourage others to work in this field. Very little is so far known about statistical theory in more than one dimension. Only a handful of problems has been brought to successful solution (of which perhaps the two most celebrated solutions are Onsager's solution of the two dimensional Ising problem with $p = q = \frac{1}{2}$, and the Fisher-Kasteleyn-Temperley solution of the two-dimensional dimer problem).

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