Alexander Iacovlevich Khinchin
(1894–1959)

Session Dedicated to the Memory of

ALEXANDER IACOVLEVICH KHINCHIN

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Ladies and Gentlemen:

We are gathered here today to give tribute to the memory of a mathematician whose achievements in formulating the fundamental ideas and methods of the contemporary theory of probability are recognized by all.

One of the characteristic peculiarities of the development of contemporary scientific thought is the rapid growth of statistical concepts in various fields of natural science, technology, and economics. It has become quite clear that the application of the methods of the theory of probability to the study of the principal problems of physics, biology, chemistry, astronomy, as well as economics, is not a whim of individual investigators and not a passing fashion, but is an inevitable consequence of the nature of the basic problems. As a result, it is now taken for granted that the laws of nature are of a statistical character determined by a discrete structure of matter. It is well known that the adoption of this point of view has led to numerous successes in all domains of science. It was natural that these successes influenced the content of the branch of mathematics that is devoted to the study of chance phenomena. The theory of probability could not remain as it was in the last century or even as it was in the first two decades of the present century. The semi-intuitive approach to the definition of the fundamental concepts of the theory of probability, characteristic of the eighteenth and nineteenth centuries, could not satisfy either mathematicians or representatives of the natural sciences. The isolated position occupied even recently by the theory of probability was sharply inconsistent with the important role which it began to play in the whole realm of knowledge. Consequently, one of the most urgent problems of the second decade of this century was the problem of transforming the theory of probability into a well-organized mathematical discipline with logically clear-cut fundamental concepts, with widely developed specific methods of investigation, and with distinctly established connections with other branches of mathematics. In order that the theory of probability should become a real tool in scientific research, the field of probabilistic problems had to be broadened, and this broadening required a deep analysis of peculiarities of mathematical formulations of the problems of science.

The role which A. I. Khinchin played in the solution of the whole complex of the above problems is exceptionally large. And, in spite of their diversity, his scientific interests are impressive by their internal unity and by their scientific coherence. In general terms, they can be characterized as a systematic study of the place and significance of statistical laws in various parts of mathematics, in the natural sciences, and in technology. I hope to be able to show this in what follows.

Alexander Iacovlevich Khinchin was born on July 19, 1894, in the village of Kondrovo in the Medynsky county of the Kaluga district, known at that time for its paper factory. His father, who was an engineer-technologist by education, was the chief engineer of the factory and was known and respected by specialists in the field of paper manufacture. Khinchin's childhood and later his vacation time, when he was at school and subsequently at the university, both in Moscow, were spent in Kondrovo. There he organized amateur theatricals with participants recruited from his buddies among the factory workers.

These years of adolescence were the years of Khinchin's entrancement with literature and of his attempts at poetry. They resulted in several little volumes of poems published between 1912 and 1917. Undoubtedly, his enthusiasm for the theater and fine literature contributed strongly to Khinchin's development into one of the most brilliant lecturers, teachers, and mathematical writers. Orally, as well as in writing, he was able to combine masterfully the best literary form with scientific depth and clarity of thought.

From 1911 to 1916 Khinchin was a student at the College of Physical and Mathematical Sciences of the Moscow University. There he joined the group of students who were captivated by the ideas of the theory of functions of a real variable and worked under the guidance of Professors D. F. Egorov and N. N. Luzin. Khinchin's first independent scientific steps belong to this period. They were stimulated by the known work of Denjoy on primitive functions. In a paper presented at a meeting of a student mathematical club, on November 6, 1914, he introduced a generalization of the concept of a derivative, a natural generalization reflecting the spirit of the metric theory of functions. This concept is now firmly entrenched in the arsenal of contemporary mathematics under the name of asymptotic derivative. The meaning of the term is as follows: if at the point x_0 there exists the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{1}$$

as x tends to x_0 over values belonging to the set E, having density 1 at the point x_0, then this limit is called the asymptotic derivative of the function $f(x)$ at the point x_0. In the above paper Khinchin showed that this derivative is invariant with respect to the choice of the set E. In other words, if two sets E_1 and E_2 have density 1 at the point x_0, and if the limit (1) exists for E_1 as well as for E_2, then the two limits are equal. The concept of asymptotic derivative and its application to the generalization of the Lebesgue integral were subjects of the first scientific articles by Khinchin. Later, the fundamental idea of the same concept was widely used by him for a many-sided study of the local behavior of measurable functions.

It is said that a property holds asymptotically at a point if it holds after the removal of a set having at this point its density equal to zero. Khinchin suggested that a function $f(x)$ be called asymptotically directed at a point x if it is asymptotically decreasing, increasing, or constant. A function $f(x)$ is asymptotically directed on a given set of positive measure if it is asymptotically directed at almost all of its points. The fundamental result characterizing the structure of asymptotically directed functions is given by the following theorem: in order that a function $f(x)$ be asymptotically directed on a given set it is necessary and sufficient that its value over this set, up to a set of arbitrarily small measure, coincide with the values of a continuous function having only a finite number of maxima and minima.

The importance of the notion of asymptotic direction is enhanced by the fact that the functions having this property also have an asymptotic derivative almost everywhere in the given set. The condition for the existence of asymptotic derivatives almost everywhere on a segment was proved by Khinchin in the note [2] published in 1917. For this it is necessary and sufficient that, over the whole segment with the possible exception of a set of arbitrarily small measure, the given function coincide with a continuous function of bounded variation. The general structure of measurable functions is clarified by the following proposition of Khinchin: with the possible exception of a set of measure zero, a measurable function either has an asymptotic derivative or else both its upper asymptotic derivatives are equal to $+\infty$ and both its lower asymptotic derivatives are equal to $-\infty$. Soon after the publication of the article "Investigations into the structure of measurable functions" in the *Matematičeskiĭ Sbornik*, giving the above analysis, it was translated into French and published in *Fundamenta Mathematicae*. This interest in the study of fundamental properties of measurable functions was not without consequence to mathematics in general as well as to the choice of direction of Khinchin's further work. The continuation of the development of Khinchin's idea, especially in the case of functions of many variables, was carried on by a number of scientists up to our time. The further work of Khinchin himself in the field of the theory of numbers, as well as the theory of probability, was influenced to a great degree by his initial interests.

The years from 1923 to 1925 were of exceptional importance in forming Khinchin as a scientist. During this period he began working in two directions of broad mathematical significance. The elements of the two directions of study were already available in the work of Borel. One of them was connected with the systematic study of metric properties of different classes of irrationals. The other consisted of the systematic application of concepts and of tools of the theory of functions and of the theory of sets to the theory of probability.

In order to get an idea of the character of the number-theoretical results of Khinchin, we give the formulation of some of his theorems. In an article [27] published in 1926, the following facts are proved.

Let $\varphi(t)$ be a positive function for which the product $t^2\varphi(t)$ is monotonically decreasing. Then the inequality

$$\left|\alpha - \frac{p}{q}\right| \leqq \varphi(q) \tag{2}$$

has, for almost all α, infinitely many solutions in integers p and q if, and only if, the integral $\int_0^\infty t\varphi(t)\,dt$ diverges.

A series of very elegant results of Khinchin has to do with the metric theory of continued fractions. We shall limit ourselves here to the formulation of two such theorems ([75] and [77]). Let $a_1, a_2, \cdots,$ be the partial quotients in the decomposition of the irrational number α in a continued fraction, and let $q_1, q_2, \cdots,$ be the denominators of the partial quotients in this decomposition. Then, for almost all α, the following limits exist

$$C = \lim_{n\to\infty} (a_1 a_2 \cdots a_n)^{1/n} \tag{3}$$

and

$$D = \lim_{n\to\infty} (q_n)^{1/n}, \tag{4}$$

where C and D are absolute constants [$C = 2.6 \cdots$; Lévy showed later that $D = \exp(\pi^2/12 \log 2)$]. From the point of view of the theory of probability these theorems can be interpreted as asymptotic properties of sums of elements of sequences of weakly dependent variables. The familiar result of Khinchin, known as the law of the iterated logarithm, belongs to the same group of ideas. In fact, in 1923 [10], he was able to sharpen the estimate of the frequency of the distribution of zeros and ones in a binary representation of real numbers obtained by Hardy and Littlewood in 1914. If we denote by $\mu(n)$ the deviation from $n/2$ of the number of units among the first n binary digits of the representation, then Hardy and Littlewood showed that for almost all numbers $\mu(n) = 0[(n \log n)^{1/2}]$. In his article [10] Khinchin was able to show that this estimate can be replaced by a sharper one as follows: for almost all numbers, $\mu(n) = 0[(n \log\log n)^{1/2}]$. A year later an article [14] appeared in which Khinchin treated this problem as a problem in the theory of probability. In terms of the theory of numbers we can state his final result as follows: for almost all numbers α, the following equality holds:

$$\overline{\lim_{n\to\infty}} \frac{\mu(n)}{(2n \log\log n)^{1/2}} = 1. \tag{5}$$

This equality is in fact the famous law of the iterated logarithm, to which were later devoted a large number of excellent investigations of many scientists. Here I wish only to remind you of a single result which sharpens the law of the iterated logarithm in the sense of the first of the two theorems in the theory of numbers which I mentioned above. This result was obtained in the early forties by Erdös ["On the law of the iterated logarithm," *Ann. of Math.*, Vol. 43 (1942), pp. 419–436] and by Feller ["The general form of the so-called law of the iterated logarithm," *Trans. Amer. Math. Soc.*, Vol. 54 (1943), pp. 373–402] based on the previous work of Petrovsky ["Zur ersten Randwertaufgabe der Wärmeleitungsgleichung," *Compositio Math.*, Vol. 1 (1935), pp. 383–419] devoted to boundary problems for the heat-transfer equation.

The question can be put as follows: find all functions $\varphi(n)$ for which the inequality

$$\mu(n) < \varphi(n) \tag{6}$$

is satisfied for almost all α and for all n with the possible exception of a finite number. From Khinchin's results it follows only that the condition

$$\underline{\lim}_{n\to\infty} \frac{\varphi(n)}{(2n \log \log n)^{1/2}} > 1 \tag{7}$$

is sufficient, while the condition

$$\overline{\lim}_{n\to\infty} \frac{\varphi(n)}{(2n \log \log n)^{1/2}} \geqq 1 \tag{8}$$

is necessary. Necessary and sufficient conditions which the function $\varphi(n)$ must satisfy consist of the convergence of the integral

$$\int_A^\infty \frac{1}{t}\, \varphi(t) \exp\left(\frac{-\varphi^2(t)}{2}\right) dt. \tag{9}$$

In discussing Khinchin's research in the theory of numbers in fields other than the metric problems we must begin by noting his work in the theory of Diophantine approximations and his theorem about the addition of sequences of integers ([53], [99], and [105]). In order to formulate this theorem, we use the symbol $\{\varphi\}$ to denote a sequence of natural numbers and let $\varphi(n)$ be the number of elements of this sequence which do not exceed n. Further, the lower bound of numbers $\varphi(n)/n$ will be called the density of the sequence $\{\varphi\}$ and will be denoted by $D(\varphi)$. Finally, for a finite collection of sequences $\{\varphi_1\}, \{\varphi_2\}, \cdots, \{\varphi_k\}$, we shall use the term sum of these sequences to denote a sequence of integers expressible as the sum $\varphi_1 + \varphi_2 + \cdots + \varphi_k$ of not more than one term from each sequence $\{\varphi_i\}$. In other words, the first term φ_1 in the above sum is either zero or an element of the sequence $\{\varphi_1\}$; the second term φ_2 is either zero or an element of $\{\varphi_2\}$, and so on. With this notation and these definitions, Khinchin showed that, if the densities of sequences $\{\varphi_i\}$ are all equal and if

$$\sum_{i=1}^{k} D(\varphi_i) \leqq 1, \tag{10}$$

then

$$D\left(\sum_{i=1}^{k} \varphi_i\right) \geqq \sum_{i=1}^{k} D(\varphi_i). \tag{11}$$

The publication of this result has attracted the attention of many mathematicians and has led to numerous attempts to extend it to sequences with different densities. However, the problem resisted all efforts for a long time. It was only in 1942 that Mann, and a year later Artin and Scherk, were able to find the complete solution.

Among the achievements of Khinchin in the theory of Diophantine approximations we can point to an important transfer principle which connects the solution of linear inequalities in integers with Diophantine approximations of the coefficients of the corresponding linear forms. In speaking of Khinchin's work in the theory of numbers one must not overlook some excellent popular books written by him at different times. Among them I would particularly like to mention two small volumes ([73] and [120]) which have been translated into a great many languages.

In spite of the considerable contribution of Khinchin to the theory of functions and the theory of numbers, his fundamental role in the progress of mathematics is connected with the theory of probability. Starting with problems connected with the theory of numbers (the law of the iterated logarithm) and the theory of functions (convergence of series of independent random variables), he gradually included in the orbit of his interests a larger and larger group of problems in the theory of probability. Moreover, he attracted many young Moscow mathematicians to the solution of the problems, thus forming the beginning of the Moscow school of the theory of probability.

Khinchin's papers devoted to the law of the iterated logarithm and the summation of series of random terms were followed by the papers pertaining to the classical problem of summation of independent random variables. From the results he obtained I should like to single out the particularly clear condition of the applicability of the law of large numbers in the case of independent and equally distributed summands, which is reduced to the existence of a finite mathematical expectation [44]. Further, one should mention the fruitful concept of the relative stability of sums [74]. This concept was found to be intimately connected with the ultimate formulation of conditions for convergence to normality of the distribution of normed sums of independent components. For the case of identically distributed components, Khinchin succeeded, simultaneously with P. Lévy and W. Feller but independently from them, in finding the necessary and sufficient conditions for convergence to the normal law [79]. Papers [47] and [48] deserve special mention because they may be considered as initiating the current studies of "large deviations."

Khinchin's construction of the general theory of limit distributions for sums of independent random variables [91] belongs to the same group of ideas. The fundamental proposition of the theory he developed can be formulated as follows: the class of limit distributions for sums of independent infinitesimal random variables coincides with the class of infinitely divisible distributions. The proof of this fact, and also of some other propositions of the theory of summation, required the development and organization of the theory of infinitely divisible laws, which were then just introduced by Bruno de Finetti and A. N. Kolmogorov.

The theory of summation inspired Khinchin to write three monographs. The first [35] was published in 1927 after the completion of a special course on summation theory given by Khinchin at Moscow University. The second monograph [65] connected the classical problems of summation with the theory of Markov processes and with the then recent investigations of Kolmogorov and Petrovsky. The third monograph [92] gave a constructive development of general limit theorems for sums of independent summands and their application to the classical problem of the convergence of normed sums to the normal law. This book was also preceded by a special course of lectures in Moscow University. This course attracted the interest of A. A. Bobrov, D. A. Raikov, and myself to the summation theory.

Khinchin's work on the arithmetic of distribution laws, in which he considered the problem of representing a distribution law as a composition (product) of distributions, are also connected with summation theory. Among the results obtained by Khinchin we note the following: every distribution can be decomposed into an infinitely divisible distribution and a convergent product of a finite or denumerable sequence of indivisible distributions. The decomposition of the distribution function is in general not unique (as is shown by an example of B. V. Gnedenko). The last proposition is connected with the work of M. G. Krein on the extension of Hermite-definite functionals. The investigations of Khinchin about the arithmetic of distribution laws brought to life a series of papers by Raikov, Dugué, and, lately, by Yu. V. Linnik.

The interest in philosophy and methodology which Khinchin exhibited even in his school years led to a number of his publications on the subject of the philosophy of mathematics ([30], [38], [43], [50], and [141]). On the other hand, his reflections about the role of mathematics in establishing the laws of nature led him into the field of statistical physics [51]. From this time on, the questions of statistical physics played an important part in his research. In fact, all his later research in the theory of probability was connected in one way or another with his investigations into statistical physics. This was also the case with his work on limit distributions for the sum of independent random variables, as well as for random processes of Markov type and his development of the theory of stationary random processes. In private conversations Khinchin always emphasized that his consideration of physical problems led him to consider that class of random processes which are now called stationary.

It is undeniable that the idea of studying random processes was one of the most fruitful in the development of the theory of probability in the last fifty years. It has not only greatly influenced the structure of this whole branch of mathematics but has established numerous deep connections with various life-science and technological disciplines. The idea of considering stationary processes was most important in the development of the theory of probability, and Khinchin's role in the formulation of the foundations of the theory of stationary processes is particularly great. To a certain extent the necessity of considering such processes was already felt in science. In considering questions of geophysics such scientists as Taylor and Keller came close to the notion of a stationary process. Slutsky, on the basis of his analysis of stationary series, came to the conclusion that processes of a certain type (which are special cases of stationary processes) are able, in a certain sense, to imitate the behavior of periodic and almost periodic processes. In his work, [52] and later [60], Khinchin established a number of theorems of the type of the law of large numbers for stationary sequences. In 1933 he gave simplified proofs of some theorems in statistical dynamics (also of the type of the law of large numbers) earlier proved by Koopman and von Neumann. His initial successes in the theory of stationary processes are, however, connected with two other articles of his. In the first he gave a wide generalization of a well-known theorem of Birkhoff, which was proved by him under some very special assumptions only for dynamic systems in a state space in the form of a finite dimensional manifold. He formulated and proved a general theorem which is now known as the Birkhoff-Khinchin theorem [58]. It can be stated in the case of discrete time as follows: if the sequence of random variables

$$\xi_1, \xi_2, \cdots, \xi_n, \cdots \tag{12}$$

is stationary, and if the mathematical expectation of ξ_n is finite, then with probability one there exists the finite limit

$$\lim_{n\to\infty} \frac{1}{n} (\xi_1 + \xi_2 + \cdots + \xi_n). \tag{13}$$

Khinchin laid the foundation of the contemporary spectral theory of stationary processes [67], giving the definitions of stationary processes in the wide and narrow sense. The results of Khinchin continue to play a central role in the theory of stationary processes in spite of the fact that more than twenty-five years have gone by.

Beginning with 1929 Khinchin repeatedly returned to the consideration of problems of statistical physics. We have already mentioned how these interests influenced his work in the theory of stationary processes. On the other hand his studies in the realm of limit theorems for sums led to the development of a method of proof of the fundamental theorems of statistical physics. Beginning with 1941 he systematically developed the idea that the fundamental mathematical problems of statistical physics may be reduced to a well-developed theory of limit theorems for sums of independent random variables. A special role is played here by local limit theorems. There has been a revival of interest in local limit theorems in the last fifteen years due to a large degree to the influence of this conviction of Khinchin. These ideas are well known from the fine monographs [113], [132], and [136], the first two of which are translated and published in many countries, among them Germany and the United States.

The last period of creative activity of Khinchin falls in the period from 1953 to 1956. Two types of problems concerned him at that time. On the one hand he was interested in a new field, namely information theory, which first arose in the works of C. Shannon. Khinchin was interested in the problem of logical improvement of proofs given by the authors of information theory without sufficiently strict foundations. The works of Khinchin, [143] and [150], are readily available for study, thanks to the German and English translations.

The second direction of his interest was connected with the theory of queues, or as Khinchin put it, the theory of mass-service. His interest in these problems arose early in the thirties when, as a result of his public services, he became closely connected with the workers of the Moscow telephone exchange. His investigations [57] and [59], which are well known by specialists, belong to this period. In these two papers Khinchin was interested not in particular questions of telephone service but in the general study of incoming calls. It is chiefly this problem that is the subject of Khinchin's last monograph [146] and of his last mathematical papers [148] and [149].

In order to complete the sketch of Khinchin as a scientist, the following trait must be mentioned: his continuous interest in questions of teaching, both in secondary schools and in universities. He played a significant role in the improvement of textbooks of elementary mathematics. He produced a sequence of articles on methodology of teaching, and wrote several exciting pamphlets for teachers and a number of excellent textbooks for students.

In his private life, Khinchin was very exacting of himself. He gave a great deal of attention to the scientific development of his students, suggesting subjects for individual study and encouraging every sign of scientific initiative and independence. I am happy to have been one of his closest pupils and to have had the opportunity to watch him in work and in life. He abhorred unfinished business and never allowed himself to put on the shoulders of others the work for which he was responsible. He did not aspire to outside honors. Even though he was a scientist known all over the world, a corresponding member of the Academy of Sciences of the Union of Soviet Socialist Republics and an academician of the Academy of Pedagogical Sciences of the Russian Soviet Federated Socialist Republic, he still continued to lead a modest life, honoring people for their inner worth rather than for their position.

Khinchin died on November 18, 1959, after a long and difficult illness. Our science has sustained a great loss. We have lost a man who had taken on his shoulders a significant part of the difficult and necessary task of building a new theory of probability. During his path of service to science he acquired the esteem of his colleagues and of the scientific youth of the whole world. At the same time he was sincerely pleased with every significant success in science and with the appearance of every new gifted student. I remember how proud he was that in our science there had appeared such a bright new representative as V. Deblin, while at the same time he mourned Deblin's untimely death at the hands of Hitler's executioners. Khinchin harmoniously combined the qualities of a classical mathematician with those of a representative of the set-theoretic culture; at the same time he saw in mathematics a powerful tool for studying the laws of the surrounding nature.

In conclusion I wish to thank all those assembled here for their attention and for their respect to the memory of my teacher.

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