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COMPOSITION OPERATORS BETWEEN WEIGHTED BERGMAN SPACES WITH ADMISSIBLE BÉKOLLÉ WEIGHTS

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ABSTRACT. We study composition operators acting between weighted Bergman spaces with admissible Békollé weights. The boundedness and compactness of composition operators are characterized in terms of the generalized Nevanlinna counting function associated with the inducing map of the composition operator and the associated weight function of Bergman space. For a special case, we also give the estimate of the essential norm.

1. INTRODUCTION

Let $\mathbb{D}$ denote the open unit disk in the complex plane and $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. For a given non-negative integrable function σ on $\mathbb{D}$, we denote $L^p(\sigma dA)$ $(p > 0)$ the space of measurable functions f with

$$\|f\|_\sigma^p = \int_{\mathbb{D}} |f(z)|^p \sigma(z) dA(z) < \infty,$$

where dA is the normalized Lebesgue measure on $\mathbb{D}$, and $A^p(\sigma dA) = L^p(\sigma dA) \cap H(\mathbb{D})$. If $\sigma(z) = (1-|z|^2)^\alpha$ $(\alpha > -1)$, then $A^p(\sigma dA)$ is the well-known weighted Bergman space A^p_α. In this paper, we will consider the space $A^p(\sigma dA)$ and study the composition operator acting on this space.

For any analytic self-map $\phi : \mathbb{D} \to \mathbb{D}$, the *composition operator* $C_\phi : H(\mathbb{D}) \to H(\mathbb{D})$ is defined by $C_\phi f = f \circ \phi$. One of interesting subjects on studies of this

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operator is to characterize its operator theoretic property in terms of the function theoretic property of ϕ. For the Hardy space H^p or the weighted Bergman space A^p_α, a consequence of the Littlewood subordination principle verifies that all composition operators are bounded on H^p or A^p_α. However, it is known that every analytic self-map of $\mathbb{D}$ does not induce a compact composition operator on H^p or A^p_α. The classical Nevanlinna counting function N_ϕ plays a key role in the study on compact composition operator on H^2. For a given analytic self-map ϕ of $\mathbb{D}$, N_ϕ is defined by

$$N_\phi(z) = \sum_{w\in\phi^{-1}(z)} \log\frac{1}{|w|} \qquad (z \in \mathbb{D}\setminus\{\phi(0)\}),$$

where we understand that $N_\phi(z) = 0$ for $z \notin \phi(\mathbb{D})$ and $w \in \phi^{-1}(z)$ is repeated according to the multiplicity of zeros of $\phi - z$. This N_ϕ was used for establishing a formula for the essential norm of C_ϕ on H^2 by Shapiro [11]. As a consequence, he proved that the compactness of C_ϕ on H^2 is characterized by the condition $N_\phi(z) = o(-\log|z|)$ as $|z| \to 1$. Furthermore, he generalized N_ϕ as follows

$$N_{\phi,\gamma}(z) = \sum_{w\in\phi^{-1}(z)} \left\{\log\frac{1}{|w|}\right\}^\gamma \qquad (\gamma > 0,\ z \in \mathbb{D}\setminus\{\phi(0)\})$$

and also characterized the compactness of C_ϕ on A^2_α.

Smith [12] used the above generalized counting function to study composition operators C_ϕ acting between different weighted Bergman spaces A^p_α. When $0 < p \le q < \infty$, Smith proved that $C_\phi : A^p_\alpha \to A^q_\beta$ is bounded (or compact) if and only if $N_{\phi,\beta+2}(z) = O((-\log|z|)^{(\alpha+2)q/p})$ (or $o((-\log|z|)^{(\alpha+2)q/p})$) as $|z| \to 1$. Since this result include the cases $A^p_{-1} = H^p$ or $A^q_{-1} = H^q$, he also characterized the boundedness and compactness of C_ϕ acting between the Hardy space and the Bergman space. Pérez-González, Rättyä and Vukotić [7] also considered the same characterization problem for the compactness of $C_\phi : A^p_\alpha \to A^q_\beta$ ($0 < p \le q < \infty$). They proved that the condition $d\mu(z) = N_{\phi,\beta+2}(z)dA(z)$ is a vanishing $2 + q(\alpha+2)/p$-Carleson measure also characterizes the compactness of $C_\phi : A^p_\alpha \to A^q_\beta$.

On the other hand, Smith and Yang [13] considered the case $C_\phi : A^p_\alpha \to A^q_\beta$ when $0 < q < p$. The boundedness of this case has a relation to the Carleson-type embedding theorem. By using a method which is based on Khinchine's inequality and the atomic decomposition of $f \in A^p_\alpha$, they proved that $C_\phi : A^p_\alpha \to A^q_\beta$ ($q < p$) is bounded if and only if the function

$$\mathbb{D} \ni z \mapsto \frac{N_{\phi,\beta+2}(z)}{(1-|z|^2)^{2+\alpha}}$$

belongs to the space $L^{\frac{p}{p-q}}(dA_\alpha)$. They also showed that every bounded composition operator $C_\phi : A^p_\alpha \to A^q_\beta$ ($q < p$) is also compact.

These results suggest a problem that what conditions on ϕ characterize the boundedness and compactness of C_ϕ acting between different weighted Bergman spaces $A^p(\sigma dA)$ with more general weight functions σ. Recently, Constantin [4] studied this problem. Constantin's characterizations are based on Carleson-type measure conditions for the pull-back measure induced by ϕ (see Theorem 2.11 below). Our aim in the present paper is to give another type characterization for the boundedness and compactness of C_ϕ acting between different spaces $A^p(\sigma_1 dA)$ and $A^q(\sigma_2 dA)$. In order to investigate this problem, we shall need the following counting function.

Definition. Let ϕ be an analytic self-map of $\mathbb{D}$ and σ a weight function on $\mathbb{D}$. We define the function $\mathcal{N}_{\phi,\sigma}$ as follows.

$$\mathcal{N}_{\phi,\sigma}(z) = \sum_{w\in\phi^{-1}(z)} \sigma(w) \qquad (z \in \mathbb{D}\setminus\{\phi(0)\}).$$

As in the classical Nevanlinna counting function N_ϕ, we understand that $\mathcal{N}_{\phi,\sigma}(z) = 0$ for $z \notin \phi(\mathbb{D})$ and $w \in \phi^{-1}(z)$ is repeated according to the multiplicity of zeros of $\phi - z$. Conventionally, we consider that $\mathcal{N}_{\phi,\sigma}(z) = 0$ if $z = \phi(0)$. When $\sigma(z) = -\log|z|$, this $\mathcal{N}_{\phi,\sigma}$ coincides with N_ϕ. So we call $\mathcal{N}_{\phi,\sigma}$ a generalized Nevanlinna counting function associated to ϕ and σ.

This generalized Nevanlinna counting function $\mathcal{N}_{\phi,\sigma}$ was first introduced by Kellay and Lefèvre in [8]. They used $\mathcal{N}_{\phi,\sigma}$ to study the compactness of C_ϕ on the weighted Dirichlet-type space $\mathcal{H}_\sigma$ which consists of all analytic functions f on $\mathbb{D}$ such that $\int_{\mathbb{D}} |f'(z)|^2\sigma(z)dA(z) < \infty$. They gave the characterization for the compactness of C_ϕ on the space $\mathcal{H}_\sigma$ by the growth condition of the generalized Nevanlinna counting function $\mathcal{N}_{\phi,\sigma}$. Their results inspired us to study the composition operator on $A^p(\sigma dA)$ and gave suggestions for the method of characterizations for the boundedness and compactness of $C_\phi : A^p(\sigma_1 dA) \to A^p(\sigma_2 dA)$.

The main result of the paper is to characterize the bounded and compact composition operator from $A^p(\sigma_1 dA)$ into $A^q(\sigma_2 dA)$ in terms of the behavior of the above generalized Nevanlinna counting function. In section 3, we will consider the operator $C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ when $0 < p \le q < \infty$. The first result in this section says that the boundedness is characterized by the growth condition of this counting function. The second one is to estimate the essential norm of C_ϕ under some restricted assumptions on p and q. Once this estimate is accomplished, we get as a consequence the characterization of the compactness for $0 < p \le q < \infty$. In section 4, we will investigate the case $0 < q < p < \infty$. Applications of well-known Khinchine's inequality and the Hardy–Littlewood maximal function play an important role in our argument of this section. The result shows that the integrability condition of the generalized counting function characterizes the boundedness of C_ϕ. Furthermore we prove that the bounded composition operator $C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ is also compact when $0 < q < p < \infty$.

Throughout this paper, the notation $a \lesssim b$ means that there exists a positive constant C such that $a \le Cb$. Moreover, if both $a \lesssim b$ and $a \gtrsim b$ hold, then one says that $a \approx b$.

2. Preliminaries

In this section, we introduce an admissible Békollé weight function σ and a composition operator on weighted Bergman spaces $A^p(\sigma dA)$. We shall need some lemmas on the space $A^p(\sigma dA)$ or the weight σ, so we also describe them.

For each $a \in \mathbb{D}$, let $\varphi_a(z)$ be the Möbius transformation of $\mathbb{D}$ interchanging a and 0, that is $\varphi_a(z) = (a - z)/(1 - \overline{a}z)$. Recall that the pseudohyperbolic metric ρ on $\mathbb{D}$ is defined by $\rho(z, a) = |\varphi_a(z)|$, and the pseudohyperbolic disk $E(a, r)$ is the set

$$E(a, r) = \{z \in \mathbb{D} : \rho(z, a) < r\},$$

for $a \in \mathbb{D}$ and $r \in (0, 1)$.

2.1. **Admissible Békollé weight.** For each $\alpha > -1$, let dA_α denote the normalized measure on $\mathbb{D}$ defined by $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$. For $p > 1$ and $\alpha > -1$, the class $B_p(\alpha)$ consists of all weight functions σ with the property that there is a constant $C > 0$ such that for every $S(a) = \{\varphi_a(z) : \mathrm{Re}(z\overline{a}) \le 0\}$, $a \in \mathbb{D}$,

$$\left(\int_{S(a)} \sigma dA_\alpha\right) \cdot \left(\int_{S(a)} \sigma^{-\frac{p'}{p}} dA_\alpha\right)^{\frac{p}{p'}} \le C\{A_\alpha(S(a))\}^p,$$

where p' is the conjugate exponent of p. Note that we put $S(0) = \mathbb{D}$.

Békollé [2] proved that this condition characterizes the boundedness of the Bergman projection P_α defined by

$$P_\alpha f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - \overline{w}z)^{\alpha+2}} dA_\alpha(w)$$

on $L^p(\sigma dA)$.

Theorem 2.1 ([2] Békollé)**.** *Let $1 < p < \infty$ and $\alpha > -1$. For a weight function σ, the following conditions are equivalent:*

(i) *P_α is a bounded projection from $L^p(\sigma dA)$ onto $A^p(\sigma dA)$.*

(ii) *The sublinear operator $\widetilde{P_\alpha}$ defined by*

$$\widetilde{P_\alpha} f(z) = \int_{\mathbb{D}} \frac{|f(w)|}{|1 - \overline{w}z|^{\alpha+2}} dA_\alpha(w)$$

is bounded on $L^p(\sigma dA)$.

(iii) *The z-variable function $\sigma(z)/(1 - |z|^2)^\alpha$ belongs to $B_p(\alpha)$.*

This result is very useful to study on the space $A^p(\sigma dA)$. For instance, Luecking [9] showed the following dual relation of $A^p(\sigma dA)$.

Theorem 2.2 ([9] Luecking)**.** *Let $1 < p < \infty$ and $\alpha > -1$. If a weight function σ satisfies the condition $\sigma(z)/(1-|z|^2)^\alpha \in B_p(\alpha)$, then the dual space of $A^p(\sigma dA)$ can be identified with $A^{p'}(\sigma^{-\frac{p'}{p}} dA_{\alpha p'})$ under the integral pairing*

$$\langle f, g\rangle_\alpha = \int_{\mathbb{D}} f(z)\overline{g(z)} dA_\alpha(z), \tag{2.1}$$

where p' is the conjugate exponent of p.

As an application of Theorem 2.2, we have the following lemma.

Lemma 2.3. *Suppose that $1 < p < \infty$, $\alpha > -1$ and a weight function σ satisfies $\sigma(z)/(1 - |z|^2)^\alpha \in B_p(\alpha)$. If a bounded sequence $\{f_j\}$ in $A^p(\sigma dA)$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, then $\{f_j\}$ also converges to 0 weakly in $A^p(\sigma dA)$.*

Proof. Take a function $h \in A^{p'}(\sigma^{-\frac{p'}{p}} dA_{\alpha p'})$ where p' is the conjugate exponent of p. For any $\epsilon > 0$, there is a continuous function g with compact support such that

$$\left[\int_{\mathbb{D}} |\overline{h(z)} - g(z)|^{p'} \sigma(z)^{-\frac{p'}{p}} (1 - |z|^2)^{\alpha p'} dA(z) \right]^{\frac{1}{p'}} < \epsilon. \tag{2.2}$$

By the integral pairing (2.1) in Theorem 2.2, we have that

$$\begin{aligned} |\langle f_j, h \rangle_\alpha| &= \left| \int_{\mathbb{D}} \{ f_j(z)\overline{h(z)} - f_j(z)g(z) + f_j(z)g(z) \} dA_\alpha(z) \right| \\ &\le \int_{\mathbb{D}} |f_j(z)||\overline{h(z)} - g(z)| dA_\alpha(z) + \int_{\mathbb{D}} |f_j(z)||g(z)| dA_\alpha(z) \\ &= (I) + (II). \end{aligned} \tag{2.3}$$

By applying the Hölder's inequality to (I), (2.2) gives that

$$\begin{aligned} (I) &\le \left[\int_{\mathbb{D}} |f_j|^p \sigma dA \right]^{\frac{1}{p}} \cdot \left[\int_{\mathbb{D}} |\overline{h(z)} - g(z)|^{p'} \sigma(z)^{-\frac{p'}{p}} (1 - |z|^2)^{\alpha p'} dA(z) \right]^{\frac{1}{p'}} \\ &\le \epsilon \cdot \|f_j\|_\sigma, \end{aligned} \tag{2.4}$$

for any $j \ge 1$.

On the other hand, since g has a compact support $\mathrm{supp}(g)$ and $f_j \to 0$ uniformly on $\mathrm{supp}(g)$ as $j \to \infty$, we see that

$$(II) = \int_{\mathrm{supp}(g)} |f_j(z)||g(z)| dA_\alpha(z) \to 0 \quad (j \to \infty). \tag{2.5}$$

Thus (2.3), (2.4) and (2.5) show that

$$\limsup_{j \to \infty} |\langle f_j, h \rangle_\alpha| \le \sup_{j \ge 1} \|f_j\|_\sigma \, \epsilon,$$

for each $h \in A^{p'}(\sigma^{-\frac{p'}{p}} dA_{\alpha p'})$. Since $\{f_j\}$ is bounded and $\epsilon > 0$ is arbitrarily, this implies that $\{f_j\}$ converges to 0 weakly in $A^p(\sigma dA)$. □

In this paper, σ denotes a non-negative continuous function on $[0, 1)$ such that $\sigma(r) \le 1$ for $r \in [0, 1)$. For $z \in \mathbb{D}$ we write $\sigma(z) = \sigma(|z|)$ and call such σ a weight function on $\mathbb{D}$. Our arguments in proofs of main results are based on the above results and require some growth conditions on σ, so we will consider the following conditions.

Definition. A weight function σ is called *an admissible Békollé weight* if σ satisfies

(W_1) $\dfrac{\sigma(z)}{(1 - |z|^2)^\alpha} \in B_{p_0}(\alpha)$ for some $p_0 > 1$ and $\alpha > -1$,
(W_2) σ is non-increasing on $[0, 1)$,
(W_3) $\sigma(r)/(1 - r^2)^{1+\delta}$ is non-decreasing on $[0, 1)$ for some $\delta > 0$.

Lemma 2.4. *Let $p > 0$ and σ be an admissible Békollé weight function. Then for each $f \in A^p(\sigma dA)$,*

(i) $$|f(z)| \lesssim \frac{\|f\|_\sigma}{\sigma(z)^{1/p}(1-|z|^2)^{2/p}},$$

(ii) $$|f'(z)| \lesssim \frac{\|f\|_\sigma}{\sigma(z)^{1/p}(1-|z|^2)^{1+2/p}}.$$

Proof. Since $\sigma(z)/(1-|z|^2)^\alpha \in B_{p_0}(\alpha)$, it follows from Lemma (3.1) in [9] that

$$|f(z)| \lesssim \left(\int_{E(z,r)} \sigma(w)dA(w)\right)^{-\frac{1}{p}} \|f\|_\sigma \tag{2.6}$$

for $r \in (0,1)$ and $z \in \mathbb{D}$. For $w \in E(z,r)$, we may assume that $|z| \le |w|$ without loss of generality. Since σ is non-increasing, we have that

$$\sigma(w) \le \sigma(z).$$

Since $\sigma(r)/(1-r^2)^{1+\delta}$ is non-decreasing for some $\delta > 0$, on the other hand, we have that

$$(1-|z|^2)^{1+\delta}\sigma(w) \ge (1-|w|^2)^{1+\delta}\sigma(z).$$

Combining this with the relation $1-|z|^2 \approx 1-|w|^2$ for $w \in E(z,r)$, we obtain that

$$(1-|z|^2)^{1+\delta}\sigma(w) \gtrsim (1-|z|^2)^{1+\delta}\sigma(z),$$

and so $\sigma(w) \gtrsim \sigma(z)$. These imply that $\sigma(w) \approx \sigma(z)$ for $w \in E(z,r)$. By (2.6), we have that

$$|f(z)| \lesssim \{\sigma(z)A(E(z,r))\}^{-\frac{1}{p}} \|f\|_\sigma \approx \left\{\sigma(z)(1-|z|^2)^2\right\}^{-\frac{1}{p}} \|f\|_\sigma,$$

and so we get the first estimate (i).

Furthermore, the subharmonicity of $|f'|^p$ gives that

$$|f'(z)|^p \lesssim \frac{1}{(1-|z|^2)^2}\int_{E(z,r)} |f'(w)|^p dA(w).$$

By noting that $\sigma(w) \approx \sigma(z)$ and $1-|z|^2 \approx 1-|w|^2$ for $w \in E(z,r)$, we have that

$$|f'(z)|^p \lesssim \frac{1}{\sigma(z)(1-|z|^2)^{2+p}}\int_{E(z,r)} |f'(w)|^p(1-|w|^2)^p\sigma(w)dA(w). \tag{2.7}$$

From [1, Theorem 3.1], we see that

$$\|f\|_\sigma^p \approx |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^p\sigma(z)dA(z). \tag{2.8}$$

By applying (2.8) to the last integral in (2.7), we obtain the second estimate (ii). □

In the above proof, we needed an equivalent norm (2.8) for $\|f\|_\sigma$. We also have another formula for $\|f\|_\sigma$. Note that the following lemma holds for any weight function σ without conditions $(W_1) \sim (W_3)$.

Lemma 2.5. *Let $p > 0$ and σ be a weight function. Then it holds that*

$$\|f\|_\sigma^p \approx |f(0)|^p + \int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2 \left\{\int_{|z|}^1 \left(\log\frac{r}{|z|}\right)\sigma(r)rdr\right\} dA(z),$$

for $f \in H(\mathbb{D})$.

Proof. Recall that if g is in the Hardy space H^p, then $|g|^p$ has the least harmonic majorant and it is equal to the Poisson integral

$$P[|g^*|^p](z) = \int_0^{2\pi} P(z, e^{i\theta})|g^*(e^{i\theta})|\frac{d\theta}{2\pi},$$

where $P(z, e^{i\theta}) = \mathrm{Re}\frac{e^{i\theta}+z}{e^{i\theta}-z}$ is the Poisson kernel for $\mathbb{D}$ and g^* is the radial limit of g. Combining this with the Riesz Decomposition theorem, we have that

$$|g(z)|^p = \int_0^{2\pi} P(z, e^{i\theta})|g^*(e^{i\theta})|^p\frac{d\theta}{2\pi} - \int_{\mathbb{D}} \log\left|\frac{1-\overline{w}z}{w-z}\right| d\mu_{|g|^p}(w).$$

Here $d\mu_{|g|^p}$ denotes the Riesz measure of $|g|^p$. In particular, we put $z = 0$, then we obtain that

$$\|g\|_{H^p}^p = |g(0)|^p + \int_{\mathbb{D}} \log\frac{1}{|w|} d\mu_{|g|^p}(w).$$

Since it is known that the Riesz measure of $|g|^p$ (see [14, p.1035 (3.2)]) is given by

$$d\mu_{|g|^p}(w) = p^2|g(w)|^{p-2}|g'(w)|^2 dA(w),$$

we have that

$$\|g\|_{H^p}^p = |g(0)|^p + p^2\int_{\mathbb{D}} |g(w)|^{p-2}|g'(w)|^2 \log\frac{1}{|w|} dA(w), \tag{2.9}$$

for $g \in H^p$.

Now we take an $f \in H(\mathbb{D})$ and $r \in (0, 1)$. Since the dilated function $f_r(z) = f(rz)$ is analytic in $\mathbb{D}$ and continuous on the closure of $\mathbb{D}$, (2.9) gives that

$$\begin{aligned}\int_0^{2\pi} |f(re^{i\theta})|^p\frac{d\theta}{2\pi} &= |f(0)|^p + p^2\int_{\mathbb{D}} |f(rw)|^{p-2}|f'(rw)|^2 r^2 \log\frac{1}{|w|} dA(w)\\ &= |f(0)|^p + p^2\int_{r\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2 \log\frac{r}{|z|} dA(z).\end{aligned}$$

Multiplying the above formula by $2r\sigma(r)$, integrating with respect to r from 0 to 1 and applying Fubini's theorem, we get

$$\begin{aligned}\|f\|_\sigma^p &= 2|f(0)|^p\int_0^1 r\sigma(r)dr\\ &\quad + 2p^2\int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2 \left\{\int_{|z|}^1 \left(\log\frac{r}{|z|}\right)\sigma(r)rdr\right\} dA(z),\end{aligned}$$

which completes the proof. □

In order to formulate our results, we need to introduce another weight function. For each weight σ, we put

$$\omega_\sigma(z) = \int_{|z|}^1 (t-|z|)\sigma(t)dt \qquad (z \in \mathbb{D}).$$

Then we see that ω_σ is non-increasing, convex and $\omega_\sigma(z) \to 0$ as $|z| \to 1$. Furthermore ω_σ has the following property.

Lemma 2.6. *If σ is an admissible Békollé weight function, then it holds that*

$$\omega_\sigma(r) \approx (1-r^2)^2\sigma(r)$$

for every $r \in [0,1)$.

Proof. Since σ is non-increasing, we have that

$$\omega_\sigma(r) = \int_r^1 (t-r)\sigma(t)dt \le \frac{1}{2}\sigma(r)(1-r^2)^2.$$

Since $\sigma(r)/(1-r^2)^{1+\delta}$ is non-decreasing for some $\delta > 0$, we have that

$$\begin{aligned}\omega_\sigma(r) &\ge \frac{\sigma(r)}{(1-r^2)^{1+\delta}} \int_r^1 (t-r)(1-t)^{1+\delta}dt \\ &\ge \frac{1}{2^{3+\delta}(2+\delta)(3+\delta)}\sigma(r)(1-r^2)^2.\end{aligned}$$

We accomplish the proof. □

2.2. **Composition operators on $A^p(\sigma dA)$.** Now we show that each Möbius transformations φ_a always induce a bounded composition operator on $A^p(\sigma dA)$. This property ensures that we may consider the operator C_ϕ under the assumption $\phi(0) = 0$.

Proposition 2.7. *Let $p > 0$, $p_0 > 1$ and $\alpha > -1$. Suppose that σ is an admissible Békollé weight function. For each $a \in \mathbb{D}$, C_{φ_a} is a bounded composition operator on $A^p(\sigma dA)$.*

Proof. Equation (2.8) and the change of variables formula give that

$$\|C_{\varphi_a} f\|_\sigma^p \approx |f(a)|^p + \int_{\mathbb{D}} |f'(z)|^p |\varphi_a'(\varphi_a(z))|^p (1-|\varphi_a(z)|^2)^p \sigma(\varphi_a(z)) J_{\varphi_a}(z) dA(z),$$

where $J_{\varphi_a}(z)$ denotes the real Jacobian of φ_a at z. By straightforward calculations we have that

$$|\varphi_a'(\varphi_a(z))|^p (1-|\varphi_a(z)|^2)^{p-2} J_{\varphi_a}(z) = (1-|z|^2)^{p-2}.$$

Lemma 2.6 shows that

$$(1-|\varphi_a(z)|^2)^2\sigma(\varphi_a(z)) \approx \omega_\sigma(\varphi_a(z)) \quad \text{and} \quad \omega_\sigma(z) \approx (1-|z|^2)^2\sigma(z).$$

Since it follows from [8, Lemma 2.1] that

$$\omega_\sigma(z) \approx \omega_\sigma(\varphi_a(z)) \quad (z \in \mathbb{D}), \tag{2.10}$$

by an application of equation (2.8) once again, we obtain that

$$\|C_{\varphi_a} f\|_\sigma^p \lesssim |f(a)|^p + \|f\|_\sigma^p,$$

for each $f \in A^p(\sigma dA)$. This implies that $C_{\varphi_a}(A^p(\sigma dA)) \subset A^p(\sigma dA)$. The closed graph theorem shows that C_{φ_a} is bounded on $A^p(\sigma dA)$. □

The following result is an immediate consequence of Proposition 2.7.

Corollary 2.8. *Let $p > 0$, $p_0 > 1$ and $\alpha > -1$. Suppose that σ is an admissible Békollé weight function. Then any analytic self-map ϕ of $\mathbb{D}$ induces a bounded composition operator C_ϕ on $A^p(\sigma dA)$.*

The following change of variables formula help us in the arguments in our main results.

Lemma 2.9. *Let $p > 0$, ϕ be an analytic self-map of $\mathbb{D}$ and σ a weight function. Then it holds that*

$$\|f \circ \phi\|_\sigma^p \approx |f(\phi(0))|^p + \int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2 \left\{ \int_0^1 N_\phi(r,z)\sigma(r) r dr \right\} dA(z),$$

for $f \in H(\mathbb{D})$. Here $N_\phi(r,z)$ denotes the partial counting function for ϕ defined by

$$N_\phi(r,z) = \sum_{w \in \phi^{-1}(z),\, |w| \le r} \log \frac{r}{|w|} \qquad (z \in \mathbb{D} \setminus \{\phi(0)\},\, r \in (0,1)).$$

Proof. Recall Stanton's formula ([5, Theorem 2]) for integral means of subharmonic functions on $\mathbb{D}$. If u is a positive subharmonic function on $\mathbb{D}$ and ϕ is an analytic self-map of $\mathbb{D}$, then it holds that

$$\int_0^{2\pi} u(\phi(re^{i\theta}))\, \frac{d\theta}{2\pi} = u(\phi(0)) + \int_{r\mathbb{D}} N_\phi(r,z)\, d\mu_u(z) \qquad (r \in (0,1)),$$

where $d\mu_u$ is the Riesz measure of u. Applying this formula to the positive subharmonic function $z \mapsto |f(z)|^p$, we obtain the desired formula. □

Next we formulate the following sub-mean value property for the generalized counting function $\mathcal{N}_{\phi,\omega_\sigma}$. We will need this property in the proofs of our results.

Lemma 2.10. *Let $t \in (0,1)$ be fixed and σ a weight function. For any analytic self-map ϕ of $\mathbb{D}$ with $\phi(0) = 0$, it holds that*

$$\mathcal{N}_{\phi,\omega_\sigma}(z) \lesssim \frac{1}{t^2(1-|z|^2)^2} \int_{E(z,t)} \mathcal{N}_{\phi,\omega_\sigma}(w)\, dA(w) \qquad (t < |z| < 1).$$

Proof. Fix $z \in \mathbb{D} \setminus t\overline{\mathbb{D}}$ and $r \in (0,1)$. Since $N_\phi(r, \varphi_z(\cdot))$ is subharmonic on $\mathbb{D} \setminus \{\varphi_z{}^{-1}(0)\} = \mathbb{D} \setminus \{a\}$ and $t\overline{\mathbb{D}} \subset \mathbb{D} \setminus \{a\}$, we have that

$$N_\phi(r,z) = N_\phi(r, \varphi_z(0)) \lesssim \frac{1}{t^2} \int_{t\overline{\mathbb{D}}} N_\phi(r, \varphi_z(w)) dA(w).$$

By the change of variables, we have that

$$\int_{t\overline{\mathbb{D}}} N_\phi(r, \varphi_z(w)) dA(w) = \int_{E(z,t)} N_\phi(r,w) \frac{(1-|z|^2)^2}{|1-\overline{z}w|^4} dA(w).$$

Since $|1-\overline{z}w|^4 \approx (1-|z|^2)^4$ for $w \in E(z,t)$, we obtain that

$$N_\phi(r,z) \lesssim \frac{1}{t^2(1-|z|^2)^2} \int_{E(z,t)} N_\phi(r,w) dA(w)$$

for $r \in (0,1)$. Multiplying the above inequality by $\sigma(r)$ and integrating with respect to r from 0 to 1, we get

$$\int_0^1 N_\phi(r,z)\sigma(r) dr \lesssim \frac{1}{t^2(1-|z|^2)^2} \int_{E(z,t)} dA(w) \int_0^1 N_\phi(r,w)\sigma(r) dr. \qquad (2.11)$$

Now we fix $w \in E(z,t)$. Since $t < |z| < 1$, we see that $0 \notin \overline{E(z,t)}$, and so $c := \inf\{|v| : v \in \overline{E(z,t)}\} > 0$ where $\overline{E(z,t)} = \{w \in \mathbb{D} : |\rho(w,z)| \leq t\}$. Since $\phi(0) = 0$, Schwarz's lemma shows that each $u \in \mathbb{D}$ with $w = \phi(u)$ satisfies $c \leq |w| \leq |u|$. Thus we have the following inequalities

$$\log \frac{r}{|u|} \leq \frac{1}{|u|}(r - |u|) \leq \frac{1}{c}(r - |u|)$$

for $|u| < r < 1$. These give that

$$\int_0^1 N_\phi(r,w)\sigma(r)dr = \sum_{w=\phi(u)} \int_{|u|}^1 \log \frac{r}{|u|}\sigma(r)dr \leq \frac{1}{c}\mathcal{N}_{\phi,\omega_\sigma}(w). \tag{2.12}$$

On the other hand, the inequality $r - |u| < \log \frac{r}{|u|}$ for $|u| < r < 1$ gives that

$$\mathcal{N}_{\phi,\omega_\sigma}(z) \leq \int_0^1 N_\phi(r,z)\sigma(r)dr. \tag{2.13}$$

By inequalities (2.11) $\sim$ (2.13), we obtain the desired inequality. □

At the end of this section, we quote the results on characterizations for the boundedness and compactness of $C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ by O. Constantin.

Recently, Constantin [3, 4] obtained some properties of $A^p(\sigma dA)$ under condition (W_1). In [4], Constantin proved the Carleson-type embedding theorem for $A^p(\sigma dA)$. For a given weight σ and an analytic self-map ϕ of $\mathbb{D}$, we define a positive Borel measure μ by

$$\mu(E) = \int_{\phi^{-1}(E)} \sigma(z)dA(z)$$

for any Borel set E of $\mathbb{D}$. Since it holds that

$$\|C_\phi f\|_\sigma^p = \int_{\mathbb{D}} |f \circ \phi|^p \, \sigma dA = \int_{\mathbb{D}} |f|^p d\mu,$$

Constantin's Carleson-type embedding theorem [4, Theorems 3.1∼3.3] indicate the following results.

Theorem 2.11 ([4] Constantin)**.** *Let $p_0 > 1$, $\alpha > -1$ and σ_j $(j = 1,2)$ be weight functions. Suppose that $\sigma_1(z)/(1-|z|^2)^\alpha \in B_{p_0}(\alpha)$. For $\lambda \in \mathbb{D}$ and $r \in (0,1)$, let $D_{\lambda,r}$ denote the disk $\{z \in \mathbb{D} : |z - \lambda| < r(1-|\lambda|)\}$. For any analytic self-map ϕ of $\mathbb{D}$, the followings hold.*

(i) *For $0 < p \leq q < \infty$, $C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ is bounded if and only if*

$$\int_{\phi^{-1}(D_{\lambda,r})} \sigma_2 \, dA = O\left(\left\{\int_{D_{\lambda,r}} \sigma_1 \, dA\right\}^{q/p}\right) \quad (|\lambda| \to 1),$$

and $C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ is compact if and only if

$$\int_{\phi^{-1}(D_{\lambda,r})} \sigma_2 \, dA = o\left(\left\{\int_{D_{\lambda,r}} \sigma_1 \, dA\right\}^{q/p}\right) \quad (|\lambda| \to 1),$$

for some $r \in (0,1)$.

(ii) *For* $0 < q < p < \infty$, $C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ *is bounded if and only if*

$$\mathbb{D} \ni \lambda \mapsto \frac{\int_{\phi^{-1}(D_{\lambda,r})} \sigma_2 \, dA}{\int_{D_{\lambda,r}} \sigma_1 \, dA}$$

belongs to $L^{p/(p-q)}(\sigma_1 dA)$ *for some* $r \in (0,1)$. *In this case* C_ϕ *is also compact.*

Remark 2.12. In the above theorem, we see from the proof of it that the choice of $\{p, q\}$ is independent of p_0.

These results have the following corollary. It plays an important role in the proof of our results below, but its proof is very easy. Thus we state the result without the proof.

Corollary 2.13. *Let* m *be a positive integer. Under the same assumptions in Theorem 2.11,* $C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ *is bounded (or compact) if and only if* $C_\phi : A^{mp}(\sigma_1 dA) \to A^{mq}(\sigma_2 dA)$ *is bounded (or compact, respectively).*

3. The case $0 < p \leq q < \infty$

Theorem 3.1. *Let* σ_1 *be an admissible Békollé weight function,* σ_2 *a weight function and* $0 < p \leq q < \infty$. *For any analytic self-map* ϕ *of* $\mathbb{D}$, $C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ *is bounded if and only if*

$$\mathcal{N}_{\phi,\omega_{\sigma_2}}(z) = O(\omega_{\sigma_1}(z)^{\frac{q}{p}}) \qquad (|z| \to 1). \tag{3.1}$$

Proof. First suppose that (3.1) holds and consider the case $q \geq 2$ and $\phi(0) = 0$. By condition (3.1), we can choose a constant $K > 0$ and $r_0 \in [1/2, 1)$ such that

$$\mathcal{N}_{\phi,\omega_{\sigma_2}}(z) \leq K \omega_{\sigma_1}(z)^{\frac{q}{p}}, \qquad z \in \mathbb{D} \setminus r_0 \overline{\mathbb{D}}. \tag{3.2}$$

For fixed $f \in A^p(\sigma_1 dA)$, by Lemma 2.9, we have that

$$\|C_\phi f\|_{\sigma_2}^q \lesssim |f(0)|^q + \int_{\mathbb{D}} |f(z)|^{q-2} |f'(z)|^2 \left\{ \int_0^1 N_\phi(r, z) \sigma_2(r) r dr \right\} dA(z). \tag{3.3}$$

Put

$$I_1(f) = \int_{r_0 \overline{\mathbb{D}}} |f(z)|^{q-2} |f'(z)|^2 \left\{ \int_0^1 N_\phi(r, z) \sigma_2(r) r dr \right\} dA(z),$$

$$I_2(f) = \int_{\mathbb{D} \setminus r_0 \overline{\mathbb{D}}} |f(z)|^{q-2} |f'(z)|^2 \left\{ \int_0^1 N_\phi(r, z) \sigma_2(r) r dr \right\} dA(z).$$

By noting our assumption $q - 2 \geq 0$, Lemmas 2.4 (i) and 2.6 give that

$$|f(z)|^{q-2} \lesssim \frac{\|f\|_{\sigma_1}^{q-2}}{\{\sigma_1(z)(1-|z|^2)^2\}^{\frac{q-2}{p}}} \approx \frac{\|f\|_{\sigma_1}^{q-2}}{\omega_{\sigma_1}(z)^{\frac{q-2}{p}}}. \tag{3.4}$$

By Lemmas 2.4 (ii) and 2.6, we have that

$$|f'(z)| \lesssim \frac{\|f\|_{\sigma_1}^2}{\{\sigma_1(z)(1-|z|^2)^2\}^{\frac{2}{p}}(1-|z|^2)^2} \approx \frac{\|f\|_{\sigma_1}^2}{\omega_{\sigma_1}(z)^{\frac{2}{p}}(1-|z|^2)^2}. \tag{3.5}$$

Hence it follows from (3.4) and (3.5) that

$$I_1(f) \lesssim \max_{|z|\le r_0} \frac{1}{\omega_{\sigma_1}(z)^{\frac{q}{p}}(1-|z|^2)^2}\|f\|_{\sigma_1}^q \int_{r_0\overline{\mathbb{D}}} \left\{ \int_0^1 N_\phi(r,z)\sigma_2(r)rdr \right\} dA(z). \quad (3.6)$$

Now we consider the function $g(z) = z+1$. Then $\|C_\phi g\|_{\sigma_2}^q = \|\phi+1\|_{\sigma_2}^q \le 2^q$. By an application of Lemma 2.9 to g, we have that

$$\int_{r_0\overline{\mathbb{D}}} \left\{ \int_0^1 N_\phi(r,z)\sigma_2(r)rdr \right\} dA(z) \lesssim 2^q.$$

Combining this with (3.6), we have that

$$I_1(f) \lesssim \max_{|z|\le r_0} \frac{2^q}{\omega_{\sigma_1}(z)^{\frac{q}{p}}(1-|z|^2)^2}\|f\|_{\sigma_1}^q. \quad (3.7)$$

Fix $z \in \mathbb{D}\setminus r_0\overline{\mathbb{D}}$. As in inequality (2.12), we obtain that

$$\int_0^1 N_\phi(r,z)\sigma_2(r)dr \le \frac{1}{r_0}\mathcal{N}_{\phi,\omega_{\sigma_2}}(z).$$

Combining this with (3.2), we have that

$$I_2(f) \lesssim K \int_{\mathbb{D}\setminus r_0\overline{\mathbb{D}}} |f(z)|^{q-2}|f'(z)|^2\omega_{\sigma_1}(z)^{\frac{q}{p}}dA(z).$$

By Lemma 2.4 and 2.6, we have that

$$|f(z)|^{q-2} \lesssim \frac{\|f\|_{\sigma_1}^{q-p}}{\omega_{\sigma_1}(z)^{\frac{q-p}{p}}}|f(z)|^{p-2}.$$

So we obtain that

$$I_2(f) \lesssim K\|f\|_{\sigma_1}^{q-p} \int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2\omega_{\sigma_1}(z)dA(z).$$

Since it holds that

$$\omega_{\sigma_1}(z) = \int_{|z|}^1 \frac{1}{r}(r-|z|)\sigma_1(r)rdr \le \int_{|z|}^1 \left(\log\frac{r}{|z|}\right)\sigma_1(r)rdr, \quad (3.8)$$

Lemma 2.5 gives that

$$\int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2\omega_{\sigma_1}(z)dA(z) \lesssim \|f\|_{\sigma_1}^p,$$

and so we obtain that $I_2(f) \lesssim K\|f\|_{\sigma_1}^q$. Combining this with (3.3) and (3.7), we see that $C_\phi(A^p(\sigma_1 dA)) \subset A^q(\sigma_2 dA)$, that is $C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ is bounded by the closed graph theorem.

When the case $0 < q < 2$, we choose a positive integer m such that $mq \ge 2$. Since the condition (3.1) implies that

$$\mathcal{N}_{\phi,\omega_{\sigma_2}}(z) = O(\omega_{\sigma_1}(z)^{\frac{mq}{mp}}) \qquad (|z|\to 1),$$

the above arguments show that $C_\phi : A^{mp}(\sigma_1 dA) \to A^{mq}(\sigma_2 dA)$ is bounded. Thus it follows from Corollary 2.13 that $C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ is also bounded.

For the case $\phi(0) \neq 0$, we may consider the composition map $\Phi = \varphi_{\phi(0)} \circ \phi$. Then $\Phi(0) = 0$. Equation (2.10) shows that

$$\frac{\mathcal{N}_{\Phi,\omega_{\sigma_2}}(z)}{\omega_{\sigma_1}(z)^{q/p}} = \frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}(\varphi_{\phi(0)}(z))}{\omega_{\sigma_1}(z)^{q/p}} \approx \frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}(\varphi_{\phi(0)}(z))}{\omega_{\sigma_1}(\varphi_{\phi(0)}(z))^{q/p}},$$

for any $z \in \mathbb{D}$. Since $|\varphi_{\phi(0)}(z)| \to 1$ as $|z| \to 1$, we see that Φ also satisfies the condition (3.1), and so $C_\Phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ is bounded. By Proposition 2.7, $C_\phi = C_\Phi C_{\varphi_{\phi(0)}}$ is also bounded from $A^p(\sigma_1 dA)$ into $A^q(\sigma_2 dA)$.

Finally, we prove that (3.1) is a necessary condition for the boundedness of $C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$. Fix $z \in \mathbb{D}$ with $|z| > 1/3$ and put

$$f_z(w) = \frac{(1-|z|^2)^{\alpha+2-2/p}}{\sigma_1(z)^{\frac{1}{p}}(1-\overline{z}w)^{\alpha+2}}, \quad w \in \mathbb{D}.$$

By [4, Lemma 2.1] (see also [3, Lemma 3.1]), it holds that

$$\int_{\mathbb{D}} \frac{1}{|1-\overline{z}w|^{p(\alpha+2)}} \sigma_1(w) dA(w) \approx \frac{\sigma_1(z)}{(1-|z|^2)^{p(\alpha+2)-2}},$$

for $z \in \mathbb{D}$. So this implies that $f_z \in A^p(\sigma_1 dA)$ and $\|f_z\|_{\sigma_1} \lesssim 1$. It follows from Lemma 2.9 that

$$\begin{aligned} \|C_\phi f_z\|^q_{\sigma_2} &\gtrsim \int_{\mathbb{D}} |f_z(w)|^{q-2} |f_z'(w)|^2 \left\{ \int_0^1 N_\phi(r,w)\sigma_2(r) r dr \right\} dA(w) \\ &\geq \int_{\mathbb{D}} |f_z(w)|^{q-2}|f_z'(w)|^2 \mathcal{N}_{\phi,\omega_{\sigma_2}}(w) dA(w) \\ &\geq \int_{E(z,\frac{1-|z|}{2})} |f_z(w)|^{q-2}|f_z'(w)|^2 \mathcal{N}_{\phi,\omega_{\sigma_2}}(w) dA(w). \end{aligned} \tag{3.9}$$

Since $|1-\overline{z}w| \approx 1-|z|^2$ for $w \in E(z, \frac{1-|z|}{2})$, Lemma 2.6 gives

$$|f_z(w)|^{q-2}|f_z'(w)|^2 \approx \frac{(\alpha+2)^2|z|^2}{\omega_{\sigma_1}(z)^{\frac{q}{p}}(1-|z|^2)^2}.$$

Combining this with (3.9), we have that

$$\frac{1}{\omega_{\sigma_1}(z)^{\frac{q}{p}}} \frac{1}{(1-|z|^2)^2} \int_{E(z,\frac{1-|z|}{2})} \mathcal{N}_{\phi,\omega_{\sigma_2}}(w) dA(w) \lesssim 1, \tag{3.10}$$

for any $z \in \mathbb{D}$ with $|z| > 1/3$. Since it holds that $|z| > \frac{1-|z|}{2}$ for $|z| > 1/3$, Lemma 2.10 and (3.10), we obtain that

$$\frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}(z)}{\omega_{\sigma_1}(z)^{\frac{q}{p}}} \lesssim 1, \quad \text{for } |z| > 1/3.$$

This implies that (3.1) is true. We accomplish the proof. □

Next we will show that the compactness of $C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ is characterized by the following condition

$$N_{\phi,\omega_{\sigma_2}}(z) = o(\omega_{\sigma_1}(z)^{\frac{q}{p}}) \qquad (|z| \to 1).$$

To prove this, we estimate the essential norm $\|C_\phi\|_e$ for the case $p \ge p_0$, $q \ge 2$ and $\phi(0) = 0$. Since the essential norm $\|C_\phi\|_e$ is defined to be the distance from C_ϕ to the closed ideal of compact operators, that is

$$\|C_\phi\|_e = \inf\{\|C_\phi - K\| \,:\, K \text{ is compact from } A^p(\sigma_1 dA) \text{ into } A^q(\sigma_2 dA)\},$$

where $\|C_\phi - K\|$ denotes the operator norm of $C_\phi - K$, the compactness of C_ϕ is characterized by the condition $\|C_\phi\|_e = 0$. Hence our object is to estimate $\|C_\phi\|_e$ in terms of $N_{\phi,\omega_{\sigma_2}}(z)/\omega_{\sigma_1}(z)^{q/p}$. For our aim, we need some preliminary results.

For the Taylor series expansion of $f \in H(\mathbb{D})$ and any integer $n \ge 1$ we put

$$R_n f(z) = \sum_{k=n}^{\infty} a_k z^k, \qquad z \in \mathbb{D},$$

and $K_n = I - R_n$ where $If = f$ is the identity operator. The following Lemma 3.2 and Corollary 3.3 hold for any weight function σ.

Lemma 3.2. *If $1 < p < \infty$ and $f \in A^p(\sigma dA)$, then $\|K_n f - f\|_\sigma \to 0$ as $n \to \infty$.*

Proof. For $f \in A^p(\sigma dA)$ and $r \in (0,1)$, the dilated function f_r is in the Hardy space H^p. Since $1 < p < \infty$, Proposition 1 and Corollary 3 in [15] imply that there exists a constant $C > 0$ such that

$$\int_0^{2\pi} |K_n(f_r)(e^{i\theta})|^p \frac{d\theta}{2\pi} \le C \int_0^{2\pi} |f_r(e^{i\theta})|^p \frac{d\theta}{2\pi},$$

for all $r \in (0,1)$ and $n \ge 1$. Multiplying both sides by $2r\sigma(r)$ and integrating with respect to r from 0 to 1 give $\|K_n f\|_\sigma^p \le C\|f\|_\sigma^p$, and so $\sup\{\|K_n\| : n \ge 1\} < \infty$. By applying [15, Proposition 1] once again, we have that $\|K_n f - f\|_\sigma \to 0$ as $n \to \infty$ for each $f \in A^p(\sigma dA)$. □

Corollary 3.3. *If $1 < p < \infty$, then $\|R_n f\|_\sigma \to 0$ as $n \to \infty$ for each $f \in A^p(\sigma dA)$. Moreover, $\sup\{\|R_n\| : n \ge 1\} < \infty$.*

Proof. The second assertion of this corollary is verified by the principle of uniform boundedness. So we omit the details of the proof. □

Lemma 3.4. *Suppose that $1 < p < \infty$ with $p \ge p_0$, $\alpha > -1$ and σ is the weight function satisfied $\sigma(z)/(1-|z|^2)^\alpha \in B_{p_0}(\alpha)$. For each $f \in H(\mathbb{D})$, it holds that*

$$|R_n f(z)| \lesssim \frac{\|f\|_\sigma}{\left(\int_{\mathbb{D}} \sigma dA\right)^{1/p}} \sum_{j=n}^{\infty} \frac{\Gamma(j+\alpha+2)}{j!\Gamma(\alpha+2)} |z|^j,$$

for all $z \in \mathbb{D}$ and $n \ge 1$. Here $\Gamma(x)$ is the well-known Gamma function.

Proof. First note that the assumption $p \ge p_0$ implies that $\sigma(z)/(1-|z|^2)^\alpha \in B_p(\alpha)$ by Hölder's inequality. Since it follows from Theorem 2.1 that the Bergman projection P_α is bounded from $L^p(\sigma dA)$ onto $A^p(\sigma dA)$, we have that

$$R_n f(z) = P_\alpha(R_n f)(z) = \int_{\mathbb{D}} \frac{R_n f(w)}{(1-\overline{w}z)^{\alpha+2}} dA_\alpha(w).$$

Furthermore, by using the self-adjointness of R_n and the expansion

$$\frac{1}{(1-\overline{w}z)^{\alpha+2}} = \sum_{j=0}^{\infty} \frac{\Gamma(j+\alpha+2)}{j!\Gamma(\alpha+2)} (\overline{w}z)^j,$$

we obtain that

$$|R_n f(z)| \le \sum_{j=n}^{\infty} \frac{\Gamma(j+\alpha+2)}{j!\Gamma(\alpha+2)} |z|^j \int_{\mathbb{D}} |f(w)||w|^j dA_\alpha(w). \tag{3.11}$$

Hölder's inequality and the condition $\sigma(z)/(1-|z|^2)^\alpha \in B_p(\alpha)$ show that

$$\begin{aligned} \int_{\mathbb{D}} |f(w)||w|^j dA_\alpha(w) &\le \left[\int_{\mathbb{D}} |f(w)|^p \sigma(w) dA(w)\right]^{\frac{1}{p}} \\ &\qquad \times \left[\int_{\mathbb{D}} |w|^{jp'} \{\sigma(w)\}^{-\frac{p'}{p}} (1-|z|^2)^{\alpha p'} dA(w)\right]^{\frac{1}{p'}} \\ &\lesssim \frac{\|f\|_\sigma}{\left(\int_{\mathbb{D}} \sigma dA\right)^{1/p}}. \end{aligned} \tag{3.12}$$

By (3.11) and (3.12) we obtain the desired estimation. □

Theorem 3.5. *Let σ_1 be an admissible Békollé weight function, σ_2 a weight function, $p_0 \le p \le q < \infty$ and $q \ge 2$. Suppose that ϕ is an analytic self-map of $\mathbb{D}$ with $\phi(0) = 0$ and $C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ is bounded. Then it holds that*

$$\|C_\phi\|_e^q \approx \limsup_{|z|\to 1} \frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}(z)}{\omega_{\sigma_1}(z)^{\frac{q}{p}}}. \tag{3.13}$$

Proof. First we prove the upper estimate. Since $C_\phi = C_\phi R_n + C_\phi K_n$ and $C_\phi K_n$ is compact, it holds that

$$\|C_\phi\|_e \le \liminf_{n\to\infty} \|C_\phi R_n\|. \tag{3.14}$$

Take $f \in A^p(\sigma_1 dA)$ with $\|f\|_{\sigma_1} \le 1$ and fix $t \in (1/2, 1)$, arbitrarily. By Lemma 3.4 and the assumption $\phi(0) = 0$ we have $|R_n f(\phi(0))| = 0$, and so an application of Lemma 2.9 to $R_n f$ gives that

$$\begin{aligned} \|C_\phi R_n f\|_{\sigma_2}^q \lesssim &\int_{\{|z|\le t\}} |R_n f(z)|^{q-2} |(R_n f)'(z)|^2 \left\{\int_0^1 N_\phi(r,z)\sigma_2(r) r dr\right\} dA(z) \\ &+ \int_{\{|z|>t\}} |R_n f(z)|^{q-2} |(R_n f)'(z)|^2 \left\{\int_0^1 N_\phi(r,z)\sigma_2(r) r dr\right\} dA(z). \end{aligned} \tag{3.15}$$

It follows from Lemmas 2.4, 2.6 and 3.4 that

$$\begin{aligned} &|R_n f(z)|^{q-2} |(R_n f)'(z)|^2 \\ \lesssim &\frac{\|f\|_{\sigma_1}^q}{\left(\int_{\mathbb{D}} \sigma_1 dA\right)^{(q-2)/p}} \left(\sum_{j=n}^{\infty} \frac{\Gamma(j+\alpha+2)}{j!\Gamma(\alpha+2)} t^j\right)^{q-2} \max_{|z|\le t} \frac{1}{\omega_{\sigma_1}(z)^{2/p}(1-|z|^2)^2} \|R_n\|^2, \end{aligned}$$

for $|z| \leq t$. Since $\sup_{n\geq 1} \|R_n\| < \infty$ by Corollary 3.3 and

$$\int_{\{|z|\leq t\}} \left\{ \int_0^1 N_\phi(r,z)\sigma_2(r) r dr \right\} dA(z) < \infty$$

by the proof of Theorem 3.1, we see that the first integral in (3.15) converges to 0 as $n \to \infty$ uniformly on the unit ball of $A^p(\sigma_1 dA)$.

On the other hand, the same argument as in the proof of Theorem 3.1 shows that

$$\int_0^1 N_\phi(r,z)\sigma_2(r) r dr \leq \frac{1}{t} \mathcal{N}_{\phi,\omega_{\sigma_2}}(z), \quad t \leq |z| < 1,$$

and so we have that

$$\begin{aligned}
&\int_{\{|z|>t\}} |R_n f(z)|^{q-2} |(R_n f)'(z)|^2 \left\{ \int_0^1 N_\phi(r,z)\sigma_2(r) r dr \right\} dA(z) \\
&\leq t \int_{\{|z|>t\}} |R_n f(z)|^{q-2} |(R_n f)'(z)|^2 \mathcal{N}_{\phi,\omega_{\sigma_2}}(z) dA(z) \\
&\leq t \sup_{|z|>t} \frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}(z)}{\omega_{\sigma_1}(z)^{\frac{q}{p}}} \int_{\{|z|>t\}} |R_n f(z)|^{q-2} |(R_n f)'(z)|^2 \omega_{\sigma_1}(z)^{\frac{q}{p}} dA(z). \qquad (3.16)
\end{aligned}$$

By Lemmas 2.4 and 2.6 we have that

$$|R_n f(z)|^{q-2} \lesssim \frac{\|R_n f\|_{\sigma_1}^{q-p}}{\omega_{\sigma_1}(z)^{\frac{q-p}{p}}} |R_n f(z)|^{p-2}.$$

Combining this with (3.16) and (3.8) and using Lemma 2.5, we obtain that

$$\begin{aligned}
&\int_{\{|z|>t\}} |R_n f(z)|^{q-2} |(R_n f)'(z)|^2 \left\{ \int_0^1 N_\phi(r,z)\sigma_2(r) r dr \right\} dA(z) \\
&\lesssim \sup_{n\geq 1} \|R_n\|^q \, t \sup_{|z|>t} \frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}(z)}{\omega_{\sigma_1}(z)^{\frac{q}{p}}},
\end{aligned}$$

for any $n \geq 1$ and $t \in (1/2, 1)$.

Taking the supremum over the unit ball of $A^p(\sigma_1 dA)$ and letting $n \to \infty$ in (3.15), we have that

$$\liminf_{n\to\infty} \|C_\phi R_n\|^q \lesssim t \sup_{|z|>t} \frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}(z)}{\omega_{\sigma_1}(z)^{\frac{q}{p}}},$$

for all $t \in (1/2, 1)$. Letting $t \to 1$ in the above inequality, by (3.14), we obtain that

$$\|C_\phi\|_e^p \lesssim \limsup_{|z|\to 1} \frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}(z)}{\omega_{\sigma_1}(z)^{\frac{q}{p}}}.$$

Next we prove the lower estimate in (3.13). For any sequence $\{z_j\} \subset \mathbb{D}$ with $|z_j| \to 1$ as $j \to \infty$, we put

$$f_{z_j}(w) = \frac{(1-|z_j|^2)^{\alpha+2-2/p}}{\sigma_1(z_j)^{\frac{1}{p}} (1-\overline{z_j} w)^{\alpha+2}}, \quad w \in \mathbb{D}.$$

Then we see that $\{f_{z_j}\}$ is bounded in $A^p(\sigma_1 dA)$ as in the proof of Theorem 3.1, and converges to 0 uniformly on compact subsets on $\mathbb{D}$ as $j \to \infty$. Lemma 2.3 implies that $\{f_{z_j}\}$ converges to 0 weakly in $A^p(\sigma_1 dA)$, and so we see that $\|\mathcal{K} f_{z_j}\|_{\sigma_2} \to 0$ as $j \to \infty$ for all compact operators $\mathcal{K} : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$. These facts give that

$$\|C_\phi\|_e^q \gtrsim \lim_{j\to\infty} \|C_\phi f_{z_j}\|_{\sigma_2}^q. \tag{3.17}$$

As in the proof of (3.9), Lemma 2.9 gives that

$$\|C_\phi f_{z_j}\|_{\sigma_2}^q \gtrsim \int_{E(z_j, \frac{1-|z_j|}{2})} |f_{z_j}(w)|^{q-2} |f'_{z_j}(w)|^2 \mathcal{N}_{\phi,\omega_{\sigma_2}}(w) dA(w), \tag{3.18}$$

for all $j \geq 1$. By Lemma 2.6, it follows that

$$|f_{z_j}(w)|^{q-2} |f'_{z_j}(w)|^2 \approx \frac{(\alpha+2)^2 |z_j|^2}{\omega_{\sigma_1}(z_j)^{\frac{q}{p}} (1-|z_j|^2)^2}, \tag{3.19}$$

for $j \geq 1$ and $w \in E(z_j, \frac{1-|z_j|}{2})$. Since $|z_j| > \frac{1-|z_j|}{2}$ if j is sufficiently large, Lemma 2.10 gives that

$$\mathcal{N}_{\phi,\omega_{\sigma_2}}(z_j) \lesssim \frac{1}{(1-|z_j|^2)^2} \int_{E(z_j, \frac{1-|z_j|}{2})} \mathcal{N}_{\phi,\omega_{\sigma_2}}(w) dA(w). \tag{3.20}$$

By (3.18), (3.19) and (3.20) we obtain that

$$|z_j|^2 \frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}(z_j)}{\omega_{\sigma_1}(z_j)^{\frac{q}{p}}} \lesssim \|C_\phi f_{z_j}\|_{\sigma_2}^q,$$

for sufficiently large j. Combining this with (3.17), we have that

$$\lim_{j\to\infty} \frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}(z_j)}{\omega_{\sigma_1}(z_j)^{\frac{q}{p}}} \lesssim \|C_\phi\|_e^q.$$

Since $\{z_j\} \subset \mathbb{D}$ with $|z_j| \to 1$ is arbitrarily, we get the desired lower estimate of $\|C_\phi\|_e$. This completes the proof. □

Corollary 3.6. *Let σ_1 be an admissible Békollé weight function, σ_2 a weight function and $0 < p \leq q < \infty$. Suppose that ϕ is an analytic self-map of $\mathbb{D}$ which $C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ is bounded. Then $C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ is compact if and only if*

$$\mathcal{N}_{\phi,\omega_{\sigma_2}}(z) = o(\omega_{\sigma_1}(z)^{\frac{q}{p}}) \qquad (|z| \to 1). \tag{3.21}$$

Proof. In the view of the proof of Theorem 3.1, we can assume that $\phi(0) = 0$ without loss of generality. Since the case $p \geq p_0$ and $q \geq 2$ is an immediate consequence of Theorem 3.5, it is enough to prove that the case $0 < p < p_0$ and $0 < q < 2$ because the rest of cases are verified by quite the same argument. Since we can choose a positive integer $m = m(p,q)$ such that $mp \geq p_0$ and $mq \geq 2$, we see that $C_\phi : A^{mp}(\sigma_1 dA) \to A^{mq}(\sigma_2 dA)$ is compact if and only if

$$\mathcal{N}_{\phi,\omega_{\sigma_2}}(z) = o(\omega_{\sigma_1}(z)^{\frac{mq}{mp}}) \qquad (|z| \to 1).$$

So, by an application of Corollary 2.13, we see that condition (3.21) also characterizes the compactness of $C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ for the case $p < p_0$ and $q < 2$. □

4. The case $0 < q < p < \infty$

In the proof of Theorem 4.1 below, we shall need Khinchine's inequality and the Hardy–Littlewood maximal function. Recall that the Rademacher functions $\{r_j(t)\}$ are defined by

$$r_0(t) = \begin{cases} 1 & 0 \le t - [t] < \frac{1}{2}, \\ -1 & \frac{1}{2} \le t - [t] < 1, \end{cases}$$
$$r_j(t) = r_0(2^j t) \qquad (j \ge 1).$$

Khinchine's Inequality. Let $0 < p < \infty$. There are constants $0 < A_p \le B_p < \infty$ such that, for any positive integer m and any complex numbers $\{c_j\}_{j=1}^m$, it holds that

$$A_p \left(\sum_{j=1}^m |c_j|^2 \right)^{\frac{p}{2}} \le \int_0^1 \left| \sum_{j=1}^m c_j r_j(t) \right|^p dt \le B_p \left(\sum_{j=1}^m |c_j|^2 \right)^{\frac{p}{2}}.$$

The Hardy-Littlewood Maximal Function. Let $\mathcal{M}[f]$ denote the Hardy-Littlewood maximal function for f, that is

$$\mathcal{M}[f](z) = \sup_{\delta > 0} \frac{1}{A(B(z,\delta))} \int_{B(z,\delta)} |f| \, dA,$$

where $B(z,\delta) = \{w \in \mathbb{D} : |w - z| < \delta\}$. Since we can find a positive constant c such that $E(z, \frac{1}{2}) \subset B(z, c(1 - |z|^2))$ for $z \in \mathbb{D}$, it holds that

$$\frac{1}{(1-|z|^2)^2} \int_{E(z,\frac{1}{2})} |f| \, dA \lesssim \mathcal{M}[f](z) \quad (z \in \mathbb{D}). \tag{4.1}$$

Moreover the Hardy-Littlewood maximal theorem (see [6, Theorem 4.3]) says that $\mathcal{M}[f] \in L^p$ and $\|\mathcal{M}[f]\|_{L^p} \lesssim \|f\|_{L^p}$ for $f \in L^p$ $(1 < p < \infty)$.

In our proof of Theorem 4.1, we adapt Luecking's approach in [10] or the method by Smith and Yang in [13] to weighted Bergman space with admissible Békollé weight. By using the same modification of Luecking's method and a c-adic decomposition of the disk $\mathbb{D}$, Constantin [4] proved the Carleson-type embedding theorem for $A^p(\sigma dA)$. In order to construct a suitable test function, however, we will use an ε-separated sequence of $\mathbb{D}$ instead of a c-adic decomposition of $\mathbb{D}$.

Theorem 4.1. *Let σ_1 be an admissible Békollé weight function, σ_2 a weight function and $0 < q < p < \infty$. For any analytic self-map ϕ of $\mathbb{D}$, $C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ is bounded if and only if*

$$\frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}}{\omega_{\sigma_1}} \in L^{\frac{p}{p-q}}(\sigma_1 dA). \tag{4.2}$$

Moreover, if $C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ is bounded, then it is also compact in this case.

Proof. By the same argument in proofs of Theorem 3.1 or Corollary 3.6, we may only prove the case $\phi(0)=0$, $p \geq p_0$ and $q \geq 2$.

First suppose that condition (4.2) holds and prove that $C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ is bounded. By Lemma 2.5, it is enough to prove that

$$\int_{\mathbb{D}} |f(z)|^{q-2}|f'(z)|^2 \left\{ \int_0^1 N_\phi(r,z)\sigma_2(r)rdr \right\} dA(z) < \infty$$

for any $f \in A^p(\sigma_1 dA)$. To prove this, we will divide the integral over $\mathbb{D}$ into two integrals over $\frac{1}{4}\overline{\mathbb{D}}$ and $\mathbb{D}\setminus\frac{1}{4}\overline{\mathbb{D}}$. As in the argument on inequality (3.7) in the proof of Theorem 3.1, however, we see that

$$\begin{aligned}&\int_{\frac{1}{4}\overline{\mathbb{D}}} |f(z)|^{q-2}|f'(z)|^2 \left\{ \int_0^1 N_\phi(r,z)\sigma_2(r)rdr \right\} dA(z)\\ &\lesssim \max_{|z|\leq\frac{1}{4}} \frac{1}{\omega_{\sigma_1}(z)^{\frac{q}{p}}(1-|z|^2)^2}\|f\|_{\sigma_1}^q.\end{aligned}$$

Hence we may only consider the integral over $\mathbb{D}\setminus\frac{1}{4}\overline{\mathbb{D}}$. Since $\phi(0)=0$, it follows from inequality (2.12) that

$$\int_0^1 N_\phi(r,z)\sigma_2(r)rdr \lesssim \mathcal{N}_{\phi,\omega_{\sigma_2}}(z)$$

for $z \in \mathbb{D}\setminus\frac{1}{4}\overline{\mathbb{D}}$. Combining this with Lemma 2.10, we have that

$$\begin{aligned}&\int_{\mathbb{D}\setminus\frac{1}{4}\overline{\mathbb{D}}} |f(z)|^{q-2}|f'(z)|^2 \left\{ \int_0^1 N_\phi(r,z)\sigma_2(r)rdr \right\} dA(z)\\ &\lesssim \int_{\mathbb{D}} |f(z)|^{q-2}|f'(z)|^2 \frac{1}{(1-|z|^2)^2} \left\{ \int_{E(z,\frac{1}{4})} \mathcal{N}_{\phi,\omega_{\sigma_2}}(w)dA(w) \right\} dA(z).\end{aligned}$$

By noting that $\chi_{E(z,\frac{1}{4})}(w) = \chi_{E(w,\frac{1}{4})}(z)$ and $1-|z|^2 \approx 1-|w|^2$ for $w \in E(z,\frac{1}{4})$, and applying Fubini's theorem, we have that

$$\begin{aligned}&\int_{\mathbb{D}} |f(z)|^{q-2}|f'(z)|^2 \frac{1}{(1-|z|^2)^2} \left\{ \int_{E(z,\frac{1}{4})} \mathcal{N}_{\phi,\omega_{\sigma_2}}(w)dA(w) \right\} dA(z)\\ &\approx \int_{\mathbb{D}} \left\{ \int_{E(w,\frac{1}{4})} |f(z)|^{q-2}|f'(z)|^2 dA(z) \right\} \frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}(w)}{(1-|w|^2)^2} dA(w).\end{aligned}$$

Since [13, Lemma 2.4] gives that

$$\int_{E(w,\frac{1}{4})} |f|^{q-2}|f'|^2 dA \lesssim \frac{1}{(1-|w|^2)^2} \int_{E(w,\frac{1}{2})} |f|^q dA,$$

we obtain that

$$\begin{aligned}&\int_{\mathbb{D}\setminus\frac{1}{4}\overline{\mathbb{D}}} |f(z)|^{q-2}|f'(z)|^2 \left\{ \int_0^1 N_\phi(r,z)\sigma_2(r)rdr \right\} dA(z)\\ &\lesssim \int_{\mathbb{D}} \left\{ \int_{E(w,\frac{1}{2})} |f(z)|^q dA(z) \right\} \frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}(w)}{(1-|w|^2)^4} dA(w). \qquad (4.3)\end{aligned}$$

By applying Fubini's theorem to the last formula in (4.3) once again, we have that

$$\int_{\mathbb{D}\setminus\frac{1}{4}\overline{\mathbb{D}}} |f(z)|^{q-2}|f'(z)|^2 \left\{\int_0^1 N_\phi(r,z)\sigma_2(r)rdr\right\} dA(z) \lesssim \int_{\mathbb{D}} |f(z)|^q H(z)dA(z), \tag{4.4}$$

where

$$H(z) = \int_{E(z,\frac{1}{2})} \frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}(w)}{(1-|w|^2)^4} dA(w).$$

Furthermore Hölder's inequality gives that

$$\begin{aligned}&\int_{\mathbb{D}} |f(z)|^q H(z)dA(z)\\ &\leq \left[\int_{\mathbb{D}} |f(z)|^p \sigma_1(z)dA(z)\right]^{\frac{q}{p}} \cdot \left[\int_{\mathbb{D}} H(z)^{\frac{p}{p-q}}\sigma_1(z)^{-\frac{q}{p-q}}dA(z)\right]^{\frac{p-q}{p}}.\end{aligned} \tag{4.5}$$

Since $\sigma_1(z) \approx \sigma_1(w)$ and $1-|z|^2 \approx 1-|w|^2$ for $w \in E(z,\frac{1}{2})$, it follows from Lemma 2.6 and inequality (4.1) that

$$\begin{aligned} H(z) &\lesssim \frac{\sigma_1(z)}{(1-|z|^2)^2}\int_{E(z,\frac{1}{2})} \frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}(w)}{\sigma_1(w)(1-|w|^2)^2}dA(w)\\ &\lesssim \frac{\sigma_1(z)}{(1-|z|^2)^2}\int_{E(z,\frac{1}{2})} \frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}(w)}{\omega_{\sigma_1}(w)}dA(w)\\ &\lesssim \sigma_1(z)\,\mathcal{M}\left[\frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}}{\omega_{\sigma_1}}\right](z) \qquad (z\in\mathbb{D}).\end{aligned}$$

Thus the Hardy-Littlewood maximal theorem shows that

$$\begin{aligned}\left[\int_{\mathbb{D}} H(z)^{\frac{p}{p-q}}\sigma_1(z)^{-\frac{q}{p-q}}dA(z)\right]^{\frac{p-q}{p}} &\lesssim \left\|\mathcal{M}\left[\frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}}{\omega_{\sigma_1}}\right]\right\|_{L^{\frac{p}{p-q}}(\sigma_1 dA)}\\ &\lesssim \left\|\frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}}{\omega_{\sigma_1}}\right\|_{L^{\frac{p}{p-q}}(\sigma_1 dA)}.\end{aligned} \tag{4.6}$$

Inequalities (4.3) $\sim$ (4.6) and condition (4.2) imply that

$$\int_{\mathbb{D}} |f(z)|^{q-2}|f'(z)|^2 \left\{\int_0^1 N_\phi(r,z)\sigma_2(r)rdr\right\} dA(z) < \infty,$$

and so $C_\phi f \in A^q(\sigma_2 dA)$ for any $f \in A^p(\sigma_1 dA)$. This indicates the boundedness of $C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$.

Conversely, we will prove that the boundedness of $C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ gives condition (4.2). To do this, we choose an ε-separated sequence $\{z_j\} \subset \mathbb{D}$, that is

$$\inf\{|\rho(z_j,z_k)| \,:\, j\neq k\} = \varepsilon > 0.$$

By this condition, we can assume $\inf_{j\geq 1}|z_j|>0$ without loss of generality. Now we put

$$g_j(z)=\frac{(1-|z_j|^2)^{\alpha+2-\frac{2}{p}}}{\sigma_1(z_j)^{\frac{1}{p}}}\cdot\frac{1}{(1-\overline{z_j}z)^{\alpha+2}}\quad(z\in\mathbb{D},j\geq 1),$$

$$f_t(z)=\sum_{j=1}^{\infty}\frac{c_j r_j(t)}{z_j}g_j(z)\quad(z\in\mathbb{D},\,t\in[0,1))$$

for some $\{c_j\}\in l^p$ and the Rademacher functions $\{r_j(t)\}$. Then these functions f_t are in $A^p(\sigma_1 dA)$ and

$$\sup_{t\in[0,1)}\|f_t\|_{\sigma_1}\lesssim\left(\sum_{j=1}^{\infty}|c_j|^p\right)^{\frac{1}{p}}. \tag{4.7}$$

This inequality is verified by Theorems 2.1 and 2.2. In fact, Theorem 2.2 shows that

$$\|f_t\|_{\sigma_1}\approx\sup\{|\langle f_t,h\rangle_\alpha|\,:\,h\in A^{p'}({\sigma_1}^{-\frac{p'}{p}}dA_{\alpha p'}),\,\|h\|\leq 1\}.$$

Here $\|\cdot\|$ denotes the norm of the space $A^{p'}({\sigma_1}^{-\frac{p'}{p}}dA_{\alpha p'})$. Since

$$\frac{\sigma_1(z)}{(1-|z|^2)^\alpha}\in B_p(\alpha)\quad\Longleftrightarrow\quad\frac{{\sigma_1}^{-\frac{p'}{p}}(z)(1-|z|^2)^{\alpha p'}}{(1-|z|^2)^\alpha}\in B_{p'}(\alpha),$$

Theorem 2.1 shows that $P_\alpha: L^{p'}({\sigma_1}^{-\frac{p'}{p}}dA_{\alpha p'})\to A^{p'}({\sigma_1}^{-\frac{p'}{p}}dA_{\alpha p'})$ is bounded and $h(w)=P_\alpha h(w)$ for $w\in\mathbb{D}$. Hence we have that

$$\langle f_t,h\rangle_\alpha=\sum_{j=1}^{\infty}\frac{c_j r_j(t)}{z_j}\cdot\frac{(1-|z_j|^2)^{\alpha+2-\frac{2}{p}}}{\sigma_1(z_j)^{\frac{1}{p}}}\cdot\overline{h(z_j)}. \tag{4.8}$$

As in the proof of Lemma 2.4, it holds that

$$\left[\int_{E(z_j,r)}\sigma_1(z)^{-\frac{p'}{p}}dA_{\alpha p'}(z)\right]^{\frac{1}{p'}}\approx\frac{(1-|z_j|^2)^{\alpha+2-\frac{2}{p}}}{\sigma_1(z_j)^{\frac{1}{p}}} \tag{4.9}$$

for each $r\in(0,1)$ and $j\geq 1$. By (4.8), (4.9) and Hölder's inequality, we have that

$$\begin{aligned}|\langle f_t,h\rangle_\alpha|&\lesssim\sum_{j=1}^{\infty}|c_j||h(z_j)|\left[\int_{E(z_j,r)}\sigma_1(z)^{-\frac{p'}{p}}dA_{\alpha p'}(z)\right]^{\frac{1}{p'}}\\&\leq\left[\sum_{j=1}^{\infty}|c_j|^p\right]^{\frac{1}{p}}\cdot\left[\sum_{j=1}^{\infty}|h(z_j)|^{p'}\int_{E(z_j,r)}\sigma_1(z)^{-\frac{p'}{p}}dA_{\alpha p'}(z)\right]^{\frac{1}{p'}}\end{aligned}$$

Since $\{z_j\}$ is ε-separated, [9, Theorem (3.12)] gives that

$$\left[\sum_{j=1}^{\infty}|h(z_j)|^{p'}\int_{E(z_j,r)}\sigma_1(z)^{-\frac{p'}{p}}dA_{\alpha p'}(z)\right]^{\frac{1}{p'}}\lesssim\|h\|,$$

and so we obtain the desired inequality (4.7).

Since $C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ is bounded, (4.7) shows that

$$\int_{\mathbb{D}} \left| \sum_{j=1}^{\infty} \frac{c_j r_j(t)}{z_j} (g_j \circ \phi) \right|^q \sigma_2 dA \leq \|C_\phi\|^q \|f_t\|_{\sigma_1}^q \lesssim \left(\sum_{j=1}^{\infty} |c_j|^p \right)^{\frac{q}{p}}.$$

By integrating the above inequalities from 0 to 1 with respect to t, and applying Fubini's theorem and Khinchine's inequality, we get

$$\int_{\mathbb{D}} \left(\sum_{j=1}^{\infty} \left| \frac{c_j}{z_j} (g_j \circ \phi) \right|^2 \right)^{\frac{q}{2}} \sigma_2 \, dA \lesssim \int_{\mathbb{D}} \left(\int_0^1 \left| \sum_{j=1}^{\infty} \frac{c_j r_j(t)}{z_j} (g_j \circ \phi) \right|^q dt \right) \sigma_2 \, dA.$$

Our assumption $q \geq 2$ shows that

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{|c_j|^q}{|z_j|^q} \int_{\mathbb{D}} |g_j \circ \phi|^q \sigma_2 \, dA &= \int_{\mathbb{D}} \sum_{j=1}^{\infty} \left| \frac{c_j}{z_j} (g_j \circ \phi) \right|^q \sigma_2 \, dA \\ &\leq \int_{\mathbb{D}} \left(\sum_{j=1}^{\infty} \left| \frac{c_j}{z_j} (g_j \circ \phi) \right|^2 \right)^{\frac{q}{2}} \sigma_2 \, dA. \end{aligned}$$

Combining these inequalities with Lemma 2.9, we obtain that

$$\sum_{j=1}^{\infty} \frac{|c_j|^q}{|z_j|^q} \int_{\mathbb{D}} |g_j|^{q-2} |g_j'|^2 \mathcal{N}_{\phi, \omega_{\sigma_2}} dA \lesssim \left(\sum_{j=1}^{\infty} |c_j|^p \right)^{\frac{q}{p}}.$$

It holds that

$$\frac{|z_j|^2}{\sigma_1(z_j)^{\frac{q}{p}} (1 - |z_j|^2)^{2 + 2\frac{q}{p}}} \lesssim |g_j|^{q-2} |g_j'|^2$$

on $E(z_j, \frac{1}{2})$, and so

$$\sum_{j=1}^{\infty} |c_j|^q \frac{\int_{E(z_j, \frac{1}{2})} \mathcal{N}_{\phi, \omega_{\sigma_2}}(z) dA(z)}{\sigma_1(z_j)^{\frac{q}{p}} (1 - |z_j|^2)^{2 + 2\frac{q}{p}}} \lesssim \left(\sum_{j=1}^{\infty} |c_j|^p \right)^{\frac{q}{p}} < \infty.$$

This inequality implies that the sequence

$$\left\{ \frac{\int_{E(z_j, \frac{1}{2})} \mathcal{N}_{\phi, \omega_{\sigma_2}}(z) dA(z)}{\sigma_1(z_j)^{\frac{q}{p}} (1 - |z_j|^2)^{2 + 2\frac{q}{p}}} \right\}_{j \geq 1}$$

belongs to the dual of $l^{\frac{p}{q}}$. Hence we see that

$$\sum_{j=1}^{\infty} \left[\frac{\int_{E(z_j, \frac{1}{2})} \mathcal{N}_{\phi, \omega_{\sigma_2}}(z) dA(z)}{\sigma_1(z_j)^{\frac{q}{p}} (1 - |z_j|^2)^{2 + 2\frac{q}{p}}} \right]^{\frac{p}{p-q}} < \infty.$$

To derive the integrability condition (4.2) from this, we choose an ε-separated sequence $\{z_j\}$ in $\mathbb{D}$ such that the disks $E(z_j, \frac{1}{4})$ cover $\mathbb{D}$. By Lemma 2.10, we have that

$$\mathcal{N}_{\phi, \omega_{\sigma_2}}(z) \lesssim \frac{1}{(1 - |z|^2)^2} \int_{E(z, \frac{1}{4})} \mathcal{N}_{\phi, \omega_{\sigma_2}}(w) dA(w)$$

for $z \in \mathbb{D} \setminus \frac{1}{4}\overline{\mathbb{D}}$. Then

$$\int_{\mathbb{D}\setminus\frac{1}{4}\overline{\mathbb{D}}} \left(\frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}}{\omega_{\sigma_1}}\right)^{\frac{p}{p-q}} \sigma_1 dA \lesssim \int_{\mathbb{D}\setminus\frac{1}{4}\overline{\mathbb{D}}} \left(\frac{\int_{E(z,\frac{1}{4})} \mathcal{N}_{\phi,\omega_{\sigma_2}}(w)dA(w)}{\omega_{\sigma_1}(z)(1-|z|^2)^2}\right)^{\frac{p}{p-q}} \sigma_1(z)dA(z)$$
$$\leq \sum_{j=1}^{\infty} \int_{E(z_j,\frac{1}{4})} \left(\frac{\int_{E(z,\frac{1}{4})} \mathcal{N}_{\phi,\omega_{\sigma_2}}(w)dA(w)}{\omega_{\sigma_1}(z)(1-|z|^2)^2}\right)^{\frac{p}{p-q}} \sigma_1(z)dA(z)$$

Since $1-|z|^2 \approx 1-|z_j|^2$, $\sigma_1(z) \approx \sigma_1(z_j)$, and so $\omega_{\sigma_1}(z) \approx \omega_{\sigma_1}(z_j)$ for $z \in E(z_j, \frac{1}{2})$, we have that

$$\left(\frac{\int_{E(z,\frac{1}{4})} \mathcal{N}_{\phi,\omega_{\sigma_2}}(w)dA(w)}{\omega_{\sigma_1}(z)(1-|z|^2)^2}\right)^{\frac{p}{p-q}} \sigma_1(z) \approx \left(\frac{\int_{E(z,\frac{1}{4})} \mathcal{N}_{\phi,\omega_{\sigma_2}}(w)dA(w)}{\omega_{\sigma_1}(z_j)(1-|z_j|^2)^2}\right)^{\frac{p}{p-q}} \sigma_1(z_j)$$
$$\leq \left(\frac{\int_{E(z_j,\frac{1}{2})} \mathcal{N}_{\phi,\omega_{\sigma_2}}(w)dA(w)}{\omega_{\sigma_1}(z_j)(1-|z_j|^2)^2}\right)^{\frac{p}{p-q}} \sigma_1(z_j)$$

for $z \in E(z_j, \frac{1}{4})$. By noting that $A(E(z_j, \frac{1}{4})) \approx (1-|z_j|^2)^2$ and applying Lemma 2.6, we obtain that

$$\int_{\mathbb{D}\setminus\frac{1}{4}\overline{\mathbb{D}}} \left(\frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}}{\omega_{\sigma_1}}\right)^{\frac{p}{p-q}} \sigma_1 dA \lesssim \sum_{j=1}^{\infty} \left(\frac{\int_{E(z_j,\frac{1}{2})} \mathcal{N}_{\phi,\omega_{\sigma_2}}(w)dA(w)}{\omega_{\sigma_1}(z_j)(1-|z_j|^2)^2}\right)^{\frac{p}{p-q}} \sigma_1(z_j)(1-|z_j|^2)^2$$
$$\approx \sum_{j=1}^{\infty} \left(\frac{\int_{E(z_j,\frac{1}{2})} \mathcal{N}_{\phi,\omega_{\sigma_2}}(w)dA(w)}{\sigma_1(z_j)^{\frac{q}{p}}(1-|z_j|^2)^{2+\frac{2q}{p}}}\right)^{\frac{p}{p-q}}$$
$$< \infty.$$

Since the integrability on $\frac{1}{4}\overline{\mathbb{D}}$ is clear by the inequality

$$\mathcal{N}_{\phi,\omega_{\sigma_2}}(z) \lesssim \int_0^1 N_\phi(z,r)r\sigma_2(r)dr \leq \log\frac{1}{|z|},$$

we obtain that

$$\frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}}{\omega_{\sigma_1}} \in L^{\frac{p}{p-q}}(\sigma_1 dA).$$

Finally we show that $C_\phi : A^p(\sigma_1 dA) \to A^q(\sigma_2 dA)$ is also compact. Take a bounded sequence $\{f_j\}$ in $A^p(\sigma_1 dA)$ which converges to 0 uniformly on compact subsets of $\mathbb{D}$. By Lemma 2.3, we see that $\{f_j\}$ also converges to 0 weakly in $A^p(\sigma_1 dA)$, and so it is enough to show that $\|C_\phi f_j\|_{\sigma_2} \to 0$ as $j \to \infty$. By Lemma 2.9, this is equivalent to

$$\lim_{j\to\infty} \int_{\mathbb{D}} |f_j(z)|^{q-2}|f_j'(z)|^2 \left\{\int_0^1 N_\phi(r,z)\sigma_2(r)rdr\right\} dA(z) = 0. \tag{4.10}$$

Since it holds that

$$\int_{r\overline{\mathbb{D}}} \left\{\int_0^1 N_\phi(r,z)\sigma_2(r)rdr\right\} dA(z) < \infty$$

for any $r \in (0,1)$, we get that

$$\lim_{j\to\infty}\int_{r\overline{\mathbb{D}}} |f_j(z)|^{q-2}|f_j'(z)|^2 \left\{\int_0^1 N_\phi(r,z)\sigma_2(r)rdr\right\} dA(z) = 0$$

for any $r \in (0,1)$. As in the arguments in (4.3) $\sim$ (4.6), we have that

$$\begin{aligned}
&\int_{\mathbb{D}\setminus r\overline{\mathbb{D}}} |f_j(z)|^{q-2}|f_j'(z)|^2 \left\{\int_0^1 N_\phi(r,z)\sigma_2(r)rdr\right\} dA(z)\\
\lesssim &\int_{\mathbb{D}\setminus r\overline{\mathbb{D}}} |f_j(z)|^q \int_{E(z,2r)} \frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}(w)}{(1-|w|^2)^4} dA(w)dA(z)\\
\lesssim &\|f_j\|_{\sigma_1}^q \left[\int_{\mathbb{D}\setminus r\overline{\mathbb{D}}} \left(\frac{\mathcal{N}_{\phi,\omega_{\sigma_2}}(w)}{\omega_{\sigma_1}(w)}\right)^{\frac{p}{p-q}} \sigma_1(w)dA(w)\right]^{\frac{p-q}{p}}
\end{aligned}$$

for $r \in (0,1)$ and $j \geq 1$. Since the boundedness of C_ϕ implies that the last integral above can be made arbitrarily small by choosing r sufficiently close to 1, we obtain (4.10), and so the proof is complete. □

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