Banach J. Math. Anal. 6 (2012), no. 1, 90–100

Banach Journal of Mathematical Analysis
ISSN: 1735-8787 (electronic)
www.emis.de/journals/BJMA/

ALMOST AUTOMORPHIC SOLUTIONS OF HYPERBOLIC EVOLUTION EQUATIONS

BRUNO DE ANDRADE[1], CLAUDIO CUEVAS[2]* AND ERWIN HENRÍQUEZ[3]

Communicated by M. S. Moslehian

Abstract. In this work we deals with almost automorphic behavior of solutions of a class of semilinear evolution equations. To achieve our goal we use interpolation theory and fixed point theory. As application, we examine sufficient conditions for existence of almost automorphic solutions of equations of the heat conduction theory.

1. Introduction

In recent years, the theory of almost automorphic functions has been developed extensively and consequently there has been a considerable interest in the existence of almost automorphic solutions of various kinds of evolution equations, see for instance [2, 3, 5, 8, 10, 11, 12] and the references therein. In this work we deals with a class of abstract semilinear evolution equations described in the form

$$u'(t) = Au(t) + f(t, u(t)),\ t \in \mathbb{R}, \tag{1.1}$$

with $A : D(A) \subset X \to X$ generator of a hyperbolic semigroup on X and $f : \mathbb{R} \times X_\alpha \to X$ a suitable continuous function, where for $\alpha \in (0, 1)$, X_α is an intermediate space between $D(A)$ and X.

We remark that the existence of a hyperbolic semigroup on a Banach space X give us a nice algebraic information about this vectorial space. In fact, let $(T(t))_{t\geq 0}$ be a hyperbolic semigroup on X. Then there are $(T(t))_{t\geq 0}$-invariant

Date: Received: 23 September 2011; Accepted: 24 October 2011.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 34G20; Secondary 47A55.

Key words and phrases. Analytic semigroup, hyperbolic semigroup, almost automorphic function, semilinear evolution equation.

closed subspaces X_s and X_u such that $X = X_s \bigoplus X_u$. Furthermore, the restricted semigroups $(T_s(t))_{t\geq 0}$ on X_s and $(T_u(t))_{t\geq 0}$ on X_u have the following properties:

(i) The semigroup $(T_s(t))_{t\geq 0}$ is uniformly exponentially stable on X_s.

(ii) The operators $T_u(t)$ are invertible on X_u, and $(T_u(t)^{-1})_{t\geq 0}$ is uniformly exponentially stable on X_u.

This algebraic decomposition will be very important in our approach. The main difficulty in this work is the fact that the semigroup generated by the linear operator A in Equation (1.1) is not stable.

Our existence results are contained in Section 3. They basically say that if the nonlinearity f is almost automorphic, in some sense, and verifies Lipschitz conditions, then we will have existence and uniqueness of almost automorphic mild solutions for Problem (1.1), see Theorem 3.2. In the Theorem 3.6 we cover situations where the nonlinearity f satisfies only locally Lipschitz conditions. We close this work with concretes examples where we examine sufficient conditions for existence of almost automorphic solutions.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a Banach space. In this work $C_b(\mathbb{R}; X)$ denotes the space consisting of the continuous and bounded functions from $\mathbb{R}$ into X endowed with the norm of the uniform convergence which is denoted for $\|\cdot\|_\infty$.

2.1. Almost automorphic functions.

We begin by recalling the notion of almost automorphic function[1].

Definition 2.1. A continuous function $f : \mathbb{R} \mapsto X$ is called almost automorphic if for every sequence of real numbers $(s_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s'_n)_{n\in\mathbb{N}} \subset (s_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n,m\to\infty} \|f(t + s_n - s_m) - f(t)\| = 0.$$

Example 2.2. Concrete examples of almost automorphic functions are given by the functions

$$a(t) = \cos\left(\frac{1}{\cos(t) + \cos(\sqrt{2}t)}\right) \ \textit{and}\ b(t) = \sin\left(\frac{1}{\sin(t) + \sin(\sqrt{5}t)}\right),$$

$t \in \mathbb{R}$.

Definition 2.3. Let X and Y be two Banach spaces. A continuous function $f : \mathbb{R} \times Y \mapsto X$ is said to be almost automorphic if $f(t, x)$ is almost automorphic in $t \in \mathbb{R}$ uniformly for all $x \in K$, where K is any bounded subset of Y.

In this work we use the notation $AA(X)$ to represent the subset of $C_b(\mathbb{R}; X)$ formed by the almost automorphic functions. We observe that $(AA(X), \|\cdot\|_\infty)$ is a Banach space. Furthermore, if $f \in AA(X)$ then the set $\{f(t) : t \in \mathbb{R}\}$ is a relatively compact subset of X. Similarly, we set $AA(Y; X)$ to represent the set of all functions almost automorphic in t uniformly for $x \in Y$.

The following result is standard in the theory of almost automorphic functions.

[1] This definition is due to Bochner (see [1]).

Lemma 2.4 ([7])**.** *If $f : \mathbb{R} \times Y \mapsto X$ is almost automorphic, and $h \in AA(Y)$, and assume that $f(t, \cdot)$ is uniformly continuous on each bounded subset $K \subset Y$ uniformly for $t \in \mathbb{R}$, that is for any $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in K$ and $\|x - y\| < \delta$ imply that $\|f(t, x) - f(t, y)\| < \epsilon$ for all $t \in \mathbb{R}$, then the function $f(\cdot, h(\cdot)) \in AA(X)$.*

2.2. **Hyperbolic semigroups, sectorial operators and intermediate spaces.** Let $(X, \|\cdot\|)$ be a Banach space. This subsection deals, first of all, with the notion of hyperbolic semigroups.

Definition 2.5. A Semigroup $(T(t))_{t\geq 0}$ on X is said to be hyperbolic if there is a projection P and constants $M, \delta > 0$ such that each $T(t)$ commutes with P, $\mathrm{Ker}P$ is invariant with respect to $T(t)$, $T(t) : \mathrm{Im}Q \to \mathrm{Im}Q$ is invertible and for every $x \in X$

$$\|T(t)Px\| \leq Me^{-\delta t}\|x\|, \quad \text{for } t \geq 0; \tag{2.1}$$

$$\|T(t)Qx\| \leq Me^{\delta t}\|x\|, \quad \text{for } t \leq 0; \tag{2.2}$$

where $Q := I - P$ and, for $t < 0$, $T(t) = T(-t)^{-1}$.

Remark 2.6. The existence of a hyperbolic semigroup on a Banach space X give us a nice algebraic information about this vectorial space. In fact, let $(T(t))_{t\geq 0}$ be a hyperbolic semigroup on X. Then there are $(T(t))_{t\geq 0}$-invariant closed subspaces X_s and X_u such that $X = X_s \bigoplus X_u$. Furthermore, the restricted semigroups $(T_s(t))_{t\geq 0}$ on X_s and $(T_u(t))_{t\geq 0}$ on X_u have the following properties:

(i) The semigroup $(T_s(t))_{t\geq 0}$ is uniformly exponentially stable on X_s.
(ii) The operators $T_u(t)$ are invertible on X_u, and $(T_u(t)^{-1})_{t\geq 0}$ is uniformly exponentially stable on X_u.

Special examples of hyperbolic semigroups are those generated by sectorial operators. We remember that a linear operator $A : D(A) \subset X \to X$ is said to be sectorial[2] if there are constants $\omega \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$ and $M > 0$ such that

$$\begin{cases} (i) \quad \rho(A) \supset S_{\theta,\omega} := \{\lambda \in \mathbb{C} : \lambda \neq \omega, \ |\arg(\lambda - \omega)| < \theta\}; \\ (ii) \quad \|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda-\omega|}, \ \lambda \in S_{\theta,\omega}. \end{cases} \tag{2.3}$$

It is well known that a sectorial operator A is the generator of an analytic semigroup $(T(t))_{t\geq 0}$ on $(0, \infty)$ to $\mathcal{L}(X)$. Furthermore, there are constants M_0 and M_1 such that

$$\|T(t)\| \leq M_0 e^{\omega t}, \ t > 0, \tag{2.4}$$

$$\|t(A - \omega I)T(t)\| \leq M_1 e^{\omega t}, \ t > 0, \tag{2.5}$$

where ω is the constant of assumption (2.3) (see, for example, [9]). Finally, $(T(t))_{t\geq 0}$ is a hyperbolic semigroup if, and only if, $\sigma(A) \cap i\mathbb{R} = \emptyset$ (see [4]).

Now, we recalling the notion of intermediate spaces. The interpolation spaces theory is crucial in the theory of regularity for abstract equations. We follows the

[2]Sectorial operator are well studied in the literature. For a recent reference including several examples and properties we refer the reader to Haase [6].

terminology used in Lunardi's book [9] which gives a self-contained exposition of interpolation theory and its applications. We only present the basic results which will be used in this work.

Let X and Z be Banach spaces, with norms $\|\cdot\|$, $\|\cdot\|_Z$ respectively, and suppose that Z is continuously embedded in X, that is, $Z \hookrightarrow X$.

Definition 2.7. Let $0 \leq \alpha \leq 1$. A Banach space Y such that $Z \hookrightarrow Y \hookrightarrow X$ is said to belong to the class J_α between X and Z if there is a constant $c > 0$ such that

$$\|x\|_Y \leq c\|x\|^{1-\alpha}\|x\|_Z^\alpha \quad (x \in Z). \tag{2.6}$$

In this case we write $Y \in J_\alpha(X, Z)$.

Definition 2.8. Let $A : D(A) \subset X \to X$ be a sectorial operator. A Banach space $(X_\alpha, \|\cdot\|_\alpha)$, $\alpha \in (0, 1)$, is said to be an intermediate space between X and $D(A)$ if $X_\alpha \in J_\alpha(X, D(A))$.

Examples of intermediate spaces between X and $D(A)$ are the domains of the fractional powers $D(-A^\alpha)$ and the interpolation spaces $D_A(\alpha, \infty)$, defined as follows

$$\begin{cases} D_A(\alpha, \infty) = \{x \in X : [x]_\alpha = \sup_{0<t\leq 1} \|t^{1-\alpha} AT(t)x\| < +\infty\}, \\ \|x\|_{D_A(\alpha,\infty)} = \|x\| + [x]_\alpha. \end{cases}$$

Remark 2.9. It is important to note that $D_A(\alpha, \infty)$ do not depend explicitly on the operator A, but only on $D(A)$ and on the graph norm of A.

We close this subsection with a result on hyperbolic analytic semigroups and intermediate spaces.

Lemma 2.10. *Let $(T(t))_{t\geq 0}$ be a hyperbolic analytic semigroup on X with generator A. For $\alpha \in (0, 1)$, let $(X_\alpha, \|\cdot\|_\alpha)$ be intermediate spaces between X and $D(A)$. Then there are positive constants $C(\alpha)$, $M(\alpha)$, δ and γ such that*

$$\|T(t)Px\|_\alpha \leq M(\alpha)t^{-\alpha}e^{-\gamma t}\|x\| \quad (t > 0)$$

and

$$\|T(t)Qx\|_\alpha \leq C(\alpha)e^{\delta t}\|x\| \quad (t \leq 0).$$

Proof. The demonstration of this assertion follows immediately from inequalities (2.1), (2.2), (2.4), (2.5) and (2.6). See, for example, [2]. □

2.3. **Linear problem.** In this subsection we consider the inhomogeneous linear evolution equation described by

$$u'(t) = Au(t) + g(t), \quad t \in \mathbb{R}, \tag{2.7}$$

where $A : D(A) \subset X \to X$ is the generator of a hyperbolic analytic semigroup $(T(t))_{t\geq 0}$ and $g : \mathbb{R} \to X$ is a continuous function.

Definition 2.11. A bounded continuous function $u : \mathbb{R} \to X$ is called a mild solution of (2.7) if

$$u(t) = T(t - s)u(s) + \int_s^t T(t - r)g(r)dr,$$

for all $t \geq s$ and all $s \in \mathbb{R}$.

We recall the following existence result of almost automorphic solutions for Equation (2.7) (see [2]). For the reader convenience we will give the proof.

Theorem 2.12. Suppose that $g \in AA(X)$. Then there is a unique mild solution of Equation (2.7) given by

$$u(t) = \int_{-\infty}^{t} T(t-s)Pg(s)ds - \int_{t}^{\infty} T(t-s)Qg(s)ds, \tag{2.8}$$

for all $t \in \mathbb{R}$, where P and Q are the projections associated to operator A. Furthermore, this mild solution belongs to $AA(X_\alpha)$.

Proof. Let u be given by (2.8). It is clear that u is a mild solution of Equation (2.7). Furthermore, if v is another mild solution of Equation (2.7) then we have that

$$Pv(t) = T(t-s)Pv(s) + \int_{s}^{t} T(t-r)Pg(r)dr, \quad \text{for } t \geq s,\ t, s \in \mathbb{R}.$$

By the estimate (2.1) and by letting $s \to -\infty$ we obtain

$$Pv(t) = \int_{-\infty}^{t} T(t-s)Pg(s)ds, \quad t \in \mathbb{R}. \tag{2.9}$$

On the other hand, since

$$Qv(t) = T(t-s)Qv(s) + \int_{s}^{t} T(t-r)Pg(r)dr, \quad \text{for } t \geq s,\ t, s \in \mathbb{R}$$

follows by the estimate (2.2) and by letting $s \to \infty$ that

$$Qv(t) = -\int_{t}^{\infty} T(t-r)Qg(r)dr, \quad t \in \mathbb{R}. \tag{2.10}$$

Hence, by (2.9), (2.10) and the decomposition of the space X (see Remark 2.6) we have that $v = u$.

Now, to prove that $u \in AA(X_\alpha)$ let us take a sequence $(s'_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$. Since $g \in AA(X)$, there is a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n,n\to\infty} \|g(t+s_n-s_m) - g(t)\| = 0, \tag{2.11}$$

Furthermore,

$$\begin{aligned} u(t+s_n-s_m)-u(t) &= \int_{-\infty}^{t+s_n-s_m} T(t+s_n-s_m-r)Pg(r)dr \\ &\quad -\int_{-\infty}^{t} T(t-r)Pg(r)dr \\ &\quad +\int_{t+s_n-s_m}^{+\infty} T(t+s_n-s_m-r)Qg(r)dr \\ &\quad -\int_{t}^{+\infty} T(t-r)Qg(r)dr \\ &= \int_{-\infty}^{0} T(-r)P[g(r+t+s_n-s_m)-g(r+t)]dr \\ &\quad +\int_{0}^{+\infty} T(-r)Q[g(r+t+s_n-s_m)-g(r+t)]dr. \end{aligned}$$

Then, we obtain

$$\begin{aligned} \|u(t+s_n-s_m)-u(t)\|_\alpha \le & \int_{-\infty}^{0} \|T(-r)P[g(r+t+s_n-s_m)-g(r+t)]\|_\alpha \, dr \\ & + \int_{0}^{+\infty} \|T(-r)Q[g(r+t+s_n-s_m)-g(r+t)]\|_\alpha \, dr. \end{aligned}$$

Hence, by Lemma 2.10 we deduce

$$\begin{aligned} & \|u(t+s_n-s_m)-u(t)\|_\alpha \\ & \le c(\alpha)\int_{-\infty}^{0} e^{\delta r}\|g(r+t+s_n-s_m)-g(r+t)\|\, dr \\ & + M(\alpha)\int_{0}^{+\infty} r^{-\alpha}e^{-\gamma r}\|g(r+t+s_n-s_m)-g(r+t)\|\, dr. \end{aligned}$$

Finally, the result follows by (2.11) and the Lebesgue's dominated convergence theorem. □

In Section 3, we will consider the semilinear version of Equation 2.7. We show how this technique considerably is useful to study situations with globally and locally Lischitz conditions.

3. Almost automorphic solutions

In this section we consider the semilinear abstract differential equation described in the form

$$u'(t) = Au(t) + f(t, u(t)), \ t \in \mathbb{R}, \tag{3.1}$$

where $A : D(A) \subset X \to X$ is the generator of a hyperbolic analytic semigroup $(T(t))_{t\geq 0}$. The function $f : \mathbb{R}\times X_\alpha \to X$ is a suitable continuous function, where, for $\alpha \in (0, 1)$, X_α is a intermediate space between $D(A)$ and X.

Similarly to previous subsection we remember the concept of mild solution for Equation (3.1).

Definition 3.1. A bounded continuous function $u : \mathbb{R} \to X_\alpha$ is called a mild solution of (2.7) if

$$u(t) = T(t-s)u(s) + \int_s^t T(t-r)f(r, u(r))dr,$$

for all $t \geq s$ and all $s \in \mathbb{R}$.

Theorem 3.2. *Let $f \in AA(X_\alpha; X)$ and suppose that there is a locally integrable function $L : \mathbb{R} \to [0, \infty)$ such that*

$$\|f(t,x) - f(t,y)\| \leq L(t)\|x - y\|_\alpha, \quad (x, y \in X_\alpha,\ t \in \mathbb{R}). \tag{3.2}$$

Let $\theta(t) = M(\alpha)\int_{-\infty}^t (t-s)^{-\alpha}e^{-\gamma(t-s)}L(s)ds + C(\alpha)\int_t^\infty e^{\delta(t-s)}L(s)ds$. If there is a constant $k > 0$ such that $\theta(t) \leq k < 1$, where $M(\alpha)$ and $C(\alpha)$ are the constants given in Lemma 2.10, then Equation (3.1) has a unique mild solution $u \in AA(X_\alpha)$.

Proof. We define the operator Λ on the space $AA(X_\alpha)$ by

$$\Lambda u(t) = \int_{-\infty}^t T(t-s)Pf(s, u(s))ds - \int_t^\infty T(t-s)Qf(s, u(s))ds. \tag{3.3}$$

Follows from Lemma 2.4 that $f(\cdot, u(\cdot)) \in AA(X)$ for every $u \in AA(X_\alpha)$. Furthermore, by demonstration of Theorem 2.12 we have that the operator $\Lambda : AA(X_\alpha) \to AA(X_\alpha)$ is well defined. We next proof that Λ is a k-contraction. In fact, if $u, v \in AA(X_\alpha)$ we have that

$$\begin{aligned}
\|\Lambda u(t) - \Lambda v(t)\|_\alpha &\leq \int_{-\infty}^t \|T(t-s)P\big(f(s,u(s)) - f(s,v(s))\big)\|_\alpha ds \\
&+ \int_t^\infty \|T(t-s)Q\big(f(s,v(s)) - f(s,u(s))\big)\|_\alpha ds \\
&\leq M(\alpha)\int_{-\infty}^t (t-s)^{-\alpha}e^{-\gamma(t-s)}L(s)\|u(s) - v(s)\|_\alpha ds \\
&+ C(\alpha)\int_t^\infty e^{\delta(t-s)}L(s)\|u(s) - v(s)\|_\alpha ds
\end{aligned}$$

$$\begin{aligned}
&\leq \left(M(\alpha)\int_{-\infty}^t (t-s)^{-\alpha}e^{-\gamma(t-s)}L(s)ds + C(\alpha)\int_t^\infty e^{\delta(t-s)}L(s)ds\right)\|u - v\|_{\alpha,\,\infty} \\
&\leq \theta(t)\|u - v\|_{\alpha,\,\infty} \leq k\|u - v\|_{\alpha,\,\infty}.
\end{aligned}$$

Hence, by the Banach's fixed point theorem we conclude the proof. ☐

The next results are immediate consequences of Theorem 3.2.

Corollary 3.3. *Let $f \in AA(X_\alpha; X)$ and suppose that f satisfies the Lipschitz condition (3.2) with L a bounded continuous function. Let*

$$\theta(t) = M(\alpha)\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\gamma(t-s)}L(s)ds + C(\alpha)\int_{t}^{\infty} e^{\delta(t-s)}L(s)ds.$$

If there is a constant $k > 0$ such that $\theta(t) \le k < 1$, where $M(\alpha)$ and $C(\alpha)$ are the constants given in Lemma 2.10, then Equation (3.1) has a unique mild solution $u \in AA(X_\alpha)$.

Corollary 3.4. *Let $f \in AA(X_\alpha; X)$ and suppose that f satisfies the Lipschitz condition*

$$\|f(t,x) - f(t,y)\| \le k\|x-y\|_\alpha$$

for all $x, y \in X_\alpha$ and $t \in \mathbb{R}$. If $k\big(M(\alpha)\gamma^{\alpha-1}\Gamma(1-\alpha) + C(\alpha)\delta^{-1}\big) < 1$, where $M(\alpha)$ and $C(\alpha)$ are the constants given in Lemma 2.10, then Equation (3.1) has a unique mild solution $u \in AA(X_\alpha)$.

We now move on to the problem of locally Lipschitz perturbations for Equation (3.1). In this sense, the following theorem is the main result of this section.

Remark 3.5. In the following result we will use the notion of locally bounded function, that is, we consider a function $L : X_\alpha \times X_\alpha \to [0,\infty)$ such that for every $r \ge 0$ there is a constant $k(r) \ge 0$ such that $L(x,y) \le k(r)$, for all $x, y \in X_\alpha$ with $\|x\|_\alpha \le r$ and $\|y\|_\alpha \le r$.

Theorem 3.6. *Let $f \in AA(X_\alpha; X)$ and assume that there is a locally bounded function $L : X_\alpha \times X_\alpha \to [0,\infty)$ such that for every $x, y \in X_\alpha$ we have*

$$\|f(t,x) - f(t,y)\| \le L(x,y)\big(1 + \|x\|_\alpha^{l-1} + \|y\|_\alpha^{l-1}\big)\|x-y\|_\alpha \quad (t \in \mathbb{R}),$$

where $l \ge 1$. If there is $R > 0$ such that

$$\left(K(R) + \frac{\|f(\cdot,0)\|_\infty}{R}\right)\big(M(\alpha)\gamma^{\alpha-1}\Gamma(1-\alpha) + C(\alpha)\delta^{-1}\big) < 1,$$

where

$$K(R) := k(R)(1 + 2R^{l-1}),$$

with $k(R)$ as in Remark 3.5, and $M(\alpha)$ and $C(\alpha)$ are the constants given in Lemma 2.10. Then Equation (3.1) has a unique almost automorphic mild solution.

Proof. Define the operator Λ by expression (3.3). Consider $R > 0$ such that

$$\big(RK(R) + \|f(\cdot,0)\|_\infty\big)\big(M(\alpha)\gamma^{\alpha-1}\Gamma(1-\alpha) + C(\alpha)\delta^{-1}\big) < R.$$

Let B_R be the close ball

$$B_R = \{u \in AA(X_\alpha) : \|u\|_{\alpha,\infty} \le R\} \subset AA(X_\alpha).$$

We observe that if $u \in B_R$ then

$$\begin{aligned}
\|\Lambda u(t)\|_\alpha &\leq \int_{-\infty}^{t} \|T(t-s)Pf(s,u(s))\|_\alpha ds + \int_{t}^{\infty} \|T(t-s)Qf(s,u(s))ds\|_\alpha ds \\
&\leq M(\alpha)\int_{-\infty}^{t}(t-s)^{\alpha}e^{-\gamma(t-s)}L(u(s),0)\big(1+\|u(s)\|_\alpha^{l-1}\big)\|u(s)\|_\alpha ds \\
&+ C(\alpha)\int_{t}^{\infty} e^{\delta(t-s)}L(u(s),0)\big(1+\|u(s)\|_\alpha^{l-1}\big)\|u(s)\|_\alpha ds \\
&+ \left(M(\alpha)\int_{-\infty}^{t}(t-s)^{\alpha}e^{-\gamma(t-s)}ds + C(\alpha)\int_{t}^{\infty} e^{\delta(t-s)}ds\right)\|f(\cdot,0)\|_\infty
\end{aligned}$$

$$\leq \big(M(\alpha)\gamma^{\alpha-1}\Gamma(1-\alpha)+C(\alpha)\delta^{-1}\big)\big(RK(R)+\|f(\cdot,0)\|_\infty\big) \leq R.$$

Therefore, $\Lambda(B_R) \subset B_R$. It remains show that Λ is a contraction. But this follows from estimate

$$\begin{aligned}
\|\Lambda u(t) - \Lambda v(t)\|_\alpha &\leq \int_{-\infty}^{t} \|T(t-s)P\big(f(s,u(s))-f(s,v(s))\big)\|_\alpha ds \\
&+ \int_{t}^{\infty} \|T(t-s)Q\big(f(s,v(s))-f(s,u(s))\big)\|_\alpha ds
\end{aligned}$$

$$\leq M(\alpha)\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\gamma(t-s)}L(u(s),v(s))\big(1+\|u(s)\|_\alpha^{l-1}+\|v(s)\|_\alpha^{l-1}\big)ds\|u-v\|_{\alpha,\infty}$$

$$+C(\alpha)\int_{t}^{\infty} e^{\delta(t-s)}L(u(s),v(s))\big(1+\|u(s)\|_\alpha^{l-1}+\|v(s)\|_\alpha^{l-1}\big)ds\|u-v\|_{\alpha,\infty}$$

$$\leq \big(M(\alpha)\gamma^{\alpha-1}\Gamma(1-\alpha)+C(\alpha)\delta^{-1}\big)K(R)\|u-v\|_{\alpha,\infty}$$

Since $\big(M(\alpha)\gamma^{\alpha-1}\Gamma(1-\alpha)+C(\alpha)\delta^{-1}\big)K(R) < 1$, the assertion is consequence of the contraction mapping principle and the proof is completed. □

The next results are an immediate consequence of Theorem 3.6.

Corollary 3.7. *Let $f \in AA(X_\alpha; X)$ and assume that there is a constant $c \geq 0$ such that for every $x, y \in X_\alpha$ we have*

$$\|f(t,x)-f(t,y)\| \leq c\big(1+\|x\|_\alpha^{l-1}+\|y\|_\alpha^{l-1}\big)\|x-y\|_\alpha \quad (t \in \mathbb{R}),$$

where $l \geq 1$. If c is small enough then Equation (3.1) has a unique almost automorphic mild solution.

Corollary 3.8. *Let $f \in AA(X_\alpha; X)$ and assume that for every $r \geq 0$ there is a constant $L(r) \geq 0$ such that for every $x, y \in X_\alpha$, with $\|x\|_\alpha \leq r$ and $\|y\|_\alpha \leq r$, we have*

$$\|f(t,x)-f(t,y)\| \leq L(r)\|x-y\|_\alpha \quad (t \in \mathbb{R}).$$

If there is $R > 0$ such that

$$\left(L(R)+\frac{\|f(\cdot,0)\|_\infty}{R}\right)\big(M(\alpha)\gamma^{\alpha-1}\Gamma(1-\alpha)+C(\alpha)\delta^{-1}\big) < 1,$$

where $M(\alpha)$ and $C(\alpha)$ are the constants given in Lemma 2.10 Then Equation (3.1) has a unique almost automorphic mild solution.

To finishes this work we will consider two simple examples.

Example 3.9. Given $k > 0$, consider the scalar system

$$\left\{\begin{array}{lcl} x'(t) & = & -kx(t) + f_1(t), \\ y'(t) & = & ky(t) + f_2(t), \end{array}\right. \tag{3.4}$$

where $f_1(t) = \sin\left(\frac{1}{\sin(t)+\sin(\sqrt{5}t)}\right)$ and $f_2(t) = \cos\left(\frac{1}{\cos(t)+\cos(\sqrt{2}t)}\right)$, $t \in \mathbb{R}$. The system (3.4) can be rewrite as

$$z'(t) = Az(t) + g(t), \quad t \in \mathbb{R}, \tag{3.5}$$

where $g(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$ and $A = \begin{pmatrix} -k & 0 \\ 0 & k \end{pmatrix}$. Clearly $\sigma(A) \cap i\mathbb{R} = \emptyset$ and therefore

$$e^{At} = \begin{pmatrix} e^{-kt} & 0 \\ 0 & e^{kt} \end{pmatrix}, \quad t \geq 0,$$

is a hyperbolic semigroup with projection

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

constants $M_1 = 1$ and $\delta = k$ (see Definition 2.5). Furthermore, in the point of view of Remark 2.6 we have $X = \mathbb{R}^2$, $X_s = \{(x, 0) : x \in \mathbb{R}\}$ and $X_u = \{(0, y) : y \in \mathbb{R}\}$. Finally, follows from Theorem 2.12 that Equation (3.5) has an almost automorphic solution given by the function

$$z(t) = \int_{-\infty}^{t} e^{-k(t-s)} f_1(s)ds + \int_{t}^{\infty} e^{k(t-s)} f_2(s)ds.$$

Example 3.10. Given $b > 0$, consider the problem

$$\left\{\begin{array}{ll} u_t = u_{xx} + bu + f(t, u_x), & \mathbb{R} \times [0, 1], \\ u(t, 0) = u(t, 1) = 0, & t \in \mathbb{R}, \end{array}\right. \tag{3.6}$$

where f is a continuous function defined on $\mathbb{R}^2$. Let $A : D(A) \subset C([0, 1]) \to C([0, 1])$ be the operator given by $Au = u'' + bu$, with domain

$$D(A) = C_0^2([0, 1]) = \{u \in C^2([0, 1]) : u(0) = u(1) = 0\}.$$

It is well known that A is a sectorial operator and therefore is the generator of an analytic semigroup $(T(t))_{t\geq 0}$ on $C([0, 1])$. Furthermore, we have that the spectrum of A is a discrete set. Indeed,

$$\sigma(A) = \{b - n^2\pi^2 : n \in \mathbb{N}\}.$$

Hence, by taking b such that $\frac{\sqrt{b}}{\pi} \in (1, \infty)\backslash\mathbb{Q}$, follows that $(T(t))_{t\geq 0}$ is a hyperbolic analytic semigroup.

In the other hand, if $\alpha \neq 1/2$, consider $X_\alpha = D_A(\alpha, \infty) = C_0^{2\alpha}([0, 1])$. In (3.6), we suppose, for example, a nonlinear term $f : \mathbb{R} \times C_0^{2\alpha}([0, 1]) \to C([0, 1])$ given by

$$f(t, \phi)(x) = f(t, \phi'(x)) := \frac{ka(t)}{1 + |\phi'(x)|},$$

where $a \in AA(\mathbb{R})$ and $k > 0$. A straightforward computation shows that $f \in AA(X_\alpha, X)$ and verifies the Lipschitz condition (3.2) with L a bounded continuous function. Then follows from Corollary 3.3 that if k is small enough, Problem (3.6) has a unique almost automorphic mild solution.

Acknowledgement. The authors would like to thank the referee very much for helpful comments and suggestions, which enable us improve this work. Bruno de Andrade is partially supported by CNPQ/Brazil under Grant 100994/2011-3. Claudio Cuevas is partially supported by CNPQ/Brazil under Grant 300365/2008-0.

References

1. S. Bochner, *Continuous mapping of almost automorphic and almost periodic functions*, Proc. nat. Acad. Sci. USA **52** (1964), 907–910.
2. S. Boulite, L. Maniar and G. N'Guérékata, *Almost automorphic solutions for hyperbolic semilinear evolution equations*. Semigroup Forum **71** (2005), no. 2, 231–240.
3. T. Diagana, G.N. NǴuérékata and N.V. Minh, *Almost automorphic solutions of evolution equations*, Proc. Amer. Math. Soc. **132** (2004), no. 11, 3289–3298.
4. K.J. Engel and R. Nagel, *One-parameter semigroup for linear evolution equations, in: Graduate Texts in Mathematics*, Springer-Verlag, 2001, p. 195.
5. G.A. Goldstein and G.M. NǴuérekata, *Almost automorphic solutions of semilinear evolution equations*, Proc. Amer. Math. Soc. **133** (2005), no. 8, 2401–2408.
6. M. Haase, *The Functional Calculus for Sectorial Operators. Operator Theory: Advances and Applications*, 169, Birkhauser-Verlag, Basel, 2006.
7. J. Liang, J. Zhang and T.J. Xiao, *Composition of pseudo almost automorphic and asymptotically almost automorphic functions*, J. Math. Anal. Appl. **340** (2008), 1493–1499.
8. J. Liu and X. Song, *Almost automorphic and weighted pseudo almost automorphic solutions of semilinear evolution equations*, J. Funct. Anal. **258** (2010), 196–207.
9. A. Lunardi, *Analytic Semigroup and Optimal Regularity in Parabolic Problems*, Progr. Nonlinear Differential Equations Appl., 16, Birkhäuser-Verlag, Besel, 1995.
10. N.V. Minh, T. Naito and G.M. NǴuérékata, *A spectral countability condition for almost automorphy of solutions of differential equations*, Proc. Amer. Math. Soc. **134** (2006), no. 11, 3257–3266.
11. G.M. N'Guérékata, *Almost Automorphic and Almost Periodic Functions in Abstract Spaces,*Kluwer Acad/Plenum, New York-Boston-Moscow-London, 2001.
12. G.M. NǴuérékata, *Topics in Almost Automorphy*, Springer-Verlag, New York, 2005.

[1] Universidade de So Paulo, Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, CEP. 13569-970 , São Carlos-SP, Brazil.
E-mail address: bruno00luis@gmail.com

[2] Departamento de Matemática, Universidade Federal de Pernambuco, Recife-PE, CEP. 50540-740, Brazil.
E-mail address: cch@dmat.ufpe.br

[3] Departamento de Matemática e Estadística, Universidad de La Frontera, Cosilla 54D Temuco, Chile.
E-mail address: ehenri@ufro.cl