Banach J. Math. Anal. 5 (2011), no. 1, 70–82

Banach Journal of Mathematical Analysis
ISSN: 1735-8787 (electronic)
www.emis.de/journals/BJMA/

ON CONVERSE THEOREMS OF TRIGONOMETRIC APPROXIMATION IN WEIGHTED VARIABLE EXPONENT LEBESGUE SPACES

RAMAZAN AKGÜN[1*] AND VAKHTANG KOKILASHVILI[2]

Communicated by L.-E. Persson

Abstract. In this work we prove improved converse theorems of trigonometric approximation in variable exponent Lebesgue spaces with some Muckenhoupt weights.

1. Introduction and the main results

In Approximation Theory there are converse estimates of trigonometric approximation determining membership of a function in some smoothness class (for example Lipschitz class) in terms of the rate of approximation. As is well-known the converse inequality

$$\omega_r\left(f,\frac{1}{n}\right)_p \leq \frac{c_1}{n^r}\left\{\sum_{\nu=0}^{n}(\nu+1)^{r-1}E_\nu(f)_p\right\} \tag{1.1}$$

of trigonometric approximation holds on Lebesgue spaces $L^p(\boldsymbol{T})$, $1\leq p<\infty$, or $C(\boldsymbol{T})$ (of continuous functions on $\boldsymbol{T}$) for $p=\infty$, ([19] $p=\infty$, [21] $p<\infty$) where $\boldsymbol{T}:=[0,2\pi)$, $f\in L^p(\boldsymbol{T})$, $1\leq p\leq\infty$, $r,n\in\mathbb{N}:=\{1,2,3,\dots\}$, $T_hf(\circ):=f(\circ+h)$ is translation operator, $\omega_r(f,\delta)_p:=\sup\left\{\|(T_h-I)^r f\|_p : 0<h\leq\delta\right\}$ is the *rth moduli of smoothness* of the function f, I is identity operator, $\mathcal{T}_n$ is

Date: Received: 17 March 2010; Accepted: 9 June 2010.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 42A10; Secondary 26A33, 41A17, 41A20, 41A25, 41A27.

Key words and phrases. weighted fractional moduli of smoothness, converse theorem, fractional derivative.

the class of trigonometric polynomials of degree not greater than n, $E_n(f)_p := \inf\left\{\|f-T\|_p : T \in \mathcal{T}_n\right\}$ and c_1 is a constant depending only on r, p. Later various generalizations and applications of (1.1) were obtained [16, 17, 18, 20]. In 1958 Timan proved [22] that an improvement of (1.1) also holds:

If $1 < p < \infty$, $f \in L^p(\boldsymbol{T})$, n, $r \in \mathbb{N}$, $q = \min\{2, p\}$ then

$$\omega_r\left(f, \frac{1}{n}\right)_p \leq \frac{c_2}{n^r}\left\{\sum_{\nu=1}^{n} \nu^{rq-1} E_{\nu-1}^q(f)_p\right\}^{1/q} \tag{1.2}$$

where c_2 is a constant depending only on r and p.

It was observed that the value $\min\{2, p\}$ in (1.2) is optimal [23]. See also [4, 9, 10].

Considering similar problems in weighted function spaces (for example weighted Lebesgue spaces L^p_ω, weighted variable exponent space, ...) we will need a different moduli of smoothness. Moduli of this type was considered first by Hadjieva [6] in Lebesgue space with Muckenhoupt A_p, $1 < p < \infty$, (see definition below) weights: Let $\omega \in A_p$, $1 < p < \infty$, $f \in L^p_\omega$, r, $n \in \mathbb{N}$ and let

$$\sigma_h f(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(t)\, dt \text{ for } h \in \mathbb{R} \text{ and } x \in \boldsymbol{T}.$$

In this case defining the modulus

$$\Omega_r(f, \delta)_{p,\omega} := \sup_{0 \leq h_i \leq \delta} \left\| \prod_{i=1}^{r} (I - \sigma_{h_i}) f \right\|_{p(\cdot),\omega}, \quad \delta \geq 0$$

she proved [6] that

$$\Omega_r\left(f, \frac{1}{n}\right)_{p,\omega} \leq \frac{c_3}{n^{2r}}\left\{E_0(f)_{p,\omega} + \sum_{\nu=1}^{n} \nu^{2r-1} E_\nu(f)_{p,\omega}\right\} \tag{1.3}$$

where c_3 is a constant depending only on r and p.

For further results [1, 3, 8, 14, 15].

On the other hand inequality (1.3) also has an improvement [14, 15]:

If $1 < p < \infty$, $\omega \in A_p$, $f \in L^p_\omega$, r, $n \in N$, then there is a positive constant c_4 depending only on r and p such that

$$\Omega_r\left(f, \frac{1}{n}\right)_{p,\omega} \leq \frac{c_4}{n^{2r}}\left\{\sum_{\nu=1}^{n} \nu^{2qr-1} E_{\nu-1}^q(f)_{p,\omega}\right\}^{1/q} \tag{1.4}$$

holds.

Using a weighted fractional moduli of smoothness [1] it was proved [2] that (1.4) holds with $r \in \mathbb{R}^+$. For weighted variable exponent Lebesgue spaces it was proved [5] that (1.4) holds with $r \in \mathbb{R}^+$. In the present work we prove that in the right side of (1.4) $2r$ can be replaced by r (as in nonweighted case) for weighted variable exponent Lebesgue spaces. We note that nonweighted fractional moduli of smoothness in classical Lebesgue spaces was first introduced by Taberski and Butzer in 1977.

We begin with some definitions. Let $\mathcal{P}$ be the class of Lebesgue measurable functions $p : \boldsymbol{T} \to (1, \infty)$ such that $1 < p_* := \operatorname{essinf}_{x\in\boldsymbol{T}} p(x) \leq p^* := \operatorname{esssup}_{x\in\boldsymbol{T}} p(x) < \infty$. The conjugate exponent of $p(x)$ is defined as $p'(x) := p(x) / (p(x) - 1)$. We define a class $L_{2\pi}^{p(\cdot)}$ of 2π periodic measurable functions $f : \boldsymbol{T} \to \mathbb{C}$ satisfying

$$\int_{\boldsymbol{T}} |f(x)|^{p(x)} \, dx < \infty$$

for $p \in \mathcal{P}$ where $\mathbb{C}$ is the complex plane.

The class $L_{2\pi}^{p(\cdot)}$ is a Banach space with the norm

$$\|f\|_{\boldsymbol{T},p(\cdot)} := \inf \left\{ \alpha > 0 : \int_{\boldsymbol{T}} \left| \frac{f(x)}{\alpha} \right|^{p(x)} dx \leq 1 \right\}.$$

A function $\omega : \boldsymbol{T} \to [0, \infty]$ will be called a weight if ω is measurable and almost everywhere (a.e.) positive. For a 2π periodic weight ω we denote by L_ω^p the weighted Lebesgue space of 2π periodic measurable functions $f : \boldsymbol{T} \to \mathbb{C}$ such that $f\omega^{1/p} \in L^p(\boldsymbol{T})$. We set $\|f\|_{p,\omega} := \left\| f\omega^{1/p} \right\|_p$ for $f \in L_\omega^p$. We will denote by $L_\omega^{p(\cdot)}$, the class of Lebesgue measurable functions $f : \boldsymbol{T} \to \mathbb{C}$ satisfying $\omega f \in L_{2\pi}^{p(\cdot)}$. $L_\omega^{p(\cdot)}$ is called weighted Lebesgue spaces with variable exponent and is a Banach space with the norm $\|f\|_{p(\cdot),\omega} := \|\omega f\|_{\boldsymbol{T},p(\cdot)}$.

For given $p \in \mathcal{P}$ the class of weights ω satisfying the condition [7]

$$\left\|\omega^{p(x)}\right\|_{A_{p(\cdot)}} := \sup_{B\in\mathcal{B}} \frac{1}{|B|^{p_B}} \left\|\omega^{p(x)}\right\|_{L^1(B)} \left\| \frac{1}{\omega^{p(x)}} \right\|_{B,(p'(\cdot)/p(\cdot))} < \infty$$

will be denoted by $A_{p(\cdot)}$. Here $p_B := \left(\frac{1}{|B|} \int_B \frac{1}{p(x)} dx \right)^{-1}$ and $\mathcal{B}$ is the class of all intervals in $\boldsymbol{T}$.

The variable exponent $p(x)$ is said to be satisfy *Local log-Hölder continuity condition* if there is a positive constant c_5 such that

$$|p(x_1) - p(x_2)| \leq \frac{c_5}{\log(e + 1/|x_1 - x_2|)} \quad \text{for all } x_1, x_2 \in \boldsymbol{T}. \tag{1.5}$$

We will denote by $\mathcal{P}^{\log}$ the class of those $p \in \mathcal{P}$ satisfying (1.5).

Let $f \in L_\omega^{p(\cdot)}$ and

$$\mathcal{A}_h f(x) := \frac{1}{h} \int_{x-h/2}^{x+h/2} f(t) \, dt, \quad x \in \boldsymbol{T}$$

be Steklov's mean operator. If $p \in \mathcal{P}^{\log}$, then it was proved in [7] that Hardy Littlewood maximal operator $\mathcal{M}$ is bounded in $L_\omega^{p(\cdot)}$ if and only if $\omega \in A_{p(\cdot)}$.

Therefore if $p\in\mathcal{P}^{\log}$ and $\omega\in A_{p(\cdot)}$, then $\mathcal{A}_h$ is bounded in $L_{\omega}^{p(\cdot)}$. Using these facts and setting $x,h\in\boldsymbol{T}$, $0\leq r$ we define via binomial expansion that

$$\begin{aligned}\sigma_h^r f(x) &= (\mathcal{A}_h-I)^r f(x)\\ &= \sum_{k=0}^{\infty}(-1)^k\left(\begin{array}{c}r\\k\end{array}\right)\frac{1}{h^k}\int_{-h/2}^{h/2}\cdots\int_{-h/2}^{h/2} f(x+u_1+\ldots u_k)\,du_1\ldots du_k,\end{aligned}$$

where $f\in L_{\omega}^{p(\cdot)}$, $\left(\begin{array}{c}r\\k\end{array}\right):=\frac{r(r-1)\ldots(r-k+1)}{k!}$ for $k>1$, $\left(\begin{array}{c}r\\1\end{array}\right):=r$ and $\left(\begin{array}{c}r\\0\end{array}\right):=1$.

Since

$$\sum_{k=0}^{\infty}\left|\left(\begin{array}{c}r\\k\end{array}\right)\right|<\infty$$

if $p\in\mathcal{P}^{\log}$, $\omega\in A_{p(\cdot)}$ and $f\in L_{\omega}^{p(\cdot)}$, then there exists a positive constant c_6 depending only on r and p such that

$$\left\|\sigma_h^r f\right\|_{p(\cdot),\omega}\leq c_6\left\|f\right\|_{p(\cdot),\omega}<\infty \tag{1.6}$$

holds.

For $0\leq r$ now we can define the *fractional moduli of smoothness of index r* for $p\in\mathcal{P}^{\log}$, $\omega\in A_{p(\cdot)}$ and $f\in L_{\omega}^{p(\cdot)}$ as

$$\Omega_r(f,\delta)_{p(\cdot),\omega}:=\sup_{0<h_i,t\leq\delta}\left\|\prod_{i=1}^{[r]}\left(I-\mathcal{A}_{h_i}\right)\sigma_t^{r-[r]}f\right\|_{p(\cdot),\omega},\quad \delta\geq 0,$$

where $\Omega_0(f,\delta)_{p(\cdot),\omega}:=\left\|f\right\|_{p(\cdot),\omega}$; $\prod_{i=1}^{0}\left(I-\mathcal{A}_{h_i}\right)\sigma_t^r f:=\sigma_t^r f$ for $0<r<1$; and $[r]$ denotes the integer part of the real number r.

We have by (1.6) that if $p\in\mathcal{P}^{\log}$, $\omega\in A_{p(\cdot)}$ and $f\in L_{\omega}^{p(\cdot)}$, then there exist a positive constant c_7 depending only on r and p such that

$$\Omega_r(f,\delta)_{p(\cdot),\omega}\leq c_7\left\|f\right\|_{p(\cdot),\omega}.$$

If $p\in\mathcal{P}^{\log}$ and $\omega\in A_{p(\cdot)}$, then $\omega^{p(x)}\in L^1(\boldsymbol{T})$. This implies that the set of trigonometric polynomials is dense [11] in $L_{\omega}^{p(\cdot)}$. On the other hand if $p\in\mathcal{P}^{\log}$ and $\omega\in A_{p(\cdot)}$, then $L_{\omega}^{p(\cdot)}\subset L^1(\boldsymbol{T})$.

For given $f\in L^1(\boldsymbol{T})$, let

$$f(x)\backsim\frac{a_0(f)}{2}+\sum_{k=1}^{\infty}\left(a_k(f)\cos kx+b_k(f)\sin kx\right)=\sum_{k=-\infty}^{\infty}c_k(f)e^{ikx} \tag{1.7}$$

be the *Fourier series* of f with $c_k(f)=(1/2)\left(a_k(f)-ib_k(f)\right)$. We set

$$L_0^1(\boldsymbol{T}):=\left\{f\in L^1(\boldsymbol{T}):c_0(f)=0\text{ for the series in (1.7)}\right\}.$$

Let $\alpha \in \mathbb{R}^{+}$ be given. We define *fractional derivative* of a function $f \in L_0^1(\boldsymbol{T})$ as

$$f^{(\alpha)}(x) := \sum_{k=-\infty}^{\infty} c_k(f)(ik)^{\alpha} e^{ikx}$$

provided the right hand side exists, where $(ik)^{\alpha} := |k|^{\alpha} e^{(1/2)\pi i\alpha \operatorname{sign} k}$ as principal value. We will say that a function $f \in L_{\omega}^{p(\cdot)}$ has fractional derivative of order $\alpha \in \mathbb{R}^{+}$ if there exists a function $g \in L_{\omega}^{p(\cdot)}$ such that its Fourier coefficients satisfy $c_k(g) = c_k(f)(ik)^{\alpha}$. In this case we will write $f^{(\alpha)} = g$.

Let $W_{p(\cdot),\omega}^{\alpha}$, $p \in \mathcal{P}$, $\alpha > 0$ be the class of functions $f \in L_{\omega}^{p(\cdot)}$ such that $f^{(\alpha)} \in L_{\omega}^{p(\cdot)}$. $W_{p(\cdot),\omega}^{\alpha}$ becomes a Banach space with the norm

$$\|f\|_{W_{p(\cdot),\omega}^{\alpha}} := \|f\|_{p(\cdot),\omega} + \left\|f^{(\alpha)}\right\|_{p(\cdot),\omega}.$$

We set $E_n(f)_{p(\cdot),\omega} := \inf\left\{\|f-T\|_{p(\cdot),\omega} : T \in \mathcal{T}_n\right\}$ for $f \in L_{\omega}^{p(\cdot)}$.

Our main results are

Theorem 1.1. *If $p \in P^{\log}$, $\omega^{-p_0} \in A_{\left(\frac{p(\cdot)}{p_0}\right)'}$ for some $p_0 \in (1, p_*)$, $n \in \mathbb{N}$, $r \in \mathbb{R}^{+}$, $\gamma := \min\{2, p_*\}$ and $f \in L_{\omega}^{p(\cdot)}$, then there exists a positive constant c_8 depending only on r and p such that*

$$\Omega_r\left(f, \frac{1}{n}\right)_{p(\cdot),\omega} \leq \frac{c_8}{n^r}\left\{\sum_{\nu=1}^{n} \nu^{\gamma r-1} E_{\nu-1}^{\gamma}(f)_{p(\cdot),\omega}\right\}^{1/\gamma}$$

holds.

Since x^{γ} is convex for $\gamma = \min\{2, p_*\}$ we have

$$\left(\nu\nu^{r-1} E_{\nu}(f)_{p(\cdot),\omega}\right)^{\gamma} - \left((\nu-1)\nu^{r-1} E_{\nu}(f)_{p(\cdot),\omega}\right)^{\gamma} \leq$$

$$\leq \left(\sum_{\mu=1}^{\nu} \mu^{r-1} E_{\mu}(f)_{p(\cdot),\omega}\right)^{\gamma} - \left(\sum_{\mu=1}^{\nu-1} \mu^{r-1} E_{\mu}(f)_{p(\cdot),\omega}\right)^{\gamma}.$$

Summing the last inequality with $\nu = 1, 2, 3, \ldots$ we find

$$\sum_{\nu=1}^{n}\left\{\left(\nu\nu^{r-1} E_{\nu}(f)_{p(\cdot),\omega}\right)^{\gamma} - \left((\nu-1)\nu^{r-1} E_{\nu}(f)_{p(\cdot),\omega}\right)^{\gamma}\right\} \leq$$

$$\leq \sum_{\nu=1}^{n}\left\{\left(\sum_{\mu=1}^{\nu} \mu^{r-1} E_{\mu}(f)_{p(\cdot),\omega}\right)^{\gamma} - \left(\sum_{\mu=1}^{\nu-1} \mu^{r-1} E_{\mu}(f)_{p(\cdot),\omega}\right)^{\gamma}\right\}$$

and hence

$$\left\{\sum_{\nu=1}^{n} \nu^{\gamma r-1} E_{\nu-1}^{\gamma}(f)_{p(\cdot),\omega}\right\}^{1/\gamma} \leq 2\sum_{\nu=1}^{n} \nu^{r-1} E_{\nu-1}(f)_{p(\cdot),\omega}.$$

The last inequality signifies that Theorem 1.1 is a refinement of the converse theorem (see [2, 3]). Furthermore, in some cases, inequalities in Theorem 1.1 gave more precise results:

If

$$E_n(f)_{p(\cdot),\omega} = \mathcal{O}\left(\frac{1}{n^r}\right), \quad n \in \mathbb{N}$$

then we have

$$\Omega_r\left(f, \frac{1}{n}\right)_{p(\cdot),\omega} = \mathcal{O}\left(\frac{1}{n^r}\left|\log\frac{1}{n}\right|\right)$$

and from Theorem 1.1

$$\Omega_r\left(f, \frac{1}{n}\right)_{p(\cdot),\omega} = \mathcal{O}\left(\frac{1}{n^r}\left|\log\frac{1}{n}\right|^{1/\gamma}\right).$$

As a corollary of Theorem 1.1 we have the following improvements of Marchaud inequality

Corollary 1.2. *Under the conditions of Theorem 1.1 if $r, l \in \mathbb{R}^+$, $r < l$, and $0 < t \leq 1/2$, then there exists a positive constant c_9 depending only on r, l and p such that*

$$\Omega_r(f,t)_{p(\cdot),\omega} \leq c_9 t^r \left\{\int_t^1 \left[\frac{\Omega_l(f,u)_{p(\cdot),\omega}}{u^r}\right]^\gamma \frac{du}{u}\right\}^{1/\gamma}$$

hold.

Theorem 1.3. *Under the conditions of Theorem 1.1 if*

$$\sum_{k=1}^{\infty} k^{\gamma\alpha-1} E_k^\gamma(f)_{p(\cdot),\omega} < \infty \tag{1.8}$$

for some $\alpha \in \mathbb{R}^+$, then $f \in W^\alpha_{p(\cdot),\omega}$. Furthermore, for $n \in \mathbb{N}$ there exists a constant $c_{10} > 0$ depending only on α and p such that

$$E_n\left(f^{(\alpha)}\right)_{p(\cdot),\omega} \leq c_{10}\left(n^\alpha E_n(f)_{p(\cdot),\omega} + \left\{\sum_{\nu=n+1}^{\infty} \nu^{\alpha\gamma-1} E_\nu^\gamma(f)_{p(\cdot),\omega}\right\}^{1/\gamma}\right)$$

holds.

Corollary 1.4. *Under the conditions of Theorem 1.1 there exists a constant $c_{11} > 0$ depending only on r, α and p such that*

$$\begin{aligned}\Omega_r\left(f^{(\alpha)}, \frac{1}{n}\right)_{p(\cdot),\omega} \quad &\leq \quad c_{11}\left(\frac{1}{n^r}\left(\sum_{\nu=1}^{n} \nu^{\gamma(r+\alpha)-1} E_\nu^\gamma(f)_{p(\cdot),\omega}\right)^{\frac{1}{\gamma}} + \right.\\ &\left. + \left(\sum_{\nu=n+1}^{\infty} \nu^{\alpha\gamma-1} E_\nu^\gamma(f)_{p(\cdot),\omega}\right)^{\frac{1}{\gamma}}\right)\end{aligned}$$

for $n \in \mathbb{N}$ *and* $\alpha, r \in \mathbb{R}^{+}$.

2. Proofs of Theorems

We need the following [12] Littlewood–Paley type theorem:

Theorem 2.1. *Under the conditions of Theorem 1.1 there are constants* c_{12}, $c_{13} > 0$ *depending only on* r *and* p *such that*

$$c_{12}\left\|\left(\sum_{\mu=\nu}^{\infty}\left|\Delta_{\mu}\right|^{2}\right)^{1/2}\right\|_{p(\cdot),\omega} \leq \left\|\sum_{|\mu|=2^{\nu-1}}^{\infty} c_{\nu}e^{i\nu x}\right\|_{p(\cdot),\omega} \leq c_{13}\left\|\left(\sum_{\mu=\nu}^{\infty}\left|\Delta_{\mu}\right|^{2}\right)^{1/2}\right\|_{p(\cdot),\omega} \tag{2.1}$$

where

$$\Delta_{\mu} := \Delta_{\mu}(x,f) := \sum_{|\nu|=2^{\mu-1}}^{2^{\mu}-1} c_{\nu}e^{i\nu x}.$$

Lemma 2.2. *If* $p \in P^{\log}$ *and* $\omega^{-p_0} \in A_{\left(\frac{p(\cdot)}{p_0}\right)'}$ *for some* $p_0 \in (1, p_*)$, *then* $\omega \in A_{p(\cdot)}$.

Proof. Using the Extrapolation Theorem 3.2 of [12] we obtain that Hardy Littlewood maximal operator $\mathcal{M}$ is bounded in $L_{\omega}^{p(\cdot)}$. This implies [7] that $\omega \in A_{p(\cdot)}$. □

Proof of Theorem 1.1. First we note by Lemma 2.2 that under the conditions of Theorem 1.1 the condition $\omega \in A_{p(\cdot)}$ holds. On the other hand it is well-known that $\sigma_{t,h_1,h_2,\ldots,h_{[r]}}^{r} f := \prod_{i=1}^{[r]} \left(I-\sigma_{h_i}\right)\left(I-\sigma_t\right)^{r-[r]} f$ has Fourier series

$$\sigma_{t,h_1,h_2,\ldots,h_{[r]}}^{r} f(\cdot) \sim \sum_{\nu=-\infty}^{\infty}\left(1-\frac{\sin\nu t}{\nu t}\right)^{r-[r]}\left(1-\frac{\sin\nu h_1}{\nu h_1}\right)\ldots\left(1-\frac{\sin\nu h_{[r]}}{\nu h_{[r]}}\right)c_{\nu}e^{i\nu\cdot}$$

and

$$\sigma_{t,h_1,h_2,\ldots,h_{[r]}}^{r} f(\cdot) = \sigma_{t,h_1,h_2,\ldots,h_{[r]}}^{r}\left(f(\cdot)-S_{2^{m-1}}(\cdot,f)\right)+\sigma_{t,h_1,h_2,\ldots,h_{[r]}}^{r} S_{2^{m-1}}(\cdot,f).$$

From $E_n(f)_{p(\cdot),\omega} \downarrow 0$ we have

$$\begin{aligned}\left\|\sigma_{t,h_1,h_2,\ldots,h_{[r]}}^{r}\left(f(\cdot)-S_{2^{m-1}}(\cdot,f)\right)\right\|_{p(\cdot),\omega} &\leq c_{14}(r,p)\left\|f(\cdot)-S_{2^{m-1}}(\cdot,f)\right\|_{p(\cdot),\omega}\\ &\leq c_{15}(r,p)\,E_{2^{m-1}}(f)_{p(\cdot),\omega}\\ &\leq \frac{c_{16}(r,p)}{n^{r}}\left\{\sum_{\nu=1}^{n}\nu^{\gamma r-1}E_{\nu-1}^{\gamma}(f)_{p(\cdot),\omega}\right\}^{1/\gamma}.\end{aligned}$$

On the other hand from (2.1) we get

$$\left\|\sigma_{t,h_1,h_2,\ldots,h_{[r]}}^{r} S_{2^{m-1}}(\cdot,f)\right\|_{p(\cdot),\omega} \leq c_{17}(r,p)\left\|\left\{\sum_{\mu=1}^{m}\left|\delta_{\mu}\right|^{2}\right\}^{1/2}\right\|_{p(\cdot),\omega}$$

where

$$\delta_\mu := \sum_{|\nu|=2^{\mu-1}}^{2^\mu-1} \left(1-\frac{\sin \nu t}{\nu t}\right)^{r-[r]} \left(1-\frac{\sin \nu h_1}{\nu h_1}\right) \dots \left(1-\frac{\sin \nu h_{[r]}}{\nu h_{[r]}}\right) c_\nu e^{i\nu x}.$$

We know that [13]

$$\left\| \left\{ \sum_{\mu=1}^{m} |\delta_\mu|^2 \right\}^{1/2} \right\|_{p(\cdot),\omega} \le \left\{ \sum_{\mu=1}^{m} \|\delta_\mu\|_{p(\cdot),\omega}^{\gamma_1} \right\}^{1/\gamma_1}.$$

We estimate $\|\delta_\mu\|_{p(\cdot),\omega}$. Since

$$\|\delta_\mu\|_{p(\cdot),\omega} = \left\| \sum_{|\nu|=2^{\mu-1}}^{2^\mu-1} \left[|\nu|^r \left(1-\frac{\sin \nu t}{\nu t}\right)^{r-[r]} \left(1-\frac{\sin \nu h_1}{\nu h_1}\right) \dots \left(1-\frac{\sin \nu h_{[r]}}{\nu h_{[r]}}\right) \right] \cdot \right.$$
$$\left. \cdot \left[\frac{1}{|\nu|^r} c_\nu e^{i\nu x} \right] \right\|_{p(\cdot),\omega}$$

using Abel's transformation we get

$$\|\delta_\mu\|_{p(\cdot),\omega} \le \sum_{|\nu|=2^{\mu-1}}^{2^\mu-2} \left| \nu^r \left(1-\frac{\sin \upsilon t}{\upsilon t}\right)^{r-[r]} \left(1-\frac{\sin \upsilon h_1}{\upsilon h_1}\right) \dots \left(1-\frac{\sin \upsilon h_{[r]}}{\upsilon h_{[r]}}\right) - \right.$$

$$- (\nu+1)^r \left(1-\frac{\sin(\nu+1)t}{(\nu+1)t}\right)^{r-[r]} \left(1-\frac{\sin(\nu+1)h_1}{(\nu+1)h_1}\right) \dots$$
$$\left. \dots \left(1-\frac{\sin(\nu+1)h_{[r]}}{(\nu+1)h_{[r]}}\right) \right| \cdot \left\| \sum_{|l|=2^{\mu-1}}^{\nu} \frac{1}{|l|^r} \left|c_l e^{ilx}\right| \right\|_{p(\cdot),\omega} +$$

$$+ \left| (2^\mu-1)^r \left(1-\frac{\sin(2^\mu-1)t}{(2^\mu-1)t}\right)^{r-[r]} \left(1-\frac{\sin(2^\mu-1)h_1}{(2^\mu-1)h_1}\right) \dots \right.$$
$$\left. \dots \left(1-\frac{\sin(2^\mu-1)h_{[r]}}{(2^\mu-1)h_{[r]}}\right) \right| \left\| \sum_{|l|=2^{\mu-1}}^{2^\mu-1} \frac{1}{|l|^r} \left|c_l e^{ilx}\right| \right\|_{p(\cdot),\omega}.$$

We have

$$\begin{aligned} \left\| \sum_{|l|=2^{\mu-1}}^{2^\mu-1} \frac{1}{|l|^r} \left|c_l e^{ilx}\right| \right\|_{p(\cdot),\omega} &\le \frac{c_{18}(r,p)}{|2^{\mu-1}|^r} \left\| \sum_{|l|=2^{\mu-1}}^{2^\mu-1} \left|c_l e^{ilx}\right| \right\|_{p(\cdot),\omega} \\ &= \frac{c_{18}(r,p)}{|2^{\mu-1}|^r} \left\| \sum_{|l|=2^{\mu-1}}^{2^\mu-1} e^{-i\arg(c_l e^{ilx})} \left|c_l e^{ilx}\right| \right\|_{p(\cdot),\omega} \end{aligned}$$

$$= \frac{c_{18}(r,p)}{\left|2^{\mu-1}\right|^r}\left\|\sum_{|l|=2^{\mu-1}}^{2^{\mu}-1} c_l e^{ilx}\right\|_{p(\cdot),\omega} \leq \frac{c_{19}(r,p)}{2^{\mu r}} E_{2^{\mu-1}-1}(f)_{p,\omega}$$

and similarly

$$\left\|\sum_{|l|=2^{\mu-1}}^{\nu} \frac{1}{|l|^r}\left|c_l e^{ilx}\right|\right\|_{p(\cdot),\omega} \leq \frac{c_{19}(r,p)}{2^{\mu r}} E_{2^{\mu-1}-1}(f)_{p,\omega}.$$

Since $x^r\left(1-\frac{\sin x}{x}\right)^r$ is non decreasing and $\left(1-\frac{\sin x}{x}\right) \leq x$ for $x>0$ we obtain

$$\|\delta_\mu\|_{p(\cdot),\omega} \leq \frac{c_{20}(r,p)\,2^{-\mu r}}{t^{r-[r]}h_1\ldots h_{[r]}}\left[\sum_{|\nu|=2^{\mu-1}}^{2^{\mu}-2}\left|(\nu t)^{r-[r]}\left(1-\frac{\sin \upsilon t}{\upsilon t}\right)^{r-[r]}(\upsilon h_1)\cdot\right.\right.$$

$$\left(1-\frac{\sin \upsilon h_1}{\upsilon h_1}\right)\ldots\left(\upsilon h_{[r]}\right)\left(1-\frac{\sin \upsilon h_{[r]}}{\upsilon h_{[r]}}\right)-((\nu+1)\,t)^{r-[r]}\left(1-\frac{\sin(\nu+1)\,t}{(\nu+1)\,t}\right)^{r-[r]}\cdot$$

$$\left.\left.\cdot\,((\upsilon+1)\,h_1)\left(1-\frac{\sin(\nu+1)\,h_1}{(\nu+1)\,h_1}\right)\ldots\left((\nu+1)\,h_{[r]}\right)\left(1-\frac{\sin(\nu+1)\,h_{[r]}}{(\nu+1)\,h_{[r]}}\right)\right|\right]\cdot$$

$$\cdot E_{2^{\mu-1}-1}(f)_{p(\cdot),\omega}+c_{20}(r,p)\,2^{-\mu r}\left|((2^\mu-1)\,t)^{r-[r]}\left(1-\frac{\sin(2^\mu-1)\,t}{(2^\mu-1)\,t}\right)^{r-[r]}\cdot\right.$$

$$\left.\cdot\,(2^\mu-1)\,h_1\left(1-\frac{\sin(2^\mu-1)\,h_1}{(2^\mu-1)\,h_1}\right)\ldots(2^\mu-1)\,h_{[r]}\left(1-\frac{\sin(2^\mu-1)\,h_{[r]}}{(2^\mu-1)\,h_{[r]}}\right)\right|\cdot$$

$$\cdot E_{2^{\mu-1}-1}(f)_{p(\cdot),\omega} \leq c_{21}(r,p)\left(1-\frac{\sin(2^\mu-1)\,t}{(2^\mu-1)\,t}\right)^{r-[r]}\left(1-\frac{\sin(2^\mu-1)\,h_1}{(2^\mu-1)\,h_1}\right)\ldots$$

$$\ldots\left(1-\frac{\sin(2^\mu-1)\,h_{[r]}}{(2^\mu-1)\,h_{[r]}}\right)E_{2^{\mu-1}-1}(f)_{p(\cdot),\omega} \leq$$

$$\leq c_{22}(r,p)\,.2^{\mu r}t^{r-[r]}h_1\ldots h_{[r]}E_{2^{\mu-1}-1}(f)_{p(\cdot),\omega}$$

and therefore

$$\|\delta_\mu\|_{p(\cdot),\omega} \leq c_{22}(r,p)\,2^{\mu r}t^{(r-[r])}h_1\ldots h_{[r]}E_{2^{\mu-1}-1}(f)_{p(\cdot),\omega}.$$

Then

$$\left\|\sigma^r_{t,h_1,h_2,\ldots,h_{[r]}}S_{2^{m-1}}(\cdot,f)\right\|_{p(\cdot),\omega} \leq$$

$$\leq c_{23}(r,p)\,t^{(r-[r])}h_1\ldots h_{[r]}\left\{\sum_{\mu=1}^{m}2^{2\mu r\gamma}E^\gamma_{2^{\mu-1}-1}(f)_{p(\cdot),\omega}\right\}^{1/\gamma}$$

$$\leq c_{24}(r,p)\,t^{(r-[r])}h_1\ldots h_{[r]}\left\{2^{\gamma_1 r}E^\gamma_0(f)_{p(\cdot),\omega}\right\}^{1/\gamma}+$$

$$+c_{25}(r,p)\,t^{(r-[r])}h_1\ldots h_{[r]}\left\{\sum_{\mu=2}^{m}\sum_{\nu=2^{\mu-2}}^{2^{\mu-1}-1}\nu^{\gamma r-1}E^\gamma_{\nu-1}(f)_{p(\cdot),\omega}\right\}^{1/\gamma}$$

$$\leq c_{26}\left(r,p\right)t^{\left(r-\left[r\right]\right)}h_{1}\ldots h_{\left[r\right]}\left\{\sum_{\nu=1}^{2^{m-1}-1}\nu^{\gamma r-1}E_{\nu-1}^{\gamma}\left(f\right)_{p(\cdot),\omega}\right\}^{1/\gamma}.$$

The last inequality implies that

$$\Omega_{r}\left(f,\frac{1}{n}\right)_{p(\cdot),\omega}\leq\frac{c_{27}\left(r,p\right)}{n^{r}}\left\{\sum_{\nu=1}^{n}\nu^{\gamma r-1}E_{\nu-1}^{\gamma}\left(f\right)_{p(\cdot),\omega}\right\}^{1/\gamma}.$$

Theorem 1.1 is proved. □

Proof of Theorem 1.3. Let T_n be a polynomial of class $\mathcal{T}_n$ such that $E_n\left(f\right)_{p(\cdot),\omega}=\left\Vert f-T_n\right\Vert_{p(\cdot),\omega}$ and we set

$$\mathcal{U}_{0}\left(x\right):=T_{1}\left(x\right)-T_{0}\left(x\right);\ \mathcal{U}_{\nu}\left(x\right):=T_{2^{\nu}}\left(x\right)-T_{2^{\nu-1}}\left(x\right),\quad\nu=1,2,3,\ldots.$$

Hence

$$T_{2^{N}}\left(x\right)=T_{0}\left(x\right)+\sum_{\nu=0}^{N}\mathcal{U}_{\nu}\left(x\right),\quad N=0,1,2,\ldots..$$

For a given $\varepsilon>0$, by (1.8) there exists $\eta\in\mathbb{N}$ such that

$$\sum_{\nu=2^{\eta}}^{\infty}\nu^{\gamma\alpha-1}E_{\nu}^{\gamma}\left(f\right)_{p(\cdot),\omega}<\varepsilon. \tag{2.2}$$

From fractional Bernstein's inequality [3]

$$\left\Vert T_{n}^{(\alpha)}\right\Vert_{p(\cdot),\omega}\leq c_{28}\left(\alpha,p\right)n^{\alpha}\left\Vert T_{n}\right\Vert_{p(\cdot),\omega},\quad\alpha\in\mathbb{R}^{+}$$

we have

$$\left\Vert\mathcal{U}_{\nu}^{(\alpha)}\right\Vert_{p(\cdot),\omega}\leq c_{29}\left(\alpha,p\right)2^{\nu\alpha}\left\Vert\mathcal{U}_{\nu}\right\Vert_{p(\cdot),\omega}\leq c_{30}\left(\alpha,p\right)2^{\nu\alpha}E_{2^{\nu-1}}\left(f\right)_{p(\cdot),\omega},\quad\nu\in\mathbb{N}.$$

On the other hand it is easily seen that

$$2^{\nu\alpha}E_{2^{\nu-1}}\left(f\right)_{p(\cdot),\omega}\leq c_{31}\left(\alpha,p\right)\left\{\sum_{\mu=2^{\nu-2}+1}^{2^{\nu-1}}\mu^{\gamma\alpha-1}E_{\mu}^{\gamma}\left(f\right)_{p(\cdot),\omega}\right\}^{1/\gamma},\quad\nu=2,3,4,\ldots.$$

For the positive integers satisfying $K<N$

$$T_{2^{N}}^{(\alpha)}\left(x\right)-T_{2^{K}}^{(\alpha)}\left(x\right)=\sum_{\nu=K+1}^{N}U_{\nu}^{(\alpha)}\left(x\right),\quad x\in\boldsymbol{T}$$

and hence if K,N are large enough we obtain from (2.2)

$$\left\Vert T_{2^{N}}^{(\alpha)}\left(x\right)-T_{2^{K}}^{(\alpha)}\left(x\right)\right\Vert_{p(\cdot),\omega}\leq\sum_{\nu=K+1}^{N}\left\Vert\mathcal{U}_{\nu}^{(\alpha)}\left(x\right)\right\Vert_{p(\cdot),\omega}$$

$$\leq c_{31}\left(\alpha,p\right)\sum_{\nu=K+1}^{N}2^{\nu\alpha}E_{2^{\nu-1}}\left(f\right)_{p(\cdot),\omega}$$

$$\leq c_{32}(\alpha,p)\sum_{\nu=K+1}^{N}\left\{\sum_{\mu=2^{\nu-2}}^{2^{\nu-1}}\mu^{\gamma\alpha-1}E_{\mu}^{\gamma}(f)_{p(\cdot),\omega}\right\}^{1/\gamma}\leq$$

$$\leq c_{33}(\alpha,p)\left\{\sum_{\mu=2^{K-1}+1}^{2^{N-1}}\mu^{\gamma\alpha-1}E_{\mu}^{\gamma}(f)_{p(\cdot),\omega}\right\}^{1/\gamma}\leq c_{34}(\alpha,p)\,\varepsilon^{1/\gamma}.$$

Therefore $\left\{T_{2^N}^{(\alpha)}\right\}$ is a Cauchy sequence in $L_{\omega}^{p(\cdot)}$. Then there exists a $\varphi\in L_{\omega}^{p(\cdot)}$ satisfying

$$\left\|T_{2^N}^{(\alpha)}-\varphi\right\|_{p(\cdot),\omega}\to 0,\quad \text{as } N\to\infty.$$

On the other hand we have [3, Theorem 5]

$$\left\|T_{2^N}^{(\alpha)}-f^{(\alpha)}\right\|_{p(\cdot),\omega}\to 0,\quad \text{as } N\to\infty.$$

Then $f^{(\alpha)}=\varphi$ *a.e.* Therefore $f\in W_{p(\cdot),\omega}^{\alpha}$.

We note that

$$E_n\left(f^{(\alpha)}\right)_{p(\cdot),\omega}\leq\left\|f^{(\alpha)}-S_nf^{(\alpha)}\right\|_{p(\cdot),\omega}$$

$$\leq\left\|S_{2^{m+2}}f^{(\alpha)}-S_nf^{(\alpha)}\right\|_{p(\cdot),\omega}+\left\|\sum_{k=m+2}^{\infty}\left[S_{2^{k+1}}f^{(\alpha)}-S_{2^k}f^{(\alpha)}\right]\right\|_{p(\cdot),\omega}. \tag{2.3}$$

By the fractional Bernstein's inequality we get for $2^m<n<2^{m+1}$

$$\begin{aligned}\left\|S_{2^{m+2}}f^{(\alpha)}-S_nf^{(\alpha)}\right\|_{p(\cdot),\omega} &\leq c_{35}(\alpha,p)\,2^{(m+2)\alpha}E_n(f)_{p(\cdot),\omega}\\ &\leq c_{36}(\alpha,p)\,n^{\alpha}E_n(f)_{p(\cdot),\omega}.\end{aligned} \tag{2.4}$$

By (2.1) we find

$$\left\|\sum_{k=m+2}^{\infty}\left[S_{2^{k+1}}f^{(\alpha)}-S_{2^k}f^{(\alpha)}\right]\right\|_{p(\cdot),\omega}\leq$$

$$\leq c_{37}(\alpha,p)\left\|\left\{\sum_{k=m+2}^{\infty}\left|\sum_{|\nu|=2^k+1}^{2^{k+1}}(i\nu)^{\alpha}c_{\nu}e^{i\nu x}\right|^2\right\}^{1/2}\right\|_{p(\cdot),\omega}$$

and therefore

$$\left\|\sum_{k=m+2}^{\infty}\left[S_{2^{k+1}}f^{(\alpha)}-S_{2^k}f^{(\alpha)}\right]\right\|_{p(\cdot),\omega}\leq$$

$$\leq c_{38}(\alpha,p)\left(\sum_{k=m+2}^{\infty}\left\|\sum_{|\nu|=2^k+1}^{2^{k+1}}(i\nu)^{\alpha}c_{\nu}e^{i\nu x}\right\|_{p(\cdot),\omega}^{\gamma}\right)^{1/\gamma}.$$

Putting

$$|\delta_\nu^*| := \sum_{|\nu|=2^k+1}^{2^{k+1}} (i\nu)^\alpha c_\nu e^{i\nu x} = \sum_{\nu=2^k+1}^{2^{k+1}} \nu^\alpha 2\mathrm{Re}\left(c_\nu e^{i(\nu x+\alpha\pi/2)}\right)$$

we have

$$\|\delta_\nu^*\|_{p(\cdot),\omega} = \left\| \sum_{\nu=2^k+1}^{2^{k+1}} \nu^\alpha U_\nu(x) \right\|_{p(\cdot),\omega}$$

where $U_\nu(x) = 2\mathrm{Re}\left(c_\nu e^{i(\nu x+\alpha\pi/2)}\right)$. Using Abel's transformation we get

$$\begin{aligned} \|\delta_\nu^*\|_{p(\cdot),\omega} \leq & \sum_{\nu=2^k+1}^{2^{k+1}-1} |\nu^\alpha - (\nu+1)^\alpha| \left\| \sum_{l=2^k+1}^{\nu} U_l(x) \right\|_{p(\cdot),\omega} \\ & + \left|\left(2^{k+1}\right)^\alpha\right| \left\| \sum_{l=2^k+1}^{2^{k+1}-1} U_l(x) \right\|_{p(\cdot),\omega} . \end{aligned}$$

For $2^k+1 \leq \nu \leq 2^{k+1}$, $(k \in \mathbb{N})$ we have

$$\left\| \sum_{l=2^k+1}^{\nu} U_l(x) \right\|_{p(\cdot),\omega} \leq c_{39}(\alpha,p) E_{2^k}(f)_{p(\cdot),\omega}$$

and since

$$(\nu+1)^\alpha - \nu^\alpha \leq \begin{cases} \alpha(\nu+1)^{\alpha-1} & , \alpha \geq 1, \\ \alpha\nu^{\alpha-1} & , 0 \leq \alpha < 1, \end{cases}$$

we obtain

$$\|\delta_\nu^*\|_{p(\cdot),\omega} \leq c_{40}(\alpha,p) 2^{k\alpha} E_{2^k-1}(f)_{p(\cdot),\omega} .$$

Therefore

$$\begin{aligned} \left\| \sum_{k=m+2}^{\infty} \left[S_{2^{k+1}} f^{(\alpha)} - S_{2^k} f^{(\alpha)}\right] \right\|_{p(\cdot),\omega} & \leq c_{41}(\alpha,p) \left\{ \sum_{k=m+2}^{\infty} 2^{k\alpha\gamma} E_{2^k-1}^\gamma(f)_{p(\cdot),\omega} \right\}^{1/\gamma} \\ & \leq c_{42}(\alpha,p) \left\{ \sum_{\nu=n+1}^{\infty} \nu^{\gamma\alpha-1} E_\nu^\gamma(f)_{p(\cdot),\omega} \right\}^{1/\gamma} \qquad (2.5) \end{aligned}$$

and using (2.3), (2.4) and (2.5) Theorem 1.3 is proved. □

Acknowledgments. Authors are indebted to referees for valuable suggestions.

References

[1] R. Akgün, *Approximating polynomials for functions of weighted Smirnov-Orlicz spaces*, J. Funct. Spaces Appl. (to appear).

[2] —, *Sharp Jackson and converse theorems of trigonometric approximation in weighted Lebesgue spaces*, Proc. A. Razmadze Math. Inst. **152** (2010), 1–18.

[3] —, *Polynomial approximation of functions in weighted Lebesgue and Smirnov spaces with nonstandard growth*, Georgian Math. J. (to appear).

[4] R. Akgün and D.M. Israfilov, *Approximation and moduli of fractional orders in Smirnov-Orlicz classes*, Glas. Mat. Ser. III **43(63)** (2008), no. 1, 121–136.
[5] R. Akgün and V. Kokilashvili, *The refined direct and converse inequalities of trigonometric approximation in weighted variable exponent Lebesgue spaces*, Georgian Math. J. (to appear).
[6] E.A. Hadjieva, *Investigation of the properties of functions with quasimonotone Fourier coefficients in generalized Nikolskii-Besov spaces*, Author's summary of dissertation, Tbilisi, 1986, (Russian).
[7] P. Hästö and L. Diening, *Muckenhoupt weights in variable exponent spaces*, Preprint, Albert Ludwings Universität Freiburg, Mathematische Fakultät, http://www.helsinki.fi/~pharjule/varsob/publications.shtml.
[8] D.M. Israfilov, V. Kokilashvili and S.G. Samko, *Approximation in weighted Lebesgue spaces and Smirnov spaces with variable exponents*, Proc. A. Razmadze Math. Inst. **143** (2007) 25–35.
[9] V. Kokilashvili, *The converse theorem of constructive theory of functions in Orlicz spaces*, Soobshch. Akad. Nauk Gruzin. SSR **37** (1965), No. 2, 263–270 (Russian).
[10] —, *On approximation of periodic functions*, Soobshch Akad. Nauk. GruzSSR **34** (1968), 51–81. (Russian)
[11] V. Kokilashvili and S.G. Samko, *Singular integrals weighted Lebesgue spaces with variable exponent*, Georgian M. J. **10** (2003), No:1, 145–156.
[12] —, *Operators of Harmonis Analysis in weighted spaces with non-standard growth*, J. Math. Anal. Appl. **352** (2009), 15–34.
[13] —, *A refined inverse inequality of approximation in weighted variable exponent Lebesgue spaces*, Proc. A. Razmadze Math. Inst. **151** (2009), 134–138.
[14] V. Kokilashvili and Y.E. Yildirir, *On the approximation in weighted Lebesgue spaces*, Proc. A. Razmadze Math. Inst. **143** (2007), 103–113.
[15] —, *The Estimation of High Order Generalized Modulus of Cotinuity in L^p_ω*, Proc. A. Razmadze Math. Inst. **143** (2007), 135–137.
[16] N. Korneĭchuk, *Exact constants in approximation theory*, Encyclopedia of Mathematics and its Applications, 38, Cambridge University Press, Cambridge, 1991.
[17] P.P. Petrushev and V.A. Popov, *Rational approximation of real functions*, Encyclopedia of Mathematics and its Applications, vol. 28, Cambridge University Press, Cambridge, New York, New Rochelle, Melbourne and Sydney, 1987.
[18] Bl. Sendov and V.A. Popov, *The averaged moduli of smoothness with applications in numerical methods and approximation,* John Wiley & Sons, New York, 1988.
[19] S.B. Stechkin, *On the order of the best approximation of continuous functions*, Izv. Akad. Nauk. SSSR, Ser. Mat. **15** (1951), 219–242.
[20] A.F. Timan, *Theory of approximation of functions of a real variable*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960 (Russian).
[21] A.F. Timan and M.F. Timan, *The generalized modulus of continuity and best mean approximation*(Russian), Doklady Akad. Nauk SSSR (N.S.) **71** (1950), 17–20.
[22] M.F. Timan, *Inverse theorems of the constructive theory of functions in the spaces L_p*, Mat. Sb. **46**(88) (1958), 125–132. (Russian)
[23] A. Zygmund, *A remark on the integral modulus of continuity*, Univ. Nac. Tucumán. Revista A. **7** (1950), 259–269.

[1] Department of Mathematics, Faculty of Arts and Sciences, Balikesir University, 10145, Balikesir, Turkey.
E-mail address: rakgun@balikesir.edu.tr

[2] A. Razmadze Mathematical Institute, I. Javakhisvili State University, Tbilisi, Georgia.
E-mail address: kokil@rmi.acnet.ge