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WEYL'S THEOREM FOR ALGEBRAICALLY ABSOLUTE-(p, r)-PARANORMAL OPERATORS

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ABSTRACT. An operator $T \in B(H)$ is said to be absolute-(p, r)-paranormal if $\||T|^p|T^*|^r x\|^r\|x\| \geq \||T^*|^r x\|^{p+r}$ for all $x \in H$ and for positive real number $p > 0$ and $r > 0$, where $T = U|T|$ is the polar decomposition of T. In this paper, we discuss some properties of absolute-(p, r)-paranormal operators and show that Weyl's theorem holds for algebraically absolute-(p, r)-paranormal operators.

1. Introduction and preliminaries

Let H be an infinite dimensional complex Hilbert space and $B(H)$ denote the algebra of all bounded linear operators acting on H. Every operator T can be decomposed into $T = U|T|$ with a partial isometry U, where $|T| = \sqrt{T^*T}$. In this paper, $T = U|T|$ denotes the polar decomposition satisfying the kernel condition $N(U) = N(|T|)$. Furuta, Ito and Yamazaki [10] introduced class $A(k)$ and absolute-k-paranormal operators for $k > 0$ as generalizations of class A and paranormal operators, respectively. An operator T belongs to class $A(k)$ if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2$ and T is said to be absolute-k-paranormal if $\||T|^k Tx\| \geq \|Tx\|^{k+1}$ for every unit vector x. On other hand Fujii, Izumino and Nakamoto [7] introduced p-paranormal operators for $p > 0$ as another generalization of paranormal operators. An operator T is said to be p-paranormal if $\||T|^p U|T|^p x\| \geq \||T|^p x\|^2$ for every unit vector x, where the polar decomposition of T is $T = U|T|$.

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Fujii, Jung, S.H. Lee, M.Y. Lee and Nakamoto [8] introduced class $A(p,r)$ as a further generalization of class $A(k)$. An operator $T \in A(p,r)$ for $p > 0$ and $r > 0$ if $(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r}$ and class $AI(p,r)$ is class of all invertible operators which belong to class $A(p,r)$. Yamazaki and Yanagida [18] introduced the notion of absolute-(p,r)-paranormal operator. It is a further generalization of the classes of both absolute-k-paranormal operators and p-paranormal operators as a parallel concept of class $A(p,r)$. An operator T is said to be absolute-(p,r)-paranormal if $\||T|^p|T^*|^rx\|^r \geq \||T^*|^rx\|^{p+r}$ for every unit vector x or equivalently $\||T|^p|T^*|^rx\|^r\|x\| \geq \||T^*|^rx\|^{p+r}$ for all $x \in H$ and for positive real numbers $p > 0$ and $r > 0$.

2. On absolute-(p,r)-Paranormal operator

In this section, we obtain a characterization of absolute-(p,r)-paranormal operators using the polar decomposition $T = U|T|$ of T i.e., $T = U|T|$ is absolute-(p,r)-paranormal operator for $p > 0$ and $r > 0$ if and only if $r|T|^rU^*|T|^{2p}U|T|^r - (p+r)\lambda^p|T|^{2r} + p\lambda^{p+r}I \geq 0$ for all real λ. Using this characterization, we also obtain some properties for absolute-(p,r)- paranormal operators.

Theorem 2.1. [9] *: Let $T_1 = U_1P_1$ and $T_2 = U_2P_2$ be the polar decomposition of T_1 and T_2, respectively. Then the following are equivalent:*
(1) T_1 doubly commutes with T_2.
(2) U_1^*, U_1 and P_1 commutes with U_2^*, U_2 and P_2.
(3) $[P_1,P_2] = 0$, $[U_1,P_2] = 0$, $[P_1,U_2] = 0$, $[U_1,U_2] = 0$ and $[U_1^*,U_2] = 0$.

Theorem 2.2. [9] *:Let $T_1 = U_1P_1$ and $T_2 = U_2P_2$ be the polar decomposition of T_1 and T_2, respectively. If T_1 doubly commutes with T_2, then $T_1T_2 = U_1U_2P_1P_2$ is also the polar decomposition of T_1T_2, that is, U_1U_2 is partial isometry with $N(U_1U_2) = N(P_1P_2)$ and $P_1P_2 = |T_1T_2|$.*

In [18], Yamazaki and Yanagida gave proof in terms of operator inequalities. Here we give the proof using polar decomposition.

Lemma 2.3. *Let an operator $T \in B(H)$ have the polar decomposition $T = U|T|$. Then T is absolute-(p,r)-paranormal for $p > 0$, $r > 0$ if and only if*

$$r|T|^rU^*|T|^{2p}U|T|^r - (p+r)\lambda^p|T|^{2r} + p\lambda^{p+r}I \geq 0 \tag{2.1}$$

for all real λ.

Proof. Suppose that (2.1) holds for all real λ. Then this inequality is equivalent to

$$\||T|^pU|T|^rx\|^{2r} - 2p^{\frac{1}{2}}\lambda^{\frac{p+r}{2}}\||T|^rx\|^{p+r} + p\lambda^{p+r} \geq 0$$

for all real λ and $x \in H$. This is equivalent to

$$\||T|^pU|T|^rx\|^{2r} \geq \||T|^rx\|^{2(p+r)}, x \in H$$

$$\text{i.e., } \||T|^pU|T|^rx\|^r \geq \||T|^rx\|^{p+r}, x \in H$$

Hence T is absolute-(p,r)-paranormal. ∎

Theorem 2.4. *Let $T = U|T|$ be invertible absolute-(p,r)-paranormal for $p > 0$, $r > 0$. Then T^{-1} is absolute-(r,p)-paranormal.*

Proof. Suppose that $T = U|T|$ is an invertible absolute-(p,r)-paranormal operator. Then $U|T|^{-r} = |T^*|^{-r}U$ and $|T^*|^{-r} = U|T|^{-r}U^*$ for all $p > 0$ and $r > 0$. Since T is absolute-(p,r)-paranormal, from Lemma 2.3, we have

$$r|T|^rU^*|T|^{2p}U|T|^r - (p+r)\lambda^p|T|^{2r} + p\lambda^{p+r}I \geq 0\,.$$

Since T is invertible, taking inverse,

$\Longrightarrow pI - (p+r)\lambda^r|T^{-1}|^{2r} - r\lambda^{(p+r)}|T^{-1}|^rU|T^{-1}|^{2p}U^*|T^{-1}|^r \geq 0$

$\Longrightarrow pI - (p+r)\lambda^rU|T^{-1}|^{2r}U^* - r\lambda^{p+r}U|T|^{-r}U|T|^{-2p}U^*|T|^{-r}U^* \geq 0$

$\Longrightarrow U|T|^{-r}U|T|^{-p}[p|T|^pU^*|T|^{2r}U|T|^p - (p+r)\lambda^r|T|^{2p} + r\lambda^{p+r}I]|T|^{-p}U^*|T|^{-r}U^*$

is positive for all real λ. Therefore by Lemma 2.3, T^{-1} is absolute-(r,p)-paranormal. ☐

Theorem 2.5. *An operator unitarily equivalent to absolute-(p,r)-paranormal operator is absolute-(p,r)-paranormal for all $p > 0$ and $r > 0$.*

Proof. Let $T_1 = W|T_1|$ be absolute-(p,r)-paranormal, W be unitary and $T_2 = W^*T_1W$. Then $|T_2|^r = W^*|T_1|^rW$ and $|T_2|^{2p} = W^*|T_1|^{2p}W$ for every $p > 0$ and $r > 0$. Then by Theorem 2.1 and Theorem 2.2, we have $T_2 = W^*T_1W = W^*U|T_1|W = W^*UWW^*|T_1|W$ and $N(W^*UW) = N(W^*|T_1|W)$. Hence $T_2 = (W^*UW)(W^*|T_1|W)$ is the polar decomposition of T_2. Thus, we have,

$r|T_2|^r(W^*UW)^*|T_2|^{2p}(W^*UW)|T_2|^r - (p+r)\lambda^p|T_2|^{2r} + p\lambda^{p+r}I$

Since $|T_2|^r = W^*|T_1|^rW$ and $|T_2|^{2p} = W^*|T_1|^{2p}W$, we get

$rW^*|T_1|^rU^*|T_1|^{2p}U|T_1|^rW - (p+r)\lambda^pW^*|T_1|^{2r}W + p\lambda^{p+r}I$

$= W^*[r|T_1|^rU^*|T_1|^{2p}U|T_1|^r - (p+r)\lambda^p|T_1|^{2r} + p\lambda^{p+r}I]W$

$= W^*[r|T_1|^rW^*|T_1|^{2p}W|T_1|^r - (p+r)\lambda^p|T_1|^{2r} + p\lambda^{p+r}I]W$

is true for all real λ. Since $T_1 = W|T_1|$ is the polar decomposition of T_1, So T_2 is also absolute-(r,p)-paranormal. ☐

Remark 2.6. The above theorem is not true for similarly equivalent operators.

Theorem 2.7. *If $T \in A(p,r)$ then T is absolute-(p,r)-paranormal.*

Proof. If $T \in A(p,r)$ for any $p > 0$ and $r > 0$, then $(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r}$ for every unit vector $x \in H$ and $T = U|T|$ is the polar decomposition of T. Then,

$\||T|^rx\|^{p+r} = (|T|^rx, x)^{p+r}$

$= (U^*|T^*|^rUx, x)^{p+r}$

$\leq (U^*(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{2(p+r)}}Ux, x)^{p+r}$(using the definition of class A(p, r)).

$= ((U^*|T^*|^r|T|^{2p}|T^*|^rU)^{\frac{r}{2(p+r)}}x, x)^{p+r}$(By Hansen inequality)

$\leq (U^*|T^*|^r|T|^{2p}|T^*|^rUx, x)^{\frac{r}{2(p+r)}.(p+r)}$(using Holder's Mc carthy inequality)

$\leq (U^*|T^*|^r|T|^{2p}|T^*|^rUx, x)^{\frac{r}{2}}$

$\leq (|T|^rU^*|T|^{2p}U|T|^rx, x)^{\frac{r}{2}}$

$\leq (|T|^pU|T|^r, |T|^pU|T|^r)^{\frac{r}{2}}$

$= \||T|^pU|T|^rx\|^r$

Therefore T is absolute-(p,r)-paranormal. □

3. Weyl's theorem for algebraically absolute-(p,r)-paranormal operators

If $T \in B(H)$, we write $N(T)$ and $R(T)$ for null space and range of T, respectively. Let $\alpha(T)$=dim $N(T)$, $\beta(T)$=dim $N(T^*)$ and let $\sigma(T), \sigma_a(T)$ and $\pi_0(T)$ denote the spectrum, approximate point spectrum and point spectrum of T, respectively. An operator $T \in B(H)$ is called Fredholm if it has closed range, finite dimensional null space and its range has finite co- dimension. The index of a Fredholm operator is given by $i(T) = \alpha(T) - \beta(T)$. T is called Weyl if it Fredholm of index zero and Browder if it is Fredholm of finite ascent and descent. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $w(T)$ and the Browder spectrum $\sigma_b(T)$ of T are defined by

$\sigma_e(T) = \{\lambda \in C : T - \lambda \text{ is not Fredholm }\}$
$w(T) = \{\lambda \in C : T - \lambda \text{ is not Weyl }\}$
$\sigma_b(T) = \{\lambda \in C : T - \lambda \text{ is not Browder }\}$, respectively [11, 12].

Evidently $\sigma_e(T) \subseteq w(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \text{acc}\sigma(T)$, where acc$K$ is accumulation points of $K \subseteq C$. Let $\pi_{00}(T) = \{\lambda \in \text{iso}\sigma(T) : 0 < \alpha(T - \lambda) < \infty\}$ and $P_{00}(T) = \sigma(T) \backslash \sigma_b(T)$. We say that Weyl's theorem holds for T if $\sigma(T) \backslash w(T) = \pi_{00}(T)$ and that Browder's theorem holds for T if $\sigma(T) \backslash w(T) = P_{00}(T)$. Berkani [2] says that generalized Weyl's theorem holds for T provided $\sigma(T) \backslash \sigma_{BW}(T) = E(T)$, where$E(T)$ and $\sigma_{BW}(T)$ denote the isolated point of the spectrum which are eigenvalues (no restriction on multiplicity) and the set of complex numbers λ for which $T - \lambda I$ fails to be Weyl, respectively. An operator $T \in B(H)$ is called B-Fredholm if there exists $n \in N$ for which the induced operator $T_n : T^n(H) \to T^n(H)$ is Fredholm in the usual sense and B-Weyl if in addition T_n has index zero. Note that, if the generalized Weyl's theorem holds for T, then so does Weyl's theorem. We say T is algebraically absolute-(p,r)-paranormal if there exists a non constant complex polynomial p such that $p(T)$ is absolute-(p,r)- paranormal.

Lemma 3.1. *Let T be invertible and absolute-(p,r)-paranormal, $\lambda \in C$ and assume that $\sigma(T) = \{\lambda\}$ then $T = \lambda$.*

Proof. Case (i): $\lambda = 0$
Since T is absolute-(p,r)-paranormal, T is normaloid by [18, Theorem 8]. Therefore $T = 0$.
Case (ii): $\lambda \neq 0$
Since T is invertible and T is absolute-(p,r)-paranormal, we have T is normaloid by [18, Theorem 8]. But T^{-1} is absolute-(r,p)-paranormal by Theorem 2.4. Therefore T^{-1} is also normaloid by [18, Theorem 8]. But $\sigma(T^{-1}) = \{\frac{1}{\lambda}\}$ then $\|T\|\|T^{-1}\| = |\lambda||\frac{1}{\lambda}| = 1$. Then by [17], T is convexoid. So $w(T) = \{\lambda\}$. Therefore $T = \lambda$. □

Lemma 3.2. *Let T be invertible and quasi-nilpotent algebraically absolute-(p,r)-paranormal. Then T is nilpotent.*

Proof. Suppose that $p(T)$ is absolute-(p,r)-paranormal for some non constant polynomial p. Since $\sigma(p(T)) = p(\sigma(T))$, the operator $p(T) - p(0)$ is quasi-nilpotent. From above Lemma 3.1, we have that

$$CT^m(T-\lambda_1)(T-\lambda_2)\cdots.(T-\lambda_n) \equiv p(T) - p(0) = 0$$

where $m \geq 1$. Since $T - \lambda_i$ is invertible for every $\lambda_i \neq 0$ and So therefore $T^m = 0$. □

Theorem 3.3. *Let T be an invertible algebraically absolute-(p,r)-paranormal operator. Then T is isoloid.*

Proof. Let $\lambda \in \text{iso}\sigma(T)$ and let $P = \frac{1}{2\pi i}\int_{\partial D}(\mu - T)^{-1}d\mu$ be the associated Riesz idempotent, where D is a closed disk centered at λ which contains no other points of $\sigma(T)$. We can then represent T as the direct sum $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T)/\{\lambda\}$. Since T is algebraically absolute-(p,r)-paranormal, $p(T)$ is absolute-(p,r)-paranormal for some non constant polynomial p. since $\sigma(T_1) = \{\lambda\}$, we must have $\sigma(p(T_1)) = p(\sigma(T_1)) = \{p(\lambda)\}$. Therefore $p(T_1) - p(\lambda)$ is quasi-nilpotent.
Since $p(T_1)$ is absolute-(p,r)-paranormal, it follows from Lemma 3.1, that $p(T_1) - p(\lambda) = 0$. Put $q(z) = p(z) - p(\lambda)$. Then $q(T_1) = 0$ and hence T_1 is algebraically absolute-(p,r)-paranormal. Since $T_1 - \lambda$ is quasi-nilpotent and algebraically absolute-(p,r)-paranormal, it follows from Lemma 3.2, that $T_1 - \lambda$ is nilpotent. Therefore $\lambda \in \pi_0(T_1)$ and hence $\lambda \in \pi_0(T)$. This shows that T is isoloid. □

Lemma 3.4. *Let T be an algebraically absolute-(p,r)-paranormal operator. Then T has SVEP (the single-valued extension property).*

Proof. We first show that if T is absolute-(p,r)-paranormal, then T has SVEP. Suppose that T is absolute-(p,r)-paranormal. If $\pi_0(T) = \phi$, then clearly T has SVEP. Suppose that $\pi_0(T) \neq \phi$. Let $\triangle(T) = \{\lambda \in \pi_0(T) : N(T-\lambda) \subseteq N(T^* - \overline{\lambda})\}$. Since T is absolute-(p,r)-paranormal and $\pi_0(T) \neq \phi$, $\triangle(T) \neq \phi$. Let M be the closed linear span of the subspaces $N(T-\lambda)$ with $\lambda \in \triangle(T)$. Then M reduces T, and so we can write T as $T_1 \oplus T_2$ on $H = M \oplus M^{\perp}$. Clearly, T_1 is normal and $\pi_0(T_2) = \phi$. Since T_1 and T_2 have both SVEP, T has SVEP. Suppose now that T is algebraically absolute-(p,r)-paranormal. Then $p(T)$ is absolute-(p,r)-paranormal for some non constant polynomial p. Since $p(T)$ has SVEP, it follows from [14, Theorem 3.3.9] that T has SVEP. □

Let $H(\sigma(T))$ be the set of all analytic functions in an open neighborhood of $\sigma(T)$.

Theorem 3.5. *Let T be an algebraically absolute-(p,r)-paranormal operator. Then Weyl's theorem holds for T.*

Proof. Suppose that $\lambda \in \sigma(T)\backslash w(T)$. Then $T - \lambda$ is Weyl and not invertible. We claim that $\lambda \in \partial\sigma(T)$. Assume that λ is an interior point of $\sigma(T)$. Then there exists a neighborhood U of λ, such that $\dim N(T-\mu) > 0$ for all $\mu \in U$. It follows from [6, Theorem 10] that T doesnot have SVEP. On the other hand, Since $p(T)$ is absolute-(p,r)-paranormal for some non constant polynomial p, it follows from

Lemma 3.4 that T has SVEP. It is a contradiction. Therefore, $\lambda \in \partial\sigma(T)\backslash w(T)$ and it follows from the punctured neighborhood theorem that $\lambda \in \pi_{00}(T)$.
Conversely, suppose that $\lambda \in \pi_{00}(T)$. Using the Riesz idempotent $E = \frac{1}{2\pi i}\int_{\partial D}(\mu - T)^{-1}d\mu$ for λ, we can represent T as the direct sum $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$, where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T)\backslash\{\lambda\}$. Now we consider two cases:
Case(i) : $\lambda = 0$: Then T_1 is algebraically absolute-(p,r)-paranormal and quasi-nilpotent. It follows from Lemma 3.2 that T_1 is nilpotent. We claim that $\dim R(E) < \infty$. For, if $N(T_1)$ is infinite dimensional, then $0 \notin \pi_{00}(T)$. It is a contradiction. Therefore T_1 is an operator on the finite dimensional space $R(E)$. So it follows that T_1 is Weyl. But since T_2 is invertible, we can conclude that T is Weyl. Therefore $0 \in \sigma(T)\backslash w(T)$.

Case(ii) : $\lambda \neq 0$: Then by the proof of Theorem 3.3, $T_1 - \lambda$ is nilpotent. Since $\lambda \in \pi_{00}(T)$, $T_1 - \lambda$ is an operator on the finite dimensional space $R(E)$. So $T_1 - \lambda$ is Weyl. Since $T_2 - \lambda$ is invertible, $T - \lambda$ is Weyl.
By Case (i) and Case (ii), Weyl's theorem holds for T. This completes the proof. □

Theorem 3.6. *Let T be an algebraically absolute-(p,r)-paranormal operator. Then Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Proof. Let $f \in H(\sigma(T))$. Since it generally holds $w(f(T)) \subseteq f(w(T))$, it suffices to show that $f(w(T)) \subseteq w(f(T))$. Suppose $\lambda \notin w(f(T))$, then $f(T) - \lambda$ is Weyl and

$$f(T) - \lambda = c(T - \alpha_1)(T - \alpha_2)(T - \alpha_3)\cdots.(T - \alpha_n)g(T) \tag{3.1}$$

where $c, \alpha_1, \alpha_2, \alpha_3, \cdots \alpha_n \in C$ and $g(T)$ is invertible. Since the operators in the right side of (3.1) commute, every $T - \alpha_i$ is Fredholm. Since T is algebraically absolute-(p,r)-paranormal, T has SVEP by Lemma 3.4. It follows from [1, Theorem 2.6] that $\operatorname{ind}(T - \alpha_i) \leq 0$ for each $i = 1, 2, 3, \cdots n$. Therefore $\lambda \notin f(w(T))$ and hence $f(w(T)) = w(f(T))$.
Now by [16], that if T is isoloid, then

$$f(\sigma(T)\backslash\pi_{00}(T)) = \sigma(f(T))\backslash\pi_{00}(f(T)) \text{ for every } f \in H(\sigma(T))$$

Since T is isoloid by Theorem 3.3 and Weyl's theorem holds for T by Theorem 3.5,

$$\sigma(f(T))\backslash\pi_{00}(f(T)) = f(\sigma(T)\backslash\pi_{00}(T)) = f(w(T)) = w(f(T))$$

which imples that Weyl's theorem holds for $f(T)$. This completes the proof. □

Theorem 3.7. *Let T be an algebraically absolute-(p,r)-paranormal operator. Then generalized Weyl's theorem holds for T.*

Proof. Assume that $\lambda \in \sigma(T)\backslash\sigma_{BW}(T)$. Then $T-\lambda I$ is B-Weyl and not invertible. We claim that $\lambda \in \partial\sigma(T)$. Assume to the contrary that λ is an interior point of $\sigma(T)$. Then there exists a neighborhood U of λ such that $\dim(T - \mu) > 0$ for all $\mu \in U$. It follows from [6, Theorem 10], that T doesnot have SVEP. On the other hand, since $p(T)$ is absolute-(p,r)-paranormal for non constant polynomial p, it follows from Lemma 3.4 that $p(T)$ has SVEP. Hence by [14, Theorem 3.3.9], T is

SVEP, a contradiction. Therefore $\lambda \in \partial\sigma(T)$. Conversely, assume that $\lambda \in E(T)$, then λ is isolated in $\sigma(T)$. From [13, Theorem 7.1], we have $X = M \oplus N$, where M, N are closed subspaces of X, $U = (T - \lambda I)|_N$ is an invertible operator and $V = (A - \lambda I)|_N$ is a quasi- nilpotent operator. Since T is algebraically absolute-(p, r)-paranormal,V is also algebraically absolute-(p, r)-paranormal, from Lemma 3.2, V is nilpotent. Therefore $T - \lambda I$ is Drazin invertible [5, Proposition 19] and [15, Corollary 2.2]. By [3, Lemma 4.1], $T - \lambda I$ is a B-Fredholm operator of index 0. □

Let $\sigma_{BF}(T) = \{\lambda \in C : T - \lambda I$ is not a B-Fredholm operator$\}$ be the B-Fredholm spectrum of T and $\rho_{BF}(T) = C \backslash \sigma_{BF}(T)$, the resolvent set of T.

Definition 3.8. Let $T \in B(H)$, we say that T is of stable index if for each $\lambda, \mu \in \rho_{BF}(T)$, ind$(T - \lambda I)$, ind$(T - \mu I)$ have the same sign index.

Lemma 3.9. *Let $T \in B(H)$ be absolute-(p, r)-paranormal, then T is of stable index.*

Proof. If T is absolute-(p, r)-paranormal, then $\||T|^p|T^*|^r x\|^r \|x\| \geq \||T^*|^r x\|^{p+r}$ for all $x \in H$. So $N(T) \subset N(T^*) = R(T)^\perp$. Since $N(T^2)/N(T) \approx N(T) \cap R(T)$, implies that $N(T^2) = N(T)$. Moreover, if T is also B- Fredholm, then there exists an integer n, such that $R(T^n)$ is closed and such that $T_n : R(T^n) \to R(T^n)$ is a Fredholm operator. We have,

$$\begin{aligned} ind(T) &= ind(T_n) \\ &= dim N(T) \cap R(T^n) - dim R(T^n)/R(T^{n+1}) \\ &= -dim R(T^n)/R(T^{n+1}). \end{aligned}$$

Hence it follows that ind$(T) \leq 0$.
Further, if $\lambda \in \rho_{BF}(T)$, then $T - \lambda I$ is a B-Fredholm operator and $T - \lambda I$ is also absolute-(p, r)- paranormal. By the same way as above, we have ind$(T - \lambda I) \leq 0$. Therefore T is of stable index. □

Theorem 3.10. *Let T be an invertible algebraically absolute-(p, r)-paranormal operator. Then generalized Weyl's theorem holds for $f(T)$ for every function f analytic on a neighborhood of $\sigma(T)$.*

Proof. Assume that T be an algebraically absolute-(p, r)-paranormal operator. We prove that $f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$ for every function f analytic on a neighborhood of $\sigma(T)$. Let f be an analytic function on a neighborhood of $\sigma(T)$. Since $\sigma_{BW}(f(T)) \subseteq f(\sigma_{BW}(T))$ with no restriction on T, it is sufficient to prove that $f(\sigma_{BW}(T)) \subseteq \sigma_{BW}(f(T))$.
Assume that $\lambda \notin \sigma_{BW}(f(T))$. Then $f(T) - \lambda$ is B-Weyl and

$$f(T) - \lambda = C(T - \alpha_1 I)(T - \alpha_2 I) \cdots (T - \alpha_n I)g(t)$$

where $c, \alpha_1, \alpha_2, \cdots, \alpha_n \in C$ and $g(T)$ is invertible. Since $f(T) - \lambda I$ is a B-Fredholm operator from [2, Theorem 3.4], it follows that for each i, $1 \leq i \leq n$, $T - \alpha_i I$ is a B-Fredholm operator. Moreover, since ind$(f(T) - \lambda I) = 0$ and T is of stable sign index by Lemma 3.9, from [3, Theorem 3.2], we have for each

i, $1 \leq i \leq n$, $\text{ind}(T - \alpha_i I) = 0$. So for each i, $1 \leq i \leq n$, $\alpha_i \notin \sigma_{BW}(T)$. If $\lambda \in f(\sigma_{BW}(T))$, there exists $\alpha \in \sigma_{BW}(T)$ such that $\lambda = f(\alpha)$. Hence $0 = f(\alpha) - \lambda = (\alpha - \alpha_1)(\alpha - \alpha_2) \cdots (\alpha - \alpha_n) g(\alpha)$. This implies that $\alpha \in \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. Hence, there exists i, $1 \leq i \leq n$, such that $\alpha_i \in \sigma_{BW}(T)$, contradiction. Hence $\lambda \notin f(\sigma_{BW}(T))$. It is known [4, Lemma 2.9] that if T is isoloid then

$$f(\sigma(T) \backslash E(T)) = \sigma(f(T)) \backslash E(f(T))$$

for every analytic function on a neighborhood of $\sigma(T)$. Since T is isoloid, by Theorem 3.3, and generalized Weyl's theorem holds for T by Theorem 3.5,

$$\sigma(f(T)) \backslash E(f(T)) = f(\sigma(T) \backslash E(T))$$

$$= f(\sigma_{BW}(T)) = \sigma_{BW}(f(T)) \text{ by [4, Theorem 2.10].} \qquad \square$$

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