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ON A CLASS OF UNIVALENT FUNCTIONS DEFINED BY SĂLĂGEAN DIFFERENTIAL OPERATOR

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Abstract. By using a certain operator S^n, we introduce a class of holomorphic functions $S_n(\beta)$, and obtain some subordination results. We also show that the set $S_n(\beta)$ is convex and obtain some new differential subordinations related to certain integral operators.

1. Introduction and preliminaries

Denote by U the unit disc of the complex plane :

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U and let

$$\mathcal{A}_n = \{f \in \mathcal{H}(U) : \ f(z) = z + a_{n+1}z^{n+1} + \cdots , \ z \in U\}$$

with $\mathcal{A}_1 = \mathcal{A}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : \ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots , \ z \in U\}.$$

Let

$$K = \left\{ f \in \mathcal{A} : \ \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, \ z \in U \right\},$$

denote the class of normalized convex functions in U.

A function f, analytic in U, is said to be convex if it is univalent and $f(U)$ is convex.

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If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $\omega(0) = 0$, $|\omega(z)| < 1$, for all $z \in U$ such that $f(z) = g[\omega(z)]$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.
Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ be a function and let h be univalent in U. If p is analytic in U and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \quad (z \in U) \tag{i}$$

then p is called a solution of the differential subordination.
The univalent function q is called a dominant of the solution of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (i).
A dominant $\widetilde{q}$, which satisfies $\widetilde{q} \prec q$ for all dominants q of (i) is said to be the best dominant of (i). (Note that the best dominant is unique up to a rotation of U).

In order to prove the original results we use the following lemmas.

Lemma 1.1. [3] *Let h be a convex function, with $h(0) = a$ and let $\gamma \in \mathbb{C}^*$ be a complex number with $\operatorname{Re}\gamma \geq 0$. If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z) \quad (z \in U)$$

then

$$p(z) \prec q(z) \prec h(z) \quad (z \in U)\,,$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\frac{\gamma}{n}-1}dt \quad (z \in U).$$

Function q is convex in U and is the best dominant.

Lemma 1.2. [1] *Let $\operatorname{Re} r > 0$ and let*

$$\omega = \frac{k^2 + |r|^2 - |k^2 - r^2|}{4k\operatorname{Re} r}.$$

Let h be an analytic function in U with $h(0) = 1$ and suppose that

$$\operatorname{Re}\left(\frac{zh''(z)}{h'(z)} + 1\right) > -\omega.$$

If

$$p(z) = 1 + p_n z^n + p_{n+1}z^{n+1} + \cdots$$

is analytic in U and

$$p(z) + \frac{1}{r} zp'(z) \prec h(z),$$

then $p(z) \prec q(z)$, where q is a solution of the differential equation

$$q(z) + \frac{n}{r} zq'(z) = h(z), \quad q(0) = 1,$$

given by

$$q(z) = \frac{r}{nz^{r/n}} \int_0^z t^{\frac{r}{n}-1} h(t)dt.$$

Moreover q is the best dominant.

Definition 1.3. [2] For $f \in \mathcal{A}$, $n \in \mathbb{N}^* \cup \{0\}$, the operator $S^n f$ is defined by $S^n : \mathcal{A} \to \mathcal{A}$

$$\begin{aligned} &S^0 f(z) = f(z) \\ &S^1 f(z) = z f'(z) \\ &\dots \\ &S^{n+1} f(z) = z[S^n f(z)]' \ (z \in U). \end{aligned}$$

Remark 1.4. [1] If $f \in \mathcal{A}$,

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

then

$$S^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j \quad (z \in U).$$

Definition 1.5. [1] If $0 \le \beta < 1$ and $n \in \mathbb{N}$, we let $S_n(\beta)$ stand for the class of functions $f \in \mathcal{A}$, which satisfy the inequality

$$\operatorname{Re}(S^n f)'(z) > \beta \quad (z \in U).$$

2. Main results

We start this section with the following theorem.

Theorem 2.1. *The set $S_n(\beta)$ is convex.*

Proof. Let the function

$$f_i(z) = z + \sum_{k=2}^{\infty} a_{ki} z^k, \quad i = 1, 2 \quad (z \in U)$$

be in the class $S_n(\beta)$. It is sufficient to show that the function

$$h(z) = \mu_1 f_1(z) + \mu_2 f_2(z)$$

with μ_1 and μ_2 nonnegative and $\mu_1 + \mu_2 = 1$ is in $S_n(\beta)$.
Since

$$h(z) = z + \sum_{k=2}^{\infty} (\mu_1 a_{k1} + \mu_2 a_{k2}) z^k \quad (z \in U)$$

then

$$S^n h(z) = z + \sum_{k=2}^{\infty} k^n (\mu_1 a_{k1} + \mu_2 a_{k2}) z^k \quad (z \in U). \tag{2.1}$$

Differentiating (2.1), we get

$$[S^n h(z)]' = 1 + \sum_{k=2}^{\infty} k^{n+1} (\mu_1 a_{k1} + \mu_2 a_{k2}) z^{k-1},$$

whence

$$\operatorname{Re}[S^n h(z)]' = \operatorname{Re}\left[1 + \sum_{k=2}^{\infty} k^{n+1} (\mu_1 a_{k1} + \mu_2 a_{k2}) z^{k-1}\right] \tag{2.2}$$

$$= 1 + \operatorname{Re}\left[\mu_1 \sum_{k=2}^{\infty} k^{n+1} a_{k1} z^{k-1}\right] + \operatorname{Re}\left[\mu_2 \sum_{k=2}^{\infty} k^{n+1} a_{k2} z^{k-1}\right].$$

Since $f_1, f_2 \in S_n(\beta)$, we obtain

$$\operatorname{Re}\left[\mu_i \sum_{k=2}^{\infty} k^{n+1} a_{ki} z^{k-1}\right] > \mu_i(\beta - 1) \quad (i = 1, 2). \tag{2.3}$$

Using (2.3) in (2.2), we obtain

$$\operatorname{Re}\,[S^n h(z)]' > 1 + \mu_1(\beta - 1) + \mu_2(\beta - 1) \quad (z \in U),$$

and since $\mu_1 + \mu_2 = 1$, we deduce

$$\operatorname{Re}\,[S^n h(z)]' > \beta$$

i.e. $S_n(\beta)$ is convex. □

Theorem 2.2. *Let q be a convex function in U, with $q(0) = 1$, and let*

$$h(z) = q(z) + \frac{1}{c+2} z q'(z) \quad (z \in U),$$

where c is a complex number, with $\operatorname{Re} c > -2$.
If $f \in S_n(\beta)$ and $F = I_c(f)$, where

$$F(z) = I_c(f)(z) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t)dt, \quad \operatorname{Re} c > -2, \tag{2.4}$$

then

$$[S^n f(z)]' \prec h(z) \quad (z \in U), \tag{2.5}$$

implies

$$[S^n F(z)] \prec q(z) \quad (z \in U),$$

and this result is sharp.

Proof. From (2.4), we deduce

$$z^{c+1} F(z) = (c+2) \int_0^z t^c f(t)dt, \quad \operatorname{Re} c > -2 \quad (z \in U). \tag{2.6}$$

Differentiating (2.6), with respect to z, we obtain

$$(c+1)F(z) + zF'(z) = (c+2)f(z) \quad (z \in U)$$

and

$$(c+1)S^n F(z) + z[S^n F(z)]' = (c+2)S^n f(z) \quad (z \in U). \tag{2.7}$$

Differentiating (2.7), we get

$$[S^n F(z)]' + \frac{z}{c+2}[S^n F(z)]'' = [S^n f(z)]' \quad (z \in U). \tag{2.8}$$

Using (2.8), the differential subordination (2.5) becomes

$$[S^n F(z)]' + \frac{1}{c+2} z[S^n F(z)]'' \prec h(z) = q(z) + \frac{1}{c+2} z q'(z). \tag{2.9}$$

Let

$$p(z) = [S^n F(z)]' = \left[z + \sum_{j=2}^{\infty} j^n a_j z^j \right]' \tag{2.10}$$

$$= 1 + p_1 z + p_2 z^2 + \cdots, \quad p \in \mathcal{H}[1,1].$$

Using (2.10) in (2.9), we have

$$p(z) + \frac{1}{c+2} z p'(z) \prec h(z) = q(z) + \frac{1}{c+2} z q'(z) \quad (z \in U).$$

Using Lemma 1.1, we obtain $p(z) \prec q(z)$, i.e.

$$[S^n F(z)]' \prec q(z) \quad (z \in U),$$

and q is the best dominant. $\square$

Example 2.3. If we take $c = 1 + i$ and $q(z) = \dfrac{1}{1-z}$, then

$$h(z) = \frac{3 + i - z(2+i)}{(3+i)(1-z)^2}$$

and from Theorem ,we deduce that if $f \in S_n(\beta)$ and F is given by

$$F(z) = \frac{3+i}{z^{2+i}} \int_0^z t^{1+i} f(t) dt \tag{2.11}$$

then

$$z^{2+i} F(z) = (3+i) \int_0^z t^{1+i} f(t) dt \quad (z \in U). \tag{2.12}$$

Differentiating (2.12) with respect to z, we obtain

$$(2+i) F(z) + z F'(z) = (3+i) f(z)$$

and

$$(2+i) S^n F(z) + z [S^n F(z)]' = (3+i) S^n f(z) \quad (z \in U). \tag{2.13}$$

Differentiating (2.13) we have

$$[S^n F(z)]' + \frac{z}{3+i} [S^n F(z)]'' = [S^n f(z)]' \quad (z \in U)$$

and we deduce

$$[S^n f(z)]' \prec \frac{3 + i - z(2+i)}{(3+i)(1-z)^2} \quad (z \in U)$$

implies

$$[S^n F(z)]' \prec \frac{1}{1-z} \quad (z \in U),$$

where F is given by (2.11).

Theorem 2.4. *Let* $\mathrm{Re}\, c > -2$ *and let*

$$\omega = \frac{1 + |c+2|^2 - |c^2 + 4c + 3|}{4\mathrm{Re}\,(c+2)} \tag{2.14}$$

Let h *be an analytic function in* U *with* $h(0) = 1$ *and suppose that*

$$\mathrm{Re}\, \frac{zh''(z)}{h'(z)} + 1 > -\omega.$$

If $f \in S_n(\beta)$ *and* $F = I_c(f)$, *where* F *is defined by* (2.4), *then*

$$[S^n f(z)]' \prec h(z) \quad (z \in U), \tag{2.15}$$

implies

$$[S^n F(z)]' \prec q(z) \quad (z \in U)\,,$$

where q *is the solution of the differential equation*

$$q(z) + \frac{1}{c+2} zq'(z) = h(z), \quad h(0) = 1,$$

given by

$$q(z) = \frac{c+2}{z^{c+2}} \int_0^z t^{c+1} h(t) dt \quad (z \in U).$$

Moreover q *is the best dominant.*

Proof. In order to prove Theorem 2.4 we will use Lemma 1.2. The value of ω is given by (2.14). From (2.10) we have

$$p(z) = [S^n F(z)]' = 1 + p_1 z + p_2 z^2 + \cdots, \quad p \in \mathcal{H}[1,1] \quad (z \in U).$$

Using Lemma 1.2, we deduce $k = 1$. Using (2.8) and (2.10), the differential subordination (2.15) becomes

$$p(z) + \frac{1}{c+2} zp'(z) \prec h(z) = q(z) + \frac{1}{c+2} zq'(z) \quad (z \in U). \tag{2.16}$$

From subordination (2.16), by using Lemma 1.2, we deduce $r = c + 2$ and

$$p(z) \prec q(z) \quad (z \in U)\,,$$

where

$$q(z) = \frac{c+2}{z^{c+2}} \int_0^z t^{c+1} h(t) dt \quad (z \in U),$$

i.e.

$$[S^n F(z)]' \prec q(z) = \frac{c+2}{z^{c+2}} \int_0^z t^{c+1} h(t) dt \quad (z \in U).$$

Moreover it is the best dominant. □

Remark 2.5. If we put

$$h(z) = \frac{1 + (2\beta - 1)z}{1+z}$$

in Theorem 2.4, we obtain the following interesting result.

Corollary 2.6. *If $0 \le \beta < 1$, $n \in \mathbb{N}$, $\operatorname{Re} c > -2$ and I_c is defined by (2.4), then*
$$I_c[S_n(\beta)] \subset S_n(\delta)\,,$$
where $\delta = \min_{|z|=1} \operatorname{Re} q(z) = \delta(c,\beta)$ and this results is sharp. Moreover
$$\delta = \delta(c,\beta) = 2\beta - 1 + (c+2)(2-2\beta)\sigma(c)\,, \tag{2.17}$$
where
$$\sigma(x) = \int_0^z \frac{t^{x+1}}{1+t} dt. \tag{2.18}$$

Proof. If we let
$$h(z) = \frac{1+(2\beta-1)z}{1+z},$$
then h is convex and by Theorem 2.4, we deduce
$$\begin{aligned}[S^n F(z)]' \prec q(z) &= \frac{c+2}{z^{c+2}} \int_0^z t^{c+1} \cdot \frac{1+(2\beta-1)t}{1+t} dt \\ &= 2\beta - 1 + \frac{(c+2)(2-2\beta)}{z^{c+2}} \int_0^z \frac{t^{c+1}}{1+t} dt \\ &= 2\beta - 1 + \frac{(c+2)(2-2\beta)}{z^{c+2}} \sigma(c)\,,\end{aligned} \tag{2.19}$$
where σ is given by (2.18).

If $\operatorname{Re} c > -2$, then from the convexity of q and the fact that $q(U)$ is symmetric with respect to the real axis, we deduce
$$\begin{aligned}\operatorname{Re}[S^n F(z)]' &\ge \min_{|z|=1} \operatorname{Re} q(z) = \operatorname{Re} q(1) = \delta(c,\beta) \\ &= 2\beta - 1 + (c+2)(2-2\beta)\sigma(c)\,,\end{aligned}$$
where σ is given by (2.18).
From (2.19), we deduce
$$I_c[S_n(\beta)] \subset S_n(\delta)\,,$$
where δ is given by (2.17). □

References

[1] G. Oros and G.I. Oros, *A Class of Holomorphic Function II*, Libertas Math., **23** (2003), 65-68.

[2] G.S. Sălăgean, *Subclasses of univalent functions*, Complex Analysis–Fift Romanian–Finnish Seminar, Part 1 (Bucharest, 1981), 362–372, Lecture Notes in Math., 1013, Springer, Berlin 1983.

[3] D.J. Hallenbeck and S. Ruscheweyh, *Subordination by convex functions*, Proc. Amer. Math. Soc., **52** (1975), 191–195.

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