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ISOMETRIC ADDITIVE MAPPINGS IN GENERALIZED QUASI-BANACH SPACES

CHUN-GIL PARK[1*] AND THEMISTOCLES M. RASSIAS[2]

Submitted by K. Ciesielski

Abstract. In this paper, we prove the generalized Hyers–Ulam stability of the isometric additive mappings in generalized quasi-Banach spaces, and prove the generalized Hyers–Ulam stability of the isometric additive mappings in generalized p-Banach spaces.

1. Introduction and preliminaries

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.1. ([5, 43]) Let X be a linear space. A *quasi-norm* is a real-valued function on X satisfying the following:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(3) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on X.

A *quasi-Banach space* is a complete quasi-normed space.

A quasi-norm $\|\cdot\|$ is called a *p-norm* $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a *p-Banach space*.

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*Corresponding author.

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Given a p-norm, the formula $d(x,y) := \|x-y\|^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz theorem [43] (see also [5]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms.

In [26], the author generalized the concept of quasi-normed spaces.

Definition 1.2. Let X be a linear space. A *generalized quasi-norm* is a real-valued function on X satisfying the following:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(3) There is a constant $K \geq 1$ such that $\|\sum_{j=1}^{\infty} x_j\| \leq \sum_{j=1}^{\infty} K\|x_j\|$ for all $x_1, x_2, \cdots \in X$.

The pair $(X, \|\cdot\|)$ is called a *generalized quasi-normed space* if $\|\cdot\|$ is a generalized quasi-norm on X. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$.

A *generalized quasi-Banach space* is a complete generalized quasi-normed space.

A generalized quasi-norm $\|\cdot\|$ is called a *p-norm* $(0 < p \leq 1)$ if

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a generalized quasi-Banach space is called a *generalized p-Banach space*.

Let X and Y be metric spaces. A mapping $f : X \to Y$ is called an isometry if f satisfies

$$d_Y\big(f(x), f(y)\big) = d_X(x,y)$$

for all $x, y \in X$, where $d_X(\cdot,\cdot)$ and $d_Y(\cdot,\cdot)$ denote the metrics in the spaces X and Y, respectively. For some fixed number $r > 0$, suppose that f preserves distance r; i.e., for all x, y in X with $d_X(x,y) = r$, we have $d_Y\big(f(x), f(y)\big) = r$. Then r is called a conservative(or preserved) distance for the mapping f. Let $(X, ||\cdot||)$ and $(Y, \|\cdot\|)$ be normed spaces. A mapping $L : X \to Y$ is called an *isometry* if $\|L(x) - L(y)\| = ||x-y||$ for all $x, y \in X$. Aleksandrov [1] posed the following problem:

Remark 1.3. **Aleksandrov problem.** Examine whether the existence of a single conservative distance for some mapping T implies that T is an isometry.

The isometric problems have been investigated in several papers (see [3, 9, 12, 13, 19, 20, 21, 35, 39, 41, 42]).

The stability problem of functional equations originated from a question of S.M. Ulam [46] concerning the stability of group homomorphisms: Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x*y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $H(x * y) = H(x) \diamond H(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

D.H. Hyers [14] gave a first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f : X \to Y$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

for all $x, y \in X$ and some $\varepsilon \ge 0$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \varepsilon$$

for all $x \in X$.

Let X and Y be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Th.M. Rassias [33] introduced the following inequality: Assume that there exist constants $\theta \ge 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \theta(||x||^p + ||y||^p)$$

for all $x, y \in X$. Th.M. Rassias [33] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{2\theta}{2 - 2^p} ||x||^p$$

for all $x \in X$. The above inequality has provided a lot of influence in the development of what is now known as *generalized Hyers–Ulam stability* of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Găvruta [11] following Th.M. Rassias approach for the stability of the linear mapping between Banach spaces obtained a generalization of Th.M. Rassias' Theorem. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2, 4, 6, 7, 8, 10, 11, 15, 16, 17, 18, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 34, 36, 37, 38, 40, 44]).

In this paper, we prove the generalized Hyers–Ulam stability of the isometric Cauchy mapping and the isometric Jensen mapping in generalized quasi-Banach spaces, and prove the generalized Hyers–Ulam stability of the isometric Cauchy mapping and the isometric Jensen mapping in generalized p-Banach spaces.

2. Stability of the isometric additive mappings in generalized quasi-Banach spaces

Throughout this section, assume that X is a generalized quasi-normed vector space with generalized quasi-norm $||\cdot||$ and that Y is a generalized quasi-Banach space with generalized quasi-norm $\|\cdot\|$. Let K be the modulus of concavity of $\|\cdot\|$.

Theorem 2.1. *Let $r > 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(||x||^r + ||y||^r), \tag{2.1}$$

$$|\ \|f(x)\| - ||x||\ | \leq 2\theta ||x||^r \tag{2.2}$$

for all $x, y \in X$. Then there exists a unique isometric Cauchy additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2K\theta}{2^r - 2} ||x||^r \tag{2.3}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.1), we get

$$\|f(2x) - 2f(x)\| \leq 2\theta ||x||^r \tag{2.4}$$

for all $x \in X$. So

$$\|f(x) - 2f(\frac{x}{2})\| \leq \frac{2\theta}{2^r} ||x||^r$$

for all $x \in X$. Hence

$$\|2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m})\| \leq K \sum_{j=l+1}^{m} \frac{2^j \theta}{2^{jr}} ||x||^r \tag{2.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.5) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$.

By (2.1),

$$\begin{aligned} \|A(x+y) - A(x) - A(y)\| &= \lim_{n\to\infty} 2^n \|f(\frac{x+y}{2^n}) - f(\frac{x}{2^n}) - f(\frac{y}{2^n})\| \\ &\leq \lim_{n\to\infty} \frac{2^n \theta}{2^{nr}} (||x||^r + ||y||^r) = 0 \end{aligned}$$

for all $x, y \in X$. So

$$A(x+y) = A(x) + A(y)$$

for all $x, y \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.5), we get (2.3).

Now, let $A' : X \to Y$ be another Cauchy additive mapping satisfying (2.3). Then we have

$$\begin{aligned} \|A(x) - A'(x)\| &= 2^n \|A(\frac{x}{2^n}) - A'(\frac{x}{2^n})\| \\ &\leq 2^n K (\|A(\frac{x}{2^n}) - f(\frac{x}{2^n})\| + \|A'(\frac{x}{2^n}) - f(\frac{x}{2^n})\|) \\ &\leq \frac{2^{n+1} K^2 \theta}{(2^r - 2) 2^{nr}} ||x||^r, \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $A(x) = A'(x)$ for all $x \in X$. This proves the uniqueness of A.

It follows from (2.2) that

$$| \|2^n f(\frac{x}{2^n})\| - ||x|| \, | = 2^n | \, \|f(\frac{x}{2^n})\| - ||\frac{x}{2^n}|| \, | \leq 2\theta \frac{2^n}{2^{nr}} ||x||^r,$$

which tends to zero as $n \to \infty$ for all $x \in X$. So

$$\|A(x)\| = \lim_{n\to\infty} \|2^n f(\frac{x}{2^n})\| = ||x||$$

for all $x \in X$. Since A is additive,

$$\|A(x) - A(y)\| = \|A(x-y)\| = ||x-y||$$

for all $x, y \in X$, as desired. □

Theorem 2.2. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.1) *and* (2.2)*. Then there exists a unique isometric Cauchy additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2K\theta}{2 - 2^r} ||x||^r$$

for all $x \in X$.

Proof. It follows from (2.4) that

$$\|f(x) - \frac{1}{2} f(2x)\| \leq \theta ||x||^r$$

for all $x \in X$. So

$$\|\frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x)\| \leq K \sum_{j=l}^{m-1} \frac{2^{jr}\theta}{2^j} ||x||^r \tag{2.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.6) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.3. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying* (2.2) *such that*

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\| \leq \theta(||x||^r + ||y||^r) \tag{2.7}$$

for all $x, y \in X$. Then there exists a unique isometric Jensen additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{(3 + 3^r)K^2\theta}{3 - 3^r} ||x||^r$$

for all $x \in X$.

Proof. Letting $y = -x$ in (2.7), we get

$$\| - f(x) - f(-x)\| \le 2\theta||x||^r$$

for all $x \in X$. Letting $y = 3x$ and replacing x by $-x$ in (2.7), we get

$$\|2f(x) - f(-x) - f(3x)\| \le (3^r + 1)\theta||x||^r$$

for all $x \in X$. Thus

$$\|3f(x) - f(3x)\| \le K(3^r + 3)\theta||x||^r \tag{2.8}$$

for all $x \in X$. So

$$\|\frac{1}{3^l}f(3^l x) - \frac{1}{3^m}f(3^m x)\| \le K^2\frac{3^r+3}{3}\sum_{j=l}^{m-1}\frac{3^{jr}\theta}{3^j}||x||^r \tag{2.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.9) that the sequence $\{\frac{1}{3^n}f(3^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{3^n}f(3^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{3^n}f(3^n x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. $\square$

Theorem 2.4. *Let $r > 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying* (2.7) *and* (2.2)*. Then there exists a unique isometric Jensen additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{(3^r + 3)K^2\theta}{3^r - 3}||x||^r$$

for all $x \in X$.

Proof. It follows from (2.8) that

$$\|f(x) - 3f(\frac{x}{3})\| \le \frac{K(3^r + 3)\theta}{3^r}||x||^r$$

for all $x \in X$. So

$$\|3^l f(\frac{x}{3^l}) - 3^m f(\frac{x}{3^m})\| \le K^2\frac{3^r+3}{3^r}\sum_{j=l}^{m-1}\frac{3^j\theta}{3^{jr}}||x||^r \tag{2.10}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.10) that the sequence $\{3^n f(\frac{x}{3^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{3^n f(\frac{x}{3^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 3^n f(\frac{x}{3^n})$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. $\square$

3. Stability of the isometric additive mappings in generalized p-Banach spaces

Throughout this section, assume that X is a generalized quasi-normed vector space with generalized quasi-norm $||\cdot||$ and that Y is a generalized p-Banach space with generalized quasi-norm $\|\cdot\|$.

The following two results except for isometries are given by Tabor [45]. The proofs of isometries are similar to the proof of Theorem 2.1.

Theorem 3.1. ([45]) *Let $r > 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.1) *and* (2.2). *Then there exists a unique isometric Cauchy additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{(2^{pr} - 2^p)^{\frac{1}{p}}}||x||^r$$

for all $x \in X$.

Remark 3.2. The result for the case $K = 1$ in Theorem 2.1 is the same as the result for the case $p = 1$ in Theorem 3.1.

Theorem 3.3. ([45]) *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.1) *and* (2.2). *Then there exists a unique isometric Cauchy additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{(2^p - 2^{pr})^{\frac{1}{p}}}||x||^r$$

for all $x \in X$.

Remark 3.4. The result for the case $K = 1$ in Theorem 2.2 is the same as the result for the case $p = 1$ in Theorem 3.3.

Theorem 3.5. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying* (2.7) *and* (2.2). *Then there exists a unique isometric Jensen additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{K(3 + 3^r)\theta}{(3^p - 3^{pr})^{\frac{1}{p}}}||x||^r \tag{3.1}$$

for all $x \in X$.

Proof. It follows from (2.8) that

$$\|f(x) - \frac{1}{3}f(3x)\| \leq \frac{K(3^r + 3)\theta}{3}||x||^r \tag{3.2}$$

for all $x \in X$. Since Y is a generalized p-Banach space,

$$\begin{aligned}
\|\frac{1}{3^l}f(3^l x) - \frac{1}{3^m}f(3^m x)\|^p &\leq \sum_{j=l}^{m-1} \|\frac{1}{3^j}f(3^j x) - \frac{1}{3^{j+1}}f(3^{j+1}x)\|^p \\
&\leq \frac{K^p(3^r + 3)^p\theta^p}{3^p} \sum_{j=l}^{m-1} \frac{3^{prj}}{3^{pj}}||x||^{pr} \qquad (3.3)
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.3) that the sequence $\{\frac{1}{3^n} f(3^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{3^n} f(3^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{3^n} f(3^n x)$$

for all $x \in X$.

By (2.7),

$$\begin{aligned}
&\|2A(\frac{x+y}{2}) - A(x) - A(y)\| \\
&= \lim_{n\to\infty} \frac{1}{3^n} \|2f(3^n \cdot \frac{x+y}{2}) - f(3^n x) - f(3^n y)\| \\
&\le \lim_{n\to\infty} \frac{3^{rn}}{3^n} \theta(||x||^r + ||y||^r) = 0
\end{aligned}$$

for all $x, y \in X$. So

$$2A(\frac{x+y}{2}) = A(x) + A(y)$$

for all $x, y \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.3), we get (3.1).

Now, let $A' : X \to Y$ be another Jensen additive mapping satisfying (3.1). Then we have

$$\begin{aligned}
\|A(x) - A'(x)\|^p &= \frac{1}{3^{pn}} \|A(3^n x) - A'(3^n x)\|^p \\
&\le \frac{1}{3^{pn}} (\|A(3^n x) - f(3^n x)\|^p + \|A'(3^n x) - f(3^n x)\|^p) \\
&\le 2 \cdot \frac{3^{prn}}{3^{pn}} \cdot \frac{K^p (3+3^r)^p \theta^p}{3^p - 3^{pr}} ||x||^{pr},
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $A(x) = A'(x)$ for all $x \in X$. This proves the uniqueness of A.

The rest of the proof is similar to the proof of Theorem 2.1. □

Remark 3.6. The result for the case $K = 1$ in Theorem 2.3 is the same as the result for the case $p = 1$ in Theorem 3.5.

Theorem 3.7. *Let $r > 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying* (2.7) *and* (2.2). *Then there exists a unique isometric Jensen additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{K(3^r + 3)\theta}{(3^{pr} - 3^p)^{\frac{1}{p}}} ||x||^r$$

for all $x \in X$.

Proof. It follows (3.2) that

$$\|f(x) - 3f(\frac{x}{3})\| \le \frac{K(3^r + 3)\theta}{3^r} ||x||^r$$

for all $x \in X$. Since Y is a generalized p-Banach space,

$$\begin{aligned} \|3^l f(\frac{x}{3^l}) - 3^m f(\frac{x}{3^m})\|^p &\leq \sum_{j=l}^{m-1} \|3^j f(\frac{x}{3^j}) - 3^{j+1} f(\frac{x}{3^{j+1}})\|^p \\ &\leq \frac{K^p(3^r+3)^p\theta^p}{3^{pr}} \sum_{j=l}^{m-1} \frac{3^{pj}}{3^{prj}} ||x||^{pr} \end{aligned} \tag{3.4}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.4) that the sequence $\{3^n f(\frac{x}{3^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{3^n f(\frac{x}{3^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 3^n f(\frac{x}{3^n})$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 2.1 and 3.5. □

Remark 3.8. The result for the case $K = 1$ in Theorem 2.4 is the same as the result for the case $p = 1$ in Theorem 3.7.

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[1]Department of Mathematics, Hanyang University, Seoul 133-791, Republic of Korea.
E-mail address: baak@hanyang.ac.kr

[2]Department of Mathematics, National Technical University of Athens, Zografou Campus, 15780 Athens, Greece.
E-mail address: trassias@math.ntua.gr