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Measuring the roughness of random paths by increment ratios

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A statistic based on increment ratios (IR's) and related to zero crossings of an increment sequence is defined and studied for the purposes of measuring the roughness of random paths. The main advantages of this statistic are robustness to smooth additive and multiplicative trends and applicability to infinite variance processes. The existence of the IR statistic limit (which we shall call the *IR-roughness*) is closely related to the existence of a tangent process. Three particular cases where the IR-roughness exists and is explicitly computed are considered. First, for a diffusion process with smooth diffusion and drift coefficients, the IR-roughness coincides with the IR-roughness of a Brownian motion and its convergence rate is obtained. Second, the case of rough Gaussian processes is studied in detail under general assumptions which do not require stationarity conditions. Third, the IR-roughness of a Lévy process with an α-stable tangent process is established and can be used to estimate the fractional parameter $\alpha \in (0, 2)$ following a central limit theorem.

Keywords: diffusion processes; estimation of the local regularity function of stochastic process; fractional Brownian motion; Hölder exponent; Lévy processes; limit theorems; multifractional Brownian motion; tangent process; zero crossings

1. Introduction and statement of main results

It is well known that random functions are typically "rough" (non-differentiable), which raises the question of determining and measuring "roughness". Probably, the most widely studied roughness measures are the Hausdorff dimension and the p-variation index. There exists a considerable body of literature on statistical estimation of these and related quantities from a discrete grid. Hence, different estimators of the Hausdorff dimension have been studied, such as the box-counting estimator; see [26] for stationary Gaussian processes or [31] for Gaussian processes with stationary increments. To the best of our knowledge, the H-variation estimator, where H is a measurable function, was first proposed by Guyon and Leon [25] for stationary Gaussian processes where central and non-central limit theorems were established following the Hermite rank of H and the asymptotic local properties of the variogram and its second derivative. Further studies provided a continuation of this seminal paper in different ways. Istas and Lang [29] studied generalized quadratic variations of Gaussian processes with stationary increments. Coeurjolly [12,13] studied ℓ^p-variations of fractional Brownian motion and ℓ^2-variations of multifractional Brownian motion. Coeurjolly [14] discussed L-variations based on linear combinations of em-

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pirical quantiles for Gaussian locally self-similar processes. An estimator counting the number of level crossings was investigated by Feuerverger *et al.* [21] for stationary Gaussian processes.

In the present paper, we introduce a new characteristic of roughness, defined as a sum of ratios of consecutive increments. For a real-valued function $f = (f(t), t \in [0,1])$, define recursively

$$\Delta_j^{1,n} f := f\left(\frac{j+1}{n}\right) - f\left(\frac{j}{n}\right),$$
$$\Delta_j^{p,n} f := \Delta_j^{1,n} \Delta_j^{p-1,n} f = \sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} f\left(\frac{j+i}{n}\right), \tag{1.1}$$

so that $\Delta_j^{p,n} f$ denotes the p-order increment of f at $\frac{j}{n}$, $p = 1, 2, \ldots$, $j = 0, 1, \ldots, n-p$. Let

$$R^{p,n}(f) := \frac{1}{n-p} \sum_{k=0}^{n-p-1} \frac{|\Delta_k^{p,n} f + \Delta_{k+1}^{p,n} f|}{|\Delta_k^{p,n} f| + |\Delta_{k+1}^{p,n} f|} \tag{1.2}$$

with the convention that $\frac{0}{0} := 1$. In particular,

$$R^{1,n}(f) = \frac{1}{n-1} \sum_{k=0}^{n-2} \frac{|f((k+1)/n) - f(k/n) + f((k+2)/n) - f((k+1)/n)|}{|f((k+1)/n) - f(k/n)| + |f((k+2)/n) - f((k+1)/n)|}. \tag{1.3}$$

Note that the ratio on the right-hand side of (1.2) is either 1 or less than 1, depending on whether the consecutive increments $\Delta_k^{p,n} f$ and $\Delta_{k+1}^{p,n} f$ have the same sign or different signs, respectively; moreover, in the latter case, this ratio is generally small whenever the increments are similar in magnitude ("cancel each other out"). Clearly, $0 \le R^{p,n}(f) \le 1$ for any f, n, p. Thus, if $\lim R^{p,n}(f)$ exists when $n \to \infty$, the quantity $R^{p,n}(f)$ can be used to estimate this limit which represents the "mean roughness of f", also called the pth *order IR-roughness of f* below. We will show that these definitions can be extended to sample paths of very general random processes, for instance, stationary processes, processes with stationary and non-stationary increments and even $\mathbb{L}^q$-processes with $q < 1$.

Let us describe the main results of this paper. Section 2 derives some general results on asymptotic behavior of this estimator. Proposition 2.1 states that, for a sufficiently smooth function f, the limit $\lim_{n\to\infty} R^{p,n}(f) = 1$. In the majority of the paper, $f = X$ is a random process. Following Dobrushin [18], we say that $X = (X_t, t \in \mathbb{R})$ has a *small scale limit* $Y^{(t_0)}$ *at point* $t_0 \in \mathbb{R}$ if there exists a normalization $A^{(t_0)}(\delta) \to \infty$ when $\delta \to 0$ and a random process $Y^{(t_0)} = (Y_\tau^{(t_0)}, \tau \ge 0)$ such that

$$A^{(t_0)}(\delta)(X_{t_0+\tau\delta} - X_{t_0}) \xrightarrow[\delta\to 0]{\text{f.d.d.}} Y_\tau^{(t_0)}, \tag{1.4}$$

where $\xrightarrow{\text{f.d.d.}}$ stands for weak convergence of finite-dimensional distributions. A related definition is given in [19,20], where the limit process $Y^{(t_0)}$ is called a *tangent process* (at t_0); see also [9]. In many cases, the normalization $A^{(t_0)}(\delta) = \delta^{H(t_0)}$, where $0 < H(t_0) < 1$ and the limit tangent process $Y^{(t_0)}$ is self-similar with index $H(t_0)$ ([20] or [18]). Proposition 2.2 states that if X satisfies a condition similar to (1.4), then the statistic $R^{p,n}(X)$ converges to the integral

$$R^{p,n}(X) \xrightarrow[n\to\infty]{\mathcal{P}} \int_0^1 \mathrm{E}\left[\frac{|\Delta_0^p Y^{(t)} + \Delta_1^p Y^{(t)}|}{|\Delta_0^p Y^{(t)}| + |\Delta_1^p Y^{(t)}|}\right] \mathrm{d}t, \tag{1.5}$$

where $\Delta_j^p Y^{(t)} = \Delta_j^{p,1} Y^{(t)} = \sum_{i=0}^p (-1)^{p-i} \binom{p}{i} Y_{j+i}^{(t)}$, $j = 0, 1$, is the corresponding increment of the tangent process $Y^{(t)}$ at $t \in [0, 1)$. In the particular case where X has stationary increments, relation (1.5) becomes

$$R^{p,n}(X) \xrightarrow[n\to\infty]{\mathcal{P}} \mathrm{E}\left[\frac{|\Delta_0^p Y + \Delta_1^p Y|}{|\Delta_0^p Y| + |\Delta_1^p Y|}\right]. \tag{1.6}$$

Section 3 discusses the convergence in (1.5) for *diffusion processes* X admitting a stochastic differential $\mathrm{d}X = a_t\,\mathrm{d}B(t) + b_t\,\mathrm{d}t$, where B is a standard Brownian motion and $(a_t), (b_t)$ are random (adapted) functions. It is clear that under general regularity conditions on the diffusion and drift coefficients $(a_t), (b_t)$, the process X admits the same local Hölder exponent as B at each point $t_0 \in (0, 1)$ and therefore the IR-roughness of X in (1.5) should not depend on these coefficients and should coincide with the corresponding limit for $X = B$. This is indeed the case since the tangent process of X at t is easily seen to be $Y^{(t)} = a_t B$ and the multiplicative factor a_t cancels in the numerator and the denominator of the fraction inside the expectation in (1.5). See Theorem 3.1 for details, where the convergence rate $\mathrm{O}(n^{1/3})$ (a.s.) in (1.5) with explicit limit values $\Lambda_p(1/2)$ is established for diffusions X and $p = 1, 2$.

Considerable attention is given to the asymptotic behavior of the statistic $R^{p,n}(X)$ for *"fractal" Gaussian processes* (see Section 4). In such a framework, *fractional Brownian motion* (fBm) is a typical example. Indeed, if X is an fBm with parameter $H \in (0, 1)$, then X is also its own tangent process for any $t \in [0, 1]$ and (see Section 4)

$$R^{p,n}(X) \xrightarrow[n\to\infty]{\text{a.s.}} \Lambda_p(H), \qquad p = 1, 2, \tag{1.7}$$

$$\sqrt{n}\big(R^{p,n}(X) - \Lambda_p(H)\big) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \Sigma_p(H)) \qquad \text{if } \begin{cases} p = 1, & 0 < H < 3/4, \\ p = 2, & 0 < H < 1, \end{cases} \tag{1.8}$$

where $\xrightarrow[n\to\infty]{\mathcal{D}}$ stands for weak convergence of probability distributions. The expressions for $\Lambda_p(H)$ and $\sqrt{\Sigma_p(H)}$ $(p = 1, 2)$ are given in (4.19) and (4.22), respectively, and their graphs in Figures 1 and 2, respectively.

The difference in the range of the parameter H for $p = 1$ and $p = 2$ in the central limit theorem in (1.8) is due to the fact that the second-order increment process $(\Delta_j^2 B_H, j \in \mathbb{Z})$ is a short-memory stationary Gaussian process for any $H \in (0, 1)$, in contrast to the first-order increment process $(\Delta_j^1 B_H, j \in \mathbb{Z})$ which has long memory for $H > 3/4$.

Generalizations of (1.7) and (1.8) to Gaussian processes having non-stationary increments are proposed in Section 4. Roughly speaking, $R^{p,n}(X)$, $p = 1, 2$, converge a.s. and satisfy a central limit theorem, provided that for any $t \in [0, 1]$, the process X admits an fBm with parameter $H(t)$ as a tangent process (more precise assumptions (A.1), (A.1)$'$ and (A.2)$_p$ are provided in

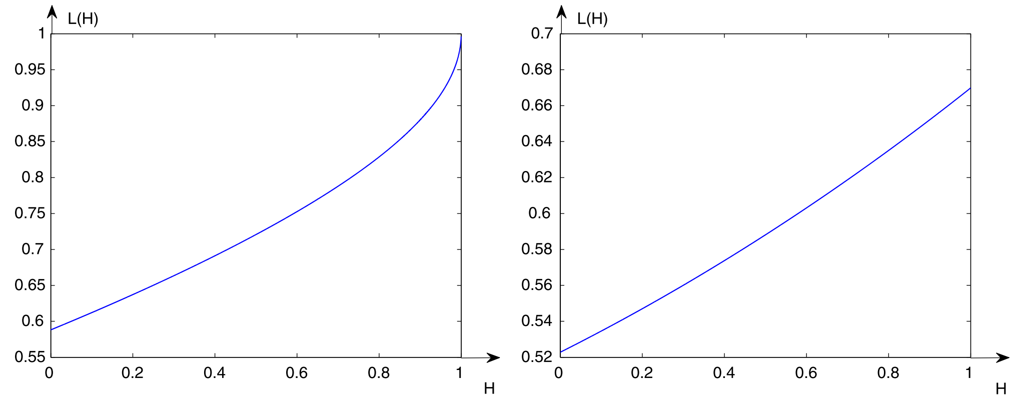

Figure 1. The graphs of $\Lambda_1(H)$ (left) and $\Lambda_2(H)$ (right).

Section 4). In such frameworks, the limits in (1.7) are $\int_0^1 \Lambda_p(H(t))\,\mathrm{d}t$ instead of $\Lambda_p(H)$ and the asymptotic variances in (1.8) also change. The case of Gaussian processes with stationary increments is discussed in detail and the results are used to define a $\sqrt{n}$-consistent estimator of H, under semi-parametric assumptions on the asymptotic behavior of the variogram or the spectral density. Bardet and Surgailis [6] study a punctual estimator of $H(t_0)$ obtained from a localization around $t_0 \in (0, 1)$ of the statistic $R^{2,n}(X)$.

The main advantages of estimators of the type (1.2) involving a scaling invariant function of increments seem to be the following. First, the estimator $R^{p,n}(X)$ essentially depends on the local regularity of the process X and not on possible "multiplicative and additive factors" such

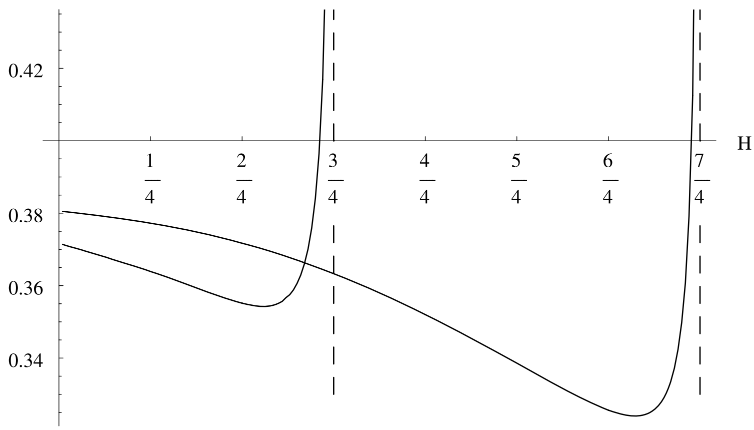

Figure 2. The graphs of $\sqrt{\Sigma_p(H)}$, $p = 1$ (with a pole at $3/4$) and $p = 2$ (with a pole at $7/4$) (from [36], with kind permission of the authors).

as diffusion and drift coefficients in Section 3 or smoothly multiplicative and additive trended Gaussian processes; see Proposition 4.1 of Section 4. This property is important when dealing with financial data involving heteroscedasticity and volatility clustering. Such a robustness property (also satisfied by the estimators based on generalized quadratic variations of wavelet coefficients) represents a clear advantage over classical parametric Whittle or semi-parametric log-periodogram estimators. Second, the estimators in (1.2) are bounded functionals and have finite moments of any order. Section 5 discusses jump Lévy processes, with the Lévy measure regularly varying with fractional index $\alpha \in (0,2)$ at the origin. Using a modification of (1.2), we define a $\sqrt{n}$-consistent estimator of α, together with a central limit theorem, in a very general semi-parametric framework. This result is new and interesting because there exist very few papers providing consistent estimators of α (to the best of our knowledge, the only comparable results have been established in [7] and [1], in a financial and somewhat different context). Finally, in the Gaussian case, using the approximation formula provided in Remark 4.3, an estimator of H based on $R^{2,n}(X)$ can be extremely simply computed:

$$\widehat{H}_n^{(2)} \simeq \frac{1}{0.1468} \times \left(\frac{1}{n-2}\sum_{k=0}^{n-3} \frac{|X_{(k+2)/n} - 2X_{(k+1)/n} + X_{k/n} + X_{(k+3)/n} - 2X_{(k+2)/n} + X_{(k+1)/n}|}{|X_{(k+2)/n} - 2X_{(k+1)/n} + X_{k/n}| + |X_{(k+3)/n} - 2X_{(k+2)/n} + X_{(k+1)/n}|} - 0.5174\right).$$

In the R language, if `X` is the vector $(X_{1/n}, X_{2/n}, \dots, X_1)$, then

$$\widehat{H}_n^{(2)} \simeq \Big(\mathtt{mean}\Big(\mathtt{abs}\big(\mathtt{diff(diff(X[-1]))} + \mathtt{diff(diff(X[-length(X)]))}\big) \\ /\big(\mathtt{abs(diff(diff(X[-1])))} + \mathtt{abs(diff(diff(X[-length(X)])))}\big)\Big) \\ - \mathtt{0.5174}\Big)/\mathtt{0.1468}.$$

Therefore, its computation is very fast and does not require any tuning parameters such as the scales for estimators based on quadratic variations or wavelet coefficients. The convergence rate of our estimator is $\sqrt{n}$, as for the parametric Whittle or the generalized quadratic variation estimators and hence it is more accurate than most of other well-known semi-parametric estimators (log-periodogram, local Whittle or wavelet-based estimators).

Estimators of the form (1.2) can also be applied to discrete-time (sequences) instead of continuous-time processes (functions). For instance, Surgailis *et al.* [38] extended the statistic $R^{2,n}(X)$ to discrete-time processes and used it to test for $I(d)$ behavior $(-1/2 < d < 5/4)$ of observed time series. Vaičiulis [40] considered estimation of the tail index of i.i.d. observations using an increment ratio statistic.

Remark 1.1. The referee noted that the IR-roughness might be connected to the level crossing index; see [21]. To our surprise, such a connection indeed exists, as explained below.

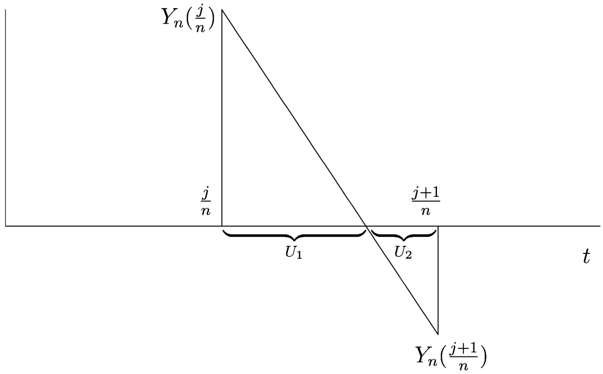

Figure 3. The proof of (1.9): this follows by $\frac{|Y_n(j/n)+Y_n((j+1)/n)|}{|Y_n(j/n)|+|Y_n((j+1)/n)|} = n|U_1 - U_2|$.

Let $Y_n(t), t \in [0, 1 - \frac{1}{n}]$, be the linear interpolation of the "differenced" sequence $\Delta_j^{1,n} X = X(\frac{j+1}{n}) - X(\frac{j}{n})$, $j = 0, 1, \ldots, n-1$:

$$Y_n(t) = n\left[\left(\frac{j+1}{n} - t\right)\Delta_j^{1,n} X + \left(t - \frac{j}{n}\right)\Delta_{j+1}^{1,n} X\right], \qquad t \in \left[\frac{j}{n}, \frac{j+1}{n}\right),$$

$j = 0, 1, \ldots, n-2$. Then, using Figure 3 as a proof,

$$\begin{aligned} R^{1,n}(X) &= \frac{n}{n-1} \sum_{j=0}^{n-2} \left| \operatorname{meas}\left\{t \in \left[\frac{j}{n}, \frac{j+1}{n}\right) : Y_n(t) > 0\right\} \right. \\ &\qquad \left. - \operatorname{meas}\left\{t \in \left[\frac{j}{n}, \frac{j+1}{n}\right) : Y_n(t) < 0\right\} \right| \qquad (1.9) \\ &= \frac{n}{n-1} \sum_{j=0}^{n-2} \left| \int_{j/n}^{(j+1)/n} \left(\mathbf{1}\big(Y_n(t) > 0\big) - \mathbf{1}\big(Y_n(t) < 0\big)\right) \mathrm{d}t \right|. \end{aligned}$$

Let $\psi(x_1, x_2) := |x_1 + x_2|/(|x_1| + |x_2|)$, $\psi_0(x_1, x_2) := \mathbf{1}(x_1 x_2 \geq 0)$. Clearly, the two quantities $1 - \psi(Y_n(\frac{j}{n}), Y_n(\frac{j+1}{n}))$ and $1 - \psi_0(Y_n(\frac{j}{n}), Y_n(\frac{j+1}{n}))$ are both strictly positive if and only if Y_n crosses the zero level in the interval $[\frac{j}{n}, \frac{j+1}{n})$, but the former quantity measures not only the fact but also the "depth" of the crossing so that $1 - \psi(Y_n(\frac{j}{n}), Y_n(\frac{j+1}{n}))$ attains its maximal value 1 in the case of a "perfect" crossing in the middle of the interval $[\frac{j}{n}, \frac{j+1}{n})$; see Figure 3.

It seems that similar asymptotic results can be obtained for $R_0^{p,n}(X) := \frac{1}{n-p} \times \sum_{k=0}^{n-p-1} \psi_0(\Delta_k^{p,n} X, \Delta_{k+1}^{p,n} X)$, measuring the number of zero crossings of the increment sequence $\Delta_k^{p,n} X$, $k = 0, 1, \ldots, n-p$, and other similar statistics obtained by replacing the functions ψ or ψ_0 by other scaling invariant functions. Let us note that $R_0^{1,n}(X)$ is related to the zero-crossings' counting statistic studied in [27] for stationary Gaussian time series. Also, note that the Hermite rank of ψ_0 is 2 and that the corresponding limit function $\lambda_0(r) = \frac{1}{\pi}\arccos(-r)$ is strictly increasing on the interval $(-1, 1)$ similarly to the function $\lambda(r)$ in (4.20). On the other hand, while the statistic $R_0^{p,n}(X)$ is certainly of interest, the statistic $R^{p,n}(X)$ seems preferable to it for the reasons explained above. In particular, in the case of symmetric Lévy processes X with independent increments studied in Section 5, the latter statistic leads to an estimator of the fractional index, while the former statistic can be easily shown to converge to $1/2$.

The paper is organized as follows. Section 2 discusses some general (consistency) properties of the estimators $R^{p,n}(X)$. Section 3 deals with the case where X is a diffusion. The case of Gaussian processes X is considered in Section 4, while the case of Lévy processes is studied in Section 5. The Appendix contains sketches of the proofs of Theorems 4.1 and 4.2 and other derivations. Complete proofs can be found in the extended version of the paper on arXiv (see http://arxiv.org/abs/0802.0489).

Below, we write C for generic constants, the value of which may change from line to line.

2. Some asymptotic results

The definition of $R^{p,n} f$ in (1.2) can be extended to more general increments (the so-called generalized variations). Consider a filter $a := (a_0, \dots, a_q) \in \mathbb{R}^{q+1}$ such that there exists $p \in \mathbb{N}$, $p \le q$, satisfying

$$\sum_{\ell=0}^{q} \ell^i a_\ell = 0 \qquad \text{for } i = 0, \dots, p-1 \text{ and } \sum_{\ell=0}^{q} \ell^p a_\ell \neq 0. \tag{2.1}$$

The class of such filters will be denoted by $\mathcal{A}(p,q)$. For $n \in \mathbb{N}^* := \{1, 2, \dots\}$ and a function $f = (f(t), t \in [0,1])$, define the generalized variations of f by

$$\Delta_j^{a,n} f := \sum_{\ell=0}^{q} a_\ell f\left(\frac{j+\ell}{n}\right), \qquad j = 0, 1, \dots, n-q. \tag{2.2}$$

A particular case of (2.2) corresponding to $q = p \ge 1$, $a_\ell = (-1)^{p-\ell}\binom{p}{\ell}$ is the p-order increment $\Delta_j^{p,n} f$ in (1.1). For a filter $a \in \mathcal{A}(p,q)$, let

$$R^{a,n}(f) := \frac{1}{n-q} \sum_{k=0}^{n-q-1} \frac{|\Delta_k^{a,n} f + \Delta_{k+1}^{a,n} f|}{|\Delta_k^{a,n} f| + |\Delta_{k+1}^{a,n} f|}. \tag{2.3}$$

It is easy to prove that $R^{1,n}(f) \underset{n\to\infty}{\longrightarrow} 1$ if f is continuously differentiable on $[0,1]$ and the derivative f' does not vanish on $[0,1]$, except perhaps for a finite number of points. Moreover, it is obvious that $R^{1,n}(f) = 1$ if f is monotone on $[0,1]$: the IR-roughness of a monotone function is the same as that of a smooth function, which is not surprising since a similar fact holds for other measures of roughness, such as the p-variation index or the Hausdorff dimension.

We conjecture that $R^{p,n}(f) \to 1$ and $R^{a,n}(f) \to 1$ for any $q \ge p \ge 1, a \in \mathcal{A}(p,q)$ and $f:[0,1] \to \mathbb{R}$ which is $(p-1)$ times differentiable, and the derivative $f^{(p-1)}$ has bounded variation on $[0,1]$ with the support $\mathrm{supp}(f^{(p-1)}) = [0,1]$. However, we can prove a weaker result.

Proposition 2.1. *Let f be $(p-1)$-times continuously differentiable ($p \ge 1$) with $f^{(p-1)}$ being absolutely continuous on $[0,1]$ and having the Radon–Nikodym derivative $g = (f^{(p-1)})'$. Assume that $g \neq 0$ a.e. in $[0,1]$. Then, $R^{p,n}(f) \underset{n\to\infty}{\longrightarrow} 1$ and $R^{a,n}(f) \underset{n\to\infty}{\longrightarrow} 1$ for any $a \in \mathcal{A}(p,q)$, $q \ge p$.*

Proof. We restrict the proof to the case $p = 2$ since the general case is analogous. Using summation by parts, we can rewrite $\Delta_j^{a,n} f$ as

$$\Delta_j^{a,n} f = \sum_{i=0}^{q} b_i \Delta_{i+j}^{2,n} f, \tag{2.4}$$

where $b_i := \sum_{k=0}^{i} \sum_{\ell=0}^{k} a_\ell, i = 0, 1, \dots, q, b_{q-1} = b_q = 0$ and $\bar{b} := \sum_{i=0}^{q} b_i = \frac{1}{2} \sum_{i=1}^{q} i^2 a_i \neq 0$ in view of the assumption $a \in \mathcal{A}(2,q)$.

Assume that n is large enough and, for a given $t \in (0,1)$, let $k_n(t) \in \{0, \dots, n-2\}$ be chosen so that $t \in [k_n(t)/n, (k_n(t)+1)/n)$ and therefore $k_n(t) = [nt] - 1$. We claim that for a.e. $t \in (0,1)$,

$$\lim_{n\to\infty} n^2 \Delta_{k_n(t)}^{a,n} f = \bar{b} g(t), \qquad \lim_{n\to\infty} n^2 \Delta_{k_n(t)+1}^{a,n} f = \bar{b} g(t). \tag{2.5}$$

Using the fact that the function $(x_1, x_2) \mapsto \frac{|x_1+x_2|}{|x_1|+|x_2|}$ is continuous on $\mathbb{R}^2 \backslash \{(0,0)\}$, we obtain

$$h^{a,n}(t) := \frac{|n^2 \Delta_{k_n(t)}^{a,n} f + n^2 \Delta_{k_n(t)+1}^{a,n} f|}{|n^2 \Delta_{k_n(t)}^{a,n} f| + |n^2 \Delta_{k_n(t)+1}^{a,n} f|} \underset{n\to\infty}{\longrightarrow} \frac{|\bar{b} g(t) + \bar{b} g(t)|}{|\bar{b} g(t)| + |\bar{b} g(t)|} = 1 \tag{2.6}$$

for a.e. $t \in (0,1)$, where we have used the fact that $\bar{b} g(t) \neq 0$ a.e. Since, for $n \ge q$, $R^{a,n}(f)$ can be written as $R^{a,n}(f) = \frac{n}{n-q} \int_0^1 h^{a,n}(t)\,\mathrm{d}t$, relation $R^{a,n}(f) \underset{n\to\infty}{\longrightarrow} 1$ follows by the dominated convergence theorem and the fact that $0 \le h^{a,n}(t) \le 1$.

Relations (2.5) can be proven using the Lebesgue–Vitali theorem (see [35], Chapter 4, Section 10, Theorem 1), as follows. Consider the signed measure μ on Borel subsets of $[0,1/2]^2$ given by

$$\mu(A) = \int_A g(x_1 + x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2.$$

Note that $\Delta_k^{2,n} f = \mu((k/2n, (k+2)/2n] \times (k/2n, (k+2)/2n]), k = 0, \dots, n-2$. Since rectangles $[x_1, x_1+h] \times [x_2, x_2+h], 0 \le x_i < x_i + h \le 1/2, i = 1, 2$, form a Vitali system on $[0,1/2]^2$, the aforementioned Lebesgue–Vitali theorem implies that

$$\phi_n(t_1, t_2) := n^2 \mu\left(\left(\frac{k_n(t_1)}{2n}, \frac{k_n(t_1)+2}{2n}\right] \times \left(\frac{k_n(t_2)}{2n}, \frac{k_n(t_2)+2}{2n}\right]\right) \underset{n\to\infty}{\longrightarrow} g\left(\frac{t_1+t_2}{2}\right) \tag{2.7}$$

a.e. in $[0,1]^2$. Taking into account the form of the measure μ and the limiting function in (2.7), it follows the convergence $n^2\Delta^{2,n}_{k_n(t)} f = \phi_n(t,t) \underset{n\to\infty}{\longrightarrow} g(t)$ a.e. on $[0,1]$. Next, for any fixed $i = 0,1,\dots,$ the sequence of rectangles $(\frac{k_n(t_1)+i}{2n}, \frac{k_n(t_1)+i+2}{2n}] \times (\frac{k_n(t_2)+i}{2n}, \frac{k_n(t_2)+i+2}{2n}], n = 1,2,\dots,$ is regularly contracting to $(t_1,t_2)\in(0,1)^2$ in the sense of [35], Chapter 4, Section 10. Hence, using the lemma on page 214 of that monograph, it follows that $n^2\mu((\frac{k_n(t_1)+i}{2n}, \frac{k_n(t_1)+i+2}{2n}] \times (\frac{k_n(t_2)+i}{2n}, \frac{k_n(t_2)+i+2}{2n}]) \underset{n\to\infty}{\longrightarrow} g(\frac{t_1+t_2}{2})$ a.e. in $[0,1]^2$, implying that

$$n^2\Delta^{2,n}_{k_n(t)+i} f \underset{n\to\infty}{\longrightarrow} g(t) \qquad \text{a.e. on } [0,1], \text{ for any } i = 0,1,\dots.$$

Together with (2.4), this proves (2.5) and the proposition. □

Let us now turn to the case when $f(t) = X_t$, $t\in[0,1]$ is a random process. Now and hereafter, $R^{p,n}(X)$, $R^{a,n}(X)$ are denoted $R^{p,n}$, $R^{a,n}$, respectively. Below, we formulate a general condition for the convergence of $R^{p,n}$ and $R^{a,n}$ to a deterministic limit.

Assumption (A). *For a.e. pairs* $(t_1,t_2)\in(0,1)^2$, $t_1\neq t_2$, *for* $i=1,2$, *there exist*:

(i) *normalizations* $A^{(t_i)}(\delta)\to\infty$ $(\delta\to 0)$;
(ii) (*mutually*) *independent random processes* $Y^{(t_i)} = (Y^{(t_i)}(\tau), \tau\in[0,1])$,

such that for $\delta\to 0$, $s_1\to t_1$, $s_2\to t_2$

$$\big(A^{(t_1)}(\delta)(X_{s_1+\delta\tau} - X_{s_1}), A^{(t_2)}(\delta)(X_{s_2+\delta\tau} - X_{s_2})\big) \overset{f.d.d.}{\longrightarrow} \big(Y^{(t_1)}(\tau), Y^{(t_2)}(\tau)\big). \tag{2.8}$$

Remark 2.1. Relation (2.8) implies the existence of a joint small scale limit $(Y^{(t_1)}, Y^{(t_2)})$ at a.e. pair $(t_1,t_2)\in(0,1)$, with independent components $Y^{(t_1)}, Y^{(t_2)}$. Note that Assumption (A) and Proposition 2.2 below are very general, in the sense that they do not assume any particular structure or distribution of X.

Proposition 2.2. *Let* $a = (a_0,\dots,a_q)\in\mathcal{A}(p,q)$, $1\le p\le q$, *be a filter and let* X *satisfy Assumption (A). Assume, in addition, that* $\mathrm{P}(|\Delta^a_j Y^{(t)}|>0) = 1$, $j=0,1$, *for a.e.* $t\in(0,1)$, *where* $\Delta^a_j Z \equiv \Delta^{a,1}_j Z = \sum_{\ell=0}^q a_\ell Z(j+\ell)$. *Then,*

$$\mathrm{E}\bigg(R^{a,n} - \int_0^1 \mathrm{E}\bigg[\frac{|\Delta^a_0 Y^{(t)} + \Delta^a_1 Y^{(t)}|}{|\Delta^a_0 Y^{(t)}| + |\Delta^a_1 Y^{(t)}|}\bigg]\,\mathrm{d}t\bigg)^2 \underset{n\to\infty}{\longrightarrow} 0. \tag{2.9}$$

Proof. The statement follows from

$$\mathrm{E}R^{a,n} \underset{n\to\infty}{\longrightarrow} \int_0^1 \mathrm{E}\bigg[\frac{|\Delta^a_0 Y^{(t)} + \Delta^a_1 Y^{(t)}|}{|\Delta^a_0 Y^{(t)}| + |\Delta^a_1 Y^{(t)}|}\bigg]\,\mathrm{d}t \quad \text{and} \quad \mathrm{E}(R^{a,n} - \mathrm{E}R^{a,n})^2 \underset{n\to\infty}{\longrightarrow} 0. \tag{2.10}$$

Write $\mathrm{E}R^{a,n} = \frac{n}{n-q}\int_0^1 \mathrm{E}h_X^{a,n}(t)\,\mathrm{d}t$, where (cf. (2.6))

$$\begin{aligned} h_X^{a,n}(t) \;:=\; & \frac{A^{(t)}(1/n)(|\Delta^{a,n}_{k_n(t)}X + \Delta^{a,n}_{k_n(t)+1}X|)}{A^{(t)}(1/n)(|\Delta^{a,n}_{k_n(t)}X| + |\Delta^{a,n}_{k_n(t)+1}X|)} \\ & \xrightarrow[n\to\infty]{\mathcal{D}} \frac{|\Delta_0^a Y^{(t)} + \Delta_1^a Y^{(t)}|}{|\Delta_0^a Y^{(t)}| + |\Delta_1^a Y^{(t)}|} =: h_Y^a(t) \end{aligned}$$

for a.e. $t \in (0,1)$, according to Assumption (A) and the continuous mapping theorem. Using the fact that $0 \le h_X^{a,n} \le 1$ and the Lebesgue dominated convergence theorem, the first relation in (2.10) then follows. Moreover,

$$\mathrm{E}(R^{a,n} - \mathrm{E}R^{a,n})^2 = \left(\frac{n}{n-q}\right)^2 \int_0^1\int_0^1 \mathrm{E}[h_X^{a,n}(t)h_X^{a,n}(t')]\,\mathrm{d}t\,\mathrm{d}t' - (\mathrm{E}R^{a,n})^2$$

and, with the same arguments as previously and the independence of $Y^{(t)}$ and $Y^{(t')}$ when $t \neq t'$,

$$\mathrm{E}[h_X^{a,n}(t)h_X^{a,n}(t')] \underset{n\to\infty}{\longrightarrow} \mathrm{E}\left[\frac{|\Delta_0^a Y^{(t)} + \Delta_1^a Y^{(t)}|}{|\Delta_0^a Y^{(t)}| + |\Delta_1^a Y^{(t)}|}\cdot\frac{|\Delta_0^a Y^{(t')} + \Delta_1^a Y^{(t')}|}{|\Delta_0^a Y^{(t')}| + |\Delta_1^a Y^{(t')}|}\right] = \mathrm{E}h_Y^a(t)\cdot\mathrm{E}h_Y^a(t')$$

and therefore $(\frac{n}{n-q})^2\int_0^1\int_0^1 \mathrm{E}[h_X^{a,n}(t)h_X^{a,n}(t')]\,\mathrm{d}t\,\mathrm{d}t' - (\int_0^1 \mathrm{E}h_X^{a,n}(t)\,\mathrm{d}t)^2 \underset{n\to\infty}{\longrightarrow} 0$, thereby proving the second relation in (2.10) and the proposition. □

The following easy, but interesting, corollary can also be added to this result. It proves that smooth multiplicative or additive trends do not change the $\mathbb{L}^2$-asymptotic behavior of $R^{a,n}$. Let $\mathcal{C}^p[0,1]$ denote the class of all p-times continuously differentiable functions on $[0,1]$.

Corollary 2.1. *Let $a \in \mathcal{A}(p,q)$ and X satisfy the conditions of Proposition 2.2 with $A^{(t)}(\delta) = \mathrm{O}(\delta^{-1})$ $(\delta \to 0)$ for each $t \in [0,1]$. Assume that $\alpha \in \mathcal{C}^1[0,1]$, $\beta \in \mathcal{C}^p[0,1]$, $\inf_{t\in[0,1]}\alpha(t) > 0$ and $\sup_{t\in[0,1]}|X(t)| < \infty$ a.s. Define Z such that $Z_t = \alpha(t)X_t + \beta(t), t \in [0,1]$. Then, (2.9) holds with $R^{p,n} = R^{p,n}(X)$ replaced by $R^{p,n}(Z)$.*

Remark 2.2. By definition, the statistics $R^{p,n}$ and $R^{a,n}$ for $a \in \mathcal{A}(p,q)$, $1 \le p \le q$, are invariant with respect to additive polynomial trends of order less than p; in particular, $R^{3,n}$ is insensitive to a quadratic trend, while $R^{2,n}$ does not have this property. On the other hand, Corollary 2.1 (see also Proposition 4.1) states that under weak additional conditions on X, any sufficiently smooth additive or multiplicative trends do not affect the limit of $R^{p,n}$, provided that $p \ge 1$. In the important special case where the limit process $Y^{(t)} = B_H$ in Assumption (A) and (2.9) is a fractional Brownian motion with parameter $H \in (0,1)$ independent of t, the statistic $R^{p,n}$ converges in mean square to the expectation $\mathrm{E}\frac{|\Delta_0^p B_H + \Delta_1^p B_H|}{|\Delta_0^p B_H| + |\Delta_1^p B_H|} = \lambda(\rho_p(H))$; cf. (4.19)–(4.21). Numerical computations show that the correlation coefficient $\rho_p(H)$ is a monotone function of H for any $p \ge 1$ and tends to the constant value -1 on the interval $(0,1)$ as p increases. Therefore, for larger values of p, the range of $\lambda(\rho_p(H))$ is rather small and $R^{p,n}$ seems less capable of estimating H. A final reason for our concentrating on the "lower order" statistics $R^{p,n}$, $p = 1, 2$, in the rest of the paper is the fact that $R^{2,n}$ satisfies the central limit theorem in (1.8) on the whole interval $H \in (0, 1)$.

3. Diffusions

Let

$$X_t = X_0 + \int_0^t a_s \,\mathrm{d}B(s) + \int_0^t b_s \,\mathrm{d}s, \qquad t \in [0, 1], \tag{3.1}$$

be a diffusion (or Itô) process on $\mathbb{R}$. In (3.1), we assume the existence of a right-continuous filtration $\mathcal{F} = (\mathcal{F}_t, t \in [0, 1])$ and a standard Brownian motion B adapted to $\mathcal{F}$; moreover, $a_s, b_s, s \in [0, 1]$ are adapted random functions satisfying $\int_0^1 |b_s| \,\mathrm{d}s < \infty$, $\int_0^1 a_s^2 \,\mathrm{d}s < \infty$ a.s. and X_0 is an $\mathcal{F}_0$-measurable r.v. Write $\mathrm{E}_t[\cdot] = \mathrm{E}[\cdot|\mathcal{F}_t]$ for the conditional expectation. Let $\Lambda_1(1/2) = \lambda(\rho_1(1/2)) \simeq 0.7206$ and $\Lambda_2(1/2) = \lambda(\rho_2(1/2)) \simeq 0.5881$. The proof of the following Lemma 3.1 is given in the arXiv version of the paper; see http://arxiv.org/abs/0802.0489.

Lemma 3.1. *Let $\psi(x_1, x_2) := |x_1 + x_2|/(|x_1| + |x_2|)$ $(x_1, x_2 \in \mathbb{R})$ and let $Z_i, i = 1, 2$, be independent $\mathcal{N}(0, 1)$ r.v.'s. Then, for any random variables ξ_1, ξ_2,*

$$|\mathrm{E}\psi(Z_1 + \xi_1, Z_2 + \xi_2) - \mathrm{E}\psi(Z_1, Z_2)| \le 20 \max_{i=1,2} (\mathrm{E}\xi_i^2)^{1/3}. \tag{3.2}$$

Theorem 3.1. *Assume the following conditions: there exist random variables K_1, K_2 such that $0 < K_i < \infty$ a.s. and such that, for any sufficiently small $h > 0$ and any $0 \le t < t + h \le 1$, the following inequalities hold, a.s.:*

$$|a_t| \ge K_1, \qquad \mathrm{E}_t b_{t+h}^2 \le K_2 \quad \textit{and} \quad \mathrm{E}_t(a_{t+h} - a_t)^2 \le K_2 h. \tag{3.3}$$

Then,

$$R^{p,n} - \Lambda_p(1/2) = \mathrm{O}(n^{-1/3}) \qquad \textit{a.s.}\ (p = 1, 2). \tag{3.4}$$

Proof. We restrict the proof to the case $p = 1$ since the case $p = 2$ is analogous. For notational simplicity, assume that n is odd. Define

$$\eta_n(k) := \frac{|\Delta_k^{1,n} X + \Delta_{k+1}^{1,n} X|}{|\Delta_k^{1,n} X| + |\Delta_{k+1}^{1,n} X|}, \qquad \eta_n'(k) := \mathrm{E}_{k/n}[\eta_n(k)], \qquad \eta_n''(k) := \eta_n(k) - \eta_n'(k) \tag{3.5}$$

and, correspondingly, write $R^{1,n} = R_n' + R_n''$, $R_n' := (n-1)^{-1} \sum_{k=0}^{n-2} \eta_n'(k)$, $R_{n1}'' := (n-1)^{-1} \sum_{k=0}^{(n-2)/2} \eta_n''(2k)$, $R_{n2}'' := (n-1)^{-1} \sum_{k=0}^{(n-4)/2} \eta_n''(2k+1)$. As $(\eta_n''(2k), \mathcal{F}_{(2k+2)/n}, k = 0, \ldots, (n-2)/2)$ is a martingale difference sequence, by Burkholder's inequality, we have

$$\mathrm{E}(R_{n1}'')^8 \le C n^{-8} \left(\sum_{k=0}^{(n-2)/2} \mathrm{E}^{1/4} (\eta_n'(2k))^8 \right)^4 \le C n^{-4}$$

and therefore

$$\sum_{n=1}^{\infty} \mathrm{P}(|R''_{n1}| > n^{-1/3}) \le C \sum_{n=1}^{\infty} n^{8/3} n^{-4} < \infty,$$

implying that $R''_{n1} = \mathrm{O}(n^{-1/3})$ a.s. A similar fact holds for R''_{n2}. Thus, it remains to prove

$$R'_n - \Lambda_1(1/2) = \mathrm{O}(n^{-1/3}) \qquad \text{a.s.} \tag{3.6}$$

Observe that

$$\eta'_n(k) - \Lambda_1(1/2) = \mathrm{E}_{k/n}\biggl[\frac{|Z_1(k) + \xi_1(k) + Z_2(k) + \xi_2(k)|}{|Z_1(k) + \xi_1(k)| + |Z_2(k) + \xi_2(k)|}\biggr] - \mathrm{E}\biggl[\frac{|Z_1(k) + Z_2(k)|}{|Z_1(k)| + |Z_2(k)|}\biggr],$$

where

$$\begin{aligned}
Z_1(k) &:= n^{1/2} \Delta_k^{1,n} B, \qquad Z_2(k) := n^{1/2} \Delta_{k+1}^{1,n} B, \\
\xi_1(k) &:= n^{1/2} \int_{k/n}^{(k+1)/n} \biggl(\frac{a_s}{a_{k/n}} - 1\biggr) \mathrm{d}B(s) + n^{1/2} \int_{k/n}^{(k+1)/n} \frac{b_s}{a_{k/n}} \,\mathrm{d}s, \\
\xi_2(k) &:= n^{1/2} \int_{(k+1)/n}^{(k+2)/n} \biggl(\frac{a_s}{a_{k/n}} - 1\biggr) \mathrm{d}B(s) + n^{1/2} \int_{(k+1)/n}^{(k+2)/n} \frac{b_s}{a_{k/n}} \,\mathrm{d}s.
\end{aligned}$$

According to Lemma 3.1 above, $|\eta'_n(k) - \Lambda_1(1/2)| \le 36 \max_{i=1,2} (\mathrm{E}_{k/n} \xi_i^2(k))^{1/3}$ and therefore

$$|R'_n - \Lambda_1(1/2)| \le 36 \max\{(\mathrm{E}_{k/n} \xi_i^2(k))^{1/3} : i = 1, 2, k = 0, 1, \ldots, n-1\},$$

whence, (3.6) follows from the following fact: there exists an r.v. $K < \infty$, independent of n and such that for any $n \ge 1, k = 0, \ldots, n-1, i = 1, 2,$

$$\mathrm{E}_{k/n} \xi_i^2(k) \le K n^{-1} \qquad \text{a.s.} \tag{3.7}$$

Indeed, using (3.3),

$$\begin{aligned}
\mathrm{E}_{k/n} \xi_1^2(k) &= n \int_{k/n}^{(k+1)/n} \mathrm{E}_{k/n} \biggl(\frac{a_s}{a_{k/n}} - 1\biggr)^2 \mathrm{d}s + n \mathrm{E}_{k/n} \biggl(\int_{k/n}^{(k+1)/n} \frac{b_s}{a_{k/n}} \,\mathrm{d}s\biggr)^2 \\
&\le n K_1^{-2} \int_{k/n}^{(k+1)/n} \mathrm{E}_{k/n} (a_s - a_{k/n})^2 \,\mathrm{d}s + K_1^{-2} \int_{k/n}^{(k+1)/n} \mathrm{E}_{k/n} b_s^2 \,\mathrm{d}s \\
&\le K_2 K_1^{-2} n^{-1} \qquad \text{a.s.}
\end{aligned}$$

and the bound (3.7) for $i = 2$ follows similarly. This proves (3.7) as well as Theorem 3.1. $\square$

Let us present some examples of Itô processes X satisfying conditions (3.3).

Example 3.1. Let $(X_t, t \in [0,1])$ be a Markov process satisfying a stochastic equation

$$X_t = x_0 + \int_0^t a(X_s)\,\mathrm{d}B(s) + \int_0^t b(X_s)\,\mathrm{d}s, \tag{3.8}$$

where $x_0 \in \mathbb{R}$ is non-random, $a(x), b(x), x \in \mathbb{R}$, are real measurable functions and B is a standard Brownian motion. Let $\mathcal{F}_t := \sigma\{B(s), s \le t\}$, $0 \le t \le 1$, be the natural filtration. Assume that

$$|a(x) - a(y)| \le K|x-y|, \qquad |b(x) - b(y)| \le K|x-y| \qquad (x, y \in \mathbb{R}) \tag{3.9}$$

for some constant $K < \infty$. Equation (3.8) then admits a unique adapted solution; see, for example, [23]. Let $a_t = a(X_t)$, $b_t = b(X_t)$. Assume, in addition, that $|a(x)| \ge K_1$ $(x \in \mathbb{R})$ for some non-random constant $K_1 > 0$. The first inequality in (3.3) is then trivially satisfied; moreover, the second and third relations in (3.3) are also satisfied, with $K_2 = C(1 + \sup_{0\le t\le 1} X_t^2) < \infty$ and $K_3 = C$, where C is non-random and depends on the constant K in (3.9) only.

Example 3.2. Let $X_t := g(t, B(t))$, where B is a standard Brownian motion and $g(t,x)$ is a (jointly) continuous function on $[0,1] \times \mathbb{R}$, having continuous partial derivatives $g_t(t,x) := \partial g(t,x)/\partial t$, $g_x(t,x) := \partial g(t,x)/\partial x$, $g_{xx}(t,x) = \partial^2 g(t,x)/\partial x^2$. By Itô's lemma,

$$\mathrm{d}X_t = g_x(t, B(t))\,\mathrm{d}B(t) + \bigl(g_t(t, B(t)) + \tfrac{1}{2} g_{xx}(t, B(t))\bigr)\,\mathrm{d}t$$

so that X admits the representation (3.1) with $a_t = g_x(t, B(t))$, $b_t = g_t(t, B(t)) + \frac{1}{2} g_{xx}(t, B(t))$ and the same filtration as in the previous example. Assume that

$$|g_x(t,x)| \ge K_1, \qquad |g_x(s,y) - g_x(t,x)| \le K(|s-t|^{1/2} + |y-x|)$$

for all $(t,x), (s,y) \in [0,1] \times \mathbb{R}$ and some constants $0 < K_1, K < \infty$. Then, X satisfies the conditions in (3.3).

4. Gaussian processes

4.1. Assumptions

Let $X = (X_t, t \in [0,1])$ be a Gaussian process with zero mean. Without loss of generality, assume that $X_0 = 0$. Define $\sigma^2_{p,n}(k)$, the variance of $\Delta_k^{p,n} X$, and $\rho_{p,n}(k)$, the correlation coefficient between $\Delta_k^{p,n} X$ and $\Delta_{k+1}^{p,n} X$, that is,

$$\sigma^2_{p,n}(k) := \mathrm{E}[(\Delta_k^{p,n} X)^2], \qquad \rho_{p,n}(k) := \frac{\mathrm{E}[\Delta_k^{p,n} X \Delta_{k+1}^{p,n} X]}{\sigma_{p,n}(k)\sigma_{p,n}(k+1)}. \tag{4.1}$$

Let $B_H = (B_H(t), t \in \mathbb{R})$ be a fractional Brownian motion (fBm) with parameter $0 < H < 1$, that is, a Gaussian process with zero mean and covariance such that $\mathrm{E} B_H(s) B_H(t) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$. Its pth order increments $(\Delta_j^p B_H, j \in \mathbb{Z})$ form a stationary Gaussian process for any $p \geq 1$. In particular, the covariance function of $\Delta_j B_H \equiv \Delta_j^1 B_H = B_H(j+1) - B_H(j)$ and $\Delta_j^2 B_H = B_H(j+2) - 2B_H(j+1) + B_H(j)$ can be explicitly calculated:

$$\mathrm{E}[\Delta_0 B_H \Delta_j B_H] = 2^{-1}(|j+1|^{2H} + |j-1|^{2H} - 2|j|^{2H}), \tag{4.2}$$

$$\begin{aligned}\mathrm{E}[\Delta_0^2 B_H \Delta_j^2 B_H] = 2^{-1}(&-|j+2|^{2H} + 4|j+1|^{2H} - 6|j|^{2H} \\ &+ 4|j-1|^{2H} - |j-2|^{2H}).\end{aligned} \tag{4.3}$$

From a Taylor expansion,

$$\begin{aligned}&\mathrm{E}[\Delta_0 B_H \Delta_j B_H] \sim 2H(2H-1) j^{2H-2}, \\ &\mathrm{E}[\Delta_0^2 B_H \Delta_j^2 B_H] \sim 2H(2H-1)(2H-2)(2H-3) j^{2H-4}\end{aligned}$$

as $j \to \infty$ and therefore the first increment, $(\Delta_j B_H)$, has a summable covariance if and only if $0 < H < 3/4$, while the second increment, $(\Delta_j^2 B_H)$, has a summable covariance for any $0 < H < 1$.

We now introduce the following conditions:

(A.1) there exist continuous functions $H(t) \in (0,1)$ and $c(t) > 0$ for $t \in [0,1]$ such that $\forall j \in \mathbb{N}^*$,

$$\lim_{n\to\infty} \sup_{t\in(0,1)} \left| \frac{\mathrm{E}(X_{([nt]+j)/n} - X_{[nt]/n})^2}{(j/n)^{2H(t)}} - c(t) \right| = 0 \qquad \text{with} \tag{4.4}$$

$$\lim_{n\to\infty} \sup_{t\in(0,1)} \left| H(t) - H\left(t + \frac{1}{n}\right) \right| \log n = 0; \tag{4.5}$$

(A.1)′ there exist continuous functions $H(t) \in (0,1)$ and $c(t) > 0$ for $t \in [0,1]$ such that $\forall j \in \mathbb{N}^*$,

$$\lim_{n\to\infty} \sup_{t\in(0,1)} \sqrt{n} \left| \frac{\mathrm{E}(X_{([nt]+j)/n} - X_{[nt]/n})^2}{(j/n)^{2H(t)}} - c(t) \right| = 0 \qquad \text{with} \tag{4.6}$$

$$\begin{aligned}&\lim_{n\to\infty} \sup_{t\in(0,1)} \left| H(t) - H\left(t + \frac{1}{n}\right) \right| \sqrt{n} \log n = 0 \quad \text{and} \\ &\lim_{n\to\infty} \sup_{t\in(0,1)} \left| c(t) - c\left(t + \frac{1}{n}\right) \right| \sqrt{n} = 0;\end{aligned} \tag{4.7}$$

$(\mathrm{A.2})_p$ there exist $d > 0$, $\gamma > 1/2$ and $0 \leq \theta < \gamma/2$ such that for any $1 \leq k < j \leq n$ with $n \in \mathbb{N}^*$,

$$|\mathrm{E}[\Delta_k^{p,n} X \Delta_j^{p,n} X]| \leq d\sigma_{p,n}(k)\sigma_{p,n}(j) \cdot n^\theta \cdot |j-k|^{-\gamma}. \tag{4.8}$$

A straightforward application of assumption (A.1) (or (A.1)′) implies that $\sqrt{c(t)} B_{H(t)}$ is the tangent process of X for all $t \in (0,1)$ and, more precisely, the following property.

Property 4.1. *Assumptions* (A.1), (A.1)$'$ *respectively imply that, for any* $j \in \mathbb{Z}$ *and* $p = 1, 2$,

$$\lim_{n\to\infty} \sup_{t\in(0,1)} \left| \frac{\mathrm{E}[\Delta_{[nt]}^{p,n} X \Delta_{j+[nt]}^{p,n} X]}{\mathrm{E}[\Delta_0^{p,n} B_{H(t)} \Delta_j^{p,n} B_{H(t)}]} - c(t) \right| = 0, \tag{4.9}$$

$$\lim_{n\to\infty} \sqrt{n} \sup_{t\in(0,1)} \left| \frac{\mathrm{E}[\Delta_{[nt]}^{p,n} X \Delta_{j+[nt]}^{p,n} X]}{\mathrm{E}[\Delta_0^{p,n} B_{H(t)} \Delta_j^{p,n} B_{H(t)}]} - c(t) \right| = 0. \tag{4.10}$$

Moreover, for any $t \in (0, 1)$ *and* $p = 1, 2$,

$$\left(n^{H(t)} \Delta_{j+[nt]}^{p,n} X\right)_{j\in\mathbb{Z}} \underset{n\to\infty}{\overset{f.d.d.}{\longrightarrow}} \left(\sqrt{c(t)}\Delta_j^p B_{H(t)}\right)_{j\in\mathbb{Z}}.$$

The proof of Property 4.1 (as well as the proofs of the remaining statements in this section) is given in the arXiv version of the paper. Assumption (A.1) can be characterized as *uniform local self-similarity of* (X_t) (the uniformity refers to the supremum over $t \in (0, 1)$ in (4.4)). Note that for X having stationary increments and variogram $V(t) = \mathrm{E}X_t^2$, assumption (A.1) reduces to $V(t) \sim ct^{2H}$ $(c > 0,\ 0 < H < 1)$. For $j = 0, 1$, relation (4.9) implies that for any $t \in (0, 1)$, the variance and the $(1/n)$-lag correlation coefficient of $\Delta_{[nt]}^{p,n} X$ satisfy the following relations:

$$\sigma_{1,n}^2([nt]) \underset{n\to\infty}{\sim} c(t)\sigma_1^2(H(t)) = c(t)\mathrm{E}\left[\left(\Delta_0 B_{H(t)}\right)^2\right] = c(t)\left(\frac{1}{n}\right)^{2H(t)}, \tag{4.11}$$

$$\rho_{1,n}([nt]) \underset{n\to\infty}{\longrightarrow} \rho_1(H(t)) = \mathrm{corr}\left(B_{H(t)}(1), B_{H(t)}(2) - B_{H(t)}(1)\right) = 2^{2H(t)-1} - 1, \tag{4.12}$$

$$\sigma_{2,n}^2([nt]) \underset{n\to\infty}{\sim} c(t)\sigma_2^2(H(t)) = c(t)\mathrm{E}\left[\left(\Delta_0^2 B_{H(t)}\right)^2\right] = c(t)\left(4 - 4^{H(t)}\right)\left(\frac{1}{n}\right)^{2H(t)}, \tag{4.13}$$

$$\begin{aligned} \rho_{2,n}([nt]) \underset{n\to\infty}{\longrightarrow} \rho_2(H(t)) &= \mathrm{corr}\left(B_{H(t)}(2) - 2B_{H(t)}, B_{H(t)}(3) - 2B_{H(t)}(2) + B_{H(t)}(1)\right) \\ [-8pt] &= \frac{-3^{2H(t)} + 2^{2H(t)+2} - 7}{8 - 2^{2H(t)+1}}; \end{aligned} \tag{4.14}$$

see (A.15). Moreover, relations (4.11)–(4.14) hold uniformly in $t \in (0, 1)$. Condition (4.5) is a technical condition which implies (and is "almost equivalent" to) the continuity of the function $t \to H(t)$. Assumption (A.1)$'$ is a sharper convergence condition than assumption (A.1), required for establishing central limit theorems.

Condition (4.8) specifies a non-asymptotic inequality satisfied by the correlation of increments $\Delta_k^{p,n} X$. The particular case of stationary processes allows this point to be better understood. Indeed, if (X_t) has stationary increments, then the covariance of the stationary process $(\Delta_k^{p,n} X, k \in \mathbb{Z})$ is completely determined by the variogram $V(t)$, for instance,

$$\mathrm{E}[\Delta_k^{1,n} X \Delta_j^{1,n} X] = \frac{1}{2}\left\{V\left(\frac{k-j+1}{n}\right) + V\left(\frac{k-j-1}{n}\right) - 2V\left(\frac{k-j}{n}\right)\right\}. \tag{4.15}$$

In the "most regular" case, when $X = B_H$ is an fBm and therefore $V(t) = t^{2H}$, it is easy to check that assumption $(\mathrm{A.2})_2$ holds with $\theta = 0$ and $\gamma = 4 - 2H > 2$ $(0 < H < 1)$, while $(\mathrm{A.2})_1$

with $\theta = 0$, $\gamma = 2 - 2H$ is equivalent to $H < 3/4$ because of the requirement that $\gamma > 1/2$. However, for $X = B_H$, $(\mathrm{A.2})_1$ holds with appropriate $\theta > 0$ in the wider region $0 < H < 7/8$, by choosing $\theta < 2 - 2H$ arbitrarily close to $2 - 2H$ and then $\gamma < 2 - 2H + \theta$ arbitrarily close to $4 - 4H$. A similar choice of parameters θ and γ allows $(\mathrm{A.2})_p$ to be satisfied for more general X with stationary increments and variogram $V(t) \sim ct^{2H}$ $(t \to 0)$, under additional regularity conditions on $V(t)$ (see below).

Property 4.2. *Let X have stationary increments and variogram $V(t) \sim ct^{2H}$ $(t \to 0)$ with $c > 0$, $H \in (0, 1)$.*

(i) *Assume, in addition, that $0 < H < 7/8$ and $|V''(t)| \le Ct^{-\kappa}$ $(0 < t < 1)$ for some $C > 0$ and $4 - 4H > \kappa \ge 2 - 2H$, $\kappa > 1/2$. Assumption $(\mathrm{A.2})_1$ then holds.*

(ii) *Assume, in addition, that $|V^{(4)}(t)| \le Ct^{-\kappa}$ $(0 < t < 1)$ for some $C > 0$ and $8 - 4H > \kappa \ge 4 - 2H$. Assumption $(\mathrm{A.2})_2$ then holds.*

The following property provides a sufficient condition for $(\mathrm{A.2})_p$ in spectral terms, which does not require differentiability of the variogram.

Property 4.3. *Let X be a Gaussian process having stationary increments and the spectral representation (see, e.g., [15])*

$$X_t = \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}t\xi} - 1) f^{1/2}(\xi) W(\mathrm{d}\xi) \qquad \text{for all } t \in \mathbb{R}, \tag{4.16}$$

where $W(\mathrm{d}x) = \overline{W(-\mathrm{d}x)}$ is a complex-valued Gaussian white noise with zero mean and variance $\mathrm{E}|W(\mathrm{d}x)|^2 = \mathrm{d}x$ and f is a non-negative even function, called the spectral density of X, such that

$$\int_{\mathbb{R}} (1 \wedge |\xi|^2) f(\xi)\, \mathrm{d}\xi < \infty. \tag{4.17}$$

Moreover, assume that f is differentiable on (K, ∞) and that

$$f(\xi) \sim c\xi^{-2H-1} \qquad (\xi \to \infty), \qquad |f'(\xi)| \le C\xi^{-2H-2} \qquad (\xi > K) \tag{4.18}$$

for some constants $c, C, K > 0$. Then, X satisfies assumption $(\mathrm{A.2})_1$ for $0 < H < 3/4$ and assumption $(\mathrm{A.2})_2$ for $0 < H < 1$.

4.2. Limit theorems

Before establishing limit theorems for the statistics $R^{p,n}$ for Gaussian processes, we introduce the following notation:

$$\Lambda_p(H) := \lambda(\rho_p(H)), \tag{4.19}$$

$$\lambda(r) := \frac{1}{\pi}\arccos(-r) + \frac{1}{\pi}\sqrt{\frac{1+r}{1-r}}\log\left(\frac{2}{1+r}\right), \tag{4.20}$$

$$\rho_p(H) := \operatorname{corr}(\Delta_0^p B_H, \Delta_1^p B_H), \quad \text{and} \tag{4.21}$$

$$\Sigma_p(H) := \sum_{j\in\mathbb{Z}} \operatorname{cov}\left(\frac{|\Delta_0^p B_H + \Delta_1^p B_H|}{|\Delta_0^p B_H| + |\Delta_1^p B_H|}, \frac{|\Delta_j^p B_H + \Delta_{j+1}^p B_H|}{|\Delta_j^p B_H| + |\Delta_{j+1}^p B_H|}\right). \tag{4.22}$$

It can now be proven (see the Appendix) that

$$\int_0^1 \mathrm{E}\left[\frac{|\Delta_0^p B_{H(t)} + \Delta_1^p B_{H(t)}|}{|\Delta_0^p B_{H(t)}| + |\Delta_1^p B_{H(t)}|}\right] \mathrm{d}t = \int_0^1 \Lambda_p(H(t))\,\mathrm{d}t.$$

Straightforward computations show that assumptions (A.1) and $(\mathrm{A.2})_p$ imply Assumption (A) with $A^{(t)}(\delta) = \delta^{-H(t)}$, $Y^{(t)} = \sqrt{c(t)}B_{H(t)}$. Therefore, Proposition 2.2 ensures the convergence (in $\mathbb{L}^2$) of the statistics $R^{p,n}$ to $\int_0^1 \Lambda_p(H(t))\,\mathrm{d}t$. Bardet and Surgailis [5] proved a.s. convergence in Theorem 4.1 below, using a general moment bound for functions of multivariate Gaussian processes (see Lemma A.1 in the Appendix). A sketch of this proof can be found in the Appendix.

Theorem 4.1. *Let X be a Gaussian process satisfying assumptions* (A.1) *and* $(\mathrm{A.2})_p$. *Then*,

$$R^{p,n} \xrightarrow[n\to\infty]{a.s.} \int_0^1 \Lambda_p(H(t))\,\mathrm{d}t \qquad (p = 1, 2). \tag{4.23}$$

Corollary 4.1. *Assume that X is a Gaussian process having stationary increments*, *whose variogram satisfies the conditions of Properties* 4.2 *or* 4.3. *Then*,

$$R^{p,n} \xrightarrow[n\to\infty]{a.s.} \Lambda_p(H) \qquad (p = 1, 2). \tag{4.24}$$

The following Theorem 4.2 is also established in [5]. Its proof (see a sketch of this proof in the Appendix) uses a general central limit theorem for Gaussian subordinated non-stationary triangular arrays (see Theorem A.1 in the Appendix). Note that the Hermite rank of $\psi(x_1, x_2) = |x_1 + x_2|/(|x_1| + |x_2|)$ is 2 and this explains the difference between the cases $p = 1$ and $p = 2$ in Theorem 4.2: in the first case, the inequalities in (A.8) for $(\mathbf{Y}_n(k))$, as defined in (A.5)–(A.6), hold only if $\sup_{t\in[0,1]} H(t) < 3/4$, while in the latter case, these inequalities hold for $0 < \sup_{t\in[0,1]} H(t) < 1$. A similar fact is also true for the estimators based on generalized quadratic variations; see [12,28].

Theorem 4.2. *Let X be a Gaussian process satisfying assumptions* $(\mathrm{A.1})'$ *and* $(\mathrm{A.2})_p$ *with* $\theta = 0$. *Assume further that* $\sup_{t\in[0,1]} H(t) < 3/4$ *if* $p = 1$. *Then*, *for* $p = 1, 2$,

$$\sqrt{n}\left(R^{p,n} - \int_0^1 \Lambda_p(H(t))\,\mathrm{d}t\right) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\left(0, \int_0^1 \Sigma_p(H(\tau))\,\mathrm{d}\tau\right) \tag{4.25}$$

with $\Lambda_p(H)$ *and* $\Sigma_p(H)$ *given in* (4.19) *and* (4.22), *respectively.*

The following proposition shows that the previous theorems are satisfied when smooth multiplicative and additive trends are considered.

Proposition 4.1. *Let* $Z=(Z_t=\alpha(t)X_t+\beta(t), t\in[0,1])$, *where* $X=(X_t, t\in[0,1])$ *is a zero-mean Gaussian process and* α,β *are deterministic continuous functions on* $[0,1]$ *with* $\inf_{t\in[0,1]}\alpha(t)>0$.

(i) *Let* X *satisfy the assumptions of Theorem* 4.2 *and* $\alpha\in\mathcal{C}^p[0,1]$, $\beta\in\mathcal{C}^p[0,1]$. *The statement of Theorem* 4.2 *then holds with* X *replaced by* Z.

(ii) *Let* X *satisfy the assumptions of Theorem* 4.1 *and* $\alpha\in\mathcal{C}^1[0,1]$, $\beta\in\mathcal{C}^1[0,1]$. *The statement of Theorem* 4.1 *then holds with* X *replaced by* Z.

Remark 4.1. A version of the central limit theorem in (4.25) is established in [5] with $\int_0^1\Lambda_p(H(t))\,\mathrm{d}t$ replaced by $\mathrm{E}R^{p,n}$ under a weaker assumption than (A.1)$'$ or even (A.1): only properties (4.11)–(4.12) (for $p=1$) and (4.13)–(4.14) (for $p=2$), in addition to $(\mathrm{A}.2)_p$ with $\theta=0$, are required.

The particular case of Gaussian processes having stationary increments can also be studied, as we shall now see.

Corollary 4.2. *Assume that* X *is a Gaussian process having stationary increments and that there exist* $c>0$, $C>0$ *and* $0<H<1$ *such that at least one of the two following conditions hold*:

(a) *variogram* $V(t)=ct^{2H}(1+\mathrm{o}(t^{1/2}))$ *for* $t\to 0$ *and* $|V^{(2p)}(t)|\le Ct^{2H-2p}4$ *for all* $t\in(0,1]$;

(b) *spectral density* f *satisfies* (4.17), (4.18) *and* $f(\xi)=c\xi^{-2H-1}(1+\mathrm{o}(\xi^{-1/2}))$ $(\xi\to\infty)$.

Then,

$$\sqrt{n}\big(R^{p,n}-\Lambda_p(H)\big)\xrightarrow[n\to\infty]{\mathcal{D}}\mathcal{N}(0,\Sigma_p(H))\qquad \text{if}\begin{cases}p=1, & 0<H<3/4,\\ p=2, & 0<H<1.\end{cases}\tag{4.26}$$

Moreover, with the expression and graph of $s_2^2(H)$ *given in the* Appendix, *we have*

$$\sqrt{n}\big(\Lambda_2^{-1}(R^{2,n})-H\big)\xrightarrow[n\to\infty]{\mathcal{D}}\mathcal{N}(0,s_2^2(H)).\tag{4.27}$$

The proof of Corollary 4.2 can be found in the Appendix. Therefore, $\widehat{H}_n=\Lambda_2^{-1}(R^{2,n})$ is an estimator of the parameter H following a central limit theorem with a convergence rate $\sqrt{n}$ under semi-parametric assumptions. Similar results were obtained by Guyon and Leon [25] and Istas and Lang [29] for generalized quadratic variations under less general assumptions.

Remark 4.2. In the context of Corollary 4.2 and $H\in(3/4,1)$, we expect that $R^{1,n}$ follows a non-Gaussian limit distribution with convergence rate n^{2-2H}.

Remark 4.3. Figure 1 shows that $H\mapsto\Lambda_2(H)$ is nearly linear and is well approximated by $0.1468H+0.5174$. Consequently, $\int_0^1\Lambda_2(H(t))\,\mathrm{d}t\approx 0.1468\bar{H}+0.5174$, where $\bar{H}=\int_0^1 H(t)\,\mathrm{d}t$ is the mean value of the function $H(\cdot)$.

Another interesting particular case of Theorem 4.2 leads to a punctual estimator of the function $H(t)$ from a localization of the statistic $R^{2,n}$. For $t_0 \in (0,1)$ and $\alpha \in (0,1)$, we define

$$R_\alpha^{2,n}(t_0) := \frac{1}{2n^\alpha} \sum_{k=[nt_0-n^\alpha]}^{[nt_0+n^\alpha]} \frac{|\Delta_k^{2,n} X + \Delta_{k+1}^{2,n} X|}{|\Delta_k^{2,n} X| + |\Delta_{k+1}^{2,n} X|}.$$

This estimator is studied in [6] and compared to the estimator based on generalized quadratic variations discussed in [8] and [13].

4.3. Examples

Below, we provide some concrete examples of Gaussian processes which admit an fBm as the tangent process. For some examples, the hypotheses of Theorems 4.1–4.2 and the subsequent corollaries are satisfied. For other examples, the verification of our hypotheses (in particular, of the crucial covariance bound $(\mathrm{A}.2)_p$) remains an open problem and will be discussed elsewhere.

Example 4.1. *Fractional Brownian motion (fBm).* As noted above, an fBm $X = B_H$ satisfies $(\mathrm{A}.1)'$ as well as $(\mathrm{A}.2)_1$ (for $0 < H < 3/4$ if $\theta = 0$ and $0 < H < 7/8$ if $0 < \theta < 2 - 2H$ with θ arbitrary close to $2 - 2H$ so that $\gamma < 2 - 2H + \theta$ arbitrary close to $4 - 4H$ may satisfy $\gamma > 1/2$) and $(\mathrm{A}.2)_2$ (for $0 < H < 1$) with $H(t) \equiv H$, $c(t) \equiv c$. Therefore, for fBm, both Theorems 4.1 (the almost sure convergence, satisfied for $0 < H < 7/8$ when $p = 1$ and for $0 < H < 1$ when $p = 2$) and 4.2 (the central limit theorem, satisfied for $0 < H < 3/4$ when $p = 1$ and for $0 < H < 1$ when $p = 2$) apply. Obviously, an fBm also satisfies the conditions of Corollary 4.2. Thus, the rate of convergence of the estimator $\Lambda_2^{-1}(R^{2,n}) =: \widehat{H}_n$ of H is $\sqrt{n}$. However, in such a case, the self-similarity property of fBm allows the use of asymptotically efficient Whittle or maximum likelihood estimators (see [22] or [17]). However, for an fBm with continuously differentiable multiplicative and additive trends, which leads to a semi-parametric context, the convergence rate of $\widehat{H}_n$ is still $\sqrt{n}$, while parametric estimators cannot be applied.

Example 4.2. *Multiscale fractional Brownian motion* (see [4]), defined as follows: for $\ell \in \mathbb{N}^*$, an (M_ℓ)-multiscale fractional Brownian motion $X = (X_t, t \in \mathbb{R})$ ((M_ℓ)-fBm for short) is a Gaussian process having stationary increments and a spectral density f such that

$$f(\xi) = \frac{\sigma_j^2}{|\xi|^{2H_j+1}} \mathbf{1}(\omega_j \le |\xi| < \omega_{j+1}) \qquad \text{for all } \xi \in \mathbb{R} \tag{4.28}$$

with $\omega_0 := 0 < \omega_1 < \cdots < \omega_\ell < \omega_{\ell+1} := \infty$, $\sigma_i > 0$ and $H_i \in \mathbb{R}$ for $i \in \{0, \ldots, \ell\}$, with $H_0 < 1$ and $H_\ell > 0$. Therefore, condition (4.18) of Property 4.3 is satisfied with $K = \omega_\ell$ and $H = H_\ell$. Moreover, the condition $f(\xi) = c\xi^{-2H-1}(1 + \mathrm{o}(\xi^{-1/2}))$ $(\xi \to \infty)$ required in Corollary 4.2 is also checked with $H = H_\ell$. Consequently, the same conclusions as in the previous example also apply for this process, in the respective regions determined by the parameter H_ℓ at high frequencies $x > \omega_\ell$. The same result is also obtained for a more general process defined by $f(\xi) = c\xi^{-2H-1}$ for $|\xi| \ge \omega$ and condition (4.17) is only required elsewhere. Once again, such conclusions also hold in the case of continuously differentiable multiplicative and additive trends.

Example 4.3. *Multifractional Brownian motion* (mBm) (see [3]). An mBm $X = (X_t, t \in [0, 1])$ is a Gaussian process defined by

$$X_t = B_{H(t)}(t) = g(H(t)) \int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}tx} - 1}{|x|^{H(t)+1/2}} W(\mathrm{d}x), \tag{4.29}$$

where $W(\mathrm{d}x)$ is the same as in (4.16), $H(t)$ is a (continuous) function on $[0, 1]$ taking values in $(0, 1)$ and, finally, $g(H(t))$ is a normalization such that $\mathrm{E}X_t^2 = 1$. It is well known that an mBm is locally asymptotically self-similar at each point $t \in (0, 1)$ having an fBm $B_{H(t)}$ as its tangent process at t (see [9]). This example is studied in more detail in [6].

Example 4.4. *Time-varying fractionally integrated processes*. Philippe *et al.* [32,33] introduced two classes of mutually inverse time-varying fractionally integrated filters with discrete time and studied long-memory properties of the corresponding filtered white noise processes. Surgailis [37] extended these filters to continuous time and defined “multifractional” Gaussian processes $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ as follows:

$$X_t = \int_{\mathbb{R}} \left\{ \int_0^t \frac{1}{\Gamma(H(\tau) - 0.5)} (\tau - s)_+^{H(\tau)-1.5} \mathrm{e}^{A_-(s,\tau)}\, \mathrm{d}\tau \right\} \mathrm{d}B(s), \tag{4.30}$$

$$Y_t = \int_{\mathbb{R}} \frac{1}{\Gamma(H(s) + 0.5)} \left\{ (t - s)_+^{H(s)-0.5} \mathrm{e}^{-A_+(s,t)} - (-s)_+^{H(s)-0.5} \mathrm{e}^{-A_+(s,0)} \right\} \mathrm{d}B(s), \tag{4.31}$$

where $s_+^\alpha := s^\alpha \mathbf{1}(s > 0)$, B is a Brownian motion,

$$A_-(s, t) := \int_s^t \frac{H(u) - H(t)}{t - u}\, \mathrm{d}u, \qquad A_+(s, t) := \int_s^t \frac{H(s) - H(v)}{v - s}\, \mathrm{d}v \qquad (s < t)$$

and where $H(t)$, $t \in \mathbb{R}$, is a general function taking values in $(0, \infty)$ and satisfying some weak additional conditions. Surgailis [37] studied small and large scale limits of (X_t) and (Y_t) and showed that these processes resemble an fBm with Hurst parameter $H = H(t)$ at each point $t \in \mathbb{R}$ (i.e., admit an fBm as a tangent process) similarly to the mBm in the previous example. That paper also argues that these processes present a more natural generalization of fBm than the mBm and have nicer dependence properties of increments. We expect that the assumptions (A.1), (A.1)$'$, (A.2)$_p$ can be verified for (4.30), (4.31); however, this question requires further work.

5. Processes with independent increments

In this section, we assume that $X = (X_t, t \geq 0)$ is a (homogeneous) Lévy process, with a.s. right-continuous trajectories, $X_0 = 0$. It is well known that if the generating triplet of X satisfies certain conditions (in particularly, if the Lévy measure ν behaves regularly at the origin with index $\alpha \in (0, 2)$), then X has a tangent process Y which is an α-stable Lévy process. A natural question is how to estimate the parameter α with the help of the introduced statistics $R^{p,n}$. Unfortunately, the limit of these statistics, as defined in (1.5) via the tangent process, also depends on the skewness parameter $\beta \in [-1, 1]$ of the α-stable tangent process Y and so this limit cannot be used for determining α if β is unknown.

In order to avoid this difficulty, we shall slightly modify our ratio statistic, as follows. First, observe that the second differences $\Delta_k^{2,n} X$ of Lévy process have a *symmetric* distribution (in contrast to the first differences $\Delta_k^{1,n} X$, which are not necessarily symmetric). For notational simplicity, we shall assume in this section that n is *even*. The modified statistic

$$\tilde{R}^{2,n} := \frac{1}{n/2-1} \sum_{k=0}^{(n-4)/2} \psi(\Delta_{2k}^{2,n} X, \Delta_{2k+2}^{2,n} X), \qquad \psi(x,y) := \frac{|x+y|}{|x|+|y|}$$

is written in terms of "disjoint" (independent) second-order increments $(\Delta_{2k}^{2,n} X, \Delta_{2k+2}^{2,n} X)$ having a symmetric joint distribution. Instead of extending the general result of Proposition 2.2 to $\tilde{R}^{2,n}$, we shall directly obtain its convergence under suitable assumptions on X. First, note that

$$\mathrm{E}\tilde{R}^{2,n} = \mathrm{E}\psi\bigl(X_{1/n}^{(2)} - X_{1/n}^{(1)}, X_{1/n}^{(4)} - X_{1/n}^{(3)}\bigr), \tag{5.1}$$

where $X^{(i)}$, $i = 1, \ldots, 4$, are independent copies of X.

Proposition 5.1. *Let there exist a limit*

$$\lim_{n\to\infty} \mathrm{E}\tilde{R}^{2,n} = \tilde{\Lambda}. \tag{5.2}$$

Then,

$$\tilde{R}^{2,n} \xrightarrow[n\to\infty]{a.s.} \tilde{\Lambda}. \tag{5.3}$$

Proof. We write $\tilde{R}^{2,n} = \mathrm{E}\tilde{R}^{2,n} + (n/2-1)^{-1} Q_n$, where Q_n is a sum of centered 1-dependent r.v.'s which are bounded by 1 in absolute value. Therefore, $\mathrm{E}((n/2-1)^{-1} Q_n)^4 = \mathrm{O}(n^{-2})$ and the a.s. convergence $(n/2-1)^{-1} Q_n \to 0$ follows by the Chebyshev inequality. □

Next, we discuss conditions on X for the convergence in (5.2). Recall that the distribution of X_t is infinitely divisible and that its characteristic function is given by

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\theta X_t} = \exp\biggl\{t\biggl(\mathrm{i}\gamma\theta - \frac{1}{2}a^2\theta^2 + \int_{\mathbb{R}} \bigl(\mathrm{e}^{\mathrm{i}u\theta} - 1 - \mathrm{i}u\theta\mathbf{1}(|u| \le 1)\bigr)\nu(\mathrm{d}u)\biggr)\biggr\}, \qquad \theta \in \mathbb{R}, \tag{5.4}$$

where $\gamma \in \mathbb{R}$, $a \ge 0$ and ν is a measure on $\mathbb{R}$ such that $\int_{\mathbb{R}} \min(u^2, 1)\nu(\mathrm{d}u) < \infty$. The triplet (a, γ, ν) is called the generating triplet of X [34]. Let $X^{(i)}$, $i = 1, 2$, be independent copies of X. Note that $W_t := X_t^{(1)} - X_t^{(2)}$ is a Lévy process having the characteristic function

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\theta W_t} = \exp\biggl\{t\biggl(-a^2\theta^2 + 2\int_0^\infty \mathrm{Re}(1 - \mathrm{e}^{\mathrm{i}u\theta})\,\mathrm{d}K(u)\biggr)\biggr\}, \qquad \theta \in \mathbb{R}, \tag{5.5}$$

where

$$K(u) := \nu\big((-\infty, -u] \cup [u, \infty)\big)$$

is monotone non-increasing on $(0, \infty)$. We introduce the following condition: there exist $0 < \alpha \le 2$ and $c > 0$ such that

$$K(u) \sim \frac{c}{u^\alpha}, \qquad u \downarrow 0. \tag{5.6}$$

It is clear that if such number α exists, then $\alpha := \inf\{r \ge 0 : \int_{|x|\le 1} |x|^r \nu(\mathrm{d}x) < \infty\}$ is the so-called fractional order or the Blumenthal–Getoor index of the Lévy process X.

Let Z_α be a standard α-stable r.v. with characteristic function $\mathrm{E}\mathrm{e}^{\mathrm{i}\theta Z_\alpha} = \mathrm{e}^{-|\theta|^\alpha}$ and $Z_\alpha^{(i)}$, $i = 1, 2, 3$, be independent copies of Z_α.

Proposition 5.2. *Assume that either $a > 0$, or else $a = 0$ and condition (5.6) holds with $0 < \alpha \le 2$ and $c > 0$. Then, $t^{-1/\alpha}(X_t^{(1)} - X_t^{(2)}) \xrightarrow[t\to 0]{\mathcal{D}} \tilde{c} Z_\alpha$ with $\tilde{c}$ depending on c and (5.2), (5.3) hold with*

$$\tilde{\Lambda} \equiv \tilde{\Lambda}(\alpha) := \mathrm{E}\psi\big(Z_\alpha^{(1)}, Z_\alpha^{(2)}\big).$$

Moreover, with $\tilde{\sigma}^2(\alpha) := 2\operatorname{var}(\psi(Z_\alpha^{(1)}, Z_\alpha^{(2)})) + 4\operatorname{cov}(\psi(Z_\alpha^{(1)}, Z_\alpha^{(2)}), \psi(Z_\alpha^{(2)}, Z_\alpha^{(3)}))$,

$$\sqrt{n}(\tilde{R}^{2,n} - \mathrm{E}\tilde{R}^{2,n}) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \tilde{\sigma}^2(\alpha)). \tag{5.7}$$

Proof. The relation $t^{-1/\alpha} W_t = t^{-1/\alpha}(X_t^{(1)} - X_t^{(2)}) \xrightarrow[t\to 0]{\mathcal{D}} \tilde{c} Z_\alpha$ is an easy consequence of the assumptions of the proposition and the general criterion of weak convergence of infinitely divisible distributions in [34], Theorem 8.7. It implies (5.2) by the fact that ψ is a.e. continuous on $\mathbb{R}^2$. Since $\tilde{R}^{2,n}$ is a sum of 1-dependent stationary and bounded r.v.'s, the central limit theorem in (5.7) follows from convergence of the variance:

$$n \operatorname{var}(\tilde{R}^{2,n}) \to \tilde{\sigma}^2(\alpha); \tag{5.8}$$

see, for example, [10]. Rewrite $\tilde{R}^{2,n} = (n/2 - 1)^{-1} \sum_{k=0}^{(n-4)/2} \tilde{\eta}_n(k)$, $\tilde{\eta}_n(k) := \psi(\Delta_{2k}^{2,n} X, \Delta_{2k+2}^{2,n} X)$. We have

$$n \operatorname{var}(\tilde{R}^{2,n}) = \frac{n}{n/2 - 1} \operatorname{var}(\tilde{\eta}_n(0)) + \frac{2n(n/2-2)}{(n/2-1)^2} \operatorname{cov}(\tilde{\eta}_n(0), \tilde{\eta}_n(1)),$$

where $\operatorname{var}(\tilde{\eta}_n(0)) \xrightarrow[n\to\infty]{} \operatorname{var}(\psi(Z_\alpha^{(1)}, Z_\alpha^{(2)}))$, $\operatorname{cov}(\tilde{\eta}_n(0), \tilde{\eta}_n(1)) \xrightarrow[n\to\infty]{} \operatorname{cov}(\psi(Z_\alpha^{(1)}, Z_\alpha^{(2)}), \psi(Z_\alpha^{(2)}, Z_\alpha^{(3)}))$ similarly as in the proof of (5.2) above. This proves (5.8) and hence the proposition. $\square$

The graph of $\tilde{\Lambda}(\alpha)$ is given in Figure 4. Note that $\tilde{\Lambda}(2) = \Lambda_1(1/2) \simeq 0.72$: this is the case of Brownian motion.

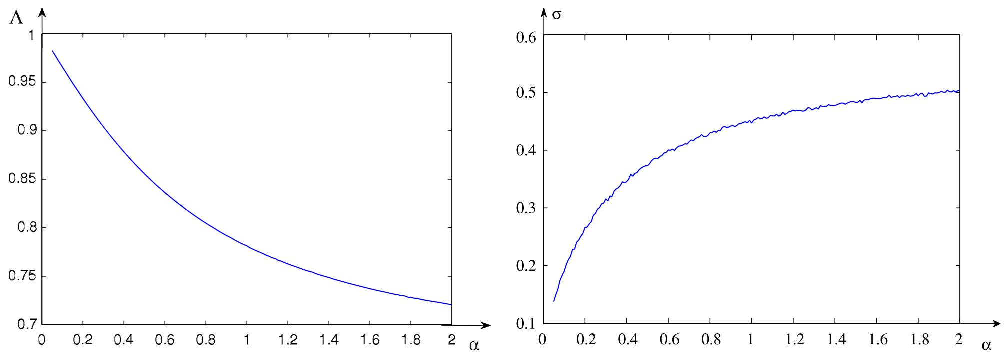

Figure 4. The graphs of $\alpha \mapsto \tilde{\Lambda}(\alpha) = \mathrm{E}\frac{|Z_\alpha^{(1)}+Z_\alpha^{(2)}|}{|Z_\alpha^{(1)}|+|Z_\alpha^{(2)}|}$ (left) and $\alpha \mapsto \tilde{\sigma}(\alpha)$ (right) for a process with independent increments.

In order to evaluate the decay rate of the bias $\mathrm{E}\tilde{R}^{2,n} - \tilde{\Lambda}(\alpha)$ we need a uniform convergence rate in Lemma 5.1 below, for

$$\|F_n - G_\alpha\|_\infty := \sup_{x\in\mathbb{R}} |F_n(x) - G_\alpha(x)|, \qquad F_n(x) := \mathrm{P}(n^{1/\alpha} W_{1/n} \le x),$$
$$G_\alpha(x) := \mathrm{P}(\tilde{Z}_\alpha \le x),$$

where $\tilde{Z}_\alpha := \tilde{c} Z_\alpha$ is the limiting α-stable r.v. in Proposition 5.2 and $(W_t, t \ge 0)$ is the symmetric Lévy process with characteristic function as in (5.5). The proof of Lemma 5.1 is given in the arXiv version of the paper.

Lemma 5.1. (i) *Let $a = 0$ and K satisfy* (5.6). *Define $K_1(u) := K(u) - cu^{-\alpha}$, $|K_1|(u) := \int_u^\infty |\mathrm{d}K_1(v)|$, the variation of K_1 on $[u,\infty)$. Moreover, assume that there exist some constants $\beta, \delta > 0$ such that*

$$|K_1|(u) = \mathrm{O}\big(u^{-(\alpha-\beta)_+}\big) \qquad (u \to 0), \qquad |K_1|(u) = \mathrm{O}(u^{-\delta}) \qquad (u \to \infty),$$

where $x_+ := \max(0,x)$. Then

$$\|F_n - G_\alpha\|_\infty = \begin{cases} \mathrm{O}(n^{-\beta/\alpha}), & \text{if } \beta < \alpha, \\ \mathrm{O}(n^{-1}\log n), & \text{if } \beta = \alpha, \\ \mathrm{O}(n^{-1}), & \text{if } \beta > \alpha. \end{cases}$$

(ii) *Let $a > 0$ and K satisfy*

$$K(u) = \mathrm{O}(u^{-\alpha}) \qquad (u \to 0), \qquad K(u) = \mathrm{O}(u^{-\delta}) \qquad (u \to \infty)$$

for some $0 \le \alpha < 2$, $\delta > 0$. Then

$$\|F_n - G_\alpha\|_\infty = \begin{cases} \mathrm{O}(n^{-1+\alpha/2}), & \text{if } \alpha > 0, \\ \mathrm{O}(n^{-1}\log n), & \text{if } \alpha = 0. \end{cases}$$

Proposition 5.3. *Assume either $a > 0$, or else $a = 0$ and condition (5.6) holds. Then, for any $\alpha \in (0, 2]$,*

$$|\mathrm{E}\tilde{R}^{2,n} - \tilde{\Lambda}(\alpha)| \le 2C\|F_n - G_\alpha\|_\infty, \qquad C := \int_0^\infty (1+z)^{-2}\,\mathrm{d}z < \infty. \tag{5.9}$$

Proof. Let $\tilde{\psi}(x, y) := |x - y|/(x + y), x, y > 0$, and let F_n, G_α be the same as in Lemma 5.1. Similarly as in the proof of [40], Theorem 1, we write

$$\begin{aligned}\mathrm{E}\tilde{R}^{2,n} - \tilde{\Lambda}(\alpha) &= 2\int_0^\infty \int_0^\infty \tilde{\psi}(x, y)\big(\mathrm{d}F_n(x)\,\mathrm{d}F_n(y) - \mathrm{d}G_\alpha(x)\,\mathrm{d}G_\alpha(y)\big)\\ &= 2(W_1 + W_2),\end{aligned}$$

where $W_1 := \int_0^\infty\int_0^\infty \tilde{\psi}(x, y)\,\mathrm{d}F_n(x)(\mathrm{d}F_n(y) - \mathrm{d}G_\alpha(y)), W_2 := \int_0^\infty\int_0^\infty \tilde{\psi}(x, y)\,\mathrm{d}G_\alpha(y) \times (\mathrm{d}F_n(x) - \mathrm{d}G_\alpha(x))$. Integrating by parts yields

$$\begin{aligned}|W_1| &= 2\int_0^\infty |x|\,\mathrm{d}F_n(x) \int_0^\infty |F_n(y) - G_\alpha(y)| \frac{\mathrm{d}y}{(x+y)^2}\\ &\le 2\|F_n - G_\alpha\|_\infty \int_0^\infty |x|\,\mathrm{d}F_n(x) \int_0^\infty \frac{\mathrm{d}y}{(x+y)^2} = C\|F_n - G_\alpha\|_\infty\end{aligned}$$

since $\int_0^\infty \mathrm{d}F_n(x) = 1/2$. A similar estimate holds for W_2. This proves (5.9). □

Propositions 5.2, 5.3 and Lemma 5.1, together with the delta method, yield the following corollary.

Corollary 5.1. *Let a and K satisfy either the assumptions of Lemma 5.1(i) with $\beta > \alpha/2$ or the assumptions of Lemma 5.1(ii). Then,*

$$\sqrt{n}\big(\tilde{R}^{2,n} - \tilde{\Lambda}(\alpha)\big) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \tilde{\sigma}^2(\alpha)).$$

Moreover, if we define $\widehat{\alpha}_n := \tilde{\Lambda}^{-1}(\tilde{R}^{2,n})$, then

$$\sqrt{n}(\widehat{\alpha}_n - \alpha) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \tilde{s}^2(\alpha)),$$

where $\tilde{s}^2(\alpha) := [\frac{\partial\tilde{\Lambda}}{\partial\alpha}(\alpha)]^{-2}\tilde{\sigma}^2(\alpha), 0 < \alpha \le 2$.

There exist very few papers concerning the estimation of α in such a semi-parametric framework. Nonparametric estimation of parameters of Lévy processes based on the empirical characteristic function has recently been considered in [30] and [24], but the convergence rates there are $(\log n)^\kappa$ with $\kappa > 0$. Aït-Sahalia and Jacod [1] have proposed an estimator of the degree of activity of jumps (which is identical to the fractional order in the case of a Lévy process) in a general semimartingale framework using small increments of high-frequency data. However, from the generality of their model, the convergence rate of the estimator is not rate-efficient (in fact, it is smaller than $n^{1/5}$). A recent paper of Belomestny [7] provides an efficient data-driven procedure to estimate α using a spectral approach, but in a different semi-parametric framework from ours. Thus, Corollary 5.1 appears as a new and interesting result since the estimator $\widehat{\alpha}_n$ follows a $\sqrt{n}$-central limit theorem.

Appendix: Proofs

Sketch of the proof of Theorem 4.1

The proof of Theorem 4.1 is based on the moment inequality in Lemma A.1 below, which extends a similar inequality in [39], Lemma 4.5, to vector-valued non-stationary Gaussian processes. The proof of Lemma A.1 uses the diagram formula and is given in [5]. To formulate this lemma, we need the following definitions. Let $\mathbf{X}$ be a standard Gaussian vector in $\mathbb{R}^\nu (\nu \geq 1)$ and let $\mathbb{L}^2(\mathbf{X})$ denote the Hilbert space of measurable functions $f:\mathbb{R}^\nu \to \mathbb{R}$ satisfying $\|f\|^2 := \mathrm{E}(f(\mathbf{X}))^2 < \infty$. Let $\mathbb{L}^2_0(\mathbf{X}) = \{f \in \mathbb{L}^2(\mathbf{X}) : \mathrm{E}f(\mathbf{X}) = 0\}$. Let $(\mathbf{X}_1, \dots, \mathbf{X}_N)$ be a collection of standardized Gaussian vectors $\mathbf{X}_t = (X_t^{(1)}, \dots, X_t^{(\nu)}) \in \mathbb{R}^\nu$ having a joint Gaussian distribution in $\mathbb{R}^{\nu N}$. Let $\varepsilon \in [0, 1]$ be a fixed number. Following Taqqu [39], we call $(\mathbf{X}_1, \dots, \mathbf{X}_N)$ *ε-standard* if $|\mathrm{E}X_t^{(u)} X_s^{(v)}| \leq \varepsilon$ for any $t \neq s$, $1 \leq t, s \leq N$ and any $1 \leq u, v \leq \nu$. Finally, $\sum'$ denotes the sum over all distinct integers $1 \leq t_1, \dots, t_p \leq N$, $t_i \neq t_j$ $(i \neq j)$.

Lemma A.1. *Let $(\mathbf{X}_1, \dots, \mathbf{X}_N)$ be an ε-standard Gaussian vector, $\mathbf{X}_t = (X_t^{(1)}, \dots, X_t^{(\nu)}) \in \mathbb{R}^\nu$ $(\nu \geq 1)$, and let $G_{j,t,N} \in \mathbb{L}^2(\mathbf{X})$, $1 \leq j \leq p$ $(p \geq 2)$, $1 \leq t \leq N$. For given integers $m, N \geq 1$, define*

$$Q_N := \max_{1 \leq t \leq N} \sum_{1 \leq s \leq N, s \neq t} \max_{1 \leq u, v \leq \nu} \left|\mathrm{E}X_t^{(u)} X_s^{(v)}\right|^m. \tag{A.1}$$

Assume that for some integer $0 \leq \alpha \leq p$, the functions $G_{1,t,N}, \dots, G_{\alpha,t,N}$ have a Hermite rank at least equal to m for any $N \geq 1$, $1 \leq t \leq N$, and that $\varepsilon < \frac{1}{\nu p - 1}$. Then,

$$\sum{}' \mathrm{E}|G_{1,t_1,N}(\mathbf{X}_{t_1}) \cdots G_{p,t_p,N}(\mathbf{X}_{t_p})| \leq C(\varepsilon, p, m, \alpha, \nu) K N^{p-\alpha/2} Q_N^{\alpha/2},$$

where the constant $C(\varepsilon, p, m, \alpha, \nu)$ depends only on $\varepsilon, p, m, \alpha, \nu$ and where $K := \prod_{j=1}^p \max_{1 \leq t \leq N} \|G_{j,t,N}\|$.

Sketch of the proof of Theorem 4.1. The convergence $\lim_{n\to\infty} \mathrm{E}R^{p,n} = \int_0^1 \Lambda_p(H(t))\,\mathrm{d}t$ is easy (see the proof of Proposition 2.2). Hence, (4.23) follows from

$$\tilde{R}^{p,n} := R^{p,n} - \mathrm{E}R^{p,n} \xrightarrow[n\to\infty]{\text{a.s.}} 0. \tag{A.2}$$

Relation (A.2) follows from the Chebyshev inequality and the following bound: there exist $C, \kappa > 1$ such that for any $n \geq 1$,

$$\mathrm{E}(\tilde{R}^{p,n})^4 \leq C n^{-\kappa}. \tag{A.3}$$

By definition, $\tilde{R}^{p,n} = \frac{1}{n-p}\sum_{k=0}^{n-p-1}\tilde{\eta}_n(k)$, where $\tilde{\eta}_n(k) := \eta_n(k) - \mathrm{E}\eta_n(k)$ and $\eta_n(k) := \psi(\Delta_k^{p,n}X, \Delta_{k+1}^{p,n}X)$, $\psi(x,y) = |x+y|/(|x|+|y|)$ are nonlinear functions of Gaussian vectors $(\Delta_k^{p,n}X, \Delta_{k+1}^{p,n}X) \in \mathbb{R}^2$ having the Hermite rank 2; however, these vectors are not ε-standard and therefore Lemma A.1 cannot be directly applied to estimate the left-hand side of (A.3) (with $p = 1, \dots, 4$, $\nu = 2$). To this end, we first need to "decimate" the sum $\tilde{R}^{p,n}$, as follows. (A similar "trick" was used in [16].) Let $\ell = [n^{\theta/\gamma}]$ be the sequence of integers increasing to ∞ (at a rate $\mathrm{o}(n^{1/2})$ by condition $\theta < \gamma/2$) and write

$$\tilde{R}^{p,n} = \sum_{j=0}^{\ell-1}\tilde{R}_\ell^{p,n}(j) + \mathrm{o}(1), \qquad \tilde{R}_\ell^{p,n}(j) := \frac{1}{n-1}\sum_{k=0}^{[(n-2-j)/\ell]}\tilde{\eta}_n(k\ell + j).$$

Then,

$$\mathrm{E}(\tilde{R}^{p,n})^4 \le \ell^4 \max_{0\le j<\ell}\mathrm{E}(\tilde{R}_\ell^{p,n}(j))^4.$$

Write $\eta_n(k)$ as a (bounded) function in standardized Gaussian variables:

$$\eta_n(k) = f_{k,n}(\mathbf{Y}_n(k)), \tag{A.4}$$

where $\mathbf{Y}_n(k) = (Y_n^{(1)}(k), Y_n^{(2)}(k)) \in \mathbb{R}^2$,

$$Y_n^{(1)}(k) := \frac{\Delta_k^{p,n}X}{\sigma_{p,n}(k)}, \tag{A.5}$$

$$Y_n^{(2)}(k) := -\frac{\Delta_k^{p,n}X}{\sigma_{p,n}(k)}\frac{\rho_{p,n}(k)}{\sqrt{1-\rho_{p,n}^2(k)}} + \frac{\Delta_{k+1}^{p,n}X}{\sigma_{p,n}(k+1)}\frac{1}{\sqrt{1-\rho_{p,n}^2(k)}} \quad \text{and} \tag{A.6}$$

$$f_{k,n}\bigl(x^{(1)}, x^{(2)}\bigr) := \psi\biggl(x^{(1)}, \frac{\sigma_{p,n}(k+1)}{\sigma_{p,n}(k)}\bigl(\rho_{p,n}(k)x^{(1)} + \sqrt{1-\rho_{p,n}^2(k)}x^{(2)}\bigr)\biggr), \tag{A.7}$$

where $\sigma_{p,n}^2(k)$, $\rho_{p,n}(k)$ are defined in (4.1). Then, for each k, $\mathbf{Y}_n(k) := (Y_n^{(1)}(k), Y_n^{(2)}(k))$ has a standard Gaussian distribution in $\mathbb{R}^2$ and $\tilde{\eta}_n(k) = f_{k,n}(\mathbf{Y}_n(k)) - \mathrm{E} f_{k,n}(\mathbf{Y}_n(k))$. Moreover, the vector $(\mathbf{Y}_n(k\ell+j), k = 0, 1, \dots, [(n-2-j)/\ell]) \in \mathbb{R}^{2([(n-2-j)/\ell]+1)}$ is ε-standard, provided that ℓ is large enough. Now, Lemma A.1 can be used and it implies the bound (A.3) using assumptions (A.1) and $(\mathrm{A.2})_p$. The details of this proof can be found in [5]. □

Sketch of the proof of Theorem 4.2

The proof of Theorem 4.1 uses the following central limit theorem for Gaussian subordinated multidimensional triangular arrays. Theorem A.1 is proved in [5]. It extends the earlier results in [11] and [2]. Below, similarly as in Lemma A.1, $\mathbf{X} \in \mathbb{R}^\nu$ designates a standard Gaussian vector.

Theorem A.1. *Let* $(\mathbf{Y}_n(k))_{1\le k\le n, n\in\mathbb{N}}$ *be a triangular array of standardized Gaussian vectors with values in* $\mathbb{R}^\nu$, $\mathbf{Y}_n(k) = (Y_n^{(1)}(k), \dots, Y_n^{(\nu)}(k))$, $\mathrm{E}Y_n^{(p)}(k) = 0$, $\mathrm{E}Y_n^{(p)}(k)Y_n^{(q)}(k) = \delta_{pq}$. *For a given integer* $m \ge 1$, *we introduce the following assumption*: *there exists a function* $\rho : \mathbb{N} \to \mathbb{R}$ *such that for any* $1 \le p, q \le \nu$,

$$\forall (j,k) \in \{1,\dots,n\}^2 \qquad \left|\mathrm{E}Y_n^{(p)}(j)Y_n^{(q)}(k)\right| \le |\rho(j-k)| \qquad \textit{with} \sum_{j\in\mathbb{Z}} |\rho(j)|^m < \infty. \tag{A.8}$$

Moreover, assume that for any $\tau \in [0,1]$ *and any* $J \in \mathbb{N}^*$,

$$\big(\mathbf{Y}_n([n\tau]+j)\big)_{-J\le j\le J} \xrightarrow[n\to\infty]{\mathcal{D}} (\mathbf{W}_\tau(j))_{-J\le j\le J}, \tag{A.9}$$

where $(\mathbf{W}_\tau(j))_{j\in\mathbb{Z}}$ *is a stationary Gaussian process taking values in* $\mathbb{R}^\nu$ *and depending on parameter* $\tau \in (0,1)$. *Let* $\tilde f_{k,n} \in \mathbb{L}_0^2(\mathbf{X})$ $(n \ge 1,\ 1 \le k \le n)$ *be a triangular array of functions all having Hermite rank at least* m. *Assume that there exists an* $\mathbb{L}_0^2(\mathbf{X})$-*valued continuous function* $\tilde\phi_\tau$, $\tau \in [0,1]$, *such that*

$$\sup_{\tau\in[0,1]} \left\| \tilde f_{[\tau n],n} - \tilde\phi_\tau \right\|^2 = \sup_{\tau\in[0,1]} \mathrm{E}\big(\tilde f_{[\tau n],n}(\mathbf{X}) - \tilde\phi_\tau(\mathbf{X})\big)^2 \xrightarrow[n\to\infty]{} 0. \tag{A.10}$$

Then, with $\sigma^2 = \int_0^1 \mathrm{d}\tau (\sum_{j\in\mathbb{Z}} \mathrm{E}[\tilde\phi_\tau(\mathbf{W}_\tau(0))\tilde\phi_\tau(\mathbf{W}_\tau(j))]) < \infty$,

$$n^{-1/2} \sum_{k=1}^n \tilde f_{k,n}(\mathbf{Y}_n(k)) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,\sigma^2). \tag{A.11}$$

Sketch of the proof of Theorem 4.2. It suffices to show that

$$\sqrt{n}\left| \mathrm{E}R^{n,p} - \int_0^1 \Lambda_p(H(t))\,\mathrm{d}t \right| \xrightarrow[n\to\infty]{} 0 \tag{A.12}$$

and

$$\sqrt{n}(R^{p,n} - \mathrm{E}R^{p,n}) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\left(0, \int_0^1 \Sigma_p(H(\tau))\,\mathrm{d}\tau\right). \tag{A.13}$$

The proof of (A.12) uses assumption (A.1)$'$ or (4.10) and the easy fact that for Gaussian vectors $(Z_n^{(1)}, Z_n^{(2)}) \in \mathbb{R}^2, n \in \mathbb{N}$, with zero mean, $\mathrm{E}Z_n^{(i)} \equiv 0$, $i = 1,2$, $n \in \mathbb{N}$ and $\mathrm{E}(Z_0^{(1)})^2 = \mathrm{E}(Z_0^{(2)})^2 = 1$, $|\mathrm{E}Z_0^{(1)}Z_0^{(2)}| < 1$

$$\left|\mathrm{E}\psi\big(Z_0^{(1)}Z_0^{(2)}\big) - \mathrm{E}\psi\big(Z_n^{(1)}, Z_n^{(2)}\big)\right| \le C \sum_{i,j=1}^2 \left|\mathrm{E}Z_0^{(i)}Z_0^{(j)} - Z_n^{(i)}Z_n^{(j)}\right|. \tag{A.14}$$

The proof of (A.13) is deduced from Theorem A.1 with the sequence of standardized Gaussian vectors $\mathbf{Y}_n(k)=(Y_n^{(1)}(k),Y_n^{(2)}(k))$ ($\nu=2$) given in (A.5)–(A.6) and the centered functions

$$\tilde f_{k,n}\big(x^{(1)},x^{(2)}\big):=f_{k,n}\big(x^{(1)},x^{(2)}\big)-\mathrm{E}f_{k,n}(\mathbf{Y}_n(k)),$$
$$\tilde\phi_\tau\big(x^{(1)},x^{(2)}\big):=\phi_\tau\big(x^{(1)},x^{(2)}\big)-\mathrm{E}\phi_\tau(\mathbf{X})$$

with $f_{k,n}:\mathbb{R}^2\to\mathbb{R}$ given in (A.7) and the (limit) function

$$\phi_\tau\big(x^{(1)},x^{(2)}\big):=\psi\big(x^{(1)},\rho_p(H(\tau))x^{(1)}+\sqrt{1-\rho_p^2(H(\tau))}x^{(2)}\big).$$

Thanks to symmetry properties of these functions, it is clear that the Hermite rank of $\tilde f_{k,n}$ (for any k and n) and $\tilde\phi_\tau$ (for any $\tau\in[0,1]$) is $m=2$. Using assumptions (A.1)$'$ and (A.2)$_p$ (with $\theta=0$, $\gamma>1/2$), we can show that the conditions of Theorem A.1 are satisfied for the above $\tilde f_{n,k},\tilde\phi_\tau$ and the limit process $(\mathbf{W}_\tau(j))_{j\in\mathbb{Z}}$ in (A.9) is written in terms of increments of fBm $(B_{H(\tau)}(j))_{j\in\mathbb{Z}}$,

$$\mathbf{W}_\tau(j):=\big(\Delta_1^pB_{H(\tau)}(j),\big(-\rho_p(H(\tau))\Delta_1^pB_{H(\tau)}(j)+\Delta_1^pB_{H(\tau)}(j+1)\big)/\sqrt{1-\rho_p^2(H(\tau))}\big),$$

having standardized uncorrelated components. The details of this proof can be found in [5]. □

Proof of Corollary 4.2

(a) The argument at the end of the proof of Property 4.1 shows that V satisfies assumption (A.1)$'$, while (A.2)$_p$ follows from Property 4.2. The central limit theorem in (4.26) then follows from Theorem 4.2.

(b) In this case, (A.2)$_p$ follows from Property 4.3. Instead of verifying (A.1)$'$, it is simpler to directly verify condition (4.10), which suffices for the validity of the statement of Theorem 4.2. Using $f(\xi)=c\xi^{-2H-1}(1+\mathrm{o}(\xi^{-1/2}))$ ($\xi\to\infty$), similarly as in the proof of Property 4.3, for $j\in\mathbb{N}^*$, we obtain

$$\begin{aligned}&\sqrt{n}\left|n^{-2H}\mathrm{E}[\Delta_0^{p,n}X\Delta_j^{p,n}X]-c2^{1+p}\int_0^\infty\big(1-\cos(x)\big)^p\cos(xj)x^{-2H-1}\,\mathrm{d}x\right|\\&\quad=2^{1+p}\left|\int_0^\infty\big(1-\cos(x)\big)^p\cos(xj)\times\sqrt{n}\big(n^{2H+1}\big(f(nx)-c(nx)^{-2H-1}\big)\big)\,\mathrm{d}x\right|\underset{n\to\infty}{\longrightarrow}0\end{aligned}$$

by the Lebesgue dominated convergence theorem since $\int_0^\infty|(1-\cos(x))^p\cos(xj)\times x^{-2H-3/2}|\,\mathrm{d}x<\infty$. Therefore, condition (4.10) is satisfied and Theorem 4.2 can be applied.

Finally, the function $H\mapsto\Lambda_2(H)$ is a $\mathcal{C}^1(0,1)$ bijective function and from the delta method (see, e.g., [41]), the central limit theorem in (4.27) follows.

Computation of $\lambda(r)$

From the definition of $\lambda(r)$ and the change of variables $x_1 = a\cos\phi$, $x_2 = a\sin\phi$, with $|r| < 1$,

$$\begin{aligned}\lambda(r) &= \frac{1}{2\pi\sqrt{1-r^2}} \int_{\mathbb{R}^2} \frac{|x_1+x_2|}{|x_1|+|x_2|} \mathrm{e}^{-(1/(2(1-r^2)))(x_1^2-2rx_1x_2+x_2^2)}\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &= \frac{\sqrt{1-r^2}}{\pi} \int_0^{\pi} \frac{|\cos\phi+\sin\phi|}{(|\cos\phi|+|\sin\phi|)(1-r\sin(2\phi))}\,\mathrm{d}\phi \\ &=: I_1 + I_2,\end{aligned}$$

where

$$\begin{aligned} I_1 &= \frac{\sqrt{1-r^2}}{\pi} \int_0^{\pi/2} \frac{1}{1-r\sin(2\phi)}\,\mathrm{d}\phi \\ &= \frac{\sqrt{1-r^2}}{\pi} \int_0^{\infty} \frac{1}{1+t^2-2rt}\,\mathrm{d}t = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{r}{\sqrt{1-r^2}}\right) = \frac{1}{\pi}\arccos(-r), \\ I_2 &= \frac{2\sqrt{1-r^2}}{\pi} \int_0^{\pi/4} \frac{\cos\phi-\sin\phi}{(\cos\phi+\sin\phi)(1+r\sin(2\phi))}\,\mathrm{d}\phi \\ &= \frac{\sqrt{1-r^2}}{\pi} \int_0^{1} \frac{1-t}{(1+t)(1+2rt+t^2)}\,\mathrm{d}t \\ &= \frac{\sqrt{1-r^2}}{\pi(1-r)} \log\left(\frac{2}{r+1}\right). \end{aligned}$$

The function $\lambda(r)$ is monotone increasing on $[-1, 1]$; $\lambda(1) = 1$, $\lambda(-1) = 0$. It is easy to check that

$$\rho_1(H) = 2^{2H-1} - 1, \qquad \rho_2(H) = \frac{-3^{2H} + 2^{2H+2} - 7}{8 - 2^{2H+1}} \tag{A.15}$$

are monotone increasing functions; $\rho_1(1) = 1$, $\rho_2(1) = 0$ so that $\Lambda_p(H) = \lambda(\rho_p(H))$ for $p = 1, 2$ is also monotone for $H \in (0, 1)$.

Expression and graph of $s_2(H)$

From the delta method, $s_2^2(H) = [\frac{\partial}{\partial x}(\Lambda_2)^{-1}(\Lambda_2(H))]^2 \Sigma_2(H)$ and therefore

$$s_2^2(H) = \left(\frac{\pi(8-2^{2H+1})^2(1-\rho_2(H))\sqrt{1-\rho_2^2(H)}}{(\log 2 - \log(1+\rho_2(H)))(2^{2H+2}9\log 2 - 3^{2H}16\log 3 + 6^{2H}4\log(3/2))} \right)^2 \Sigma_2(H),$$

with the approximated graph (using the numerical values of $\Sigma_2(H)$ from [36]) provided in Figure 5.

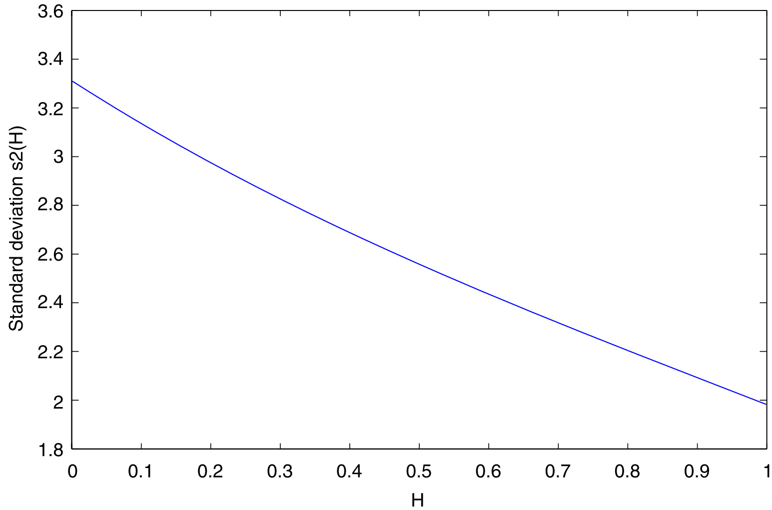

Figure 5. The graph of $\sqrt{s_2^2(H)}$.

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References

[1] Aït-Sahalia, Y. and Jacod, J. (2009). Estimating the degree of activity of jumps in high frequency financial data. *Ann. Statist.* **37** 2202–2244. MR2543690

[2] Arcones, M.A. (1994). Limit theorems for nonlinear functionals of a stationary Gaussian sequence of vectors. *Ann. Probab.* **22** 2242–2274. MR1331224

[3] Ayache, A., Benassi, A., Cohen, S. and Lévy Véhel, J. (2005). Regularity and identification of generalized multifractional Gaussian processes. In *Séminare de Probabilités XXXVIII. Lecture Notes in Math.* **1857** 290–312. Berlin: Springer. MR2126981

[4] Bardet, J.-M. and Bertrand, P. (2007). Definition, properties and wavelet analysis of multiscale fractional Brownian motion. *Fractals* **15** 73–87. MR2281946

[5] Bardet, J.-M. and Surgailis, D. (2009). A central limit theorem for triangular arrays of nonlinear functionals of Gaussian vectors. Preprint.

[6] Bardet, J.-M. and Surgailis, D. (2010). Nonparametric estimation of the local Hurst function of multifractional processes. Preprint.

[7] Belomestny, D. (2010). Spectral estimation of the fractional order of a Lévy process. *Ann. Statist.* **38** 317–351. MR2589324
[8] Benassi, A., Cohen, S. and Istas, J. (1998). Identifying the multifractal function of a Gaussian process. *Statist. Probab. Lett.* **39** 337–345. MR1646220
[9] Benassi, A., Jaffard, S. and Roux, D. (1997). Gaussian processes and pseudodifferential elliptic operators. *Rev. Mat. Iberoamericana* **13** 19–89. MR1462329
[10] Berk, K.N. (1973). A central limit theorem for m-dependent random variables with unbounded m. *Ann. Probab.* **1** 352–354. MR0350815
[11] Breuer, P. and Major, P. (1983). Central limit theorems for nonlinear functionals of Gaussian fields. *J. Multivariate Anal.* **13** 425–441. MR0716933
[12] Coeurjolly, J.-F. (2001). Estimating the parameters of a fractional Brownian motion by discrete variations of its sample paths. *Stat. Inference Stoch. Process.* **4** 199–227. MR1856174
[13] Coeurjolly, J.-F. (2005). Identification of multifractional Brownian motion. *Bernoulli* **11** 987–1008. MR2188838
[14] Coeurjolly, J.-F. (2007). Hurst exponent estimation of locally self-similar Gaussian processes using sample quantiles. *Ann. Statist.* **36** 1404–1434. MR2418662
[15] Cramér, H. and Leadbetter, M.R. (1967). *Stationary and Related Stochastic Processes. Sample Function Properties and Their Applications.* New York: Wiley. MR0217860
[16] Csörgő, M. and Mielniczuk, J. (1996). The empirical process of a short-range dependent stationary sequence under Gaussian subordination. *Probab. Theory Related Fields* **104** 15–25. MR1367664
[17] Dahlhaus, R. (1989). Efficient parameter estimation for self-similar processes. *Ann. Statist.* **17** 1749–1766. MR1026311
[18] Dobrushin, R.L. (1980). Automodel generalized random fields and their renorm group. In *Multicomponent Random Systems* (R.L. Dobrushin and Y.G. Sinai, eds.) 153–198. New York: Dekker. MR0599535
[19] Falconer, K. (2002). Tangent fields and the local structure of random fields. *J. Theoret. Probab.* **15** 731–750. MR1922445
[20] Falconer, K. (2003). The local structure of random processes. *J. London Math. Soc.* **67** 657–672. MR1967698
[21] Feuerverger, A., Hall, P. and Wood, A.T.A. (1994). Estimation of fractal index and fractal dimension of a Gaussian process by counting the number of level crossings. *J. Time Series Anal.* **15** 587–606. MR1312323
[22] Fox, R. and Taqqu, M.S. (1986). Large-sample properties of parameter estimates for strongly dependent Gaussian time series. *Ann. Statist.* **14** 517–532. MR0840512
[23] Gikhman, I.I. and Skorohod, A.V. (1969). *Introduction to the Theory of Random Processes.* Philadelphia: Saunders. MR0247660
[24] Gugushvili, S. (2008). Nonparametric estimation of the characteristic triplet of a discretely observed Lévy process. *J. Nonparametr. Stat.* **21** 321–343. MR2530929
[25] Guyon, X. and Leon, J. (1989). Convergence en loi des H-variations d'un processus gaussien stationnaire. *Ann. Inst. H. Poincaré Probab. Statist.* **25** 265–282. MR1023952
[26] Hall, P. and Wood, A. (1993). On the performance of box-counting estimators of fractal dimension. *Biometrika* **80** 246–252. MR1225230
[27] Ho, H.-C. and Sun, T.C. (1987). A central limit theorem for non-instantaneous filters of a stationary Gaussian process. *J. Multivariate Anal.* **22** 144–155. MR0890889
[28] Ibragimov, I.A. and Linnik. Y.V. (1971). *Independent and Stationary Sequences of Random Variables.* Groningen: Wolters-Noordhoff. MR0322926
[29] Istas, J. and Lang, G. (1997). Quadratic variations and estimation of the local Hölder index of a Gaussian process. *Ann. Inst. H. Poincaré Probab. Statist.* **33** 407–436. MR1465796

[30] Neumann, M. and Reiß, M. (2009). Nonparametric estimation for Lévy processes from low-frequency observations. *Bernoulli* **15** 223–248. MR2546805

[31] Peltier, R. and Lévy-Vehel, J. (1994). A new method for estimating the parameter of fractional Brownian motion, INRIA. Preprint. Available at http://hal.inria.fr/docs/00/07/42/79/PDF/RR-2396.pdf.

[32] Philippe, A., Surgailis, D. and Viano, M.-C. (2006). Invariance principle for a class of non stationary processes with long memory. *C. R. Acad. Sci. Paris Ser. 1* **342** 269–274. MR2196011

[33] Philippe, A., Surgailis, D. and Viano, M.-C. (2008). Time-varying fractionally integrated processes with nonstationary long memory. *Theory Probab. Appl.* **52** 651–673.

[34] Sato, K.-I. (1999). *Lévy Processes and Infinitely Divisible Distributions.* Cambridge: Cambridge Univ. Press. MR1739520

[35] Shilov, G.E. and Gurevich, B.L. (1967). *Integral, Measure and Derivative* (in Russian). Moscow: Nauka. MR0219686

[36] Stoncelis, M. and Vaičiulis, M. (2008). Numerical approximation of some infinite Gaussian series and integrals. *Nonlinear Anal. Modell. Control* **13** 397–415. MR2457610

[37] Surgailis, D. (2008). Nonhomogeneous fractional integration and multifractional processes. *Stochastic Process. Appl.* **118** 171–198. MR2376898

[38] Surgailis, D., Teyssière, G. and Vaičiulis, M. (2008). The increment ratio statistic. *J. Multivariate Anal.* **99** 510–541. MR2396977

[39] Taqqu, M.S. (1977). Law of the iterated logarithm for sums of non-linear functions of Gaussian variables that exhibit a long range dependence. *Z. Wahrsch. Verw. Gebiete* **40** 203–238. MR0471045

[40] Vaičiulis, M. (2009). An estimator of the tail index based on increment ratio statistics. *Lithuanian Math. J.* **49** 222–233. MR2532037

[41] Van der Vaart, A. (1998). *Asymptotic Statistics*. Cambridge: Cambridge Univ. Press. MR1652247

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