Asymptotic distributions and subsampling in spectral analysis for almost periodically correlated time series

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The aim of this article is to establish asymptotic distributions and consistency of subsampling for spectral density and for magnitude of coherence for non-stationary, almost periodically correlated time series. We show the asymptotic normality of the spectral density estimator and the limiting distribution of a magnitude of coherence statistic for all points from the bifrequency square. The theoretical results hold under α-mixing and moment conditions.

Keywords: α-mixing properties; almost periodically correlated time series; consistency; spectral analysis; subsampling

1. Introduction

The analysis of the second order structure of time series is usually based on the characterization of the autocovariance function. One possibility is that the time series is not stationary, with a periodic or an almost periodic autocovariance function. The models with this structure were applied in many fields, including telecommunications [12,23], meteorology [2], econometrics [24,25] and many others (see the review in [28] or [13]). This class of non-stationary time series was introduced by [14]. Formally, we say that a second-order real-valued time series $\{X_t : t \in \mathbb{Z}\}$ is *periodically correlated* (PC for short) if both the mean $\mu(t) = E(X_t)$ and the autocovariance function $B(t, \tau) = \text{cov}(X_t, X_{t+\tau})$ are periodic at t for every $\tau \in \mathbb{Z}$, with the same period T. A second-order real-valued time series $\{X_t : t \in \mathbb{Z}\}$ is called *almost periodically correlated* (APC for short) if both the mean $\mu(t)$ and the autocovariance function $B(t, \tau)$ are almost periodic function at t for every $\tau \in \mathbb{Z}$. The definition of almost periodic function that we use in this paper can be found in [4], page 45. It is easy to see that the class of APC time series contains the class of PC time series and the class of stationary time series.

One of the main problems is how to detect this kind of non-stationary structure in the time series. This problem was considered in the time domain by Vecchia and Ballerini [30], Dandawate and Giannakis [5], Dehay and Leśkow [7], Synowiecki [29] and, recently, by Lenart *et al.* [19]. Lenart *et al.* [19] present tests for PC structure based on the estimator of Fourier coefficient $a(\lambda, \tau)$ and the subsampling methodology. In the frequency domain, the problem of detecting periodicity was considered by Hurd and Gerr [16], Lund *et al.* [22], Broszkiewicz-Suwaj *et al.* [3], Lii and Rosenblatt [20] and Hurd and Miamee [17]. Hurd and Gerr [16] and Hurd and Miamee [17] present a graphical method for detecting stationarity and periodicity, where the testing

statistics were approximated by a beta distribution. However, this method has not yet been justified for non-Gaussian white noise or dependent random variables. Broszkiewicz-Suwaj *et al.* [3] have used the *moving block bootstrap* to construct methods for detecting periodicity in time series. Still, the consistency of the bootstrap applied for their testing statistics remains an open problem. Lii and Rosenblatt [20] present an innovative algorithm for estimating the support of the spectral measure for Gaussian APC processes. The authors considered the case where the support of the spectral measure is concentrated on a finite number of parallel lines.

In the next section, we recall the theoretical background and formulate the assumptions which are used in the sections which follow. We consider general APC time series for which the support of the spectral measure can be concentrated on an infinite number of parallel lines. Section 3 presents the consistent estimator for the magnitude of coherence and the extension of the spectral density function to the bifrequency square $(0, 2\pi]^2$. In this section, we show the asymptotic normality of the normalized spectral density estimator, the asymptotic distribution of the normalized magnitude of the spectral density estimator and the coherence statistic for all points from the bifrequency square $(0, 2\pi]^2$. We give the exact forms for these distributions. It is shown that the asymptotic distribution of the normalized magnitude of the coherence statistic strongly depends on the support of the spectral measure. This distribution is not Gaussian for the points which do not belong to the support of the spectral measure. Recall that for the time series, the asymptotic normality of the spectral density estimator was shown for the stationary case in [31] under an α-mixing condition and in [27] for linear filters. For the PC case, the asymptotic normality was established in [17] for linear filters. Asymptotic distributions from Section 3 provide a possibility to establish consistency of the subsampling for the magnitude of spectral density and coherence, which is shown in Section 4. It should be emphasized that magnitude of spectral density and coherence are broadly used as fundamental characteristics for telecommunication signals (see [11]). Section 5 presents a simulation study, where we show applications of our results. Following the ideas of Hurd and Gerr [16] and Hurd and Miamee [17], we present graphical methods for detecting periodicity in autocovariance and give theoretical justifications for these methods. Finally, the asymptotic distribution and consistency of subsampling provide a possibility to construct asymptotically consistent confidence intervals for magnitude of spectral density and coherence. All proofs can be found in the Appendix.

2. Definitions and assumptions

We will be focusing on second-order inference for time series, therefore, for simplicity, we make the following assumption:

(A1) *Assume that the real-valued time series $\{X_t : t \in \mathbb{Z}\}$ is zero-mean.*

Moreover, we will consider non-stationary, almost periodically correlated time series; therefore, we formulate the next assumption:

(A2) *Assume that the time series $\{X_t : t \in \mathbb{Z}\}$ is APC.*

In the APC case with $\mu(t) \equiv 0$ for any $\tau \in \mathbb{Z}$, the autocovariance function $B(t, \tau)$ has the Fourier representation

$$B(t, \tau) \sim \sum_{\lambda \in \Lambda_\tau} a(\lambda, \tau) e^{i\lambda t}, \tag{1}$$

where $a(\lambda, \tau)$ are the Fourier coefficients of the form $a(\lambda, \tau) = \lim_{n \to \infty} n^{-1} \sum_{j=1}^{n} B(j, \tau) e^{-i\lambda j}$ and the set $\Lambda_\tau = \{\lambda \in [0, 2\pi) : a(\lambda, \tau) \neq 0\}$ is countable (see [4]).

Define the set $\Lambda = \bigcup_{\tau \in \mathbb{Z}} \Lambda_\tau$. We make the following assumption concerning the set Λ and the Fourier coefficients $a(\lambda, \tau)$. We need this assumption in order to establish asymptotic properties of the spectral density estimator:

(A3) *Assume that for any $x \in [0, 2\pi)$, there exists a real-valued sequence $\{z_\tau(x)\}_{\tau \in \mathbb{Z}}$ such that we have*

$$\sum_{\lambda \in \Lambda_\tau \setminus \{x\}} \left| a(\lambda, \tau) \operatorname{cosec} \left(\frac{\lambda - x}{2} \right) \right| < z_\tau(x) < \infty \tag{2}$$

and $z_\tau(x) \to 0$ for $|\tau| \to \infty$.

Remark 2.1. By (A3), we have that $\sum_{\lambda \in \Lambda_\tau} |a(\lambda, \tau)| < \infty$, which implies that

$$B(t, \tau) = \sum_{\lambda \in \Lambda_\tau} a(\lambda, \tau) e^{i\lambda t}. \tag{3}$$

If $\operatorname{card}(\Lambda) < \infty$, and we assume that $|B(t, \tau)| < p_\tau$ uniformly at t and $p_\tau \to 0$ for $|\tau| \to \infty$, then (A3) holds. Note that (2) implies that for any $\lambda \in [0, 2\pi)$, we have that $|a(\lambda, \tau)| \to 0$ as $|\tau| \to \infty$.

To introduce the spectral theory for APC time series, we need the following assumption:

(A4) *Assume that the time series $\{X_t : t \in \mathbb{Z}\}$ is harmonizable (for the definition, see [21] or [32]).*

By assumptions (A2) and (A4), the spectral measure R, defined on the bifrequency square $(0, 2\pi]^2$, has support contained in the set (see [6])

$$S = \bigcup_{|\lambda| \in \Lambda} \{(\nu, \omega) \in (0, 2\pi]^2 : \omega = \nu - \lambda\}.$$

Moreover, the coefficients $a(\lambda, \tau)$ are the Fourier transforms of complex measures $r_\lambda(\cdot)$, that is, $a(\lambda, \tau) = \int_0^{2\pi} e^{i\xi\tau} r_\lambda(\mathrm{d}\xi)$. The measure r_λ can be identified with the restriction of the spectral measure R to the line $\omega = \nu - \lambda$, where $|\lambda| \in \Lambda$. We need the following assumption:

(A5) *Assume that the measure r_0 is absolutely continuous with respect to the Lebesgue measure.*

By the above assumptions, for any $|\lambda| \in \Lambda$, there exists a spectral density function $g_\lambda(\cdot)$ such that $g_\lambda(v) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} a(\lambda, \tau) e^{-iv\tau}$ (see [15] and [6]).

Now, let us define the extension of the spectral density function $P(\cdot, \cdot)$ to the bifrequency square $(0, 2\pi]^2$ via

$$P(v, \omega) = \begin{cases} g_\lambda(v) & \text{for } \omega = v - \lambda \text{ and } |\lambda| \in \Lambda, \\ 0 & \text{for } (v, \omega) \notin S. \end{cases} \tag{4}$$

Hence,

$$P(v, \omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} a(v - \omega, \tau) e^{-iv\tau} \tag{5}$$

for any $(v, \omega) \in (0, 2\pi]^2$, which follows from the fact that $a(v - \omega, \tau) = 0$ for any $|v - \omega| \in [0, 2\pi) \setminus \Lambda_\tau$ and $\tau \in \mathbb{Z}$. If we assume that $g_0(\xi) \neq 0$ for any $\xi \in (0, 2\pi]$, then for any point (v, ω) from the bifrequency square $(0, 2\pi]^2$, we define a magnitude of coherence via

$$|\gamma(v, \omega)| = \begin{cases} \dfrac{|g_\lambda(v)|}{\sqrt{g_0(v) g_0(\omega)}} & \text{for } \omega = v - \lambda \text{ and } |\lambda| \in \Lambda, \\ 0 & \text{for } (v, \omega) \notin S. \end{cases} \tag{6}$$

Note that the magnitude of coherence is a real number from the interval $[0, 1]$, while the extension of the spectral density function is a complex number.

3. Asymptotic distributions

The estimator of $P(v, \omega)$ (based on a sample $\{X_1, X_2, \ldots, X_n\}$) of the form

$$\hat{P}_n(v, \omega) = \frac{1}{2\pi n} \sum_{s=1}^{n} \sum_{t=1}^{n} X_s X_t e^{-ivs} e^{i\omega t} \tag{7}$$

is generally not consistent in the mean square sense for APC time series. The exact form of the asymptotic variance of this estimator was calculated in [18].

To obtain a consistent estimator, we consider the following class of smoothed estimators of $P(v, \omega)$ (see [31] for the stationary case):

$$\hat{G}_n(v, \omega) = \frac{1}{2\pi n} \sum_{s=1}^{n} \sum_{t=1}^{n} H_{L_n}(s - t) X_s X_t e^{-ivs} e^{i\omega t}, \tag{8}$$

where $H_{L_n}(\cdot)$ is a lag window function such that $H_{L_n}(\tau) = 0$ for $|\tau| > L_n$, $\tau \in \mathbb{Z}$ and L_n is a sequence of positive integers tending to infinity with n. Moreover, we assume that $L_n/n \to 0$. If we assume that $g_0(\xi) \neq 0$ for any $\xi \in (0, 2\pi]$, then we define a magnitude of coherence statistic based on the estimator (8) via

$$|\hat{\gamma}_n(v, \omega)| = \frac{|\hat{G}_n(v, \omega)|}{\sqrt{|\operatorname{Re}(\hat{G}_n(v, v)) \operatorname{Re}(\hat{G}_n(\omega, \omega))|}}. \tag{9}$$

The following considerations will involve the general assumption that we have a sample $\{X_{c_n+1}, X_{c_n+2}, \ldots, X_{c_n+d_n}\}$ from time series $\{X_t : t \in \mathbb{Z}\}$, where $\{c_n\}_{n \in \mathbb{Z}}$ and $\{d_n\}_{n \in \mathbb{Z}}$ are arbitrary sequences of non-negative integers such that $d_n \to \infty$. This will help us to prove the consistency of the subsampling procedure to be discussed in Section 4.

Given a sample $\{X_{c_n+1}, X_{c_n+2}, \ldots, X_{c_n+d_n}\}$, we define the estimators of $P(\nu, \omega)$ and $|\gamma(\nu, \omega)|$ via

$$\hat{P}_n^{c,d}(\nu, \omega) = \frac{1}{2\pi d_n} \sum_{s=c_n+1}^{c_n+d_n} \sum_{t=c_n+1}^{c_n+d_n} X_s X_t e^{-is\nu} e^{it\omega}, \tag{10}$$

$$\hat{G}_n^{c,d}(\nu, \omega) = \frac{1}{2\pi d_n} \sum_{s=c_n+1}^{c_n+d_n} \sum_{t=c_n+1}^{c_n+d_n} H_{L_{d_n}}(s-t) X_s X_t e^{-is\nu} e^{it\omega} \tag{11}$$

and

$$|\hat{\gamma}_n^{c,d}(\nu, \omega)| = \frac{|\hat{G}_n^{c,d}(\nu, \omega)|}{\sqrt{|\operatorname{Re}(\hat{G}_n^{c,d}(\nu, \nu)) \operatorname{Re}(\hat{G}_n^{c,d}(\omega, \omega))|}}. \tag{12}$$

Besides the fact that, in this work, we consider zero-mean time series, in the following remark, we define the estimator of the spectral density function in the case where the mean function $\mu(t)$ is an almost periodic function such that $\mu(t) \not\equiv 0$.

Remark 3.1. Assume that the mean function $\mu(t) = E(X_t)$ for APC time series is an almost periodic function such that $\mu(t) = \sum_{\gamma \in \Gamma} b(\gamma) e^{i\gamma t}$, where $b(\gamma)$ are the Fourier coefficients and the set Γ is known and finite. The natural estimator of the mean function $\mu(t)$ (for $c_n + 1 \leq t \leq c_n + d_n$) based on a sample $\{X_{c_n+1}, X_{c_n+2}, \ldots, X_{c_n+d_n}\}$ then takes the form $\hat{\mu}_n^{c,d}(t) = \sum_{\gamma \in \Gamma} \hat{b}_n^{c,d}(\gamma) e^{i\gamma t}$, where $\hat{b}_n^{c,d}(\gamma) = \frac{1}{d_n} \sum_{j=c_n+1}^{c_n+d_n} X_j e^{-i\gamma j}$ is an estimator of the Fourier coefficient $b(\gamma)$ for any $\gamma \in \Gamma$. The more general natural estimator of the spectral density $P(\nu, \omega)$ based on a sample $\{X_{c_n+1}, X_{c_n+2}, \ldots, X_{c_n+d_n}\}$ then takes the form

$$\hat{T}_n^{c,d}(\nu, \omega) = \frac{1}{2\pi d_n} \sum_{s=c_n+1}^{c_n+d_n} \sum_{t=c_n+1}^{c_n+d_n} H_{L_{d_n}}(s-t)\big(X_s - \hat{\mu}_n^{c,d}(s)\big)\big(X_t - \hat{\mu}_n^{c,d}(t)\big) e^{-i\nu s} e^{i\omega t}. \tag{13}$$

Similar modification in the PC case can be found in [17], Section 10.4.

In Theorem 3.1, we show the asymptotic covariance between normalizing estimators $\hat{G}_n^{c,d}(\nu_1, \omega_1)$ and $\hat{G}_n^{c,d}(\nu_1, \omega_1)$ for $(\nu_1, \omega_1), (\nu_1, \omega_1) \in [0, 2\pi)^2$. We need the following assumption concerning the lag window function $H_{L_n}(\cdot)$.

(B) *Assume that for the lag window function $H_{L_n}(\cdot)$, where $L_n \to \infty$, $L_n/n \to 0$, there exists a real-valued function $w(\cdot)$ such that:*

(i) *$w(x) = 0$ for $|x| > 1$ and $w(\cdot)$ is an even function and non-increasing on the interval $[0, 1]$;*

(ii) *there exists a real number $\theta \in (0, 1]$ such that for any $|x| \leq \theta$, we have $w(x) = 1$;*

(iii) *$H_{L_n}(\tau) = w(\tau/L_n)$ for any $\tau \in \mathbb{Z}$ and $n \geq 1$;*

(iv) *$w(\cdot)$ is Lipschitz continuous on the interval $[-1, 1]$ with Euclidean metric, which means that there exists a real number W such that for any $x, y \in [-1, 1]$, we have*

$$|w(x) - w(y)| \leq W|x - y|.$$

For the convenience of the reader, before Theorem 3.1, we introduce the concept of α-mixing.

Definition 3.1 ([8]). *Let $\{X_t : t \in \mathbb{Z}\}$ be a real-valued time series. The α-mixing sequence $\alpha(\cdot)$ which corresponds to the time series $\{X_t : t \in \mathbb{Z}\}$ is defined as*

$$\alpha(s) = \sup_{t \in \mathbb{Z}} \sup_{\substack{A \in \mathcal{F}_X(-\infty, t) \\ B \in \mathcal{F}_X(t+s, \infty)}} |P(A \cap B) - P(A)P(B)|,$$

where $\mathcal{F}_X(t_1, t_2)$ stands for the σ-algebra generated by $\{X(t) : t_1 \leq t \leq t_2\}$. The time series $\{X_t : t \in \mathbb{Z}\}$ is called α-mixing if $\alpha(s) \to 0$ as $s \to \infty$.

For any random variable X and positive constant p, we define the norm $\|X\|_p = (E|X|^p)^{1/p}$.

Theorem 3.1. *Assume that* (A1)–(A5) *and* (B) *hold. If, additionally, there exist $\delta > 0$, $\Delta < \infty$ and $K < \infty$ such that*:

(i) *$\sup_{t \in \mathbb{Z}} \|X_t\|_{6+3\delta} < \Delta$;*

(ii) *$\sum_{k=0}^{\infty} (k+1)^2 \alpha(k)^{\delta/(2+\delta)} \leq K$,*

then for any $(\nu_1, \omega_1), (\nu_2, \omega_2) \in (0, 2\pi]^2$, we have the convergence

$$\lim_{n \to \infty} \frac{d_n}{L_{d_n}} \operatorname{cov}(\hat{G}_n^{c,d}(\nu_1, \omega_1), \hat{G}_n^{c,d}(\nu_2, \omega_2))$$

$$= \rho\big(P(\nu_1, \nu_2)\overline{P(\omega_1, \omega_2)} + P(\nu_1, 2\pi - \omega_2)\overline{P(\nu_2, 2\pi - \omega_1)}\big),$$

where $\rho = \int_{-1}^{1} w^2(x)\,dx$.

In Theorem 3.2, we show that the estimator (11) has an asymptotically normal distribution. This result is crucial for establishing the asymptotic distribution for the magnitude of coherence statistic and proving the consistency of subsampling applied for magnitude of coherence and spectral density.

Theorem 3.2. *Assume that* (A1)–(A5) *and* (B) *hold. Additionally, assume that*:

(i) *there exists $\delta > 0$ such that $\sup_{t \in \mathbb{Z}} \|X_t\|_{6+3\delta} \leq \Delta < \infty$;*

(ii) *$L_n = O(n^\kappa)$ for some $\kappa \in (0, \delta/(4+4\delta))$;*

(iii) $\sum_{k=0}^{\infty}(k+1)^r \alpha(k)^{\delta/(r+2+\delta)} < K < \infty$, *where r is the even integer such that*

$$r > \max\left\{1 + 3\delta/2, \frac{1-\kappa}{2\kappa}, \frac{2\kappa(1+\delta)}{\delta - 2\kappa(1+\delta)}\right\}.$$

Then

$$\sqrt{\frac{d_n}{L_{d_n}}}\left(\begin{bmatrix} \operatorname{Re}(\hat{G}_n^{c,d}(v,\omega)) \\ \operatorname{Im}(\hat{G}_n^{c,d}(v,\omega)) \end{bmatrix} - \begin{bmatrix} \operatorname{Re}(P(v,\omega)) \\ \operatorname{Im}(P(v,\omega)) \end{bmatrix}\right) \xrightarrow{d} \mathcal{N}_2(0, \Sigma(v,\omega)), \tag{14}$$

where $\Sigma(v,\omega) = [\sigma_{ij}]_{i,j=1,2}$,

$$\sigma_{11} = \tfrac{1}{2}\big(g_0(v)g_0(\omega) + |P(v, 2\pi - \omega)|^2 + \operatorname{Re}\big(P(v, 2\pi - v)P(2\pi - \omega, \omega)\big)$$
$$+ [\operatorname{Re}(P(v,\omega))]^2 - [\operatorname{Im}(P(v,\omega))]^2\big),$$

$$\sigma_{12} = \sigma_{21} = -[\operatorname{Re}(P(v,\omega))]^2[\operatorname{Im}(P(v,\omega))]^2 - \tfrac{1}{2}\operatorname{Im}\big(P(v, 2\pi - v)P(2\pi - \omega, \omega)\big), \tag{15}$$

$$\sigma_{22} = \tfrac{1}{2}\big(g_0(v)g_0(\omega) + |P(v, 2\pi - \omega)|^2 - \operatorname{Re}\big(P(v, 2\pi - v)P(2\pi - \omega, \omega)\big)$$
$$- [\operatorname{Re}(P(v,\omega))]^2 + [\operatorname{Im}(P(v,\omega))]^2\big).$$

Recall that in the stationary case, for the majority of spectral windows, the optimal L_n (in the mean square sense) for the estimator $\hat{G}_n(v,\omega)$ is of order $n^{1/5}$ (see [27], pages 462–463, or [26], page 86). Note that condition (ii) of Theorem 3.2 includes the case $L_n = O(n^{1/5})$ for $\delta > 4$.

The following corollary is a natural generalization of the previous theorem to the multidimensional case. We need this result to establish the asymptotic distribution for the magnitude of coherence statistic.

Corollary 3.1. *Let* $\{X_t : t \in \mathbb{Z}\}$ *be a time series such that all the assumptions of Theorem* 3.2 *hold. We then have the convergence*

$$\sqrt{\frac{d_n}{L_{d_n}}}\left(\begin{bmatrix} \operatorname{Re}(\hat{G}_n^{c,d}(v,\omega)) \\ \operatorname{Re}(\hat{G}_n^{c,d}(v,v)) \\ \operatorname{Re}(\hat{G}_n^{c,d}(\omega,\omega)) \\ \operatorname{Im}(\hat{G}_n^{c,d}(v,\omega)) \end{bmatrix} - \begin{bmatrix} \operatorname{Re}(P(v,\omega)) \\ g_0(v) \\ g_0(\omega) \\ \operatorname{Im}(P(v,\omega)) \end{bmatrix}\right) \xrightarrow{d} \mathcal{N}_4(0, \Psi(v,\omega)), \tag{16}$$

where the covariance matrix $\Psi(v,\omega)$ *can be obtained by Theorem* 3.1.

Remark 3.2. Note that in the special case where $v = \omega$, we have $\operatorname{rank}(\Sigma(v,v)) = 1$ and $\operatorname{rank}(\Psi(v,v)) = 3$, which follow from the fact that $\operatorname{Im}(\hat{G}_n^{c,d}(v,v)) = 0$.

Let us establish the asymptotic distribution of the normalized estimator $|\hat{G}_n^{c,d}(v,\omega)|$ (under all the assumptions of Theorem 3.2). Note that if $P(v,\omega) = 0$, then by Theorem 3.2 and the continuous mapping theorem, we get

$$\sqrt{\frac{d_n}{L_{d_n}}}|\hat{G}_n^{c,d}(v,\omega)| \xrightarrow{d} \mathcal{L}(Z), \tag{17}$$

where $Z := \sqrt{S_1^2 + S_2^2}$ and the random vector (S_1, S_2) has a two-dimensional normal distribution with mean zero and a covariance matrix equal to $\Sigma(\nu, \omega)$. If we assume that trace$(\Sigma(\nu, \omega)) > 0$, then the distribution of a random variable Z is continuous. Now, assume that $P(\nu, \omega) \neq 0$. Applying the delta method for the convergence (14) and the function $f(x, y) = \sqrt{x^2 + y^2}$ that is differentiable at the point $(x_0, y_0) = (\text{Re}(P(\nu, \omega)), \text{Im}(P(\nu, \omega)))$, we get

$$\sqrt{\frac{d_n}{L_{d_n}}}\left(|\hat{G}_n^{c,d}(\nu, \omega)| - |P(\nu, \omega)|\right) \xrightarrow{d} \mathcal{N}_1(0, D_1 \Sigma(\nu, \omega) D_1^T), \tag{18}$$

where

$$D_1 = \frac{1}{|P(\nu, \omega)|}(\text{Re}(P(\nu, \omega)), \text{Im}(P(\nu, \omega))) \tag{19}$$

and M^T is the transpose of the matrix M.

Now, let us establish the asymptotic distribution of a normalized statistic $|\hat{\gamma}_n^{c,d}(\nu, \omega)|$ for $\nu \neq \omega$ (in the case $\nu = \omega$, we have $|\hat{\gamma}_n^{c,d}(\nu, \omega)| = 1$). To do this, we assume that $g_0(\nu)g_0(\omega) \neq 0$. In the first case, take $(\nu, \omega) \in (0, 2\pi]^2$ such that $P(\nu, \omega) = 0$. By Theorem 3.2, the continuous mapping theorem, consistency of the estimators $\hat{G}_n^{c,d}(\nu, \nu)$ and $\hat{G}_n^{c,d}(\omega, \omega)$ and Slutsky's lemma, we then have

$$\sqrt{\frac{d_n}{L_{d_n}}}|\gamma_n^{c,d}(\nu, \omega)| = \sqrt{\frac{d_n}{L_{d_n}}}\frac{|\hat{G}_n^{c,d}(\nu, \omega)|}{\sqrt{|\text{Re}(\hat{G}_n^{c,d}(\nu, \nu))\text{Re}(\hat{G}_n^{c,d}(\omega, \omega))|}}$$
$$\xrightarrow{d} \mathcal{L}\left(\frac{Z}{\sqrt{g_0(\nu)g_0(\omega)}}\right). \tag{20}$$

If the trace$(\Sigma(\nu, \omega)) > 0$, then the limiting distribution in (20) is continuous. As a final consideration, assume that $P(\nu, \omega) \neq 0$. In this case, we use the delta method to show the asymptotic normality of the normalized estimator $|\hat{\gamma}_n^{c,d}(\nu, \omega)|$. In doing this, we use convergence (16) and the function $f(x, y, z, t) = \frac{\sqrt{x^2 + t^2}}{\sqrt{yz}}$ that is differentiable at a point $(x_0, y_0, z_0, t_0) = (\text{Re}(P(\nu, \omega)), g_0(\nu), g_0(\omega), \text{Im}(P(\nu, \omega)))$. Applying the delta method, we conclude that

$$\sqrt{\frac{d_n}{L_{d_n}}}\left(|\hat{\gamma}_n^{c,d}(\nu, \omega)| - |\gamma(\nu, \omega)|\right) \xrightarrow{d} \mathcal{N}_1(0, D_2 \Psi(\nu, \omega) D_2^T),$$

where

$$D_2 = \frac{|P(\nu, \omega)|}{\sqrt{g_0(\nu)g_0(\omega)}}\left(\frac{\text{Re}(P(\nu, \omega))}{|P(\nu, \omega)|^2}, -\frac{1}{2g_0(\nu)}, -\frac{1}{2g_0(\omega)}, \frac{\text{Im}(P(\nu, \omega))}{|P(\nu, \omega)|^2}\right). \tag{21}$$

Remark 3.3. By elementary calculation, the assumption that trace$(\Sigma(\nu, \omega)) > 0$ is equivalent to $g_0(\nu)g_0(\omega) + |P(\nu, 2\pi - \omega)|^2 > 0$, which is true if we assume that $g_0(\xi) > 0$ for any $\xi \in (0, 2\pi]$. Moreover, $D_1 \Sigma(\nu, \omega) D_1^T > 0$ if we assume that $\det(\Sigma(\nu, \omega)) \neq 0$, and $D_2 \Psi(\nu, \omega) D_2^T > 0$ if we assume that $\det(\tilde{\Psi}(\nu, \omega)) \neq 0$.

In the corollaries which follow, we summarize considerations concerning the asymptotic distributions of normalized estimators $|\hat{G}_n^{c,d}(v, \omega)|$ and $|\hat{\gamma}_n^{c,d}(v, \omega)|$.

Corollary 3.2. *Let $\{X_t : t \in \mathbb{Z}\}$ be a time series such that all the assumptions of Theorem* 3.2 *hold and that $g_0(\xi) \neq 0$ for any $\xi \in (0, 2\pi)$. Take any $(v, \omega) \in (0, 2\pi]^2$. For the case where $P(v, \omega) \neq 0$, we require that $\det(\Sigma(v, \omega)) \neq 0$, where matrix $\Sigma(v, \omega)$ is given by Theorem* 3.2. *Under these assumptions, we have the convergence*

$$
\sqrt{\frac{d_n}{L_{d_n}}} \big(|\hat{G}_n^{c,d}(v, \omega)| - |P(v, \omega)| \big)
$$

$$
\xrightarrow{d} J^{P(v,\omega)} := \begin{cases} \mathcal{L}(Z) & \text{if } P(v, \omega) = 0, \\ \mathcal{N}_1(0, D_1 \Sigma(v, \omega) D_1^T) & \text{if } P(v, \omega) \neq 0, \end{cases}
\tag{22}
$$

where $\mathcal{L}(Z)$ and D_1 are given by (17) *and* (19), *respectively. Moreover, the distribution $J^{P(v,\omega)}$ (J^P for short) is continuous.*

Corollary 3.3. *Suppose that all the assumptions of Theorem* 3.2 *hold and that $g_0(\xi) \neq 0$ for any $\xi \in (0, 2\pi]$. Take any $(v, \omega) \in (0, 2\pi]^2$ such that $v \neq \omega$. For the case where $P(v, \omega) \neq 0$, we require that $\det(\Psi(v, \omega)) \neq 0$, where the matrix $\Psi(v, \omega)$ is given by* (16). *Then*

$$
\sqrt{\frac{d_n}{L_{d_n}}} \big(|\hat{\gamma}_n^{c,d}(v, \omega)| - |\gamma(v, \omega)| \big)
$$

$$
\xrightarrow{d} J^{\gamma(v,\omega)} := \begin{cases} \mathcal{L}(Z/\sqrt{g_0(v)g_0(\omega)}) & \text{if } P(v, \omega) = 0, \\ \mathcal{N}_1(0, D_2 \Psi(v, \omega) D_2^T) & \text{if } P(v, \omega) \neq 0, \end{cases}
\tag{23}
$$

where $\mathcal{L}(Z/\sqrt{g_0(v)g_0(\omega)})$ and D_2 are given by (20) *and* (21), *respectively. Moreover, the distribution $J^{\gamma(v,\omega)}$ (J^γ for short) is continuous.*

Note that we cannot use the asymptotic distributions from Corollaries 3.2 and 3.3 in practice if we do not know that $P(v, \omega) = 0$. Even if we know the period T for PC time series, the value of $|P(v, v - \lambda)|$ for $\lambda \in \{2k\pi/T : k = 0, 1, \ldots, T - 1\}$ can be equal to zero. For example, consider the model (29) with period $T = 12$. After calculations (we omit overly long formulae), $g_0(v) = (235 + 72\cos(v))/(16\pi)$, $g_{\pi/6}(v) \not\equiv 0$, $g_{\pi/3}(v) \not\equiv 0$, $g_{\pi/2}(v) = \frac{1}{2\pi}$, $g_{2\pi/3}(v) = \frac{1}{32\pi}$ and, crucially for this example, $g_{5\pi/6}(v) = 0$, $g_\pi(v) = 0$ for $v \in (0, 2\pi]$. Therefore, we refer to the subsampling methodology, which does not require information about $P(v, \omega)$.

4. Subsampling procedure and consistency

Let us introduce the subsampling procedure. Some of the notation is adopted from [26]. By $L_{n,b}^P$ and $L_{n,b}^\gamma$, we denote the subsampling estimators of distribution functions of $\sqrt{n/L_n}(|\hat{G}_n(v, \omega)| - $

$|P(v, \omega)|)$ and $\sqrt{n/L_n}(|\hat{\gamma}_n(v, \omega)| - |\gamma(v, \omega)|)$, respectively. For any point $x \in \mathbb{R}$, we define those estimators as

$$L_{n,b}^{P}(x) = \frac{1}{n-b+1} \sum_{t=1}^{n-b+1} \mathbf{1}\{\sqrt{b/L_b}\big(|\hat{G}_n^{t-1,b}(v, \omega)| - |\hat{G}_n(v, \omega)|\big) \leq x\}, \qquad (24)$$

$$L_{n,b}^{\gamma}(x) = \frac{1}{n-b+1} \sum_{t=1}^{n-b+1} \mathbf{1}\{\sqrt{b/L_b}\big(|\hat{\gamma}_n^{t-1,b}(v, \omega)| - |\hat{\gamma}_n(v, \omega)|\big) \leq x\}, \qquad (25)$$

where

$$\hat{G}_n^{t-1,b}(v, \omega) = \frac{1}{2\pi b} \sum_{j_1=t}^{t+b-1} \sum_{j_2=t}^{t+b-1} H_{L_b}(j_1 - j_2) X_{j_1} X_{j_2} e^{-ivj_1} e^{i\omega j_2},$$

$$|\hat{\gamma}_n^{t-1,b}(v, \omega)| = \frac{|\hat{G}_n^{t-1,b}(v, \omega)|}{\sqrt{|\operatorname{Re}(\hat{G}_n^{t-1,b}(v, v)) \operatorname{Re}(\hat{G}_n^{t-1,b}(\omega, \omega))|}}$$

and $\mathbf{1}\{B\}$ is an indicator function of the event B. Denote by $J^P(x)$ and $J^{\gamma}(x)$ the distribution functions of J^P and J^{γ}, respectively. For any $\alpha \in (0, 1)$ let $c_{n,b}^P(1 - \alpha)$ and $c_{n,b}^{\gamma}(1 - \alpha)$ be quantiles of the nominal level $1 - \alpha$ from subsampling distributions (24) and (25), respectively. The following theorems concerning consistency of subsampling then hold.

Theorem 4.1. *Under all assumptions of Corollary* 3.2:

(i) $L_{n,b}^{P}(x) \xrightarrow{P} J^P(x)$ *for any* $x \in \mathbb{R}$;

(ii) $\sup_{x \in \mathbb{R}} |L_{n,b}^{P}(x) - J^P(x)| \xrightarrow{P} 0$;

(iii) *the subsampling confidence intervals for* $|P(v, \omega)|$ *are consistent, which means that*

$$P\big(\sqrt{n/L_n}\big(|\hat{G}_n(v, \omega)| - |P(v, \omega)|\big) \leq c_{n,b}^P(1 - \alpha)\big) \longrightarrow 1 - \alpha, \qquad (26)$$

where $b = b(n) \to \infty$ *and* $b/n \to 0$.

Theorem 4.2. *Under all the assumptions of Corollary* 3.3:

(i) $L_{n,b}^{\gamma}(x) \xrightarrow{P} J^{\gamma}(x)$ *for any* $x \in \mathbb{R}$;

(ii) $\sup_{x \in \mathbb{R}} |L_{n,b}^{\gamma}(x) - J^{\gamma}(x)| \xrightarrow{P} 0$;

(iii) *the subsampling confidence intervals for* $|\gamma(v, \omega)|$ *are consistent, which means that*

$$P\big(\sqrt{n/L_n}\big(|\hat{\gamma}_n(v, \omega)| - |\gamma(v, \omega)|\big) \leq c_{n,b}^{\gamma}(1 - \alpha)\big) \longrightarrow 1 - \alpha, \qquad (27)$$

where $b = b(n) \to \infty$ *and* $b/n \to 0$.

5. Simulation study

In this section, we show possible applications of the results from Sections 3 and 4. In Section 5.1, we present graphical methods for detecting the presence of periodic autocorrelation. We consider simulated data sets. In Section 5.2, we calculate confidence intervals for the magnitude of a spectral density using subsampling and asymptotic distributions.

5.1. Graphical methods for detecting periodicity

Take any point (ν, ω) from the bifrequency square $(0, 2\pi]^2$ such that $\nu \neq \omega$. Let us consider the null hypothesis $H_0 : |P(\nu, \omega)| = 0$ and the alternative hypothesis $H_1 : |P(\nu, \omega)| \neq 0$. If we assume that $g_0(\nu)g_0(\omega) \neq 0$, then the above hypotheses H_0 and H_1 are equivalent to $H_0 : |\gamma(\nu, \omega)| = 0$ and $H_1 : |\gamma(\nu, \omega)| \neq 0$, respectively. Under the assumptions of Theorems 4.1 and 4.2, we can use the statistics $\Upsilon_n^P(\nu, \omega) = \sqrt{n/L_n}|\hat{G}_n(\nu, \omega)|$ and $\Upsilon_n^\gamma(\nu, \omega) = \sqrt{n/L_n}|\hat{\gamma}_n(\nu, \omega)|$, and critical values from the subsampling distributions (24) and (25) for the above testing problems. Under H_0, the rejection probability tends to α and under H_1, this probability tends to one, which means that the both tests are asymptotically consistent.
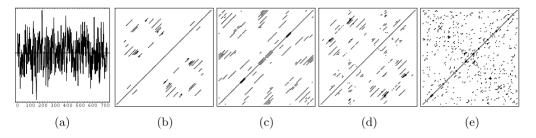
Additionally, by Corollary 3.2, we use the statistics

$$\tilde{\Upsilon}_n^P(\nu, \omega) = nL_n^{-1}\left(\left(\mathrm{Re}(\hat{G}_n(\nu, \omega))/\hat{\sigma}_n^R(\nu, \omega)\right)^2 + \left(\mathrm{Im}(\hat{G}_n(\nu, \omega))/\hat{\sigma}_n^I(\nu, \omega)\right)^2\right)$$

for the above testing problem, where $\hat{\sigma}_n^R(\nu, \omega)$ and $\hat{\sigma}_n^I(\nu, \omega)$ are estimators of the asymptotic variances of $\sqrt{n/L_n}\,\mathrm{Re}(\hat{G}_n(\nu, \omega))$ and $\sqrt{n/L_n}\,\mathrm{Im}(\hat{G}_n(\nu, \omega))$, respectively (see Theorem 3.1), obtained by replacing the unknown spectral densities by their estimates (see formula (8)). Under hypothesis H_0, the matrix $\Sigma(\nu, \omega)$ from Theorem 3.2 has non-zero elements only on the main diagonal. Therefore, under H_0, the test statistic $\tilde{\Upsilon}_n^P(\nu, \omega)$ has, asymptotically, a chi-square distribution with two degrees of freedom.

For different points (ν, ω) from the bifrequency square $(0, 2\pi]^2$, we may consider the above testing problems. If we reject hypothesis H_0 for many points for which $\omega = \nu - \lambda$, where λ is some constant different from zero, then it can be suspected that the time series $\{X_t : t \in \mathbb{Z}\}$ is not stationary (see [16], Figure 1).

We calculate values of the statistic $\Upsilon_n^P(\nu, \omega)$ for data for each point from the set $\{(\nu_s, \omega_t) = (2\pi s/120, 2\pi t/120), 1 \leq s \leq 120, 1 \leq t \leq 120, s \neq t\} \subset (0, 2\pi]^2$ and we compare these values with the critical values $c_{n,b}^P(1 - \alpha)$. If the value of $\Upsilon_n^P(\nu, \omega)$ exceeds $c_{n,b}^P(1 - \alpha)$, then we reject hypothesis H_0 and put a black mark at the point (ν, ω) on the bifrequency square. The same steps are repeated using statistics $\Upsilon_n^\gamma(\nu, \omega)$, $\tilde{\Upsilon}_n^P(\nu, \omega)$ and appropriate critical values from subsampling and chi-square distributions. We fix $n = 720$, $L_n = [n^{1/5}] = 4$, $b = [3\sqrt{n}] = 80$, $L_b = [b^{1/5}] = 2$ and $\alpha = 99\%$. We put $H_{L_n}(\tau) = \mathbf{1}\{|\tau| \leq L_n\}$, where $\mathbf{1}\{B\}$ is an indicator function of the event B. We compare our methods with the method presented in [16] and [17]. We put $M = 25$ into [16], formula (11). Note that using multiple testing procedures for various points seems to be difficult for graphical methods presented in Figures 1(b)–(e) and 2(b)–(e) because the number of different tests on each figure is 14 400 and this is bigger than the length of the sample, which is $n = 720$. If we use multiple testing procedures and we calculate the quantiles

$$(a) \qquad (b) \qquad (c) \qquad (d) \qquad (e)$$

Figure 1. Graphical methods for detecting the presence of periodicity of covariance for samples from the MA(2) model: $X_t = 2\epsilon_{t-2} + \epsilon_{t-1} + \epsilon_t$, with Normal(0, 1)-distributed innovations: (a) time series; (b) tests for $|P(\nu_s, \omega_t)|$ based on the statistic $\Upsilon_n^\gamma(\nu_s, \omega_s)$ and subsampling; (c) tests for $|P(\nu_s, \omega_t)|$ based on the statistic $\tilde{\Upsilon}_n^P(\nu_s, \omega_t)$ and central chi-square distribution with two degrees of freedom; (d) tests for $|\gamma(\nu_s, \omega_t)|$ based on the statistic $\Upsilon_n^\gamma(\nu_s, \omega_t)$ and subsampling; (e) tests for $|\gamma(\nu_s, \omega_t)|$ based on the statistic presented in [16], formula (11), page 342, and the beta distribution.

from the subsampling distribution, then the justified significant level should not be too small in comparison with the sample size. Some intuitive adjustment for the significant level in the graphical method presented in Figures 1(e) and 2(e) for the PC case was made in [17]. The theoretical results which can help in multiple testing procedures in connection with subsampling approximation can be found in [1], where the question of how well the subsampling distribution approximates the tail of an unknown distribution that we approximate was considered.

Figure 1 presents the stationary case. All methods for this case fail to show the evidence of periodic correlation. Figure 2 presents methods for the sample from the periodic moving average model of order one with period equal to $T = 4$. All methods for this case clearly reveal the presence of periodic correlation with appropriate length of period $T = 4$. In summary, all methods can successfully distinguish the stationary case from the PC case. Moreover, there is no clear differ-

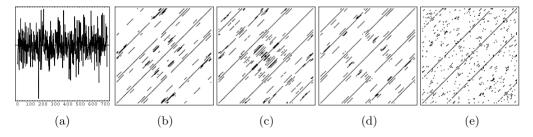$$(a) \qquad (b) \qquad (c) \qquad (d) \qquad (e)$$

Figure 2. Graphical methods for detecting the presence of periodicity of covariance for samples from the PMA(1) model: $X_t = (2 + \sin(2\pi t/4))^2 \epsilon_{t-1} + \epsilon_t$, with period $T = 4$ and Normal(0,1)-distributed innovations ϵ_t: (a) time series; (b) tests for $|P(\nu_s, \omega_t)|$ based on the statistic $\Upsilon_n^\gamma(\nu_s, \omega_s)$ and subsampling; (c) tests for $|P(\nu_s, \omega_t)|$ based on the statistic $\tilde{\Upsilon}_n^P(\nu_s, \omega_t)$ and central chi-square distribution with two degrees of freedom; (d) tests for $|\gamma(\nu_s, \omega_t)|$ based on the statistic $\Upsilon_n^\gamma(\nu_s, \omega_t)$ and subsampling; (e) tests for $|\gamma(\nu_s, \omega_t)|$ based on the statistic presented in [16], formula (11), page 342, and the beta distribution.

ence in these examples between the efficiency of the above graphical methods based on subsampling, the central chi-square distribution with two degrees of freedom and the beta distribution.

We chose the parameters L_n and b to satisfy the assumptions in Theorems 4.1 and 4.2. Note that in [26], pages 190–191, the problem of choosing b in practice in special cases was considered. It was found that the choice of b is, in practice, a delicate issue, like the choice of the bandwidth L_n in nonparametric spectral density estimation. It was also shown in [19] in simulations that the type I error and power of the test based on subsampling approximation for significance of the value $|a(\lambda, \tau)|$ strongly depends on the choice of the parameter b. It is likely that a similar problem occurs in the graphical methods presented in this work, where the choice of the parameters L_n and b is a open problem.

5.2. Confidence intervals

From Theorem 4.1, we can obtain a two-sided equal-tailed consistent confidence interval for the parameter $|P(\nu, \omega)|$ in the form

$$\left(|\hat{G}_n(\nu, \omega)| - c_{n,b}^P\left(1 - \frac{\alpha}{2}\right) \Big/ \sqrt{n/L_n}, \ |\hat{G}_n(\nu, \omega)| - c_{n,b}^P\left(\frac{\alpha}{2}\right) \Big/ \sqrt{n/L_n} \right), \qquad (28)$$

where $\alpha \in (0, 1)$ is a nominal level. A similar confidence interval can be obtained for coherence using Theorem 4.2. Consider the periodic moving average model of order one with period equal to T,

$$X_t = \theta(t - 1)\epsilon_{t-1} + \epsilon_t, \qquad (29)$$

where $\theta(t) = (2 + \sin(2\pi t/T))^2$ and ϵ_t is a Gaussian white noise with mean zero and variance equal to one. The aim of this section is to calculate confidence intervals for $|g_\lambda(\nu)|$ for time series of the form (29).

Figure 3 presents 95% confidence intervals (28) for $|P(\nu, \nu - \lambda)|$, $T = 4$, $\lambda \in \Lambda = \{0, \pi/2, \pi, \frac{3}{2}\pi\}$ and for $\nu \in \{2k\pi/120 : k = 1, 2, \ldots, 120\}$. We put $L_n = [n^{1/5}]$, $b = [3\sqrt{n}]$ and $n = 500$. We also present 95% confidence intervals based on the asymptotic distribution J^P (see the formula in Corollary 3.2) by replacing unknown values of spectral densities in the asymptotic distribution J^P by their estimates using (8). The confidence interval for $g_\lambda(\nu)$ based on the asymptotic distribution J^P has spikes on the interval $(0, 2\pi]$ being considered. It follows from the fact that asymptotic variance in the limiting distribution J^P depends (for fixed λ) on a value $P(-\nu, \nu - \lambda)$ that is non-zero for the considered $\nu \in \{2k\pi/120 : k = 1, 2, \ldots, 120\}$ only for the points $\nu \in \{\pi \pm x/2 : x \in \Lambda\}$. This can also be observed for the stationary case, where the asymptotic variance of the normalized spectral density estimator on the main diagonal, expressed as a function of ν, has a discontinuity at points $\nu_0 = \pi$ and $\nu_0 = 2\pi$ (see [27] and [31]). Note, that the distribution J^P can be used to construct a confidence interval for $|P(\nu, \omega)|$ under additional information concerning parameter $P(\nu, \omega)$. Therefore, subsampling seems to be more useful in practice when there is no rule to obtain the parameter b in our testing problem.
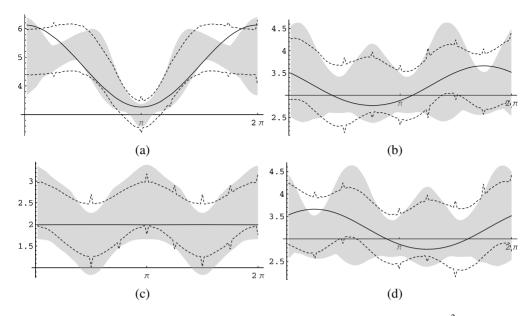
(a)

(b)

(c)

(d)

Figure 3. Confidence internals for $|P(\nu, \nu - \lambda)|$ for $\lambda \in \Lambda = \{0, \pi/2, \pi, \frac{3}{2}\pi\}$ and for $\nu \in \{2k\pi/120 : k = 1, 2, \ldots, 120\}$. Solid line: theoretical $|P(\nu, \nu - \lambda)|$; dashed line: 95% asymptotic confidence interval (based on distribution J^P from Corollary 3.3) for sample size $n = 500$; light shading: 95% subsampling confidence interval (28). (a) The case $\lambda = 0$, $g_0(\nu) = |P(\nu, \nu)| = (59 + 18\cos(\nu))/4\pi$; (b) the case $\lambda = \frac{\pi}{2}$, $|g_{\pi/2}(\nu)| = |P(\nu, \nu - \pi/2)| = \sqrt{2}(\sqrt{51 + 10\cos(\nu) - 10\sin(\nu) - \sin(2\nu)})/\pi$; (c) the case $\lambda = \pi$, $|g_\pi(\nu)| = |P(\nu, \nu - \pi)| = \sqrt{625 + 4\sin^2(\nu)}/4\pi$; (d) the case $\lambda = \frac{3}{2}\pi$, $|g_{(3/2)\pi}(\nu)| = |P(\nu, \nu - \frac{3}{2}\pi)| = \sqrt{2}\sqrt{51 + 10\cos(\nu) + 10\sin(\nu) + \sin(2\nu)}/\pi$.

6. Conclusions and open problems

In this work, we give the exact forms of the asymptotic distributions for normalized spectral density and magnitude of coherence statistics and we prove the consistency of subsampling applied for magnitude of spectral density and coherence. This research was done for α-mixing zero-mean APC time series with the support of the spectral measure concentrated on a countable set of parallel lines. By the consistency of subsampling, we construct asymptotically consistent confidence intervals for magnitude of spectral density and coherence. Importantly, this confidence intervals does not depend on the support of the spectral measure, unlike confidence intervals based on asymptotic distributions. Graphical methods for detecting the presence of periodic autocovariance were presented and their theoretical properties explored. One of the main problems is the choice of the parameter b in graphical methods, as presented in Section 5.

The next problem connected with applications is the estimation of spectral density and coherence for non-zero-mean time series. Note that the well-known differencing operation, popular in time series analysis, is useful when the mean function is a constant or a periodic function with known period. Generally, if we assume that the mean function is almost periodic, then we can-

not use the differencing operation, even if we known the set Γ (see Remark 3.1), in the Fourier representation of the mean function. To cope with this problem, we can consider spectral density estimators for non-zero-mean time series of the form (13). This, and other open problems, are currently being researched by the author.

Appendix

To begin, we formulate a few auxiliary lemmas which are crucial for proving the theorems.

Lemma A.1. *Let $\{X_t : t \in \mathbb{Z}\}$ be real-valued time series with corresponding α-mixing sequence $\alpha(\cdot)$. Assume that there exist real numbers $\delta > 0$ and $\Delta < \infty$ such that $\sup_{t \in \mathbb{Z}} \|X_t\|_{2+\delta} \leq \Delta$. For any $\lambda \in [0, 2\pi)$, $j \in \mathbb{Z}$ and $\tau \in \mathbb{Z}$, we then have the following estimates*:

  (i) $|B(j, \tau)| \leq 8\Delta^2 \alpha^{\delta/(2+\delta)}(|\tau|)$;
  (ii) $|a(\lambda, \tau)| \leq 8\Delta^2 \alpha^{\delta/(2+\delta)}(|\tau|)$.

Proof. The proof of (i) is elementary and is based on [8], Theorem 3, page 9. The proof of (ii) follows from (i) by noting that $a(\lambda, \tau) = \lim_{n \to \infty} n^{-1} \sum_{t=1}^{n} B(t, \tau) e^{-i\lambda t}$ for any $\lambda \in [0, 2\pi)$ and $\tau \in \mathbb{Z}$. $\qquad\square$

Lemma A.2 (Conclusion from [8], Theorem 2, page 26). *Let $\{X_t : t \in \mathbb{Z}\}$ be a zero-mean time series. Assume that there exist real numbers $l > 2$ and $\delta > 0$ and an even integer $r \geq l - 2$ such that $\sup_{t \in \mathbb{Z}} \|X_t\|_{l+\delta} < \infty$ and $\sum_{k=1}^{\infty} k^r \alpha^{\delta/(r+2+\delta)}(k) < \infty$. There then exists a constant $K_{\alpha,l} < \infty$ (which depends only on the sequence $\alpha(\cdot)$ and constant l) such that for any $a \in \mathbb{Z}$ and $g \in \mathbb{N}$, we have the estimate*

$$\left\| \sum_{t=a+1}^{a+g} X_t \right\|_l \leq (K_{\alpha,l} \max\{Q(l, \delta, a, g), [Q(2, \delta, a, g)]^{l/2}\})^{1/l}, \tag{30}$$

where $Q(l, \delta, a, g) = \sum_{t=a+1}^{a+g} \|X_t\|_{l+\delta}^l$.

Lemma A.3 (Multidimensional generalization of the CLT of [10]). *Let $\{Y_{n,t} = (y_{n,t,1}, y_{n,t,2}, \ldots, y_{n,t,d}) \in \mathbb{R}^d : 1 \leq t \leq a_n\}$ be a triangular array of zero-mean real-valued random vectors, where $\{a_n\}_{n \in \mathbb{N}}$ is a sequence of positive integers tending to infinity. Define*

$$S_n = \frac{1}{\sqrt{a_n}} \sum_{j=1}^{a_n} Y_{n,j}.$$

Assume that:

  (i) *there exists a constant $\delta > 0$ such that*

$$\sup\{\|y_{n,t,k}\|_{2+\delta} : n \geq 1, 1 \leq t \leq a_n, 1 \leq k \leq d\} < \Delta < \infty;$$

(ii) $\lim_{n\to\infty} \mathrm{Var}(S_n) = \Sigma$;

(iii) *there exists a sequence of positive integers* $\{h_n\}_{n\in\mathbb{N}}$ *such that* $h_n = \mathrm{O}(a_n^\kappa)$ *for some* $\kappa \in [0, \frac{\delta}{4+4\delta})$, *and a sequence* $\{\varsigma(k)\}_{k\geq 0}$ *such that* $\alpha_n(k) \leq \varsigma(k - h_n)$ *for all* $k \geq h_n$ *and*

$$\sum_{k=1}^{\infty} k^r \varsigma(k)^{\delta/(2+\delta)} < K < \infty$$

for some $r > \frac{2\kappa(1+\delta)}{\delta - 2\kappa(1+\delta)}$, *where* $\alpha_n(\cdot)$ *is a mixing sequence for a triangular array* $\{Y_{n,t} : 1 \leq t \leq a_n\}$ *defined for any* $n \geq 2$ *and* $h = 1, 2, \ldots, a_n - 1$ *as*

$$\alpha_n(h) = \sup_{1\leq t\leq a_n-h} \sup_{\substack{A\in A_{n,t} \\ B\in B_{n,t+h}}} |P(A \cap B) - P(A)P(B)|.$$

$A_{n,t}$ *stands for the* σ-*algebra generated by* $\{Y_{n,u} : 1 \leq u \leq t\}$ *and* $B_{n,t}$ *stands for the* σ-*algebra generated by* $\{Y_{n,u} : t \leq u \leq a_n\}$.

Then $S_n \xrightarrow{d} \mathcal{N}_d(0, \Sigma)$.

Proof. The proof is based on the same steps as that of the CLT of [10], and the Cramér–Wold device. □

Lemma A.4. *Let* $\{X_t : t \in \mathbb{Z}\}$ *be a zero-mean and* α-*mixing time series. Assume that there exists a real number* $\delta > 0$ *such that*:

(i) $\sup_{t\in\mathbb{Z}} \|X_t\|_{6+3\delta} \leq \Delta < \infty$;

(ii) $\sum_{k=0}^{\infty} (k + 1)^2 \alpha(k)^{\delta/(2+\delta)} \leq K < \infty$.

For any sequences $\{c_n\}_{n\in\mathbb{Z}}$ *and* $\{d_n\}_{n\in\mathbb{Z}}$ *of integers where* $d_n > 0$, *we then have*

$$\sum_{s=c_n+1}^{c_n+d_n} \sum_{t=c_n+1}^{c_n+d_n} \sum_{u=c_n+1}^{c_n+d_n} \sum_{v=c_n+1}^{c_n+d_n} |E(X_s X_t X_u X_v) - E(X_s X_t)E(X_u X_v)$$

$$- E(X_s X_u)E(X_t X_v) - E(X_s X_v)E(X_t X_u)| \leq Cd_n,$$

where the constant C *depends only on* K *and* Δ.

Proof. This proof is based on a similar technique as in the proof of [18], Theorem 4.1, and is therefore omitted. □

Lemma A.5. *Suppose that assumptions* (A1)–(A5) *hold. Using the notation from Section 3, for any* $(v, \omega) \in (0, 2\pi]^2$, *we have*

$$\lim_{n\to\infty} E[\hat{P}_n^{c,d}(v, \omega)] = P(v, \omega).$$

Proof. Take any $(v, \omega) \in (0, 2\pi]^2$ and note that $(j = t, \tau = s - t)$

$$
E[\hat{P}_n^{c,d}(v, \omega)] = \frac{1}{2\pi d_n} \sum_{s=c_n+1}^{c_n+d_n} \sum_{t=c_n+1}^{c_n+d_n} E(X_s X_t) e^{-i(v-\omega)t} e^{-iv(s-t)}
$$

$$
= \frac{1}{2\pi d_n} \sum_{j=c_n+1}^{c_n+d_n} \sum_{\tau=c_n+1-j}^{c_n+d_n-j} B(j, \tau) e^{-i(v-\omega)j} e^{-iv\tau}
$$

$$
= \frac{1}{2\pi d_n} \sum_{j=c_n+1}^{c_n+d_n} \sum_{\tau=c_n+1-j}^{c_n+d_n-j} \sum_{\lambda \in \Lambda_\tau} a(\lambda, \tau) e^{i(\lambda-(v-\omega))j} e^{-iv\tau}
$$

$$
= \frac{1}{2\pi d_n} \sum_{j=c_n+1}^{c_n+d_n} \sum_{\tau=c_n+1-j}^{c_n+d_n-j} a(v - \omega, \tau) e^{-iv\tau}
$$

$$
+ \frac{1}{2\pi d_n} \sum_{j=c_n+1}^{c_n+d_n} \sum_{\tau=c_n+1-j}^{c_n+d_n-j} \sum_{\lambda \in \Lambda_\tau \setminus \{v-\omega\}} a(\lambda, \tau) e^{i(\lambda-(v-\omega))j} e^{-iv\tau}
$$

$$
= \frac{1}{2\pi} \sum_{|\tau| < d_n} \left(1 - \frac{|\tau|}{d_n}\right) a(v - \omega, \tau) e^{-iv\tau}
$$

$$
+ \frac{1}{2\pi d_n} \sum_{j=c_n+1}^{c_n+d_n} \sum_{\tau=c_n+1-j}^{c_n+d_n-j} \sum_{\lambda \in \Lambda_\tau \setminus \{v-\omega\}} a(\lambda, \tau) e^{i(\lambda-(v-\omega))j} e^{-iv\tau}.
$$

Denote the first and second terms of the last equality by $\epsilon_{1,n}$ and $\epsilon_{2,n}$, respectively. Note that $\epsilon_{1,n}$ is a Cesáro mean for the sequence $\sum_{|\tau| < d_n} a(v - \omega, \tau) e^{-iv\tau}$. Hence, $\epsilon_{1,n}$ goes to $P(v, \omega)$. By the estimate $|\sum_{j=p}^q e^{-ixj}| \leq |\csc(x/2)|$, where $p \leq q$, $x \not\equiv 0$ modulo 2π, we have

$$
|\epsilon_{2,n}| = \frac{1}{2\pi d_n} \left| \left(\sum_{\tau=-d_n+1}^{0} \sum_{j=c_n+1-\tau}^{c_n+d_n} + \sum_{\tau=1}^{d_n-1} \sum_{j=c_n+1}^{c_n+d_n-\tau} \right) \sum_{\lambda \in \Lambda_\tau \setminus \{v-\omega\}} a(\lambda, \tau) e^{i(\lambda-(v-\omega))j} e^{-iv\tau} \right|
$$

$$
\leq \frac{1}{2\pi d_n} \sum_{\tau=-d_n+1}^{d_n-1} \sum_{\lambda \in \Lambda_\tau \setminus \{v-\omega\}} \left| a(\lambda, \tau) \csc\left((\lambda - (v - \omega))/2\right) \right| \leq \frac{1}{2\pi d_n} \sum_{\tau=-d_n+1}^{d_n-1} z_\tau(v - \omega),
$$

which means that $\epsilon_{2,n} \to 0$ (by (A3)). This completes the proof. $\square$

Lemma A.6. *Suppose that* (A1)–(A5) *and* (B) *hold. Additionally, assume that there exist* $\delta > 0$, $\Delta < \infty$ *and* $K < \infty$ *such that*:

(i) $\sup_{t \in \mathbb{Z}} \|X_t\|_{6+3\delta} < \Delta$;
(ii) $\sum_{k=0}^{\infty} (k + 1)^2 \alpha(k)^{\delta/(2+\delta)} \leq K$.

Using the notation from Section 3, for any $(\nu, \omega) \in (0, 2\pi]^2$ and any $a \in \mathbb{Z}$, we then have

$$\frac{1}{2\pi L_{d_n}} \sum_{\tau_1=-L_{d_n}}^{L_{d_n}} \sum_{\tau_2=-L_{d_n}}^{L_{d_n}} E(X_{\tau_1+a} X_{\tau_2+a}) H_{L_{d_n}}(\tau_1)^2 \mathrm{e}^{-\mathrm{i}\nu(\tau_1+a)} \mathrm{e}^{\mathrm{i}\omega(\tau_2+a)}$$

$$= \int_{-1}^{1} w(x)^2 \, \mathrm{d}x \, P(\nu, \omega) + e_n(\nu, \omega),$$

where $|e_n(\nu, \omega)| \leq \tilde{e}_n(\nu, \omega) \to 0$ and $\tilde{e}_n(\nu, \omega)$ does not depend on a.

Proof. Take any $(\nu, \omega) \in (0, 2\pi]^2$ and note that using the same steps as in the proof of Lemma A.5, we get

$$\frac{1}{2\pi L_{d_n}} \sum_{\tau_1=-L_{d_n}}^{L_{d_n}} \sum_{\tau_2=-L_{d_n}}^{L_{d_n}} E(X_{\tau_1+a} X_{\tau_2+a}) H_{L_{d_n}}(\tau_1)^2 \mathrm{e}^{-\mathrm{i}\nu(\tau_1+a)} \mathrm{e}^{\mathrm{i}\omega(\tau_2+a)}$$

$$= \frac{1}{2\pi L_{d_n}} \sum_{j=a-L_{d_n}}^{a+L_{d_n}} \sum_{\tau=a-L_{d_n}-j}^{a+L_{d_n}-j} H_{L_{d_n}}^2(j-a) a(\nu-\omega, \tau) \mathrm{e}^{-\mathrm{i}\nu\tau}$$

$$+ \frac{1}{2\pi L_{d_n}} \sum_{j=a-L_{d_n}}^{a+L_{d_n}} \sum_{\tau=a-L_{d_n}-j}^{a+L_{d_n}-j} \sum_{\lambda \in \Lambda_\tau \setminus \{\nu-\omega\}} H_{L_{d_n}}^2(j-a) a(\lambda, \tau) \mathrm{e}^{\mathrm{i}(\lambda-(\nu-\omega))j} \mathrm{e}^{-\mathrm{i}\nu\tau}$$

$$= \sum_{|\tau|<2L_{d_n}} \sum_{j=-L_{d_n}}^{L_{d_n}} \frac{w^2(j/L_{d_n})}{2\pi L_{d_n}} a(\nu-\omega, \tau) \mathrm{e}^{-\mathrm{i}\nu\tau} - \sum_{|\tau|<2L_{d_n}} \sum_{j=L_{d_n}-|\tau|}^{L_{d_n}} \frac{w^2(j/L_{d_n})}{2\pi L_{d_n}}$$

$$\times a(\nu-\omega, \tau) \mathrm{e}^{-\mathrm{i}\nu\tau}$$

$$+ \frac{1}{2\pi L_{d_n}} \sum_{\tau=1}^{2L_{d_n}} \sum_{j=a-L_{d_n}}^{a+L_{d_n}-\tau} \sum_{\lambda \in \Lambda_\tau \setminus \{\nu-\omega\}} w^2\left(\frac{j-a}{L_{d_n}}\right) a(\lambda, \tau) \mathrm{e}^{\mathrm{i}(\lambda-(\nu-\omega))j} \mathrm{e}^{-\mathrm{i}\nu\tau}$$

$$+ \frac{1}{2\pi L_{d_n}} \sum_{\tau=-2L_{d_n}}^{0} \sum_{j=a-L_{d_n}-\tau}^{a+L_{d_n}} \sum_{\lambda \in \Lambda_\tau \setminus \{\nu-\omega\}} w^2\left(\frac{j-a}{L_{d_n}}\right) a(\lambda, \tau) \mathrm{e}^{\mathrm{i}(\lambda-(\nu-\omega))j} \mathrm{e}^{-\mathrm{i}\nu\tau}$$

$$= u_0(n) - u_1(n) + u_2(n) + u_3(n),$$

where $u_0(n)$, $u_1(n)$, $u_2(n)$ and $u_3(n)$ are the first, second, third and fourth terms, respectively, of the last equation. It it easy to see that $u_0(n) \to \int_{-1}^{1} w^2(x) \, \mathrm{d}x \, P(\nu, \omega)$. Therefore, to prove the theorem, it is sufficient to show that $u_1(n) \to 0$, $u_2(n) \to 0$ and $u_3(n) \to 0$. By Lemma A.1(ii), for the term $u_1(n)$, we have

$$|u_1(n)| \leq \sum_{|\tau|<2L_{d_n}} \frac{|\tau|+1}{2\pi L_{d_n}} 8\Delta^2 \alpha^{\delta/(2+\delta)}(|\tau|) \leq \frac{32\Delta^2 K}{2\pi L_{d_n}} \to 0.$$

For the term $u_2(n)$, we have

$$|u_2(n)| = \left| \frac{1}{2\pi L_{d_n}} \sum_{\tau=1}^{2L_{d_n}} \sum_{j=-L_{d_n}}^{L_{d_n}-\tau} \sum_{\lambda \in \Lambda_\tau \setminus \{\nu-\omega\}} w^2\left(\frac{j}{L_{d_n}}\right) a(\lambda, \tau) e^{i(\lambda-(\nu-\omega))(j+a)} e^{-i\nu\tau} \right|$$

$$\leq \left| \frac{1}{2\pi L_{d_n}} \sum_{\tau=1}^{L_{d_n}} \sum_{\lambda \in \Lambda_\tau \setminus \{\nu-\omega\}} \left(\sum_{j=-L_{d_n}}^{-1} + \sum_{j=0}^{L_{d_n}-\tau} \right) w^2\left(\frac{j}{L_{d_n}}\right) e^{i(\lambda-(\nu-\omega))(j+a)} a(\lambda, \tau) e^{-i\nu\tau} \right|$$

$$+ \left| \frac{1}{2\pi L_{d_n}} \sum_{\tau=L_{d_n}+1}^{2L_{d_n}} \sum_{\lambda \in \Lambda_\tau \setminus \{\nu-\omega\}} \sum_{j=-L_{d_n}}^{L_{d_n}-\tau} w^2\left(\frac{j}{L_{d_n}}\right) e^{i(\lambda-(\nu-\omega))(j+a)} a(\lambda, \tau) e^{-i\nu\tau} \right|$$

$$\leq \frac{1}{2\pi L_{d_n}} \sum_{\tau=1}^{L_{d_n}} \sum_{\lambda \in \Lambda_\tau \setminus \{\nu-\omega\}} \underbrace{\left| \sum_{j=-L_{d_n}}^{-1} w^2\left(\frac{j}{L_{d_n}}\right) e^{i(\lambda-(\nu-\omega))j} \right|}_{f_1(n)} |a(\lambda, \tau)|$$

$$+ \frac{1}{2\pi L_{d_n}} \sum_{\tau=1}^{L_{d_n}} \sum_{\lambda \in \Lambda_\tau \setminus \{\nu-\omega\}} \underbrace{\left| \sum_{j=0}^{L_{d_n}-\tau} w^2\left(\frac{j}{L_{d_n}}\right) e^{i(\lambda-(\nu-\omega))j} \right|}_{f_2(n)} |a(\lambda, \tau)|$$

$$+ \frac{1}{2\pi L_{d_n}} \sum_{\tau=L_{d_n}+1}^{2L_{d_n}} \sum_{\lambda \in \Lambda_\tau \setminus \{\nu-\omega\}} \underbrace{\left| \sum_{j=-L_{d_n}}^{L_{d_n}-\tau} w^2\left(\frac{j}{L_{d_n}}\right) e^{i(\lambda-(\nu-\omega))j} \right|}_{f_3(n)} |a(\lambda, \tau)|.$$

Now, using the property $|\sum_{j=p}^q c_j e^{ijx}| \leq c_p |\operatorname{cosec}(x/2)|$, which is true for any $x \not\equiv 0$ modulo 2π and $c_p \geq c_{p+1} \geq \cdots \geq c_q$ (see [9], Exercise 1.2, page 10) for the terms $f_1(n)$, $f_2(n)$, $f_3(n)$, we obtain the estimate

$$|u_2(n)| \leq \frac{1}{\pi L_{d_n}} \sum_{\tau=1}^{L_{d_n}} \sum_{\lambda \in \Lambda_\tau \setminus \{\nu-\omega\}} |a(\lambda, \tau)| \left| \operatorname{cosec}\left((\lambda-(\nu-\omega))/2\right) \right|$$

$$+ \frac{1}{2\pi L_{d_n}} \sum_{\tau=L_{d_n}+1}^{2L_{d_n}} \sum_{\lambda \in \Lambda_\tau \setminus \{\nu-\omega\}} |a(\lambda, \tau)| \left| \operatorname{cosec}\left((\lambda-(\nu-\omega))/2\right) \right|$$

$$\leq \frac{1}{\pi L_{d_n}} \sum_{\tau=1}^{2L_{d_n}} z_\tau(\nu-\omega) \to 0.$$

Using the same arguments, we get that $u_3(n) \to 0$. This completes the proof. $\square$

Proof of Theorem 3.1. Using simple decomposition, we have

$$
\frac{(2\pi)^2 d_n}{L_{d_n}} \operatorname{cov}(\hat{G}_n^{c,d}(\nu_1, \omega_1), \hat{G}_n^{c,d}(\nu_2, \omega_2))
$$

$$
= \frac{1}{d_n L_{d_n}} \sum_{s=c_n+1}^{c_n+d_n} \sum_{t=c_n+1}^{c_n+d_n} \sum_{u=c_n+1}^{c_n+d_n} \sum_{v=c_n+1}^{c_n+d_n} \operatorname{cov}(X_s X_t, X_u X_v) H_{L_{d_n}}(s-t) H_{L_{d_n}}(u-v)
$$

$$
\times e^{-i(\nu_1 s - \omega_1 t - \nu_2 u + \omega_2 v)}
$$

$$
= \frac{1}{d_n L_{d_n}} \sum_{s=c_n+1}^{c_n+d_n} \sum_{t=c_n+1}^{c_n+d_n} \sum_{u=c_n+1}^{c_n+d_n} \sum_{v=c_n+1}^{c_n+d_n} \big(\operatorname{cov}(X_s X_t, X_u X_v) - E(X_s X_u) E(X_t X_v)
$$

$$
- E(X_s X_v) E(X_t X_u) \big)
$$

$$
\times H_{L_{d_n}}(s-t) H_{L_{d_n}}(u-v) e^{-i(\nu_1 s - \omega_1 t - \nu_2 u + \omega_2 v)}
$$

$$
+ \frac{1}{d_n L_{d_n}} \sum_{s=c_n+1}^{c_n+d_n} \sum_{t=c_n+1}^{c_n+d_n} \sum_{u=c_n+1}^{c_n+d_n} \sum_{v=c_n+1}^{c_n+d_n} E(X_s X_u) E(X_t X_v) H_{L_{d_n}}(s-t)
$$

$$
\times H_{L_{d_n}}(u-v) e^{-i(\nu_1 s - \omega_1 t - \nu_2 u + \omega_2 v)}
$$

$$
+ \frac{1}{d_n L_{d_n}} \sum_{s=c_n+1}^{c_n+d_n} \sum_{t=c_n+1}^{c_n+d_n} \sum_{u=c_n+1}^{c_n+d_n} \sum_{v=c_n+1}^{c_n+d_n} E(X_s X_v) E(X_t X_u)
$$

$$
\times H_{L_{d_n}}(s-t) H_{L_{d_n}}(u-v) e^{-i(\nu_1 s - \omega_1 t - \nu_2 u + \omega_2 v)}
$$

$$
= s_1(n) + s_2(n) + s_3(n),
$$

where $s_1(n), s_2(n), s_3(n)$ are the first, second and the third term, respectively, of the last equation. To prove the theorem, it is sufficient to show that $s_1(n) \to 0$, $s_2(n) \to (2\pi)^2 P(\nu_1, \nu_2) \overline{P(\omega_1, \omega_2)}$, $s_3(n) \to (2\pi)^2 P(\nu_1, 2\pi - \omega_1) \overline{P(\nu_2, 2\pi - \omega_2)}$ as $n \to \infty$. Note that by the estimate

$$
|s_1(n)| \le \frac{1}{d_n L_{d_n}} \sum_{s=c_n+1}^{c_n+d_n} \sum_{t=c_n+1}^{c_n+d_n} \sum_{u=c_n+1}^{c_n+d_n} \sum_{v=c_n+1}^{c_n+d_n} |\operatorname{cov}(X_s X_t, X_u X_v) - E(X_s X_u) E(X_t X_v)
$$

$$
- E(X_s X_v) E(X_t X_u)|
$$

and Lemma A.4, we get that $s_1(n) \to 0$. Considering the term $s_2(n)$, we get ($t = j_1, s = j_1 + \tau_1, v = j_2, u = j_2 + \tau_2$)

$$
s_2(n) = \frac{1}{d_n L_{d_n}} \sum_{s=c_n+1}^{c_n+d_n} \sum_{t=c_n+1}^{c_n+d_n} \sum_{u=c_n+1}^{c_n+d_n} \sum_{v=c_n+1}^{c_n+d_n} E(X_s X_u) E(X_t X_v)
$$

$$
\times H_{L_{d_n}}(s-t) H_{L_{d_n}}(u-v) e^{-i(\nu_1 s - \omega_1 t - \nu_2 u + \omega_2 v)}
$$

$$= \frac{1}{d_n L_{d_n}} \left(\sum_{j_1=c_n+1}^{c_n+d_n} \sum_{\tau_1=-L_{d_n}}^{L_{d_n}} - \sum_{\tau_1=-L_n}^{-1} \sum_{j_1=c_n+1}^{c_n-\tau_1} - \sum_{\tau_1=1}^{L_n} \sum_{j_1=c_n+d_n-\tau_1+1}^{c_n+d_n} \right)$$

$$\times \left(\sum_{j_2=c_n+1}^{c_n+d_n} \sum_{\tau_2=-L_{d_n}}^{L_{d_n}} - \sum_{\tau_2=-L_n}^{-1} \sum_{j_2=c_n+1}^{c_n-\tau_2} - \sum_{\tau_2=1}^{L_n} \sum_{j_2=c_n+d_n-\tau_2+1}^{c_n+d_n} \right)$$

$$\times E(X_{j_1+\tau_1} X_{j_2+\tau_2}) E(X_{j_1} X_{j_2}) H_{L_{d_n}}(\tau_1) H_{L_{d_n}}(\tau_2) \mathrm{e}^{-\mathrm{i}(\nu_1(j_1+\tau_1)-\omega_1 j_1 - \nu_2(j_2+\tau_2)+\omega_2 j_2)}$$

$$= d_0(n) - d_1(n) - d_2(n) + d_3(n),$$

where

$$d_0(n) = \frac{1}{d_n L_{d_n}} \sum_{j_1=c_n+1}^{c_n+d_n} \sum_{\tau_1=-L_{d_n}}^{L_{d_n}} \sum_{j_2=c_n+1}^{c_n+d_n} \sum_{\tau_2=-L_{d_n}}^{L_{d_n}} \varphi_{j_1 j_2 \tau_1 \tau_2},$$

$$d_1(n) = \frac{1}{d_n L_{d_n}} \sum_{j_1=c_n+1}^{c_n+d_n} \sum_{\tau_1=-L_{d_n}}^{L_{d_n}} \sum_{\tau_2=-L_n}^{-1} \sum_{j_2=c_n+1}^{c_n-\tau_2} \varphi_{j_1 j_2 \tau_1 \tau_2}$$

$$+ \frac{1}{d_n L_{d_n}} \sum_{j_1=c_n+1}^{c_n+d_n} \sum_{\tau_1=-L_{d_n}}^{L_{d_n}} \sum_{\tau_2=1}^{L_n} \sum_{j_2=c_n+d_n-\tau_2+1}^{c_n+d_n} \varphi_{j_1 j_2 \tau_1 \tau_2},$$

$$d_2(n) = \frac{1}{d_n L_{d_n}} \sum_{j_2=c_n+1}^{c_n+d_n} \sum_{\tau_2=-L_{d_n}}^{L_{d_n}} \sum_{\tau_1=-L_n}^{-1} \sum_{j_1=c_n+1}^{c_n-\tau_2} \varphi_{j_1 j_2 \tau_1 \tau_2}$$

$$+ \frac{1}{d_n L_{d_n}} \sum_{j_2=c_n+1}^{c_n+d_n} \sum_{\tau_2=-L_{d_n}}^{L_{d_n}} \sum_{\tau_1=1}^{L_n} \sum_{j_1=c_n+d_n-\tau_1+1}^{c_n+d_n} \varphi_{j_1 j_2 \tau_1 \tau_2},$$

$$d_3(n) = \frac{1}{d_n L_{d_n}} \sum_{\tau_1=-L_n}^{-1} \sum_{j_1=c_n+1}^{c_n-\tau_1} \sum_{\tau_2=-L_n}^{-1} \sum_{j_2=c_n+1}^{c_n-\tau_2} \varphi_{j_1 j_2 \tau_1 \tau_2}$$

$$+ \frac{1}{d_n L_{d_n}} \sum_{\tau_1=-L_n}^{-1} \sum_{j_1=c_n+1}^{c_n-\tau_1} \sum_{\tau_2=1}^{L_n} \sum_{j_2=c_n+d_n-\tau_2+1}^{c_n+d_n} \varphi_{j_1 j_2 \tau_1 \tau_2}$$

$$+ \frac{1}{d_n L_{d_n}} \sum_{\tau_1=1}^{L_n} \sum_{j_1=c_n+d_n-\tau_1+1}^{c_n+d_n} \sum_{\tau_2=-L_n}^{-1} \sum_{j_2=c_n+1}^{c_n-\tau_2} \varphi_{j_1 j_2 \tau_1 \tau_2}$$

$$+ \frac{1}{d_n L_{d_n}} \sum_{\tau_1=1}^{L_n} \sum_{j_1=c_n+d_n-\tau_1+1}^{c_n+d_n} \sum_{\tau_2=1}^{L_n} \sum_{j_2=c_n+d_n-\tau_2+1}^{c_n+d_n} \varphi_{j_1 j_2 \tau_1 \tau_2}$$

and $\varphi_{j_1 j_2 \tau_1 \tau_2} = E(X_{j_1} X_{j_2}) E(X_{j_1+\tau_1} X_{j_2+\tau_2}) H_{L_{d_n}}(\tau_1) H_{L_{d_n}}(\tau_2) e^{-i(\nu_1(j_1+\tau_1)-\omega_1 j_1-\nu_2(j_2+\tau_2)+\omega_2 j_2)}$.
We show that the terms $d_1(n), d_2(n), d_3(n)$ tend to zero, which means that $s_2(n)$ has the same limit as $d_0(n)$. We start with the term $d_1(n)$. By Lemma A.1(i), we have $|\varphi_{j_1 j_2 \tau_1 \tau_2}| \leq 64\Delta^4 \alpha^{\delta/(2+\delta)}(|j_1 - j_2|)\alpha^{\delta/(2+\delta)}(|j_1 - j_2 - (\tau_1 - \tau_2)|)$, which means that

$$
\begin{aligned}
|d_1(n)| \leq{} & \frac{64\Delta^4}{d_n L_{d_n}} \sum_{j_1=c_n+1}^{c_n+d_n} \sum_{\tau_1=-L_{d_n}}^{L_{d_n}} \sum_{\tau_2=-L_n}^{-1} \sum_{j_2=c_n+1}^{c_n-\tau_2} \alpha^{\delta/(2+\delta)}(|j_1 - j_2|) \\
& \qquad\qquad\qquad\qquad\qquad\qquad \times \alpha^{\delta/(2+\delta)}\big(|j_1 - j_2 - (\tau_1 - \tau_2)|\big) \\
& + \frac{64\Delta^4}{d_n L_{d_n}} \sum_{j_1=c_n+1}^{c_n+d_n} \sum_{\tau_1=-L_{d_n}}^{L_{d_n}} \sum_{\tau_2=1}^{L_n} \sum_{j_2=c_n+d_n-\tau_2+1}^{c_n+d_n} \alpha^{\delta/(2+\delta)}(|j_1 - j_2|) \\
& \qquad\qquad\qquad\qquad\qquad\qquad \times \alpha^{\delta/(2+\delta)}\big(|j_1 - j_2 - (\tau_1 - \tau_2)|\big) \\
\leq{} & \frac{64\Delta^4}{d_n L_{d_n}} \Bigg(\sum_{k=1-L_{d_n}}^{d_n-1} L_{d_n} \alpha^{\delta/(2+\delta)}(|k|) \sum_{\tau_1=-L_{d_n}}^{L_{d_n}} \sum_{\tau_2=-L_{d_n}}^{L_{d_n}} \alpha^{\delta/(2+\delta)}\big(|k - (\tau_1 - \tau_2)|\big) \\
& \qquad\qquad + \sum_{k=-d_n+1}^{L_n-1} L_{d_n} \alpha^{\delta/(2+\delta)}(|k|) \sum_{\tau_1=-L_{d_n}}^{L_{d_n}} \sum_{\tau_2=-L_{d_n}}^{L_{d_n}} \alpha^{\delta/(2+\delta)}\big(|k - (\tau_1 - \tau_2)|\big) \Bigg) \\
\leq{} & \frac{64\Delta^4}{d_n L_{d_n}} \Bigg(\sum_{k=1-L_{d_n}}^{d_n-1} L_{d_n} \alpha^{\delta/(2+\delta)}(|k|)(2L_{d_n} + 1)2K \\
& \qquad\qquad + \sum_{k=-d_n+1}^{L_n-1} L_{d_n} \alpha^{\delta/(2+\delta)}(|k|)(2L_{d_n} + 1)2K \Bigg) \\
\leq{} & \frac{64\Delta^4}{d_n L_{d_n}} (2L_{d_n} + 1)^2 8K^2 \to 0 \qquad \text{as } n \to \infty.
\end{aligned}
$$

Using the same steps, we have that the terms $d_2(n)$ and $d_3(n)$ tend to zero. For $d_0(n)$, we get

$$
\begin{aligned}
d_0(n) ={} & \frac{1}{d_n L_{d_n}} \sum_{j_1=c_n+1}^{c_n+d_n} \sum_{j_2=c_n+1}^{c_n+d_n} E(X_{j_1} X_{j_2}) e^{-i(\omega_2 j_2 - \omega_1 j_1)} \\
& \qquad\qquad\qquad \times \sum_{\tau_1=-L_{d_n}}^{L_{d_n}} \sum_{\tau_2=-L_{d_n}}^{L_{d_n}} E(X_{j_1+\tau_1} X_{j_1+\tau_2}) \\
& \qquad\qquad\qquad\qquad \times H_{L_{d_n}}(\tau_1)^2 e^{-i(\nu_1(j_1+\tau_1)-\nu_2(j_1+\tau_2))} \\
& + \frac{1}{d_n L_{d_n}} \sum_{j_1=c_n+1}^{c_n+d_n} \sum_{j_2=c_n+1}^{c_n+d_n} E(X_{j_1} X_{j_2}) e^{-i(\omega_2 j_2 - \omega_1 j_1)}
\end{aligned}
$$

$$\times \sum_{\tau_1=-L_{d_n}}^{L_{d_n}} \sum_{\tau_2=-L_{d_n}}^{L_{d_n}} \Big(E(X_{j_1+\tau_1} X_{j_2+\tau_2}) \mathrm{e}^{\mathrm{i}v_2(j_2+\tau_2)}$$

$$- E(X_{j_1+\tau_1} X_{j_1+\tau_2}) \mathrm{e}^{\mathrm{i}v_2(j_1+\tau_2)} \Big)$$

$$\times H_{L_{d_n}}(\tau_1)^2 \mathrm{e}^{-\mathrm{i}v_1(j_1+\tau_1)}$$

$$+ \frac{1}{d_n L_{d_n}} \sum_{j_1=c_n+1}^{c_n+d_n} \sum_{j_2=c_n+1}^{c_n+d_n} E(X_{j_1} X_{j_2}) \mathrm{e}^{-\mathrm{i}(\omega_2 j_2 - \omega_1 j_1)}$$

$$\times \sum_{\tau_1=-L_{d_n}}^{L_{d_n}} \sum_{\tau_2=-L_{d_n}}^{L_{d_n}} E(X_{j_1+\tau_1} X_{j_2+\tau_2})$$

$$\times H_{L_{d_n}}(\tau_1) \big(H_{L_{d_n}}(\tau_2) - H_{L_{d_n}}(\tau_1) \big)$$

$$\times \mathrm{e}^{-\mathrm{i}(v_1(j_1+\tau_1)-v_2(j_2+\tau_2))}.$$

Denote the first, second and third terms of the last equality by $c_0(n)$, $c_1(n)$ and $c_2(n)$, respectively. For the term $c_1(n)$, we have $c_1(n) = z_1(n) + z_2(n) + z_3(n)$, where

$$z_1(n) = \frac{1}{d_n L_{d_n}} \sum_{|j_1-j_2|\geq 2L_{d_n}} E(X_{j_1} X_{j_2}) \mathrm{e}^{-\mathrm{i}(\omega_2 j_2 - \omega_1 j_1)}$$

$$\times \sum_{\tau_1=-L_{d_n}}^{L_{d_n}} \sum_{\tau_2=-L_{d_n}}^{L_{d_n}} \Big(E(X_{j_1+\tau_1} X_{j_2+\tau_2}) \mathrm{e}^{\mathrm{i}v_2(j_2+\tau_2)}$$

$$- E(X_{j_1+\tau_1} X_{j_1+\tau_2}) \mathrm{e}^{\mathrm{i}v_2(j_1+\tau_2)} \Big)$$

$$\times H_{L_{d_n}}(\tau_1)^2 \mathrm{e}^{-\mathrm{i}v_1(j_1+\tau_1)},$$

$$z_2(n) = \frac{1}{d_n L_{d_n}} \sum_{0<j_1-j_2<2L_{d_n}} E(X_{j_1} X_{j_2}) \sum_{\tau_1=-L_{d_n}}^{L_{d_n}} \left(\sum_{\tau_2=j_2-j_1-L_{d_n}}^{-L_{d_n}-1} - \sum_{\tau_2=j_2-j_1+L_{d_n}+1}^{L_{d_n}} \right)$$

$$\times E(X_{j_1+\tau_1} X_{j_1+\tau_2}) H_{L_{d_n}}(\tau_1)^2$$

$$\times \mathrm{e}^{-\mathrm{i}(v_1(j_1+\tau_1)-\omega_1 j_1 - v_2(j_1+\tau_2)+\omega_2 j_2)},$$

$$z_3(n) = \frac{1}{d_n L_{d_n}} \sum_{0<j_2-j_1<2L_{d_n}} E(X_{j_1} X_{j_2}) \sum_{\tau_1=-L_{d_n}}^{L_{d_n}} \left(\sum_{\tau_2=L_{d_n}+1}^{j_2-j_1+L_{d_n}} - \sum_{\tau_2=-L_{d_n}}^{j_2-j_1-L_{d_n}-1} \right)$$

$$\times E(X_{j_1+\tau_1} X_{j_1+\tau_2}) H_{L_{d_n}}(\tau_1)^2$$

$$\times \mathrm{e}^{-\mathrm{i}(v_1(j_1+\tau_1)-\omega_1 j_1 - v_2(j_1+\tau_2)+\omega_2 j_2)}.$$

Using Lemma A.1(i) and similar steps as for the term $d_1(n)$, it can be proven that $z_1(n)$ tends to zero. For the term $z_2(n)$, we have

$$|z_2(n)| \leq \frac{64\Delta^4}{d_n L_{d_n}} \sum_{0<j_1-j_2<2L_{d_n}} \sum_{\tau_1=-L_{d_n}}^{L_{d_n}} \left(\sum_{\tau_2=j_2-j_1-L_{d_n}}^{-L_{d_n}-1} + \sum_{\tau_2=j_2-j_1+L_{d_n}+1}^{L_{d_n}} \right)$$

$$\times \alpha^{\delta/(2+\delta)}(j_1-j_2)\alpha^{\delta(2+\delta)}(|\tau_1-\tau_2|)$$

$$\leq \frac{128K\Delta^4}{d_n L_{d_n}} \sum_{0<j_1-j_2<2L_{d_n}} (j_1-j_2)\alpha^{\delta/(2+\delta)}(j_1-j_2) \to 0.$$

In the same way, we may prove that $z_3(n) \to 0$. This means that $c_1(n) \to 0$. Using Lemma A.1(i), inequality $|H_{L_{d_n}}(\tau_2) - H_{L_{d_n}}(\tau_1)| \leq W|\tau_2-\tau_1|/L_{d_n} \leq W(|\tau_2-\tau_1+(j_2-j_1)|+|j_2-j_1|)/L_{d_n}$ and similar steps as for the term $d_1(n)$, it can be proven that $c_2(n)$ tends to zero. This means that the term $d_0(n)$ has the same limit as $c_0(n)$. Now, using Lemmas A.5 and A.6 for the term $c_0(n)$, we get

$$c_0(n) = 2\pi \int_{-1}^{1} w^2(x)\,dx\, P(\nu_1,\nu_2) \frac{1}{d_n} \sum_{j_1=c_n+1}^{c_n+d_n} \sum_{j_2=c_n+1}^{c_n+d_n} B(j_1,j_2-j_1)e^{-i(-\omega_1 j_1+\omega_2 j_2)}$$

$$+ \underbrace{\frac{2\pi}{d_n} \sum_{j_1=c_n+1}^{c_n+d_n} \sum_{j_2=c_n+1}^{c_n+d_n} B(j_1,j_2-j_1)e^{-i(-\omega_1 j_1+\omega_2 j_2)} e_n(\nu_1,\nu_2)}_{y_n} \tag{31}$$

$$= (2\pi)^2 \int_{-1}^{1} w^2(x)\,dx\, P(\nu_1,\nu_2)\overline{P(\omega_1,\omega_2)} + o(1) + y_n.$$

For the term y_n, we have

$$|y_n| \leq \tilde{e}_n(\nu_1,\nu_2) \frac{2\pi}{d_n} \sum_{j_1=c_n+1}^{c_n+d_n} \sum_{j_2=c_n+1}^{c_n+d_n} 8\Delta^2 \alpha^{\delta/(2+\delta)}(|j_1-j_2|) \leq \tilde{e}_n(\nu_1,\nu_2)4K\pi \to 0.$$

Hence, $s_2(n) \to (2\pi)^2 \rho P(\nu_1,\nu_2)\overline{P(\omega_1,\omega_2)}$. Following the same steps, we get that $s_3(n) \to (2\pi)^2 \rho P(\nu_1, 2\pi-\omega_2)\overline{P(\nu_2, 2\pi-\omega_1)}$. This completes the proof. $\square$

Proof of Theorem 3.2. Let us consider the following decomposition for the estimator $\hat{G}_n^{c,d}(\nu,\omega)$:

$$\sqrt{d_n/L_{d_n}}\big(\hat{G}_n^{c,d}(\nu,\omega) - P(\nu,\omega)\big) = \sqrt{d_n/L_{d_n}}\big(\hat{G}_n^{c,d}(\nu,\omega) - E(\hat{G}_n^{c,d}(\nu,\omega))\big)$$

$$+ \sqrt{d_n/L_{d_n}}\big(E(\hat{G}_n^{c,d}(\nu,\omega)) - P(\nu,\omega)\big)$$

$$= S_n^{c,d}(\nu,\omega) + \epsilon_n^{c,d}(\nu,\omega),$$

where $S_n^{c,d}(\nu, \omega) = \sqrt{d_n/L_{d_n}}(\hat{G}_n^{c,d}(\nu, \omega) - E(\hat{G}_n^{c,d}(\nu, \omega)))$, $\epsilon_n^{c,d}(\nu, \omega) = \sqrt{d_n/L_{d_n}}(E(\hat{G}_n^{c,d}(\nu, \omega)) - P(\nu, \omega))$. We will now split the proof into two steps. In the first step, we will show that the deterministic term $\epsilon_n^{c,d}(\nu, \omega)$ tends to zero for $n \to \infty$. In the second step, we will show that

$$\begin{bmatrix} \mathrm{Re}(S_n^{c,d}(\nu, \omega)) \\ \mathrm{Im}(S_n^{c,d}(\nu, \omega)) \end{bmatrix} \xrightarrow{d} \mathcal{N}_2(0, \Sigma(\nu, \omega)). \tag{32}$$

Step 1. Without loss of generality, we may assume that $\omega \leq \nu$. Changing variables and using, sequentially, (3), (5), (4), (A3) and assumption (ii) of our theorem, we then obtain

$$|\epsilon_n^{c,d}(\nu, \omega)| = \sqrt{d_n/L_{d_n}}|E(\hat{G}_n^{c,d}(\nu, \omega)) - P(\nu, \omega)|$$

$$= \sqrt{\frac{d_n}{L_{d_n}}}\left| \frac{1}{2\pi d_n} \sum_{j=c_n+1}^{c_n+d_n} \sum_{\tau=-L_{d_n}}^{L_{d_n}} \mathbf{1}\{c_n + 1 \leq j + \tau \leq c_n + d_n\}B(j, \tau)H_{L_{d_n}}(\tau)\right.$$

$$\left. \times \mathrm{e}^{-\mathrm{i}\nu\tau}\mathrm{e}^{-\mathrm{i}(\nu-\omega)j} - P(\nu, \omega)\right|$$

$$\leq \sqrt{\frac{d_n}{L_{d_n}}}\left| \frac{1}{2\pi d_n} \sum_{j=c_n+1}^{c_n+d_n} \sum_{\tau=-L_{d_n}}^{L_{d_n}} B(j, \tau)H_{L_{d_n}}(\tau)\mathrm{e}^{-\mathrm{i}\nu\tau}\mathrm{e}^{-\mathrm{i}(\nu-\omega)j} - P(\nu, \omega)\right|$$

$$+ \sqrt{\frac{d_n}{L_{d_n}}}\frac{1}{2\pi d_n} \sum_{j=c_n+1}^{c_n+d_n} \sum_{\tau=-L_{d_n}}^{L_{d_n}} (\mathbf{1}\{j + \tau > c_n + d_n\}$$

$$+ \mathbf{1}\{j + \tau < c_n + 1\})8\Delta^2\alpha^{\delta/(2+\delta)}(|\tau|)$$

$$\leq \sqrt{\frac{d_n}{L_{d_n}}}\left| \frac{1}{2\pi} \sum_{|\tau|\leq L_{d_n}} a(\nu - \omega, \tau)H_{L_{d_n}}(\tau)\mathrm{e}^{-\mathrm{i}\nu\tau} - P(\nu, \omega)\right|$$

$$+ \sqrt{\frac{d_n}{L_{d_n}}}\left| \frac{1}{2\pi d_n} \sum_{\tau=-L_{d_n}}^{L_{d_n}} H_{L_{d_n}}(\tau) \sum_{j=c_n+1}^{c_n+d_n} \sum_{\lambda\in\Lambda_\tau\backslash\{\nu-\omega\}} a(\lambda, \tau)\mathrm{e}^{\mathrm{i}(\lambda-(\nu-\omega))j}\mathrm{e}^{-\mathrm{i}\nu\tau}\right|$$

$$+ \frac{4\Delta^2}{\pi\sqrt{d_n L_{d_n}}} \sum_{\tau=-L_{d_n}}^{L_{d_n}} |\tau|\alpha^{\delta/(2+\delta)}(|\tau|)$$

$$\leq \sqrt{\frac{d_n}{L_{d_n}}}\left| \frac{1}{2\pi}\left(\sum_{|\tau|\leq\theta L_{d_n}} + \sum_{L_{d_n}>|\tau|>\theta L_{d_n}} \right)a(\nu - \omega, \tau)H_{L_{d_n}}(\tau)\mathrm{e}^{-\mathrm{i}\nu\tau} - P(\nu, \omega)\right|$$

$$+ \sqrt{\frac{d_n}{L_{d_n}}}\frac{1}{2\pi d_n} \sum_{\tau=-L_{d_n}}^{L_{d_n}} \sum_{\lambda\in\Lambda_\tau\backslash\{\nu-\omega\}}\left| \sum_{j=c_n+1}^{c_n+d_n} a(\lambda, \tau)\mathrm{e}^{\mathrm{i}(\lambda-(\nu-\omega))j}\right| + \mathrm{o}(1)$$

$$\leq \sqrt{\frac{d_n}{L_{d_n}}} \left(\frac{1}{\pi} \sum_{|\tau|>\theta L_{d_n}} 8\Delta^2 \alpha^{\delta/(2+\delta)}(|\tau|) \right.$$

$$\left. + \frac{1}{2\pi d_n} \sum_{\tau=-L_{d_n}}^{L_{d_n}} \sum_{\lambda \in \Lambda_\tau \setminus \{\nu-\omega\}} \left| a(\lambda,\tau) \csc((\lambda-(\nu-\omega))/2) \right| \right) + o(1)$$

$$\leq \frac{8\Delta^2}{\pi} \sqrt{\frac{d_n}{L_{d_n}}} \sum_{|\tau|>\theta L_{d_n}} \alpha^{\delta/(r+2+\delta)}(|\tau|)$$

$$+ \frac{1}{2\pi\sqrt{d_n L_{d_n}}} \sum_{\tau=-L_{d_n}}^{L_{d_n}} \sum_{\lambda \in \Lambda_\tau \setminus \{\nu-\omega\}} \left| a(\lambda,\tau) \csc((\lambda-(\nu-\omega))/2) \right| + o(1)$$

$$\leq \frac{8\Delta^2}{\pi\theta^r} \sqrt{\frac{d_n}{L_{d_n}^{1+2r}}} \sum_{|\tau|>\theta L_{d_n}} \theta^r L_{d_n}^r \alpha^{\delta/(r+2+\delta)}(|\tau|)$$

$$+ \frac{1}{2\pi} \sqrt{\frac{L_{d_n}}{d_n}} \left(\frac{1}{L_{d_n}} \sum_{\tau=-L_{d_n}}^{L_{d_n}} z_\tau(\nu-\omega) \right) + o(1)$$

$$\leq \frac{8\Delta^2}{\pi\theta^r} \sqrt{\frac{d_n}{L_{d_n}^{1+2r}}} K + \frac{1}{2\pi} \sqrt{\frac{L_{d_n}}{d_n}} o(1) + o(1) = O(1) d_n^{(1-\kappa(1+2r))/2} + o(1)$$

$$= O(1) d_n^{\kappa((1-\kappa)/(2\kappa)-r)} + o(1) \to 0,$$

where $\mathbf{1}\{B\}$ is an indicator function of the event B. This completes the proof of step 1.

Step 2. In this step, we use Theorem 3.1 and Lemma A.3 to show (32). Note that

$$S_n^{c,d}(\nu,\omega) = \frac{1}{2\pi\sqrt{d_n}} \sum_{j=1}^{d_n} \big(R_{n,j}(\nu,\omega) + iI_{n,j}(\nu,\omega) \big),$$

where $R_{n,j}(\nu,\omega) = R_{n,j}$ and $I_j(\nu,\omega) = I_{n,j}$ are defined via

$$R_{n,j}(\nu,\omega) = \frac{1}{\sqrt{L_{d_n}}} \sum_{\tau=-L_{d_n}}^{L_{d_n}} H_{L_{d_n}}(\tau) \mathbf{1}\{1 \leq j+\tau \leq d_n\} Z_{j+c_n}(\tau) \cos\big(\nu\tau + (\nu-\omega)(j+c_n)\big),$$

$$I_{n,j}(\nu,\omega) = \frac{1}{\sqrt{L_{d_n}}} \sum_{\tau=-L_{d_n}}^{L_{d_n}} H_{L_{d_n}}(\tau) \mathbf{1}\{1 \leq j+\tau \leq d_n\} Z_{j+c_n}(\tau) \sin\big(-\nu\tau - (\nu-\omega)(j+c_n)\big)$$

and $Z_j(\tau) = X_j X_{j+\tau} - B(j,\tau)$. We show that for triangular array $\{(R_{n,t}, I_{n,t}) : 1 \leq t \leq d_n\}$, the assumptions of Theorem A.3 hold. Note that by the Minkowski inequality, Lemma A.1(i) and,

finally, by the Hölder inequality, we have

$$\|R_{n,j}(\nu, \omega)\|_{2+\delta}$$

$$\leq \left\| \frac{1}{\sqrt{L_{d_n}}} \sum_{\tau=-L_{d_n}}^{L_{d_n}} H_{L_{d_n}}(\tau) \mathbf{1}\{1 \leq j+\tau \leq d_n\} X_{j+c_n} X_{j+c_n+\tau} \cos(\nu\tau + (\nu-\omega)(j+c_n)) \right\|_{2+\delta}$$

$$+ \frac{8\Delta^2}{\sqrt{L_{d_n}}} \sum_{\tau=-L_{d_n}}^{L_{d_n}} \alpha^{\delta/(2+\delta)}(|\tau|)$$

$$\leq \|X_{j+c_n}\|_{6+3\delta} \left\| \frac{1}{\sqrt{L_{d_n}}} \sum_{\tau=-L_{d_n}}^{L_{d_n}} H_{L_{d_n}}(\tau) \mathbf{1}\{1 \leq j+\tau \leq d_n\} X_{j+c_n+\tau} \right.$$

$$\left. \times \cos(\nu\tau + (\nu-\omega)(j+c_n)) \right\|_{3+3\delta/2} + 16\Delta^2 K$$

$$\leq \frac{\Delta}{\sqrt{L_{d_n}}} \underbrace{\left\| \sum_{\tau=-L_{d_n}}^{L_{d_n}} H_{L_{d_n}}(\tau) \mathbf{1}\{1 \leq j+\tau \leq d_n\} X_{j+c_n+\tau} \cos(\nu\tau + (\nu-\omega)(j+c_n)) \right\|_{3+3\delta/2}}_{t_1(n)}$$

$$+ 16\Delta^2 K.$$

In the next step, we use Lemma A.2 to estimate the term $t_1(n)$. Let $l = 3 + \frac{3}{2}\delta$ and note that $l + \delta < 6 + 3\delta$, which means that assumption (i) of Lemma A.2 holds. Assumption (ii) of Lemma A.2 follows from assumption (iii) of our theorem. Therefore, using Lemma A.2 (similarly as for $I_{n,j}(\nu, \omega)$), we get

$$\|R_{n,j}(\nu, \omega)\|_{2+\delta}$$

$$\leq \Delta \left(K_{\alpha,l} \max \left\{ Q(3+3\delta/2, \delta, -L_{d_n}-1, 2L_{d_n}+1), \right.\right.$$

$$\left.\left. [Q(2, \delta, -L_{d_n}-1, 2L_{d_n}+1)]^{(3+(3/2)\delta)/2} \right\} \right)^{1/(3+(3/2)\delta)} / \sqrt{L_{d_n}} + 16\Delta^2 K$$

$$\leq \Delta K_{\alpha,l}^{1/(3+(3/2)\delta)} \left(\max\left\{ (2L_{d_n}+1)\Delta^{3+(3/2)\delta}, \left((2L_{d_n}+1)\Delta^2\right)^{(3+(3/2)\delta)/2} \right\} \right)^{1/(3+(3/2)\delta)} / \sqrt{L_{d_n}}$$

$$+ 16\Delta^2 K$$

$$\leq \Delta^2 K_{\alpha,l}^{1/(3+(3/2)\delta)} \sqrt{2L_{d_n}+1} / \sqrt{L_{d_n}} + 16\Delta^2 K$$

$$\leq \Delta^2 \left(2K_{\alpha,l}^{1/(3+(3/2)\delta)} + 16K \right),$$

which means that assumption (i) of Lemma A.3 holds. Condition (ii) of Lemma A.3 follows from Theorem 3.1 by noting that

$$\mathrm{cov}(\mathrm{Re}(S_n^{c,d}(v,\omega)), \mathrm{Re}(S_n^{c,d}(v,\omega)))$$
$$= \frac{d_n}{4L_{d_n}} \mathrm{cov}\big(\hat{G}_n^{c,d}(v,\omega) + \overline{\hat{G}_n^{c,d}(v,\omega)}, \hat{G}_n^{c,d}(v,\omega) + \overline{\hat{G}_n^{c,d}(v,\omega)}\big),$$

$$\mathrm{cov}(\mathrm{Re}(S_n^{c,d}(v,\omega)), \mathrm{Im}(S_n^{c,d}(v,\omega)))$$
$$= \frac{d_n}{4\mathrm{i}L_{d_n}} \mathrm{cov}\big(\hat{G}_n^{c,d}(v,\omega) + \overline{\hat{G}_n^{c,d}(v,\omega)}, \hat{G}_n^{c,d}(v,\omega) - \overline{\hat{G}_n^{c,d}(v,\omega)}\big),$$

$$\mathrm{cov}(\mathrm{Im}(S_n^{c,d}(v,\omega)), \mathrm{Im}(S_n^{c,d}(v,\omega)))$$
$$= \frac{d_n}{4L_{d_n}} \mathrm{cov}\big(\hat{G}_n^{c,d}(v,\omega) - \overline{\hat{G}_n^{c,d}(v,\omega)}, \hat{G}_n^{c,d}(v,\omega) - \overline{\hat{G}_n^{c,d}(v,\omega)}\big).$$

Finally, putting $h_n = 2L_{d_n}$ and $\varsigma(k) = \alpha_X(k)$, and taking into consideration assumption (iii) of our theorem, we get that condition (iii) of Lemma A.3 holds, which means that (32) holds, where $\Sigma(v,\omega)$ can be obtained by the last three equations and Theorem 3.1. The calculation of the matrix $\Sigma(v,\omega)$ is too technical to be presented here. This completes the proof. □

Proof of Theorem 4.1. By Politis *et al.* [26], Theorem 4.2.1, it is sufficient to prove that there exists a continuous distribution J such that:

(i) $\sqrt{n/L_n}(|\hat{G}_n(v,\omega)| - |P(v,\omega)|) \xrightarrow{d} J$;

(ii) for any sequence of positive integers $\{t_b\} = \{t_{b(n)}\}$ such that $b = b(n) \to \infty$, $b/n \to 0$ as $n \to \infty$, we have $J_{b,t_b}^P(x) \longrightarrow J(x)$, where $J_{b,t_b}^P(x) = P(\sqrt{b/L_b}(|\hat{G}_n^{t_b-1,b}(v,\omega)| - |P(v,\omega)|) \leq x)$ and $J(x)$ is a distribution function at the point x for the distribution J.

This follows immediately from Corollary 3.2 by setting $c_n = t_{b(n)}$ and $d_n = b(n)$. This completes the proof. □

Proof of Theorem 4.2. This proof is analogous to the proof of Theorem 4.1. The only difference is that we use Corollary 3.3 instead of Corollary 3.2.

□

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