Bernoulli **16**(4), 2010, 1114–1136
DOI: 10.3150/09-BEJ246

Compound Poisson and signed compound Poisson approximations to the Markov binomial law

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Compound Poisson distributions and signed compound Poisson measures are used for approximation of the Markov binomial distribution. The upper and lower bound estimates are obtained for the total variation, local and Wasserstein norms. In a special case, asymptotically sharp constants are calculated. For the upper bounds, the smoothing properties of compound Poisson distributions are applied. For the lower bound estimates, the characteristic function method is used.

Keywords: compound Poisson approximation; geometric distribution; local norm; Markov binomial distribution; signed compound Poisson measure; total variation norm; Wasserstein norm

1. Introduction

The closeness of a compound Poisson (CP) distribution to the Markov binomial (MB) distribution has been investigated in numerous papers; see, for example, [5,6,12,14,20,31,34,37] and the references therein. Related problems were considered in [4,7,8,13,18,19,30,33] and [38]. One would expect the MB–CP case to have been comprehensively studied. As it turns out, this is not the case. Many papers deal with the convergence facts only. Only a few of the papers dealing with the estimates of accuracy of approximation involve no assumptions about the stationarity of the Markov chain.

The aim of this paper is to discuss some compound approximations for non-stationary Markov chains. We show that for our version of the MB distribution, the natural approximation is a convolution of CP and compound binomial distributions, both having the same compounding geometric law. We outline some principles of construction of asymptotic expansions and consider second order approximations. Part of the paper is devoted to signed compound Poisson approximations which can be viewed as the second order expansions in the exponent. We obtain upper and lower bound estimates and show that under certain conditions, they are of the same order of accuracy. All estimates are proved for the total variation, local and Wasserstein norms. For the upper bound estimates, we employ a convolution technique which can be dated back to [23]. For the lower bound estimates, we use the characteristic function method. The methods of proof do not allow for reasonably small absolute constants. However, in special cases, asymptotically sharp constants are calculated.

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We now introduce some notation. Let I_k denote the distribution concentrated at an integer $k \in \mathbb{Z}$, the set of integers, and set $I = I_0$. In what follows, V and M denote two finite signed measures on $\mathbb{Z}$. Products and powers of V and M are understood in the convolution sense, that is, $VM\{A\} = \sum_{k=-\infty}^{\infty} V\{A-k\}M\{k\}$ for a set $A \subseteq \mathbb{Z}$; further, $M^0 = I$. The total variation norm, the local norm and the Wasserstein norm of M are denoted by

$$\|M\| = \sum_{k=-\infty}^{\infty} |M\{k\}|, \qquad \|M\|_\infty = \sup_{k\in\mathbb{Z}} |M\{k\}|, \qquad \|M\|_{\mathrm{W}} = \sum_{k=-\infty}^{\infty} |M\{(-\infty, k]\}|,$$

respectively. Note that $\|(I_1 - I)M\|_{\mathrm{W}} = \|M\|$. The logarithm and exponential of M are given, respectively, by

$$\ln M = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(M-I)^k \qquad (\text{if } \|M - I\| < 1), \qquad \mathrm{e}^M = \exp\{M\} = \sum_{k=0}^{\infty} \frac{1}{k!} M^k.$$

Note that

$$\|VM\|_\infty \le \|V\|\|M\|_\infty, \qquad \|VM\| \le \|V\|\|M\|, \qquad \|\mathrm{e}^M\| \le \mathrm{e}^{\|M\|}.$$

Let $\widehat{M}(t)$ $(t \in \mathbb{R})$ be the Fourier transform of M. We denote by C positive absolute constants. Θ stands for any finite signed measure on $\mathbb{Z}$ satisfying $\|\Theta\| \le 1$. The values of C and Θ can vary from line to line, or even within the same line. Sometimes, to avoid possible ambiguity, the C's are supplied with indices. For $x \in \mathbb{R}$ and $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$, we set

$$\binom{x}{k} = \frac{1}{k!} x(x-1)\cdots(x-k+1), \qquad \binom{x}{0} = 1.$$

Let $\xi_0, \xi_1, \ldots, \xi_n, \ldots$ be a Markov chain with the initial distribution

$$\mathrm{P}(\xi_0 = 1) = p_0, \qquad \mathrm{P}(\xi_0 = 0) = 1 - p_0, \qquad p_0 \in [0, 1]$$

and transition probabilities

$$\mathrm{P}(\xi_i = 1 | \xi_{i-1} = 1) = p, \qquad \mathrm{P}(\xi_i = 0 | \xi_{i-1} = 1) = q,$$
$$\mathrm{P}(\xi_i = 1 | \xi_{i-1} = 0) = \overline{q}, \qquad \mathrm{P}(\xi_i = 0 | \xi_{i-1} = 0) = \overline{p},$$
$$p + q = \overline{q} + \overline{p} = 1, \qquad p, \overline{q} \in (0, 1), \qquad i \in \mathbb{N}.$$

The distribution of $S_n = \xi_1 + \cdots + \xi_n$ $(n \in \mathbb{N})$ is called the *Markov binomial distribution*. We denote it by F_n. We should note that the definition of the Markov binomial distribution varies slightly from paper to paper; see [12,30] and [36]. We choose the definition which, on the one hand, contains the binomial distribution as a special case and, on the other hand, allows comparison to the Dobrushin's results. Dobrushin [12] assumed that $p_0 = 1$ and considered $S_{n-1} + 1$.

Later, we will need various characteristics of S_n. Let

$$\gamma_1 = \frac{q\overline{q}}{q+\overline{q}}, \qquad \gamma_2 = -\frac{q\overline{q}^2}{(q+\overline{q})^2}\left(p + \frac{q}{q+\overline{q}}\right) - \frac{\gamma_1^2}{2},$$

$$\gamma_3 = \gamma_1^2 \tilde{\gamma}_3,$$

$$\tilde{\gamma}_3 = \frac{\gamma_1}{3} + \frac{1}{q(q+\overline{q})}\left\{p^2\overline{q} + \frac{pq(2\overline{q}-q)}{q+\overline{q}} + \frac{2\overline{q}q^2}{(q+\overline{q})^2}\right\} + \frac{\overline{q}}{q+\overline{q}}\left(p + \frac{q}{q+\overline{q}}\right),$$

$$\lambda = n - p_0, \qquad \varkappa_1 = \gamma_1\left(\frac{\overline{q}-p}{q+\overline{q}} - p_0\right), \qquad \varkappa_2 = p_0\frac{pq}{q+\overline{q}}, \qquad C_1 = \ln\frac{30}{19} = 0.4567\ldots.$$

We use the following measures also:

$$G = qI_1\sum_{j=0}^{\infty} p^j I_j \qquad \left(\widehat{G}(t) = \frac{q\mathrm{e}^{\mathrm{i}t}}{1-p\mathrm{e}^{\mathrm{i}t}}\right), \qquad H = I + \varkappa_2(G-I),$$

$$H_1 = (1-\gamma_1)I + \gamma_1 G, \qquad H_1^{\lambda} = \exp\left\{\lambda\sum_{j=1}^{\infty}\frac{(-1)^j}{j}\gamma_1^j(G-I)^j\right\} \qquad \left(\widehat{H}_1^{\lambda}(t) = (\widehat{H}_1(t))^{\lambda}\right),$$

$$D_j = \exp\left\{\sum_{i=1}^{j}\gamma_i(G-I)^i\right\}, \qquad 1 \le j \le 3, \qquad D_1^{\lambda} = \exp\{\lambda\gamma_1(G-I)\}.$$

2. Known results

In this section, we discuss some of the known results on the compound Poisson approximations to the MB distribution. Many papers deal with the convergence facts only; see, for example, [14, 19,20,36]. Usually, the chain is assumed to be stationary. A typical example is Theorem 4.1 in [31] which states that if $p_0 = \overline{q}/(q+\overline{q})$ and $\tilde{\alpha} > 0$, then

$$\|F_n - \exp\{\tilde{\alpha}(G-I)\}\| \le 2|n\overline{q} - \tilde{\alpha}| + \frac{2\overline{q}(1+p+n\overline{q}(2-p))}{q+\overline{q}}. \tag{1}$$

Even if we choose $\tilde{\alpha} = n\overline{q}$, the order of accuracy in (1) is not better than $n\overline{q}^2$. A similar estimate was obtained in Theorem 5 of [37]. If we use the terminology of the book [2], we can say that the estimate (1) contains no 'magic' factor. If we turn to the papers with 'magic' factors, then we have the following results. In [5], it was proven that if $0 \le p \le C_0 < 1$, then

$$\|F_n - D_1^n\| \le C\max(p_0,\overline{q})\min\left(1, \frac{1}{\sqrt{n\overline{q}}}\right) + C\min(\overline{q}, n\overline{q}^2) + C\mathrm{e}^{-Cn}. \tag{2}$$

The accuracy can be improved, by some asymptotic expansions, to

$$\left\| F_n - D_1^n\left(I + p_0\frac{q^2(p-\overline{q})}{(q+\overline{q})^2}I_1\right)\right\| \le C\overline{q}(p+\overline{q})\min\left(1, \frac{1}{\sqrt{n\overline{q}}}\right) + C\min(\overline{q}, n\overline{q}^2) + Ce^{-Cn}. \tag{3}$$

Note that in [5] formulas (4.5), (4.12) and (4.23) contain misprints. The parameter p_0 is misplaced and should be in the brackets. If $p \to \tilde{p} = const$, $\overline{q} \to 0$ and $n\overline{q} \to \infty$, then the order of accuracy in (2) is $\max(\overline{q}, (n\overline{q})^{-1/2})$. Also, the order of accuracy in (3) is $\overline{q}$. We can hardly call (3) the second order expansion since the improvement of the accuracy was achieved due to the more precise approximation of the initial distribution of ξ_0 only.

The main idea of signed CP approximations is to leave more than one factorial cumulant in the exponent. In short, the signed CP measure has the same structure as the CP measure, but can have negative Poisson parameters. Such approximations are commonly used in insurance models and in limit theorems; see [1,9,17,21,22,25,28] and the references therein. For the MB distribution in [5] the following result is proved. If

$$p \le \tilde{C} < 1, \qquad \frac{\overline{q}}{q+\overline{q}} \le \frac{1-\tilde{C}}{30}, \tag{4}$$

then

$$\|F_n - D_2^n\| \le C(p+\overline{q})\left\{\min\left(\sqrt{\frac{\overline{q}}{n}}, n\overline{q}^2\right) + \max(p_0, \overline{q})\min\left(1, \frac{1}{\sqrt{n\overline{q}}}\right) + e^{-Cn}\right\}. \tag{5}$$

Note that $\gamma_2 < 0$ and, therefore, D_2 is a signed measure rather than a distribution. As a rule, signed CP approximations are more accurate than CP approximations. Indeed, if $n \to \infty$, then (5) gives the estimate converging to zero, even if p and $\overline{q}$ are constants. Also, (2) and (3) are non-trivial even if we only have $\overline{q} = o(1)$.

We are unaware of any lower bound estimate for the compound Poisson approximation to the Markov binomial distribution.

3. The main results

3.1. Geometric expansions

Before formulating our results, it is necessary to explain the choice of approximating measures. Dobrushin [12] proved that if $p \to \tilde{p}$, $n\overline{q} \to \tilde{\lambda}$ and $p_0 = 1$, then the limit distribution for $1 + S_{n-1}$ is the convolution $G_1 \exp\{\tilde{\lambda}(G_1 - I)\}$, where G_1 is a geometric distribution with parameter $\tilde{p}$, that is, $\widehat{G}_1(t) = (1-\tilde{p})e^{it}/(1-\tilde{p}e^{it})$. This suggests that approximation of S_n for arbitrary p_0 should also be based on expansions in powers of $G - I$.

Let F be concentrated on $\mathbb{Z}$ and have all moments finite. We can write formally (i.e., without investigating conditions needed for the convergence of series)

$$\widehat{F}(t) = 1 + \sum_{j=1}^{\infty} \frac{\nu_j}{j!}(\mathrm{e}^{\mathrm{i}t} - 1)^j = 1 + \sum_{m=1}^{\infty} \frac{\tilde{\nu}_m}{m!}\big(\widehat{G}(t) - 1\big)^m.$$

Here, ν_j $(j = 1, 2, \ldots)$ are factorial moments of F, and $\tilde{\nu}_m$ can be called the *geometric factorial moments*. Since

$$\mathrm{e}^{\mathrm{i}t} - 1 = \frac{q(\widehat{G}(t) - 1)}{1 + p(\widehat{G}(t) - 1)},$$

it is not difficult to establish a relation between ν_j and $\tilde{\nu}_m$:

$$\frac{\tilde{\nu}_m}{m!} = (-p)^m \sum_{j=1}^{m} \frac{\nu_j}{j!}\left(-\frac{q}{p}\right)^j \binom{m-1}{m-j}, \qquad m = 1, 2, \ldots. \tag{6}$$

Similar relations hold for factorial cumulants and geometric factorial cumulants. For the MB distribution, we have

$$\tilde{\nu}_1 = q\nu_1 = q\mathrm{E}S_n = n\gamma_1 + \varkappa_1 + \varkappa_2 - (\varkappa_1 + \varkappa_2)(p - \overline{q})^n. \tag{7}$$

(For the formula of the mean, see [7].) Since we will assume p and $\overline{q}$ to be small, the last summand in (7) will be neglected. As it turns out, F_n is close to some convolution $W_1\Lambda_1^n$; see (32) below. We use (6) for choosing the approximating measure for W_1. The cumulant analog of (6) is used for Λ_1^n.

3.2. Compound Poisson approximation

In this paper, we usually assume that

$$p \le \tfrac{1}{2}, \qquad \overline{q} \le \tfrac{1}{30}. \tag{8}$$

The size of the absolute constants is determined by the method of proof. We expand F_n as a series of convolutions of measures. The remainder term is usually estimated by a series containing powers of $\overline{q} + p$. If the sum $\overline{q} + p$ is sufficiently small, the series converges. Thus, although we have some freedom in the choice of magnitude of p and $\overline{q}$, the sum $p + \overline{q}$ must be small. The choice of condition (8) is determined by the fact that the CP limit occurs when $n\overline{q} \to \tilde{\lambda}$; see, for example, Table 1 in [12]. Therefore, if we expect the CP approximation to be accurate, then $\overline{q}$ should be small. On the other hand, we have included the case $\overline{q} = \textit{constant}$, that is, the case which is usually associated with the normal approximation. We choose the assumption $p \le 1/2$ instead of (4), in order to make our proofs clearer.

Theorem 3.1. *Let $p \le 1/2$. We then have*

$$\|F_n - HD_1^\lambda\| \le C\overline{q}(p+\overline{q})\min\left(1, \frac{1}{\sqrt{n\overline{q}}}\right) + C\min(\overline{q}, n\overline{q}^2) + C(p+\overline{q})\mathrm{e}^{-C_1 n}. \tag{9}$$

If, in addition, $\overline{q} \le 1/30$, *then*

$$\begin{aligned}\|F_n - HD_1^\lambda\|_\infty &\le C\overline{q}(p+\overline{q})\min\left(1, \frac{1}{n\overline{q}}\right) + C\min\left(\sqrt{\frac{\overline{q}}{n}}, n\overline{q}^2\right)\\ &\quad + C(p+\overline{q})\mathrm{e}^{-C_1 n},\end{aligned} \tag{10}$$

$$\|F_n - HD_1^\lambda\|_{\mathrm{W}} \le C\overline{q}(p+\overline{q}) + C\min\left(\overline{q}\sqrt{n\overline{q}}, n\overline{q}^2\right) + C(p+\overline{q})\mathrm{e}^{-C_1 n}. \tag{11}$$

Corollary 3.1. *If* (8) *is satisfied and* $n\overline{q} \ge 1$, *then*

$$\|F_n - HD_1^n\| \le C\overline{q}, \qquad \|F_n - HD_1^n\|_\infty \le C\sqrt{\frac{\overline{q}}{n}}, \qquad \|F_n - HD_1^n\|_{\mathrm{W}} \le C\overline{q}\sqrt{n\overline{q}}.$$

Corollary 3.2. *If* (8) *is satisfied, then*

$$\|F_n - HH_1^\lambda\| \le C(\overline{q} + p\mathrm{e}^{-C_1 n}).$$

Remark 3.1.

(i) H is compound Poisson distribution; see Lemma 5.3 below. If $\overline{q} \le p$, then H_1^λ is also a CP distribution. Thus, we see that there exist quite different forms of CP approximations with similar orders of accuracy.

(ii) The estimate (9) is slightly better than (3) for $p, \overline{q} \le \exp\{-C_1 n\}$, and more accurate than (2) for $\overline{q} \le 1/\sqrt{n\overline{q}}$ and $p_0 \ge C$.

(iii) For the closeness of F_n and $H_1 D_1^\lambda$, it suffices to assume $\overline{q} \to 0$, in considerable contrast to $n\overline{q} \to \tilde{\lambda}$, the latter being needed for the convergence to the limit CP law.

(iv) We can write $\min(\overline{q}, n\overline{q}^2) = n\overline{q}^2 \min(1, (n\overline{q})^{-1})$. The last factor, in terms of [2], page 5, can be called the 'magic' factor.

The accuracy of approximation can be improved by the second order expansion.

Theorem 3.2. *If* $p \le 1/2$, *then*

$$\left\|F_n - HD_1^\lambda\big(I + n\gamma_2(G-I)^2\big)\right\| \le C\left\{\overline{q}^2 + p\overline{q}\min\left(1, \frac{1}{\sqrt{n\overline{q}}}\right) + (p+\overline{q})\mathrm{e}^{-C_1 n}\right\}.$$

If, in addition, $\overline{q} \le 1/30$, *then*

$$\begin{aligned}&\left\|F_n - HD_1^\lambda\big(I + n\gamma_2(G-I)^2\big)\right\|_\infty\\ &\quad\le C\left\{\overline{q}^2\min\left(1, \frac{1}{\sqrt{n\overline{q}}}\right) + p\overline{q}\min\left(1, \frac{1}{n\overline{q}}\right) + (p+\overline{q})\mathrm{e}^{-C_1 n}\right\},\end{aligned}$$

$$\begin{aligned}&\big\|F_n - HD_1^{\lambda}\big(I + n\gamma_2(G-I)^2\big)\big\|_{\mathrm{W}}\\&\quad\le C\big\{\overline{q}^2\max\big(1,\sqrt{n\overline{q}}\big) + p\overline{q} + (p+\overline{q})\mathrm{e}^{-C_1 n}\big\}.\end{aligned}$$

Note that the last estimate contains $\max(1, \sqrt{n\overline{q}})$, reflecting the fact that the estimates for the Wasserstein distance are less accurate than the ones for the total variation norm. It is even more evident when $n\overline{q} \ge 1$.

Corollary 3.3. *If* (8) *is satisfied and* $n\overline{q} \ge 1$, *then the estimates in Theorem* 3.2 *are*

$$C\overline{q}\left(\overline{q} + \frac{p}{\sqrt{n\overline{q}}}\right), \qquad C\sqrt{\frac{\overline{q}}{n}}\left(\overline{q} + \frac{p}{\sqrt{n\overline{q}}}\right), \qquad C\overline{q}\sqrt{n\overline{q}}\left(\overline{q} + \frac{p}{\sqrt{n\overline{q}}}\right),$$

respectively.

We see that, in general, even the second order estimates in total variation are only meaningful for $\overline{q} = \mathrm{o}(1)$.

3.3. Signed compound Poisson approximations

The choice of a signed CP approximation, in general, means that the first term of the asymptotic expansion, unlike Theorem 3.2, is in the exponent.

Theorem 3.3. *If condition* (8) *is satisfied*, *then*

$$\|F_n - H\exp\{\varkappa_1(G-I)\}D_2^n\| \le C(p+\overline{q})\left\{\min\left(\overline{q}, \sqrt{\frac{\overline{q}}{n}}\right) + \mathrm{e}^{-C_1 n}\right\}, \tag{12}$$

$$\|F_n - H\exp\{\varkappa_1(G-I)\}D_2^n\|_\infty \le C(p+\overline{q})\left\{\min\left(\overline{q}, \frac{1}{n}\right) + \mathrm{e}^{-C_1 n}\right\},$$

$$\|F_n - H\exp\{\varkappa_1(G-I)\}D_2^n\|_{\mathrm{W}} \le C(p+\overline{q})\{\overline{q} + \mathrm{e}^{-C_1 n}\}.$$

Note that for $n\overline{q} \le 1$ and $p_0 = \textit{constant}$, (12) is more accurate that (5). More importantly, when $p = \textit{constant}$ and $\overline{q} = \textit{constant}$, the estimate (12) is of order $\mathrm{O}(n^{-1/2})$. In this sense, the signed CP approximation is comparable to the normal one and, moreover, it holds in the total variation metric. Meanwhile, for discrete distributions, the normal approximation holds in the uniform metric only. Just as in the CP case, the second order expansions can be used.

Theorem 3.4. *If* (8) *holds*, *then*

$$\begin{aligned}&\big\|F_n - H\exp\{\varkappa_1(G-I)\}D_2^n\big(I + n\gamma_3(G-I)^3\big)\big\|\\&\quad\le C(p+\overline{q})\left\{\min\left(\overline{q}, \frac{1}{n}\right) + \mathrm{e}^{-C_1 n}\right\},\end{aligned} \tag{13}$$

$$\begin{aligned}
&\bigl\| F_n - H\exp\{\varkappa_1(G-I)\}D_2^n\bigl(I+n\gamma_3(G-I)^3\bigr)\bigr\|_\infty \\
&\quad \le C(p+\overline{q})\left\{\min\left(\overline{q}, \frac{1}{n\sqrt{n\overline{q}}}\right) + \mathrm{e}^{-C_1 n}\right\},
\end{aligned} \tag{14}$$

$$\begin{aligned}
&\bigl\| F_n - H\exp\{\varkappa_1(G-I)\}D_2^n\bigl(I+n\gamma_3(G-I)^3\bigr)\bigr\|_{\mathrm{W}} \\
&\quad \le C(p+\overline{q})\left\{\min\left(\overline{q}, \sqrt{\frac{\overline{q}}{n}}\right) + \mathrm{e}^{-C_1 n}\right\}.
\end{aligned} \tag{15}$$

Corollary 3.4. *Let $n\overline{q} \ge 1$. The estimates (13)–(15) are then at least of order*

$$\frac{C(p+\overline{q})}{n}, \qquad \frac{C(p+\overline{q})}{n\sqrt{n\overline{q}}}, \qquad \frac{C(p+\overline{q})\sqrt{\overline{q}}}{\sqrt{n}},$$

respectively.

In Theorem 3.4, only a part of the asymptotic expansion is in the exponent. Therefore, the following question naturally arises. Is it possible to find a signed CP measure which, up to a constant, provides the same accuracy as in Theorem 3.4? As it follows from the following result, such a measure indeed exists.

Theorem 3.5. *If* (8) *holds*, *then*

$$\|F_n - H\exp\{\varkappa_1(G-I)\}D_3^n\| \le C(p+\overline{q})\left\{\min\left(\overline{q}, \frac{1}{n}\right) + \mathrm{e}^{-C_1 n}\right\},$$

$$\|F_n - H\exp\{\varkappa_1(G-I)\}D_3^n\|_\infty \le C(p+\overline{q})\left\{\min\left(\overline{q}, \frac{1}{n\sqrt{n\overline{q}}}\right) + \mathrm{e}^{-C_1 n}\right\},$$

$$\|F_n - H\exp\{\varkappa_1(G-I)\}D_3^n\|_{\mathrm{W}} \le C(p+\overline{q})\left\{\min\left(\overline{q}, \sqrt{\frac{\overline{q}}{n}}\right) + \mathrm{e}^{-C_1 n}\right\}.$$

3.4. Lower bound estimates

In this section, we show that in some cases, the estimates in Theorems 3.1 and 3.3 are of the correct order. We concentrate our attention on the case $n\overline{q} \ge 1$.

Theorem 3.6. *Let condition* (8) *be satisfied and let $n\overline{q} \ge 1$. Then, for some absolute constants C_2 and C_3*,

$$\|F_n - HD_1^\lambda\| \ge C_2\overline{q}\left(1 - C_3\left(\overline{q} + \frac{p}{\sqrt{n\overline{q}}}\right)\right), \tag{16}$$

$$\|F_n - HD_1^\lambda\|_\infty \ge C_2\sqrt{\frac{\overline{q}}{n}}\left(1 - C_3\left(\overline{q} + \frac{p}{\sqrt{n\overline{q}}}\right)\right), \tag{17}$$

$$\|F_n - HD_1^{\lambda}\|_W \geq C_2\overline{q}\sqrt{n\overline{q}}\left(1 - C_3\left(\overline{q} + \frac{p}{\sqrt{n\overline{q}}}\right)\right). \tag{18}$$

It is obvious that estimates (16)–(18) are non-trivial only when the expression in the brackets is positive. Let $p \leq 1/2$, $n\overline{q} \to \infty$ and $\overline{q} \to 0$. Combining Theorems 3.1 and 3.6, for sufficiently large n, we obtain

$$C_4\overline{q} \leq \|F_n - HD_1^{\lambda}\| \leq C_5\overline{q},$$
$$C_4\sqrt{\frac{\overline{q}}{n}} \leq \|F_n - HD_1^{\lambda}\|_\infty \leq C_5\sqrt{\frac{\overline{q}}{n}},$$
$$C_4\overline{q}\sqrt{n\overline{q}} \leq \|F_n - HD_1^{\lambda}\|_W \leq C_5\overline{q}\sqrt{n\overline{q}}.$$

Of course, the last estimate, as well as the one in (18), is of interest only if $\overline{q}\sqrt{n\overline{q}} \to 0$. Similar results can be obtained for the signed CP approximations.

Theorem 3.7. *Let condition* (8) *be satisfied and let* $n\overline{q} \geq 1$. *Then, for some absolute constants* C_6 *and* C_7,

$$\|F_n - H\exp\{\varkappa_1(G - I)\}D_2^n\| \geq C_6\sqrt{\frac{\overline{q}}{n}}\left(|\tilde{\gamma}_3| - C_8\frac{p + \overline{q}}{\sqrt{n\overline{q}}}\right), \tag{19}$$

$$\|F_n - H\exp\{\varkappa_1(G - I)\}D_2^n\|_\infty \geq \frac{C_6}{n}\left(|\tilde{\gamma}_3| - C_8\frac{p + \overline{q}}{\sqrt{n\overline{q}}}\right), \tag{20}$$

$$\|F_n - H\exp\{\varkappa_1(G - I)\}D_2^n\|_W \geq C_6\overline{q}\left(|\tilde{\gamma}_3| - C_8\frac{p + \overline{q}}{\sqrt{n\overline{q}}}\right). \tag{21}$$

Let $n\overline{q} \to \infty$ as $\overline{q} \to 0$ and $p \to \tilde{p}$. Also, assume that n is sufficiently large so that the right-hand estimates of (19)–(21) are positive. We then have

$$C_8\sqrt{\frac{\overline{q}}{n}} \leq \|F_n - H\exp\{\varkappa_1(G - I)\}D_2^n\| \leq C_9\sqrt{\frac{\overline{q}}{n}},$$
$$\frac{C_8}{n} \leq \|F_n - H\exp\{\varkappa_1(G - I)\}D_2^n\|_\infty \leq \frac{C_9}{n},$$
$$C_8\overline{q} \leq \|F_n - H\exp\{\varkappa_1(G - I)\}D_2^n\|_W \leq C_9\overline{q}.$$

3.5. Asymptotically sharp constants

In the previous section, we proved that upper and lower bound estimates are of the same order, provided that $n\overline{q}$ is large and $\overline{q}$ is small. As it turns out, if, in addition, p is small, then it is possible to obtain asymptotically sharp constants.

Theorem 3.8. *Let* $p \le 1/4$, $\overline{q} \le 1/30$ *and* $n\overline{q} \ge 1$. *Then*

$$\big|\|F_n - HD_1^{\lambda}\| - A_{11}\big| \le C\overline{q}\left(p + \overline{q} + \frac{1}{\sqrt{n\overline{q}}}\right), \tag{22}$$

$$\big|\|F_n - HD_1^{\lambda}\|_{\infty} - A_{12}\big| \le C\sqrt{\frac{\overline{q}}{n}}\left(p + \overline{q} + \frac{1}{\sqrt{n\overline{q}}}\right), \tag{23}$$

$$\big|\|F_n - HD_1^{\lambda}\|_{\mathrm{W}} - A_{13}\big| \le C\overline{q}\sqrt{n\overline{q}}\left(p + \overline{q} + \frac{1}{\sqrt{n\overline{q}}}\right), \tag{24}$$

where

$$A_{11} = \frac{4|\gamma_2|}{\gamma_1 q\sqrt{2\pi \mathrm{e}}}, \qquad A_{12} = \frac{|\gamma_2|}{\gamma_1\sqrt{\gamma_1}\sqrt{2\pi n q}}, \qquad A_{13} = \frac{|\gamma_2|\sqrt{2n}}{q\sqrt{\gamma_1 \pi q}}.$$

As a consequence of (22), we note that if $p \to 0$, $\overline{q} \to 0$ and $n\overline{q} \to \infty$, then

$$\|F_n - HD_1^{\lambda}\| \sim \frac{6\overline{q}}{\sqrt{2\pi \mathrm{e}}}.$$

Similar relations can be obtained for the local and Wasserstein norms as well.

4. Applications of Markov binomial models

In this section, we discuss some areas where the results of our paper can be applied.

(i) *Aggregate claim distribution in the individual model.* Consider a portfolio of n risks. Each risk produces a positive claim amount during a certain reference period. The aggregate claim of the portfolio is then

$$S^{\mathrm{ind}} = X_1 + X_2 + \cdots + X_n.$$

It is usually assumed that all X_j are independent. However, the independence of claims does not always reflect reality. For example, an accident involving a tourist group, life insurance for a husband and wife or pensions for workers of the same company are likely to produce dependent risks. For discussion of the dependence of risks and further examples, see [10,16] and [26].

Compound Poisson and signed compound Poisson approximations in the independent case of an individual model have been quite thoroughly investigated; see, for example, [15,17]. On the other hand, there are only a few results for the total variation metric for dependent risks. Dhaene and Goovaerts [10] investigated a similar model (although not explicitly Markovian) under an assumption which, in our notation, is equivalent to $\overline{q}_m = 0$. However, under such an assumption, one cannot expect the limiting law to be compound Poisson. Therefore, we have excluded this peripheral case from this paper, assuming $\overline{q}$ to be small, but not identically zero. In [16], Poisson approximation in the general setting of dependent risks was discussed. However, in our case, their result is not applicable since for small $\overline{q}$, the distribution of the approximated sum is not close to the Poisson distribution, but rather to the compound Poisson law.

Let us assume that aggregated claim amount S^{ind} of the portfolio consists of N independent groups of risks. We assume a homogeneous model for each group of risks with Markovian dependence. Let each risk have a two-point distribution. More precisely, let

$$S^{\text{ind}} = \sum_{m=1}^{N} \sum_{j=1}^{n_m} X_j^m.$$

Here, X_j^m and X_k^l are independent if $m \neq l$. We assume that each risk of the mth group can produce a claim of size a_m. Moreover, the dependence of risks of the same group is Markovian: $\mathrm{P}(X_1^m = a_m) = \overline{q}_m$, $\mathrm{P}(X_1^m = 0) = \overline{p}_m$ and

$$\mathrm{P}(X_j^m = a_m | X_{j-1}^m = 0) = \overline{q}_m, \qquad \mathrm{P}(X_j^m = 0 | X_{j-1}^m = 0) = \overline{p}_m,$$
$$\mathrm{P}(X_j^m = a_m | X_{j-1}^m = a_m) = p_m < 1/2, \qquad \mathrm{P}(X_j^m = 0 | X_{j-1}^m = a_m) = q_m,$$
$$p_m + q_m = \overline{q}_m + \overline{p}_m = 1, \qquad p_m, \overline{q}_m \in (0,1), \qquad m = 1, 2, \dots, N,\ j = 2, \dots, n_m.$$

The results of the previous sections can now easily be applied. We illustrate this with just one example. Let us define a compound Poisson variable in the following way:

$$S^{\text{cp}} = \sum_{m=1}^{N} a_m \sum_{j=0}^{N_m} Y_{jm}.$$

Here, Y_{jm} are i.i.d. geometric random variables, $\mathrm{P}(Y_{jm} = k) = q_m p_m^{k-1}$, $k = 1, 2, \dots,$ and N_m is a Poisson random variable with parameter $n_m q_m \overline{q}_m / (q_m + \overline{q}_m)$. The random variables N_m, $m = 1, 2, \dots, N$, are independent and also do not depend on Y_{jm}. Denote the distributions of S^{ind} and S^{cp} by F^{ind} and F^{cp}, respectively. The characteristic function of F^{cp} is then given by

$$\widehat{F}^{\text{cp}}(t) = \exp\left\{ \sum_{m=1}^{N} \frac{n_m q_m \overline{q}_m (\mathrm{e}^{\mathrm{i}ta_m} - 1)}{(q_m + \overline{q}_m)(1 - p_m \mathrm{e}^{\mathrm{i}ta_m})} \right\}.$$

Also, we have the following estimate of approximation:

$$\begin{aligned} \|F^{\text{ind}} - F^{\text{cp}}\| \le C \sum_{m=1}^{N} [&\overline{q}_m (p_m + \overline{q}_m) \min(1, (n_m \overline{q}_m)^{-1/2}) \\ &+ \min(\overline{q}_m, n_m \overline{q}_m^2) + (p_m + \overline{q}_m) \mathrm{e}^{-C_1 n_m}]. \end{aligned} \tag{25}$$

Note that the approximation is closer if all $\overline{q}_m$ are small.

For the proof of (25), one should use the triangle inequality, thus reducing the problem to N estimates of Markov binomial distributions concentrated on $0, a_m, 2a_m, \dots$. The total variation metric is invariant with respect to norming. Therefore, without loss of generality, one can switch to integer numbers and apply (9) N times with $p_0 = 0$.

It is obvious that the second order estimates and estimates in Wasserstein metric can be obtained in a similar way.

(ii) *System failure models.* The Markov binomial distribution naturally arises in weather and stock market trends. It is also a natural model for system failure situations. As an example, we present one model from Sahinoglu [29], who considered an electric power supply system with operating and non-operating states throughout a year-long period of operation, discretized in hours. Let M_i be the margin values at hourly steps, that is,

$$M_i = TPG - X - L_i,$$

where *TPG* denotes total power generation, L_i denotes power demand (hourly peak load forecast) and X denotes unplanned forced outages. Let Y_i be an indicator of $\{M_i < 0\}$. Then $S = Y_1 + Y_2 + \cdots + Y_n$ represents cumulated hours of negative-margin hours, that is, the unavailability of power at the nth hour. It is natural to assume that S has a Markov binomial distribution. Notably, the Markov chain $Y_1, Y_2, \ldots$ is non-stationary. Therefore, many known results about the compound Poisson approximation cannot be applied directly. Also, the results of our paper relax the assumptions on transition probabilities from Sahinoglu's model and give estimates of the accuracy of approximations. Further, as shown in Sahinoglu [29], page 49, the probabilities of the compounding geometric law under certain assumptions can be viewed as probabilities for the number of trials required to repair the system.

(iii) *Industrial applications: sampling plans.* A basic assumption in standard acceptance plans for attributes is that the characteristics of items in the lots are i.i.d. Bernoulli variables. Recently, however, the focus has been on monitoring the ongoing production process by inspecting the items sequentially. In such cases, the quality levels of successive items are statistically dependent and it has been found in practice that the Markov-dependent model is a very useful one; see [24]. Indeed, Bhat *et al.* [3] modified the standard acceptance sampling plans and proposed sequential single sampling plans for monitoring Markov-dependent production processes. Vellaisamy and Sankar [35] proposed optimal systematic sampling plans for Markov-dependent processes. We will outline some possible new research directions in this field.

5. Auxiliary results

We now introduce further notation:

$$a_1 = \gamma_1, \qquad a_2 = \gamma_2 + \frac{a_1^2}{2}, \qquad a_3 = \gamma_3 + a_1 a_2 - \frac{a_1^3}{3}, \tag{26}$$

$$Y = G - I, \qquad B = \sum_{j=0}^{\infty} (pI_1 - \overline{q}I)^j, \qquad K = \sum_{j=0}^{\infty} (pI_1 - \overline{q}I - 2\gamma_1 Y)^j, \tag{27}$$

$$L = \frac{4\overline{q}^2}{(q+\overline{q})^2} Y^2 [q^2 I + p(q+\overline{q})(I - pI_1)] K^2. \tag{28}$$

In the following two lemmas, $C(k)$ denotes an absolute positive constant depending on k. Throughout this paper, we set $0^0 = 1$.

Lemma 5.1. *Let $t > 0$, $k \in \{0, 1, \dots\}$ and $0 < p < 1$. Also, let M be a finite (signed) measure concentrated at $\mathbb{Z}$. Then, for Y defined in (27),*

$$\|Y^2 \mathrm{e}^{tY}\| \le \frac{3}{t\mathrm{e}}, \qquad \|Y^k \mathrm{e}^{tY}\| \le \left(\frac{2k}{t\mathrm{e}}\right)^{k/2}. \tag{29}$$

If $p \le 1/2$, then

$$\|Y^k \mathrm{e}^{tY}\|_\infty \le \frac{C(k)}{t^{(k+1)/2}}, \qquad \|YM\|_{\mathrm{W}} \ge \frac{2}{3}\|M\|, \qquad \|YM\| \ge \frac{2}{3}\|(I_1 - I)M\|. \tag{30}$$

Proof. The estimates in (29) follow from the properties of the total variation norm and results in [27] and [11]. The first estimate in (30) is a consequence of the inversion formula and the following inequalities:

$$\mathrm{Re}\,\widehat{Y}(t) \le -\frac{2}{1+p}\sin^2\frac{t}{2}, \qquad |\widehat{Y}(t)| \le \frac{2}{q}\left|\sin\frac{t}{2}\right|.$$

Here, Re$\{\cdot\}$ means the real part of the complex number. In view of the relation between total variation and Wasserstein norms (see the Introduction), we get

$$\|MY\|_{\mathrm{W}} = \left\|(I_1 - I)M\sum_{j=0}^{\infty} p^j I_j\right\|_{\mathrm{W}} = \left\|M\sum_{j=0}^{\infty} p^j I_j\right\|,$$

$$\|M(I_1 - I)\| = \|MY(I - pI_1)\| \le \|MY\|(1+p),$$

$$\|M\| = \left\|M\sum_{j=0}^{\infty} p^j I_j(I - pI_1)\right\| \le \left\|M\sum_{j=0}^{\infty} p^j I_j\right\|(1+p).$$

The results in (30) now follow easily. □

For our asymptotically sharp results, we need the following lemma. Set

$$\varphi_k(x) = \frac{1}{\sqrt{2\pi}}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\mathrm{e}^{-x^2/2}, \qquad \|\varphi_k\|_1 = \int_{\mathbb{R}} |\varphi_k(x)|\,\mathrm{d}x, \qquad \|\varphi_k\|_\infty = \sup_{x\in\mathbb{R}} |\varphi_k(x)|$$
$$(k = 0, 1, \dots).$$

Lemma 5.2. *Let $t > 0$ and $k = 0, 1, 2, \dots$. We then have*

$$\left|\left\|(I_1 - I)^k \mathrm{e}^{t(I_1 - I)}\right\| - \frac{\|\varphi_k\|_1}{t^{k/2}}\right| \le \frac{C(k)}{t^{(k+1)/2}},$$

$$\left|\left\|(I_1 - I)^k \mathrm{e}^{t(I_1 - I)}\right\|_\infty - \frac{\|\varphi_k\|_\infty}{t^{(k+1)/2}}\right| \le \frac{C(k)}{t^{k/2+1}},$$

$$\left|\left\|(I_1 - I)^k \mathrm{e}^{t(I_1 - I)}\right\|_{\mathrm{W}} - \frac{\|\varphi_{k-1}\|_1}{t^{(k-1)/2}}\right| \le \frac{C(k)}{t^{k/2}} \qquad (k \ne 0).$$

The proof follows from a more general Proposition 4 in [28].

Lemma 5.3. *If* $N > 0$ *and* $0 < \alpha \le p < 1$, *then* $(I + \alpha Y)^N$ *is a CP distribution.*

Proof. Note that

$$(I + \alpha Y)^N = \exp\{-N \ln(1-\alpha)(F - I)\}.$$

Here, F is a distribution concentrated on $\{1, 2, 3, \ldots\}$ with

$$F\{j\} = -\frac{1}{\ln(1-\alpha)} \frac{1}{j}\left(p^j - \left(\frac{p-\alpha}{1-\alpha}\right)^j\right).$$

The last relation obviously completes the proof. □

Before we proceed to our main lemma, we need some additional facts about F_n. Similarly to [5] (see also [7]), it is possible to check that under assumption (8), we have

$$\widehat{F}_n(t) = \widehat{\Lambda}_1^n(t)\widehat{W}_1(t) + \widehat{\Lambda}_2^n(t)\widehat{W}_2(t), \tag{31}$$

where

$$\widehat{\Lambda}_{1,2}(t) = \frac{p\mathrm{e}^{\mathrm{i}t} + \overline{p} \pm \widehat{D}^{1/2}(t)}{2},$$

$$\widehat{W}_{1,2}(t) = \frac{p_0}{2}\left(1 \pm \frac{q + \overline{q} + p(\mathrm{e}^{\mathrm{i}t} - 1)}{\widehat{D}^{1/2}(t)}\right) + \frac{1-p_0}{2}\left(1 \pm \frac{q + \overline{q} + (2\overline{q} - p)(\mathrm{e}^{\mathrm{i}t} - 1)}{\widehat{D}^{1/2}(t)}\right),$$

$$\widehat{D}(t) = (p\mathrm{e}^{\mathrm{i}t} + \overline{p})^2 + 4\mathrm{e}^{\mathrm{i}t}(\overline{q} - p).$$

This allows us to write F_n as

$$F_n = \Lambda_1^n W_1 + \Lambda_2^n W_2 \tag{32}$$

and to express $\Lambda_{1,2}$ and $W_{1,2}$ as the following series:

$$\Lambda_1 = I + a_1 Y + \frac{1}{2}\{(1 + \overline{q})I - pI_1 + 2a_1 Y\} \sum_{j=1}^{\infty} \binom{1/2}{j} (-1)^j L^j, \tag{33}$$

$$\Lambda_2 = pI_1 - \overline{q}I + (I - \Lambda_1), \tag{34}$$

$$\begin{aligned} W_{1,2} = \frac{1}{2}\Biggl\{ I \pm [(q + \overline{q})I + p(I_1 - I)]K \sum_{j=0}^{\infty} \binom{-1/2}{j} (-1)^j L^j \Biggr\} \\ \pm (1 - p_0)(\overline{q} - p)(I_1 - I)K \sum_{j=0}^{\infty} \binom{-1/2}{j} (-1)^j L^j. \end{aligned} \tag{35}$$

The following lemma is used as the main tool in the proofs.

Lemma 5.4. *If condition* (8) *is satisfied, then*

$$\Lambda_1 = I + \sum_{j=1}^{3} a_j Y^j + C\overline{q}^3(p+\overline{q})Y^4\Theta, \tag{36}$$

$$\ln \Lambda_1 = \sum_{j=1}^{3} \gamma_j Y^j + C\overline{q}^3(p+\overline{q})Y^4\Theta, \tag{37}$$

$$\ln \Lambda_1 = \gamma_1 Y + \frac{19}{60}\gamma_1 Y^2\Theta, \tag{38}$$

$$\|\Lambda_2\| \leq \frac{19}{30}, \qquad \|\Lambda_1 - I\| \leq 0.1, \tag{39}$$

$$W_1 = I + (\varkappa_1 + \varkappa_2)Y + C\overline{q}(p+\overline{q})Y^2\Theta, \tag{40}$$

$$W_2 = C(p+\overline{q})(I_1 - I)\Theta, \qquad \|W_2\| \leq 7. \tag{41}$$

For any finite signed measure M on $\mathbb{Z}$ and any $t > 0$, we have

$$\|M \exp\{t \ln \Lambda_1\}\| \leq C\|M \exp\{(t\gamma_1/30)Y\}\|, \tag{42}$$

$$\|M D_j^t\| \leq C\|M \exp\{(t\gamma_1/30)Y\}\|, \qquad j = 1, 2, 3. \tag{43}$$

Estimates (42)–(43) *also hold for the local norm.*

Proof. We have

$$a_1 = \gamma_1 \leq \frac{1}{30}, \qquad \frac{1}{q+\overline{q}} \leq \frac{1}{1-p} \leq 2, \qquad \|Y\| \leq \|G\| + 1 = 2,$$

$$\|K\| \leq \sum_{j=0}^{\infty}(p+\overline{q}+4a_1)^j \leq 3, \tag{44}$$

$$\|L\| \leq 9 \cdot 4 \cdot \overline{q}^2 \cdot 4\left(1 + \frac{p}{q+\overline{q}}(1+p)\right) \leq 0.4. \tag{45}$$

Note that

$$\left|\binom{1/2}{2}\right| = \frac{1}{8}, \qquad \left|\binom{1/2}{3}\right| = \frac{1}{16}, \qquad \left|\binom{1/2}{j}\right| \leq \frac{5}{128}, \qquad j \geq 4.$$

We have

$$\sum_{j=1}^{\infty}\left|\binom{1/2}{j}\right| \|L\|^{j-1} \leq \frac{1}{2} + \frac{0.4}{8} + \frac{(0.4)^2}{16} + \frac{5}{128}\frac{(0.4)^3}{0.6} \leq 0.5642$$

and

$$[I(1+\overline{q})-pI_1+2a_1Y]L=4\frac{\overline{q}^2Y^2}{(q+\overline{q})^2}[q^2I+p(q+\overline{q})(I-pI_1)]K$$
$$=\gamma_1Y^2\frac{12\overline{q}}{q(q+\overline{q})}\big(q^2+p(q+\overline{q})(1+p)\big)\Theta=\gamma_1Y^2\Theta.$$

Consequently,

$$\Lambda_1=I+\gamma_1Y+\tfrac{1}{2}0.5642\gamma_1Y^2\Theta=I+1.2821\gamma_1Y\Theta=I+0.1\Theta$$

and

$$\ln\Lambda_1=\Lambda_1-I+\sum_{j=2}^{\infty}\frac{(-1)^{j+1}}{j}(\Lambda_1-I)^j$$
$$=\gamma_1Y+0.2821\gamma_1Y^2\Theta+\frac{1}{2}1.2821\gamma_1^2Y^2\sum_{j=2}^{\infty}(0.1)^{j-2}\Theta\;=\gamma_1Y+\frac{19}{60}\gamma_1Y^2\Theta.$$

Moreover, $\|\Lambda_2\|\le p+\overline{q}+\|\Lambda_1-I\|\le 19/30$. Thus, we have proven (38). We use this estimate for obtaining (42). By the properties of the total variation norm, we have

$$\|M\mathrm{e}^{t\ln\Lambda_1}\|\le\left\|M\exp\left\{\frac{t\gamma_1}{30}Y\right\}\right\|\left\|\exp\left\{\frac{29t\gamma_1}{30}Y+\frac{19t\gamma_1}{60}Y^2\Theta\right\}\right\|.$$

Applying Lemma 5.1, we prove that the second norm is majorized by

$$1+\sum_{r=1}^{\infty}\frac{1}{r!}\left\|\frac{19}{60}t\gamma_1Y^2\exp\left\{\frac{29}{30r}t\gamma_1Y\right\}\right\|^r\le 1+\sum_{r=1}^{\infty}\frac{\mathrm{e}^r}{r^r\sqrt{2\pi r}}\left(\frac{57r}{58\mathrm{e}}\right)^r\le C.$$

The last two estimates obviously lead to (42). The estimate (43) is proved similarly. For the proof of (36), note that

$$\Lambda_1=I+\gamma_1Y-\frac{\overline{q}^2}{(q+\overline{q})^2}Y^2[q^2I+p(q+\overline{q})(I-pI_1)]K+C\overline{q}^4Y^4\Theta,\tag{46}$$

$$I_1-I=Y(I-pI_1),\qquad (q+\overline{q})B=I+p(I_1-I)B,\tag{47}$$

$$I_1-I=2Y\Theta,\qquad (q+\overline{q})(I-pI_1)B=qI-p\overline{q}(I-pI_1)BY,\tag{48}$$

$$B=\frac{1}{q+\overline{q}}I+\frac{pq}{(q+\overline{q})^2}Y-\frac{p^2\overline{q}}{(q+\overline{q})^2}Y(I_1-I)B,\tag{49}$$

$$K=B+2\gamma_1YKB=B-2\gamma_1B^2Y+4\gamma_1^2B^2Y^2B^2K.\tag{50}$$

Substituting (48)–(50) into (46), we obtain (36). Taking into account (39), we obtain

$$\ln \Lambda_1 = \sum_{j=1}^{3} \frac{(-1)^{j+1}}{j}(\Lambda_1 - I)^j + C(\Lambda_1 - I)^4\Theta.$$

Now, for the proof of (37), it suffices to use (36). From (50) and the first relation in (48), we get $p(I_1 - I)K = p(I_1 - I)B + Cp\overline{q}Y^2\Theta$. Moreover,

$$p(I_1 - I)B = \frac{pq}{q+\overline{q}}Y - \frac{p^2\overline{q}(I_1 - I)BY}{q+\overline{q}} = \frac{pqY}{q+\overline{q}} + Cp\overline{q}Y^2\Theta.$$

The last two equations and (35) allow us to prove (40). Since $W_1 + W_2 = I$, we easily obtain the first relation in (41). Now,

$$\|W\|_2 \le \frac{1}{2}\left(1 + (q+\overline{q})3\sum_{j=0}^{\infty}\left|\binom{-1/2}{j}\right|0.4^j\right) + |\overline{q} - p| \cdot 2 \cdot 3\sum_{j=0}^{\infty}\left|\binom{-1/2}{j}\right|0.4^j < 7.$$

Thus, Lemma 5.4 is proved. For the lower bound estimates, we need the following result. □

Lemma 5.5. *Let M be concentrated on $\mathbb{Z}$, $\alpha \in \mathbb{R}$ and $b > 1$. Then,*

$$\|M\| \ge C\left|\int_{-\infty}^{\infty} \mathrm{e}^{-t^2/2}\widehat{M}\left(\frac{t}{b}\right)\mathrm{e}^{-\mathrm{i}t\alpha}\,\mathrm{d}t\right|, \tag{51}$$

$$\|M\|_\infty \ge \frac{C}{b}\left|\int_{-\infty}^{\infty} \mathrm{e}^{-t^2/2}\widehat{M}\left(\frac{t}{b}\right)\mathrm{e}^{-\mathrm{i}t\alpha}\,\mathrm{d}t\right|. \tag{52}$$

The estimates (51) and (52) remain valid if $\mathrm{e}^{-t^2/2}$ is replaced by $t\mathrm{e}^{-t^2/2}$.

Lemma 5.5, with $\|M\|$ replaced by the uniform norm of M, was proven in [32]. Since the uniform norm is majorized by the total variation norm, (51) also holds.

Lemma 5.6. *If (8) is satisfied, then, for all $|t| \le \pi$,*

$$\left|\exp\{n\varkappa_1(\widehat{Y}(t) - \mathrm{i}t/q)\} - 1\right| \le Cn\overline{q}t^2, \tag{53}$$

$$|\widehat{D}_2^n(t)| \le 1, \qquad |\widehat{D}_2^n \exp\{-\mathrm{i}tn\gamma_1/q\} - 1| \le Cn\overline{q}t^2. \tag{54}$$

Proof of Lemma 5.6 is straightforward and therefore omitted.

Finally, let us introduce an inverse compound measure for H. Let

$$H^{-1} = \exp\left\{-\sum_{j=1}^{\infty}\frac{p^j}{j}\left(1 - \left(\frac{1 - p_0q/(q+\overline{q})}{1-\varkappa_2}\right)^j\right)(I_j - I)\right\}.$$

Lemma 5.7. *If* (8) *is satisfied, then*

$$\|H^{-1}\| \le \mathrm{e}^2 \tag{55}$$

and, for any (*signed*) *finite measure* M *concentrated at* $\mathbb{Z}$,

$$\|MH\| \ge \mathrm{e}^{-2}\|M\|, \qquad \|M\exp\{-p_0\gamma_1 Y\}\| \ge \|M\|. \tag{56}$$

The estimates in (56) *remain valid if the total variation norm is replaced by the local norm.*

Proof. Estimate (55) easily follows from the property $\|\mathrm{e}\|^M \le \mathrm{e}^{\|M\|}$; see the Introduction. Now, $\|M\| = \|MHH^{-1}\| \le \|MH\|\|H^{-1}\| \le \mathrm{e}^2\|MH\|$. Since $\exp\{p_0\gamma_1 Y\}$ is a distribution, its total variation is 1. Therefore, $\|M\| = \|M\exp\{-p_0\gamma_1 Y\}\exp\{p_0\gamma_1 Y\}\| \le \|M\exp\{-p_0\gamma_1 Y\}\|$. Estimates for the local norm are proved similarly. □

6. Proofs

For upper bound estimates, we use an adaptation of Le Cam's [23] approach which deals with convolutions of measures.

Proof of Theorem 3.1. Without loss of generality, we can assume that (8) holds. We have

$$\|F_n - HD_1^\lambda\| \le \|\Lambda_1^n - D_1^n\|\|W_1\| + \|D_1^n(W_1 - H\exp\{-p_0\gamma_1 Y\})\| + \|\Lambda_2\|^n\|W_2\|.$$

Further, in view of Lemma 5.4,

$$\begin{aligned}\|\Lambda_1^n - D_1^n\| &\le \left\| D_1^n \int_0^1 (\exp\{\tau[n\ln\Lambda_1 - n\gamma_1 Y]\})'_\tau \,\mathrm{d}\tau \right\| \\ &\le n\int_0^1 \|[\ln\Lambda_1 - \gamma_1 Y]\exp\{\tau n\ln\Lambda_1 + (1-\tau)n\gamma_1 Y\}\|\,\mathrm{d}\tau \\ &\le Cn\|[\ln\Lambda_1 - \gamma_1 Y]\exp\{(n\gamma_1/30)Y\}\| \le Cn\overline{q}^2\|Y^2\exp\{(n\gamma_1/30)Y\}\|.\end{aligned}$$

By Lemma 5.4,

$$\begin{aligned}W_1 - H\exp\{-p_0\gamma_1 Y\} &= [W_1 - I - (\varkappa_1 + \varkappa_2)Y] + [I + (\varkappa_1 + \varkappa_2)Y - (I - p_0\gamma_1 Y)H] \\ &\quad + [H(I - p_0\gamma_1 Y - \exp\{-p_0\gamma_1 Y\})] = C\overline{q}(p + \overline{q})Y\Theta.\end{aligned}$$

Taking into account the last two estimates, applying Lemma 5.1 and estimating $\|W_2\|$ and $\|\Lambda_2\|$ by (41) and (39), we complete the proof of (9). The estimates in (10) and (11) are proved similarly. □

Proof of Corollary 3.2. Following the proof of (42), one can prove the same property for H_1. Also,

$$\|HD_1^\lambda - HH_1^\lambda\| \le C\lambda\|(D_1 - H_1)\exp\{(n\gamma_1/30)Y\}\| \le Cn\overline{q}^2\|Y^2\exp\{(n\gamma_1/30)Y\}\|.$$

The rest of the proof is obvious. □

Proof of Theorem 3.2. We have

$$\begin{aligned}
&\|F_n - HD_1^{\lambda}(I + n\gamma_2 Y^2)\| \\
&\quad \le \|\Lambda_2\|^n \|W_2\| + \|W_1\| \|\Lambda_1^n - D_2^n\| \\
&\qquad + \|W_1\| \|D_1^n(\mathrm{e}^{n\gamma_2 Y^2} - I - n\gamma_2 Y^2)\| + \|D_1^n(I + n\gamma_2 Y^2)(W_1 - H\mathrm{e}^{-p_0\gamma_1 Y})\|.
\end{aligned}$$

Similarly to the proof of Theorem 3.1, and using (43), we obtain

$$\begin{aligned}
\|\Lambda_1^n - D_2^n\| &\le Cn\left\|[\ln\Lambda_1 - \gamma_1 Y - \gamma_2 Y^2]\int_0^1 \exp\{\tau n\ln\Lambda_1 + (1-\tau)[n\gamma_1 Y + n\gamma_2 Y^2]\}\,\mathrm{d}\tau\right\| \\
&\le Cn\|[\ln\Lambda_1 - \gamma_1 Y - \gamma_2 Y^2]\exp\{(n\gamma_1/30)Y\}\| \\
&\le Cn\overline{q}^2(\overline{q} + p)\|Y^3\exp\{(n\gamma_1/30)Y\}\|
\end{aligned}$$

and

$$\begin{aligned}
\|D_1^n(\mathrm{e}^{n\gamma_2 Y^2} - I - n\gamma_2 Y^2)\| &\le \left\|(n\gamma_2 Y^2)^2\int_0^1 D_1^n \mathrm{e}^{\tau n\gamma_2 Y^2}(1-\tau)\,\mathrm{d}\tau\right\| \\
&\le C(n\gamma_2)^2\|Y^4\exp\{(n\gamma_1/30)Y\}\|.
\end{aligned}$$

Note that for any signed finite measure M,

$$\|D_1^n(I + n\gamma_2 Y^2)M\| \le \|D_1^{n/2}\|M(1 + n|\gamma_2|\|Y^2 D_1^{n/2}\|) \le C\|D_1^{n/2}M\|.$$

The rest of the proof is very similar to the proof of Theorem 3.1 and is hence omitted. □

Proof of Theorems 3.3, 3.4 and 3.5. The proofs are very similar to those of Theorems 3.1 and 3.2. From Lemma 5.4 and the definition of the exponent measure, it is not difficult to show that

$$\begin{aligned}
W_1 - \mathrm{e}^{\varkappa_1 Y}H &= [W_1 - I - (\varkappa_1 + \varkappa_2)Y] + [I + (\varkappa_1 + \varkappa_2)Y - (I + \varkappa_1 Y)H] \\
&\quad + H(I + \varkappa_1 Y - \mathrm{e}^{\varkappa_1 Y}) = C\overline{q}(p + \overline{q})Y^2\Theta, \\
\|\Lambda_1^n W_1 - D_2^n H\mathrm{e}^{\varkappa_1 Y}\| &\le \|\Lambda_1^n - D_2^n\|\|W_1\| + \|D_2^n(W_1 - H\mathrm{e}^{\varkappa_1 Y})\|.
\end{aligned}$$

Now, it is not difficult to prove Theorem 3.3. Theorem 3.5 is proved similarly. For the proof of Theorem 3.4, one should use Theorem 3.5, the triangle inequality and the fact that

$$\begin{aligned}
\|D_2^n(I + n\gamma_3 Y^3) - D_3^n\| &= \left\|D_2^n\int_0^1 (1-\tau)\mathrm{e}^{\tau n\gamma_3 Y^3}(n\gamma_3 Y^3)^2\,\mathrm{d}\tau\right\| \\
&\le C(n\gamma_3)^2\|Y^6\exp\{n\gamma_1 Y^2/30\}\|.
\end{aligned}$$

For the last estimate, we have used the same argument as in the proof of (43). □

Proof of Theorem 3.6. Taking into account Theorem 3.2, (30) and (56), we get

$$\begin{aligned}\|F_n - HD_1^{\lambda}\| &\geq n|\gamma_2|\|HD_1^{\lambda}Y^2\| - C\overline{q}\left(\overline{q} + \frac{p}{\sqrt{n\overline{q}}}\right)\\ &\geq C_{10}n|\gamma_2|\|D_1^n(I_1 - I)^2\| - C_{11}\overline{q}\left(\overline{q} + \frac{p}{\sqrt{n\overline{q}}}\right).\end{aligned} \tag{57}$$

Let $z = t/(h\sqrt{n\overline{q}})$ and $\mu = n\gamma_1/q$. The constant $h > 1$ will be chosen later. Applying Lemma 5.6, we then obtain

$$J = \left|\int_{\mathbb{R}} \mathrm{e}^{-t^2/2}\widehat{D}_1^n(z)\mathrm{e}^{-\mathrm{i}z\mu}(\mathrm{e}^{\mathrm{i}z} - 1)^2\,\mathrm{d}t\right| \geq \left|\int_{\mathbb{R}} \mathrm{e}^{-t^2/2}z^2\,\mathrm{d}t\right| - J_1 - J_2. \tag{58}$$

Here,

$$\begin{aligned}J_1 &= \int_{\mathbb{R}} \mathrm{e}^{-t^2/2}z^2|\widehat{D}_1^n(z)\mathrm{e}^{-\mathrm{i}z\mu} - 1|\,\mathrm{d}t \leq Cn\overline{q}\int_{\mathbb{R}} z^4\mathrm{e}^{-t^2/2}\,\mathrm{d}t = \frac{C}{h^4 n\overline{q}},\\ J_2 &= \int_{\mathbb{R}} \mathrm{e}^{-t^2/2}|\widehat{D}_1^n(z)\mathrm{e}^{-\mathrm{i}z\mu}||(\mathrm{e}^{\mathrm{i}z} - 1)^2 - (\mathrm{i}z)^2|\,\mathrm{d}t \leq \frac{C}{h^3 n\overline{q}\sqrt{n\overline{q}}}.\end{aligned}$$

Combining the last two estimates with (58) and choosing h to be a sufficiently large absolute constant, we obtain

$$J \geq \frac{C_{12}}{h^2 n\overline{q}}\left(1 - \frac{C_{13}}{h^2} - \frac{C_{14}}{h\sqrt{n\overline{q}}}\right) \geq \frac{C_{15}}{n\overline{q}}.$$

Applying Lemma 5.5 and substituting the result into (57), we get (16). Estimates (17) and (18) are proved similarly.

For the proof of Theorem 3.7, one should use Theorem 3.4 and take $t\exp\{-t^2/2\}$ instead of $\exp\{-t^2/2\}$. The proof is then almost identical to that of Theorem 3.6 and is hence omitted. □

Proof of Theorem 3.8. We have

$$\begin{aligned}\big|\|F_n - HD_1^{\lambda}\| - A_{11}\big| &\leq \|F_n - HD_1^{\lambda}(I + n\gamma_2 Y^2)\|\\ &\quad + \|(H\mathrm{e}^{-p_0\gamma_1 Y} - I)D_1^n n\gamma_2 Y^2\| + n|\gamma_2|\left\|\left(Y^2 - \frac{1}{q^2}(I_1 - I)^2\right)D_1^n\right\|\\ &\quad + \frac{n|\gamma_2|}{q^2}\left\|(I_1 - I)^2\left(D_1^n - \exp\left\{\frac{n\gamma_1}{q}(I_1 - I)\right\}\right)\right\|\\ &\quad + \left|\frac{n|\gamma_2|}{q^2}\left\|(I_1 - I)^2\exp\left\{\frac{n\gamma_1}{q}(I_1 - I)\right\}\right\| - A_{11}\right|.\end{aligned}$$

One should now apply Theorem 3.2, (48), Lemmas 5.2, 5.1 and the following, easily verifiable, relations:

$$Y = \frac{(I_1 - I)}{q} \sum_{j=0}^{\infty} \left(\frac{p}{q}\right)^j (I_1 - I)^j = \frac{(I_1 - I)}{q} + \frac{3p}{q^2}(I_1 - I)^2\Theta$$

and

$$\begin{aligned} &D_1 - \exp\left\{\frac{\gamma_1}{q}(I_1 - I)\right\} \\ &\quad = \exp\left\{\frac{\gamma_1}{q}(I_1 - I)\right\}\left(\exp\left\{\frac{3p\gamma_1}{q^2}(I_1 - I)^2\Theta\right\} - I\right) = Cp\overline{q}(I_1 - I)^2\Theta. \end{aligned}$$

Note that

$$\begin{aligned} &\left\|(I_1 - I)^2\left(D_1^n - \exp\left\{\frac{n\gamma_1}{q}(I_1 - I)\right\}\right)\right\| \\ &\quad = \left\|(I_1 - I)^2\left(D_1 - \exp\left\{\frac{\gamma_1}{q}(I_1 - I)\right\}\right)\sum_{j=1}^{n} D_1^{n-j}\exp\left\{(j-1)\frac{\gamma_1}{q}(I_1 - I)\right\}\right\| \\ &\quad \le Cnp\overline{q}\left(\|(I_1 - I)^4 D_1^{n/3}\| + \left\|(I_1 - I)^4\exp\left\{\frac{n\gamma_1}{3q}(I_1 - I)\right\}\right\|\right). \end{aligned}$$

All other estimates are obtained similarly. □

Acknowledgements

The main part of this work was accomplished during the first author's stay at the Department of Mathematics, IIT Bombay, during November 2006. The first author would like to thank the members of the Department for their hospitality. Also, we are grateful to the referees for some useful remarks.

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Received July 2007 and revised October 2009