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A note on the Lindeberg condition for convergence to stable laws in Mallows distance

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We correct a condition in a result of Johnson and Samworth (*Bernoulli* **11** (2005) 829–845) concerning convergence to stable laws in Mallows distance. We also give an improved version of this result, setting it in the more familiar context of a Lindeberg-like condition.

Keywords: Lindeberg condition; Mallows distance; stable laws

Theorem 5.2 of [1] considers a fixed parameter $\alpha \in (0,2)$, an independent sequence of random variables $X_1, X_2, \ldots$ with $S_n = (X_1 + \cdots + X_n)/n^{1/\alpha}$ and a random variable Y with an α-stable distribution. Theorem 5.2 claims that if there exist (independent) copies $Y_1, Y_2, \ldots$ of Y satisfying

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\{|X_i - Y_i|^{\alpha}\mathbb{1}(|X_i - Y_i| > b)\} \to 0 \tag{1}$$

as $b \to \infty$, then S_n (possibly shifted) converges to Y in Mallows distance d_α. The proof given for Theorem 5.2 requires simultaneous control of b and n, which is not provided by (1) as stated. Although the result could be corrected by adding "$\sup_n$" to the beginning of (1) and with other small modifications, we instead provide a more natural Lindeberg condition. We also change the centering, providing explicit expressions for the centering sequence for the case $\alpha \in (1,2)$. This is, in fact, a coupling theorem. Indeed, for $\alpha \in [1,2)$, if the Mallows distance between the distributions F_X and F_Y of X and Y is finite, then the random variables X and Y are highly dependent, in the sense that $d_\alpha^\alpha(X,Y) = \mathbb{E}|X - Y|^\alpha$ provided the joint distribution of (X,Y) is $F_X \wedge F_Y$.

Theorem 1. *Fix $0 < \alpha < 2$. Let $(X_1, Y_1), (X_2, Y_2), \ldots$ be a sequence of independent pairs such that $Y_1, Y_2, \ldots$ are copies of an α-stable random variable Y, and such that for all $b > 0$, we have*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\{|X_i - Y_i|^{\alpha}\mathbb{1}(|X_i - Y_i| > bn^{(2-\alpha)/2\alpha})\} = 0. \tag{2}$$

Then, writing $S_n = (X_1 + \cdots + X_n)/n^{1/\alpha}$, *there exists a sequence of constants* (c_n) *such that* $\lim_{n\to\infty} d_\alpha(S_n - c_n, Y) = 0$. *Moreover, when* $\alpha \in (1,2)$, *we may take* $c_n = n^{-1/\alpha} \times \sum_{i=1}^n \mathbb{E}X_i - \mathbb{E}Y$.

Proof. By Corollary 1.2.9 of [2],

$$\frac{1}{n^{1/\alpha}} \sum_{i=1}^n Y_i \stackrel{d}{=} \begin{cases} Y + \mu n^{1-1/\alpha} - \mu, & \text{if } \alpha \neq 1, \\ Y + \dfrac{2}{\pi}\sigma\beta \log n, & \text{if } \alpha = 1. \end{cases}$$

Here, the constants $\mu \in \mathbb{R}$, $\sigma \geq 0$ and $\beta \in [-1,1]$ are, respectively, the shift, scale and skewness parameters of the stable law of Y (see, e.g., [2], page 5), so for $\alpha \in (1,2)$, we may take $\mu = \mathbb{E}Y$. We first treat the case $\alpha \in (1,2)$. With c_n defined as in the statement of the theorem,

$$S_n - c_n - Y \stackrel{d}{=} n^{-1/\alpha} \sum_{i=1}^n (U_i - \mathbb{E}U_i + V_i - \mathbb{E}V_i),$$

where, writing $\delta = \frac{2-\alpha}{2\alpha}$,

$$U_i = (X_i - Y_i)\mathbb{1}(|X_i - Y_i| \leq bn^\delta),$$
$$V_i = (X_i - Y_i)\mathbb{1}(|X_i - Y_i| > bn^\delta).$$

Using Lyapunov's inequality and the fact that $|U_i| \leq bn^\delta$, we have

$$\begin{aligned} \mathbb{E}\left\{\left|\sum_{i=1}^n (U_i - \mathbb{E}U_i)\right|^\alpha\right\} &\leq \left[\mathbb{E}\left\{\left|\sum_{i=1}^n (U_i - \mathbb{E}U_i)\right|^2\right\}\right]^{\alpha/2} = \left(\sum_{i=1}^n \operatorname{Var} U_i\right)^{\alpha/2} \\ &\leq b^\alpha n^{(1+2\delta)\alpha/2} = b^\alpha n. \end{aligned} \tag{3}$$

Similarly, a von Bahr–Esseen moment bound given as equation (12) in [1] yields

$$\mathbb{E}\left\{\left|\sum_{i=1}^n (V_i - \mathbb{E}V_i)\right|^\alpha\right\} \leq 2\sum_{i=1}^n \mathbb{E}(|V_i - \mathbb{E}V_i|^\alpha) \leq 2^{\alpha+1} \sum_{i=1}^n \mathbb{E}(|V_i|^\alpha). \tag{4}$$

Thus, by (3) and (4), we find that for $\alpha \in (1,2)$,

$$\begin{aligned} d_\alpha^\alpha(S_n - c_n, Y) &\leq \mathbb{E}\{|S_n - c_n - Y|^\alpha\} \\ &\leq \frac{2^{\alpha-1}}{n}\mathbb{E}\left\{\left|\sum_{i=1}^n (U_i - \mathbb{E}U_i)\right|^\alpha\right\} + \frac{2^{\alpha-1}}{n}\mathbb{E}\left\{\left|\sum_{i=1}^n (V_i - \mathbb{E}V_i)\right|^\alpha\right\} \\ &\leq 2^{\alpha-1} b^\alpha + \frac{2^{2\alpha}}{n} \sum_{i=1}^n \mathbb{E}\{|X_i - Y_i|^\alpha \mathbb{1}(|X_i - Y_i| > bn^\delta)\}. \end{aligned}$$

We deduce from condition (2) that $\limsup_{n\to\infty} d_\alpha^\alpha(S_n - c_n, Y) \le 2^{\alpha-1}b^\alpha$. However, $b > 0$ was arbitrary, so the result follows.

When $\alpha \in (0, 1]$ and condition (2) holds, we can find a sequence (b_n) converging to zero with

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\bigl\{|X_i - Y_i|^\alpha \mathbb{1}\bigl(|X_i - Y_i| > b_n n^{(2-\alpha)/2\alpha}\bigr)\bigr\} = 0.$$

In this case, we should define

$$c_n = \begin{cases} n^{-1/\alpha}\sum_{i=1}^{n} \mathbb{E}\{(X_i - Y_i)\mathbb{1}(|X_i - Y_i| \le b_n n^\delta)\} + \mu n^{1-1/\alpha} - \mu, & \text{for } 0 < \alpha < 1, \\ n^{-1/\alpha}\sum_{i=1}^{n} \mathbb{E}\{(X_i - Y_i)\mathbb{1}(|X_i - Y_i| \le b_n n^\delta)\} + \frac{2}{\pi}\sigma\beta\log n, & \text{for } \alpha = 1. \end{cases}$$

Then, with the same definitions of U_i and V_i, except with b replaced by b_n, we have

$$S_n - c_n - Y \stackrel{d}{=} n^{-1/\alpha}\sum_{i=1}^{n} (U_i - \mathbb{E}U_i + V_i).$$

The argument now mimics the case $\alpha \in (1, 2)$. Using analogues of the bounds (3) and (4), we find

$$d_\alpha^\alpha(S_n - c_n, Y) \le b_n^\alpha + \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\bigl\{|X_i - Y_i|^\alpha \mathbb{1}\bigl(|X_i - Y_i| > b_n n^{(2-\alpha)/2\alpha}\bigr)\bigr\} \to 0. \qquad \square$$

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References

[1] Johnson, O. and Samworth, R. (2005). Central limit theorem and convergence to stable laws in Mallows distance. *Bernoulli* **11** 829–845. MR2172843

[2] Samorodnitsky, G. and Taqqu, M.S. (1994). *Stable Non-Gaussian Random Processes*. Boca Raton, FL: Chapman & Hall. MR1280932

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