

Integrated volatility and round-off error

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We consider a microstructure model for a financial asset, allowing for price discreteness and for a diffusive behavior at large sampling scale. This model, introduced by Delattre and Jacod, consists in the observation at the high frequency n , with round-off error α_n , of a diffusion on a finite interval. We give from this sample estimators for different forms of the integrated volatility of the asset. Our method is based on variational properties of the process associated with wavelet techniques. We prove that the accuracy of our estimation procedures is $\alpha_n \vee n^{-1/2}$. Using compensated estimators, limit theorems are obtained.

Keywords: diffusion models; high frequency data; integrated volatility; microstructure noise; round-off error; variation methods; wavelets

1. Introduction

Nowadays, a massive amount of high frequency financial data is available. This large quantity of data has paradoxically complicated some problems in statistical finance. Among them, one of the most relevant is the estimation of the integrated volatility of an asset. To fix ideas, let us consider an asset whose theoretical, efficient price $(X_t)_{t \in [0,1]}$ follows an Itô process of the form

$$dX_t = \mu_t dt + \sigma_t dW_t,$$

where W_t is a Brownian motion, μ_t the drift process and σ_t^2 the instantaneous volatility. From market price observations, we wish to estimate the quantity

$$\int_0^1 g(X_t)^2 \sigma_t^2 dt,$$

where g is a known deterministic function. The case $g(x) = 1$ corresponds to the absolute integrated volatility of the asset and the case $g(x) = 1/x$ to its relative integrated volatility, that is, the integral of the squared diffusion coefficient of the logarithmic price.¹ Assume first that we observe the efficient price data with frequency n , that is, the sample

$$(X_{i/n}, i = 0, \dots, n).$$

In this setting, a common convergent estimator of the integrated volatility, with rate $n^{-1/2}$ and feasible asymptotic theory, is given by the realized volatility, that is, for the absolute integrated

¹Note that the usual notion of integrated volatility refers to the relative integrated volatility.

volatility

$$\sum_{i=1}^n (X_{i/n} - X_{(i-1)/n})^2$$

and for the relative integrated volatility

$$\sum_{i=1}^n (\log(X_{i/n}) - \log(X_{(i-1)/n}))^2,$$

see Jacod and Protter [16], Barndorff-Nielsen and Shephard [7], Meddahi [22] and Gonçalves and Meddahi [12].

However, it is a well-known fact that high frequency financial data do not behave like an Itô process. In the literature, this gap is often considered to be a “contamination” of the theoretical, efficient price and is called microstructure noise. This microstructure noise increases with the sampling frequency and is due to several reasons, one of the most obvious being price discreteness.

To get rid of this noise, the first solution is to sample our data over a longer period. But, if we imagine we collect one unit of data per second and consider five minutes as the finest period we can tolerate to make the noise insignificant, we throw away a lot of data, which is hardly acceptable. Consequently, dealing with these high frequency noisy data has become a challenging issue. Many recent papers treat this problem, especially for the purpose of estimating the integrated volatility; see in particular Barndorff-Nielsen *et al.* [6]; Bandi and Russell [4]; Jacod *et al.* [15]; Zhang [30]; Zhang, Mykland and Aït-Sahalia [31]; Hansen and Lunde [13]; Aït-Sahalia, Mykland and Zhang [1]; and Gloter and Jacod [11]. For a comparison between several estimators, see Andersen, Bollerslev and Meddahi [3]; Bandi, Russel and Yang [5]; and Gatheral and Oomen [23].

In most of these works, one observes at some deterministic times t_i^n , $i = 0, \dots, n$, a log-price $\tilde{Y}_{t_i^n}$ composed of a theoretical, efficient log-price $\tilde{X}_{t_i^n}$, coming from the classical continuous-time mathematical finance theory, contaminated by an additive microstructure noise $\varepsilon_{t_i^n}^n$, that is,

$$\tilde{Y}_{t_i^n} = \tilde{X}_{t_i^n} + \varepsilon_{t_i^n}^n,$$

where \tilde{X}_t is, for example, an Itô process. In these additive microstructure noise models, the developed technologies often aim at reducing the impact of the noise.

Nevertheless, although price discreteness is largely accepted as one of the main reasons for microstructure noise, these models rarely allow for it; see Large [19] and Robert and Rosenbaum [25,26] for models considering discrete prices. In this paper, we study the problem of estimating the integrated volatility of an asset when assuming that the efficient price data are observed with round-off error.

2. Model and results

2.1. Description of the model

We consider the model of a diffusion observed with round-off error. Let α_n be a positive decreasing sequence tending to zero as n goes to infinity and $\beta_n = \alpha_n \sqrt{n}$. On a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P})$, we consider a one-dimensional Brownian semi-martingale $(X_t)_{t \in [0,1]}$, taking values in an open interval (ν, μ) , $-\infty \leq \nu < \mu \leq +\infty$, of the form

$$X_t = x_0 + \int_0^t \sigma(X_s) dW_s + \int_0^t a_s ds, \tag{1}$$

where $(W_t)_{t \in [0,1]}$ is a (\mathcal{F}_t) -standard Brownian motion, $(a_t)_{t \in [0,1]}$ is a progressively measurable process with respect to $(\mathcal{F}_t)_{t \in [0,1]}$, $x \rightarrow \sigma(x)$ is a real deterministic function that is not known and x_0 is a real constant. We observe the sample

$$(X_{i/n}^{(\alpha_n)}, i = 0, \dots, n), \tag{2}$$

where

$$X_{i/n}^{(\alpha_n)} = \alpha_n \lfloor X_{i/n} / \alpha_n \rfloor.$$

Thus, $X_{i/n}^{(\alpha_n)}$ is the observation of $X_{i/n}$ with round-off error α_n . This model has already been studied by Delattre and Jacod [9] when β_n tends to a constant finite value and by Delattre [8] in the other cases. Based on the sample (2), our goal is to estimate the random parameter

$$\theta = \int_0^1 g(X_s)^2 \sigma(X_s)^2 ds,$$

where g is a known deterministic function on (ν, μ) .

Note that for the Black–Scholes specification of the model

$$\sigma(x) = \sigma x,$$

the problem of the estimation of the constant parameter σ has been partially treated by Li and Mykland [20] in the case where β_n tends to zero.

We denote by $\mathcal{C}^k(I)$ the set of k times continuously differentiable functions on $I \subseteq \mathbb{R}$. We write $\mathcal{C}_b^k(I)$ if all the derivatives are bounded. We will consider the following assumptions:

Assumption A.

$$\sup_{n \geq 0} \alpha_n (\log n)^2 < \infty.$$

Assumption A1. There exists $\rho > 0$ such that $\sup_{n \geq 0} \alpha_n^{1-\rho} (\log n)^2 < \infty$.

Assumption B.

- (i) For all $x \in (v, \mu)$, $\sigma(x) > 0$,
- (ii) $x \rightarrow \sigma(x) \in \mathcal{C}^2((v, \mu))$,
- (iii) $\int_0^1 a_s^2 ds < +\infty$, almost surely.

Assumption C.

$x \rightarrow g(x) \in \mathcal{C}^2((v, \mu))$ and for all $x \in (v, \mu)$, $g(x) > 0$.

Assumption C1.

- (i) $x \rightarrow g(x) \in \mathcal{C}^2((v, \mu))$ and for all $x \in (v, \mu)$, $g(x) > 0$,
- (ii) $x \rightarrow g'(x)$ is of constant sign on (v, μ) and $x \rightarrow |g'(x)|^{1/2} \in \mathcal{C}^2((v, \mu))$.

By convention, if $x \notin (v, \mu)$, we set $g(x) = g'(x) = 0$.

Note that on $(0, +\infty)$, the functions defined by $x \rightarrow 1$ and $x \rightarrow 1/x$ satisfy Assumption C1. These functions are those respectively associated to the absolute integrated volatility and to the relative integrated volatility.

2.2. First estimator

Our estimation method is based on the theory of wavelet methods for quadratic functionals estimation; see, for example, Gayraud and Tribouley [10]. Throughout the paper, for $k \in \mathbb{N}$ and $j \in \mathbb{N}$, we set

$$\begin{aligned} \mathbb{1}_{jk}(s) &= \mathbb{1}_{(k/2^j, k+1/2^j)}(s), & \psi(s) &= -\mathbb{1}_{[0, 1/2]}(s) + \mathbb{1}_{(1/2, 1]}(s), \\ \psi_{jk}(s) &= 2^{j/2} \psi(2^j s - k). \end{aligned}$$

We define the coefficients c_{j_0k} , $j_0 \in \mathbb{N}$, $k \in [0, 2^{j_0} - 1]$ and d_{jk} , $j \in \mathbb{N}$, $k \in [0, 2^j - 1]$ by

$$c_{j_0k} = 2^{j_0/2} \int \mathbb{1}_{j_0k}(s) g(X_s) \sigma(X_s) ds, \quad d_{jk} = \int \psi_{jk}(s) g(X_s) \sigma(X_s) ds.$$

Hence, the c_{j_0k} and d_{jk} are the coefficients of $s \rightarrow g(X_s) \sigma(X_s)$ in the Haar basis. Consequently, we have

$$\theta = \sum_{k=0}^{2^{j_0}-1} c_{j_0k}^2 + \sum_{j=j_0}^{+\infty} \sum_{k=0}^{2^j-1} d_{jk}^2.$$

We set

$$\hat{c}_{j_0k} = \sqrt{\frac{\pi}{2}} \frac{2^{j_0/2}}{\sqrt{n}} \sum_{i=1}^n \mathbb{1}_{j_0k}(i/n) g(X_{(i-1)/n}^{(\alpha_n)}) |X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)}|.$$

Thus, in the case $g(x) = 1$, \hat{c}_{j_0k} can be seen as a rescaled local average of the increments of the rounded diffusion over a window of size 2^{-j_0} . We define our first estimator $\tilde{\theta}_n$ of θ by

$$\tilde{\theta}_n = \sum_{k=0}^{2^{j_{0,n}-1}} \hat{c}_{j_{0,n}k}^2$$

with $j_{0,n} = \lfloor \log_2(\alpha_n^{-1} \wedge \sqrt{n}) \rfloor$.

2.3. Convergence in probability

We set $r_n = \alpha_n \vee n^{-1/2}$. We have the following theorem:

Theorem 1 (Convergence in probability). *In model (1)–(2), under Assumptions A, B and C, the sequence*

$$r_n^{-1}(\tilde{\theta}_n - \theta)$$

is tight.

2.4. Compensated estimators

It seems difficult to obtain a central limit theorem for the previous estimator (see the proofs for details). Consequently, we introduce compensated estimators. We set

$$Q_j = \sum_{k=0}^{2^j-1} d_{jk}^2$$

and define

$$\hat{d}_{jk} = \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi_{jk}(i/n) g(X_{(i-1)/n}^{(\alpha_n)}) |X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)}|.$$

We denote by \mathcal{S} the set of all triples $(a, (j_{1,n}), (j_{2,n}))$, where a is a real number with $0 < a < 1$ and $(j_{1,n}), (j_{2,n})$ are two sequences of integers such that

$$\begin{aligned} \sup_n \alpha_n^{1-a} (\log n)^2 < \infty, & \quad r_n 2^{2j_{2,n}-j_{1,n}} \rightarrow 0, & \quad r_n^{-1} 2^{j_{1,n}/2} (\alpha_n^2 \log n + 1/n) \rightarrow 0, \\ r_n 2^{j_{1,n}} \rightarrow 0, & \quad r_n^{-1} 2^{-3j_{1,n}/2} \rightarrow 0, & \quad 2^{j_{2,n}-j_{1,n}} \rightarrow 0, & \quad r_n^{-1} 2^{-(j_{1,n}+j_{2,n}/2)} \rightarrow 0. \end{aligned}$$

Under Assumption A1, the set \mathcal{S} is not empty. For example, if one takes $j_{1,n} = \lfloor \log_2(r_n^{-3/4}) \rfloor$ and $j_{2,n} = \lfloor \log_2(r_n^{-2/3}) \rfloor$, then $(\rho, (j_{1,n}), (j_{2,n})) \in \mathcal{S}$. For $S = (a, (j_{1,n}), (j_{2,n})) \in \mathcal{S}$, we set

$$\hat{Q}_{j_{2,n}} = \sum_k \hat{d}_{j_{2,n}k}^2$$

and consider

$$R_n(S) = \sum_{j=j_{1,n}}^{\lfloor (1+a)\log_2 r_n^{-1} \rfloor} 2^{j_{2,n}-j} \hat{Q}_{j_{2,n}}.$$

For $S = (a, (j_{1,n}), (j_{2,n})) \in \mathcal{S}$, our final estimator of θ is

$$\hat{\theta}_n(S) = \sum_{k=0}^{2^{j_{1,n}}-1} \hat{c}_{j_{1,n}k}^2 + R_n(S) + \alpha_n(\mathbb{1}_{g' \geq 0} - \mathbb{1}_{g' \leq 0}) \sum_{k=0}^{2^{j_{0,n}}-1} \hat{e}_{j_{0,n}k}^2,$$

where

$$\hat{e}_{j_{0,n}k} = \sqrt{\frac{\pi}{2}} \frac{2^{j_{0,n}/2}}{\sqrt{n}} \sum_{i=1}^n \mathbb{1}_{j_{0,n}k}(i/n) |g(X_{(i-1)/n}^{(\alpha_n)})g'(X_{(i-1)/n}^{(\alpha_n)})|^{1/2} |X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)}|$$

and $\mathbb{1}_{g' \geq 0}$ indicates whether $x \rightarrow g'(x)$ is non-negative or not.

2.5. Convergence in law

We state in this section some limit theorems. In this context, it is convenient to use the notion of stable convergence in law; see Rényi [24], Aldous and Eagleson [2], Jacod and Shiryaev [17] and Jacod [14].

Definition 1 (Stable convergence in law). A sequence of variable $(X_n)_{n \in \mathbb{N}}$ converges stably in law to a variable X ($X_n \rightarrow_{\mathcal{L}_s} X$) if X is defined on an appropriate extension $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ of $(\Omega, \mathcal{F}, \mathbb{P})$ and if for any \mathcal{F} -measurable bounded variable Y and any bounded continuous function g , $\mathbb{E}[Yg(X_n)] \rightarrow \mathbb{E}[Yg(X)]$.

For $\beta > 0$, we define the function Δ_β by

$$\Delta_\beta(x) = \lim_n \mathbb{E} \left[n^{-1/2} \left(\sum_{i=1}^n Z_i \right)^2 \right],$$

with

$$Z_i = \beta(\pi/2)^{1/2} \lfloor \{U + \beta^{-1}\sigma(x)W_{i-1}\} + \beta^{-1}\sigma(x)(W_i - W_{i-1}) \rfloor - \sigma(x),$$

where W is a Brownian motion and U a uniform random variable on $[0, 1]$, independent of W . From Delattre [8], we get that the function Δ_β is well defined. We have the following theorem:

Theorem 2 (Convergence in law). *In model (1)–(2), under Assumptions A1, B and C1, for $S \in \mathcal{S}$, we have the following stable convergences in law, where B is a standard Brownian motion,*

independent of \mathcal{F} :

$$\text{if } \beta_n \rightarrow 0, \quad \sqrt{n}(\hat{\theta}_n(S) - \theta) \rightarrow_{\mathcal{L}_s} \sqrt{2}(\pi - 2)^{1/2} \int_0^1 g(X_t)^2 \sigma(X_t)^2 dB_t,$$

$$\text{if } \beta_n \rightarrow \beta > 0, \quad \sqrt{n}(\hat{\theta}_n(S) - \theta) \rightarrow_{\mathcal{L}_s} 2 \int_0^1 g(X_t)^2 \sigma(X_t) [\Delta_\beta(X_t)]^{1/2} dB_t,$$

$$\text{if } \beta_n \rightarrow +\infty, \quad \alpha_n^{-1}(\hat{\theta}_n(S) - \theta) \rightarrow_{\mathcal{L}_s} \frac{2}{\sqrt{3}} \int_0^1 g(X_t)^2 \sigma(X_t) dB_t.$$

3. Discussion

3.1. Comments on the results

- Our microstructure model with round-off error is obviously built to face the problem of price discreteness. Indeed, market price increments have to be multiples of the tick size. Moreover, it is striking to see how high frequency financial data do look like diffusions with round-off error; see, for example, [28]. In particular, the well-known bid-ask bounce is reproduced in this model. Furthermore, if the sampling period becomes large, the round-off error becomes insignificant. According to the theory and the empirical studies, this is also the case on the markets where it is often admitted that low frequency financial data can be modeled as data coming from a diffusion process. Hence, this model is relevant because it is clearly linked with market observations and financial theory.
- Our point of view is different from those of an additive microstructure noise. We do not make assumptions on the difference between the observed log-price and the theoretical log-price but on the observed price itself. Hence, our method is not a denoising method. We directly use the properties of the noisy data. Moreover, Li and Mykland [21] have proved that estimators built for additive noise, like the two scales estimator of Zhang, Mykland and Ait-Sahalia [31], are not robust in the case of a “quite big” rounding error.
- The estimation rates are the same as those obtained by Delattre [8] for other procedures on this model. In particular, if the order of magnitude of the round-off error is smaller than $n^{-1/2}$, we find the classical parametric rate.
- More general forms of stochastic volatility seem difficult to treat with our wavelet technique. Indeed, our proof of the central limit theorem relies on the fact that under an equivalent measure, the process can be written as a function of a Brownian motion, which is not the case for general stochastic volatility models. Nevertheless, Theorem 1 remains true in the case where the instantaneous volatility is of the form $\sigma(x, t) = g_1(x)g_2(t)$, with g_1 and g_2 as two positive functions such that $g_1 \in \mathcal{C}^2(\mathbb{R})$ and $g_2 \in \mathcal{C}^1([0, 1])$; see [28].
- The integrated volatility cannot be recovered in a pure rounding framework where the sequence α_n is supposed to be constant as n goes to infinity. Nevertheless, it can be done for some particular rounding procedures; see Jacod *et al.* [15].

3.2. Intuition for the results and important ideas

To give some intuition for the results, introduce important ideas used in the proofs, and explain why methods based on the quadratic variation do not work here, we recall and explain an inspiring result of Delattre [8] when β_n tends to infinity.

3.2.1. The behavior of the p -variations

Let h be the density of a standard Gaussian variable and

$$\gamma_p(\sigma, \beta) = \int_0^1 du \int_{\mathbb{R}} dy h(y) |(\beta u + \sigma y)^{(\beta)}|^p.$$

It is shown in Delattre [8] that if β_n tends to infinity, that is, if the round-off error is quite big, for $p > 0$, we have

$$\alpha_n^{-p} \beta_n n^{-1} \sum_{i=1}^n |X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)}|^p - \beta_n^{1-p} \int_0^1 \gamma_p(\sigma(X_s), \beta_n) ds$$

tends to zero in probability. The stable convergence in law of this sequence normalized by α_n^{-1} has also been proved in [8].

3.2.2. Remarks and explanations

The point is to remark that if $p = 1$,

$$\beta_n^{1-p} \int_0^1 \gamma_p(\sigma(X_s), \beta_n) ds = (2/\pi)^{1/2} \int_0^1 \sigma(X_s) ds$$

and that if $p > 0$,

$$\beta_n^{1-p} \int_0^1 \gamma_p(\sigma(X_s), \beta_n) ds - (2/\pi)^{1/2} \int_0^1 \sigma(X_s) ds$$

tends to zero in probability. So, in the case β_n tends to infinity, if p is not equal to one, contrary to what happens when considering non-noisy data, $\sigma(X_t)^p$ does not appear in the limit of the sum of the rescaled rounded increments to the power p . Thus, estimating

$$\int_0^1 \sigma^2(X_s) ds,$$

seems more complicated than estimating

$$\int_0^1 \sigma(X_s) ds.$$

We now give an intuition for this surprising behavior of the p -variations through a non-rigorous argument. Introducing several important ideas, we explain why, when β_n tends to infinity,

$$\mathbb{E}_{\sigma(X_{(i-1)/n})} [|X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)}|^p] \approx \alpha_n^p \beta_n^{-1} (2/\pi)^{1/2} \sigma(X_{(i-1)/n}),$$

where $\mathbb{E}_{\sigma(X_{(i-1)/n})}$ denotes the expectation conditional on $\sigma(X_{(i-1)/n})$. We define the fractional part of X_t by $\{X_t\} = X_t - \lfloor X_t \rfloor$. First we have to remark that

$$X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)} = \alpha_n \lfloor \{X_{(i-1)/n}/\alpha_n\} + (X_{i/n} - X_{(i-1)/n})/\alpha_n \rfloor. \tag{3}$$

Kosulajeff [18] and Tukey [29] have established that when α is small, $\{X/\alpha\}$ is almost independent of X and approximately follows a uniform law on $[0, 1]$. More precisely, the following result has been shown by Delattre and Jacod [9].

Lemma 1 (The fractional part of a variable). *Let k be a function on \mathbb{R} , C^r ($r \geq 1$), integrable with integrable derivatives. Let f be a function on $\mathbb{R} \times [0, 1]$, C^r in the first variable and such that for $0 \leq l \leq r$, $M_l = \sup_x \int_0^1 |\frac{\partial^l}{\partial x^l} f(x, u)| du < +\infty$. Then*

$$\left| \int_{\mathbb{R}} k(x) \left[f(x, \{x/\alpha\}) - \int_0^1 f(x, u) du \right] dx \right| \leq (2\alpha)^r \sup_{0 \leq l \leq r} M_l \sup_{0 \leq l \leq r} \int_{\mathbb{R}} \left| \frac{\partial^l}{\partial x^l} k(x) \right| dx.$$

Thus, since

$$X_{i/n} - X_{(i-1)/n} \approx \sigma(X_{(i-1)/n})(W_{i/n} - W_{(i-1)/n}),$$

we have

$$\mathbb{E}_{\sigma(X_{(i-1)/n})} [|X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)}|^p] \approx \alpha_n^p \mathbb{E}_{\sigma(X_{(i-1)/n})} [|U + \beta_n^{-1} \sigma(X_{(i-1)/n}) Y|^p],$$

where U is a uniform variable on $[0, 1]$, independent of X , and Y is a standard Gaussian variable, independent of X and U . Therefore, if β_n tends to infinity,

$$\mathbb{E}_{\sigma(X_{(i-1)/n})} [|X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)}|^p] \approx \alpha_n^{p-1} \mathbb{E}_{\sigma(X_{(i-1)/n})} [|X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)}|].$$

We conclude our argument using the simple but nice fact that if U is a uniform variable on $[0, 1]$ and Z is independent of U , with a density with respect to the Lebesgue measure,

$$\mathbb{E}[|U + Z|] = \mathbb{E}[|Z|].$$

4. Proofs

In all the proofs we use the previously defined notation. For technical reasons, we suppose without loss of generality that for given j , $n2^{-j}$ is a positive integer. In the following, c and c_p denote constants not depending on n , j or k and that may vary from line to line.

4.1. Preliminaries for the proofs of Theorems 1 and 2

4.1.1. Localization procedure

We slightly adapt here a classical localization procedure used, for example, in Delattre [8]. It will enable us to replace Assumptions **B**, **C** and **C1** with much stronger assumptions in the proofs of Theorems 1 and 2. We fix two sequences $(\nu_q)_{q \geq 1}$ and $(\mu_q)_{q \geq 1}$ such that (ν_q) is strictly decreasing to ν , (μ_q) is strictly increasing to μ and $(\mu_q - \nu_q) > 0$. We also fix a sequence of functions $\chi_q : \mathbb{R} \rightarrow [0, 1]$ such that $\chi_q \in \mathcal{C}_b^2(\mathbb{R})$, $(\chi_q)^{1/2} \in \mathcal{C}_b^2(\mathbb{R})$ and

$$\chi_q(x) = 1 \quad \text{on } [\nu_q, \mu_q] \quad \text{and} \quad \chi_q(x) = 0 \quad \text{on } (-\infty, \nu_{q+1}] \cup [\mu_{q+1}, +\infty).$$

For $q \in \mathbb{N}$, we define on \mathbb{R} the real functions σ_q and g_q by

$$\sigma_q(x) = \sigma(x)\chi_q(x) + (1 - \chi_q(x)), \quad g_q(x) = g(\nu_q) + \int_{\nu_q}^x g'(s)\chi_q(s) ds.$$

We finally set

$$T_q = \inf \left\{ t \in [0, 1], X_t \leq \nu_q + \alpha_q \text{ or } X_t \geq \mu_q \text{ or } \int_0^t a_s^2 ds \geq q \right\} \wedge 1.$$

Under Assumption **B**, T_q tends almost surely to 1 and $\mathbb{P}(T_q = 1) \rightarrow 1$ as $q \rightarrow +\infty$. Let $(W_t^q, t \geq 0)$ be defined by $W_t^q = W_{(T_q+t) \wedge 1} - W_{T_q}$ and $(Y_t^q)_{t \geq 0}$ be the solution of

$$dY_t^q = \sigma_q(Y_t^q) dW_t^q, \quad Y_0^q = X_{T_q}.$$

Consider now the process $(X_t^q)_{t \in [0,1]}$ defined by $X_t^q = X_t$ for $t \in [0, T_q]$ and $X_t^q = Y_{t-T_q}^q$ for $t \in (T_q, 1]$. This process satisfies

$$dX_t^q = \sigma_q(X_t^q) dW_t + a_t^q dt,$$

where $a_t^q = a_t$ for $t \in [0, T_q]$ and $a_t^q = 0$ for $t \in (T_q, 1]$. The process X^q coincides with the initial process X on $[0, T_q]$. Let $q_0 = \inf\{q, \mu_q > \nu_q + \alpha_q\}$. For $q \geq q_0$, on $[0, T_q]$, $g_q(X_t^q)$ coincides with $g(X_t)$ and for $n \geq q$, $g_q(X_t^{q(\alpha_n)})$ coincides with $g(X_t^{(\alpha_n)})$. Finally, under Assumption **C1**, $(g_q)'$ is of constant sign on \mathbb{R} , $|(g_q)'|^{1/2} \in \mathcal{C}_b^2(\mathbb{R})$ and on $[0, T_q]$, $(g_q)'(X_t^q)$ coincides with $g'(X_t)$. Furthermore, for $n \geq q$, $(g_q)'(X_t^{q(\alpha_n)})$ coincides with $g'(X_t^{(\alpha_n)})$.

Hence it is enough to prove Theorems 1 and 2 for the processes X^q , for all $q \geq q_0$, and so it is enough to prove Theorem 1 under Assumptions **B'** and **C'** instead of Assumptions **B** and **C** and Theorem 2 under Assumptions **B'** and **C1'** instead of Assumptions **B** and **C1**, with Assumptions **B'**, **C'** and **C1'** defined the following way:

Assumption B'.

- (i) There exists $c > 0$ such that for all $x \in \mathbb{R}$, $\sigma(x) \geq c$,
- (ii) $x \rightarrow \sigma(x) \in \mathcal{C}_b^2(\mathbb{R})$,
- (iii) $\sup_{\omega \in \Omega} \int_0^1 a_s^2 ds < +\infty$.

Assumption C'.

$x \rightarrow g(x) \in \mathcal{C}_b^2(\mathbb{R})$ and there exists $c > 0$ such that for all $x \in \mathbb{R}$, $g(x) \geq c$.

Assumption C1'.

- (i) $x \rightarrow g(x) \in \mathcal{C}_b^2(\mathbb{R})$ and there exists $c > 0$ such that for all $x \in \mathbb{R}$, $g(x) \geq c$,
- (ii) $x \rightarrow g'(x)$ is of constant sign on \mathbb{R} and $x \rightarrow |g'(x)|^{1/2} \in \mathcal{C}_b^2(\mathbb{R})$.

4.1.2. Change of probability

Under Assumption B', by the Girsanov theorem, we can construct a probability \mathbb{P}' on (Ω, \mathcal{F}) , absolutely continuous with respect to \mathbb{P} and a Brownian motion under \mathbb{P}' , $(W'_t, t \geq 0)$ such that

$$dX_t = \sigma(X_t) dW'_t + \frac{1}{2} \sigma(X_t) \frac{\partial}{\partial x} \sigma(X_t) dt.$$

Assumption B' holds for this representation. We define the following supplementary hypothesis:

Assumption D.

$$a_t = \frac{1}{2} \sigma(X_t) \frac{\partial}{\partial x} \sigma(X_t).$$

The convergence in probability and the stable convergence in law being preserved by an absolutely continuous change of probability, it is consequently enough to prove Theorem 1 under Assumptions A, B', C' and D and Theorem 2 under Assumptions A1, B', C1' and D. Under Assumptions B' and D, $X_t = h(W_t)$ with $h : x \rightarrow S^{-1}(x + S(x_0))$ and

$$S : x \rightarrow \int_0^x \frac{1}{\sigma(y)} dy.$$

For simplicity, we suppose now that $x_0 = 0$. Note that X is a homogeneous Markov process with transition densities

$$p_t(x, y) = \sigma(y)^{-1} (2\pi t)^{-1/2} \exp[-(2t)^{-1} (S(y) - S(x))^2].$$

Moreover, the following inequalities hold; see, for example, Delattre and Jacod [9].

$$\int \left| \frac{\partial^{i+j}}{\partial x^i \partial x^j} p_t(x, y) \right| dy \leq c t^{-(i+j)/2}, \quad i + j \leq 2, \tag{4}$$

$$\int \left| \frac{\partial^i}{\partial x^i} q_t(x, y) \right| |y|^p dy \leq c_p t^{p/2}, \quad i \leq 2, \tag{5}$$

with $q_t(x, y) = p_t(x, x + y)$. We now give the proofs of Theorems 1 and 2.

4.2. The behavior of the sampling functions

We give in this section a key proposition for the proofs of Theorems 1 and 2. As in Delattre [8], we consider the following assumption:

Assumption E. Let $(x, u, y) \rightarrow f_n(x, u, y)$ be a sequence of real functions on $\mathbb{R} \times [0, 1] \times \mathbb{R}$. The sequence f_n satisfies Assumption E if the functions f_n are twice continuously differentiable with respect to the first variable and if there exists $\gamma > 0$ such that for all $n \geq 1$,

- (i) $|f_n(x, u, y)| \leq \gamma(1 + \beta_n^2)(1 + |y|^\gamma)$,
- (ii) $\int_0^1 |f_n(x, u, y)| du \leq \gamma(1 + |y|^\gamma)$,
- (iii) $|\frac{\partial^i}{\partial x^i} f_n(x, u, y)| \leq \gamma(1 + \beta_n^2)(1 + |y|^\gamma), i = 1, 2$,
- (iv) $\int_0^1 |\frac{\partial^i}{\partial x^i} f_n(x, u, y)| du \leq \gamma(1 + |y|^\gamma), i = 1, 2$.

Notation. For some sequences of real functions $x \rightarrow g_n(x)$ on \mathbb{R} and $(x, u, y) \rightarrow f_n(x, u, y)$ on $\mathbb{R} \times [0, 1] \times \mathbb{R}$, we define

$$V^{jk}(n, g_n) = \frac{2^{j/2}}{n} \sum_{i=1}^n \mathbb{1}_{jk}(i/n) g_n(X_{(i-1)/n})$$

and

$$V^{jk}(n, f_n) = \frac{2^{j/2}}{n} \sum_{i=1}^n \mathbb{1}_{jk}(i/n) f_n(X_{(i-1)/n}, \{X_{(i-1)/n}/\alpha_n\}, \sqrt{n}[X_{i/n} - X_{(i-1)/n}]).$$

Let h_σ be the density of a centered Gaussian variable with variance σ^2 . For a real function $(x, u, y) \rightarrow f(x, u, y)$ on $\mathbb{R} \times [0, 1] \times \mathbb{R}$, we set

$$mf(x, u) = \int_{\mathbb{R}} h_{\sigma(x)}(y) f(x, u, y) dy, \quad Mf(x) = \int_0^1 mf(x, u) du.$$

The following proposition is a general result on the behavior of the sampling functions.

Proposition 1 (Behavior of the sampling functions). Let $(x, u, y) \rightarrow f_n(x, u, y)$ be a sequence of real functions on $\mathbb{R} \times [0, 1] \times \mathbb{R}$ satisfying Assumption E. Under Assumptions A, B' and D,

$$\mathbb{E} \left[\left(V^{jk}(n, f_n) - 2^{j/2} \int_0^1 \mathbb{1}_{jk}(s) Mf_n(X_s) ds \right)^2 \right] \leq cr_n^2$$

for $0 \leq j \leq \lfloor \log_2 r_n^{-1} \rfloor$ and $0 \leq k \leq 2^j - 1$. This holds for $0 \leq j \leq \lfloor (1 + \rho) \log_2 r_n^{-1} \rfloor$ under Assumptions A1, B' and D.

4.3. Proof of Proposition 1

In this proof, we widely use the methods and results developed by Delattre in [8]. We set ρ to zero if only Assumptions A, B' and D are satisfied and write $\mathbb{E}_{\mathcal{F}_t}$ for the conditional expectation with respect to \mathcal{F}_t .

4.3.1. Fundamental decomposition

Notation. Let $s_{jk} = [2^{-j}nk + 1, \dots, 2^{-j}n(k + 1)]$. We use the following notation:

$$m_n f_n(x, u) = \int q_{1/n}(x, y) f_n(x, u, \sqrt{n}y) dy, \quad M_n f_n(x) = \int_0^1 m_n f_n(x, u) du,$$

$$\bar{m}_n f_n(x) = m_n f_n(x, \{x/\alpha_n\}) - M_n f_n(x), \quad l_i^n f_n(x) = \int p_{i/n}(x, y) \bar{m}_n f_n(y) dy.$$

We set

$$f_{i+1}^n = f_n(X_{i/n}, \{X_{i/n}/\alpha_n\}, \sqrt{n}[X_{(i+1)/n} - X_{i/n}]),$$

$$\eta_i^n(f_n) = f_i^n - M_n f_n(X_{(i-1)/n}),$$

$$\delta_i^n(f, l) = \sum_{z=i}^{n \wedge (i+l-1)} (\mathbb{E}_{\mathcal{F}_{i/n}}[\eta_z^n(f)] - \mathbb{E}_{\mathcal{F}_{(i-1)/n}}[\eta_z^n(f)])$$

and

$$\mathcal{M}_{jk}^n(f_n, l) = \frac{2^{j/2}}{n} \sum_{i=1}^n \mathbb{1}_{jk}(i/n) \delta_i^n(f_n, l),$$

$$H_{jk}^n(f_n, l) = \frac{2^{j/2}}{n} \sum_{i \in s_{jk}} [\bar{m}_n f_n(X_{i/n}) - \bar{m}_n f_n(X_{(i-1)/n})]$$

$$- \frac{2^{j/2}}{n} \sum_{i \in s_{jk}} l_{(n-i) \wedge (l-1)}^n f_n(X_{(i-1)/n}) \mathbb{1}_{2 \leq (n-i) \wedge (l-1)},$$

$$K_{jk}^n(f_n, l) = \frac{2^{j/2}}{n} \sum_{i \in s_{jk}} \sum_{z=1}^{(n-i-1) \wedge (l-2)} [l_z^n f_n(X_{i/n}) - l_z^n f_n(X_{(i-1)/n})].$$

Remark that for given n and $z \in s_{jk}$,

$$\mathcal{M}_y^n = \frac{2^{j/2}}{n} \sum_{i=1}^y \mathbb{1}_{jk}(i/n) \delta_i^n(f_n, l)$$

is a (\mathcal{F}_t) -martingale in y . The following fundamental decomposition will be constantly used.

Proposition 2 (Fundamental decomposition).

$$\begin{aligned} &V^{jk}(n, f_n) - 2^{j/2} \int_0^1 \mathbb{1}_{jk}(s) M f_n(X_s) ds \\ &= \mathcal{M}_{jk}^n(f_n, l) + V^{jk}(n, M_n f_n - M f_n) \\ &\quad + V^{jk}(n, M f_n) - 2^{j/2} \int_0^1 \mathbb{1}_{jk}(s) M f_n(X_s) ds - H_{jk}^n(f_n, l) - K_{jk}^n(f_n, l). \end{aligned}$$

Proof. We have

$$\begin{aligned} \delta_i^n(f_n, l) &= \eta_i^n(f_n) - M_n f_n(X_{i/n}) + M_n f_n(X_{(i-1)/n}) - \mathbb{E}_{\mathcal{F}_{(i-1)/n}}[f_i^n] + \mathbb{E}_{\mathcal{F}_{i/n}}[f_{i+1}^n] \\ &\quad - \mathbb{E}_{\mathcal{F}_{(i-1)/n}}[\eta_{i+1}^n(f_n)] + \sum_{z=i+2}^{n \wedge (i+l-1)} (\mathbb{E}_{\mathcal{F}_{i/n}}[\eta_z^n(f_n)] - \mathbb{E}_{\mathcal{F}_{(i-1)/n}}[\eta_z^n(f_n)]). \end{aligned}$$

Using that

$$\mathbb{E}_{\mathcal{F}_{i/n}}[f_{i+1}^n] = \int q_{1/n}(X_{i/n}, y) f_n(X_{i/n}, \{X_{i/n}/\alpha_n\}, \sqrt{n}y) dy,$$

we get

$$\begin{aligned} \delta_i^n(f_n, l) &= \eta_i^n(f_n) + \bar{m}_n f_n(X_{i/n}) - \bar{m}_n f_n(X_{(i-1)/n}) - \mathbb{E}_{\mathcal{F}_{(i-1)/n}}[\mathbb{E}_{\mathcal{F}_{i/n}}[\eta_{i+1}^n(f_n)]] \\ &\quad + \sum_{z=i+2}^{n \wedge (i+l-1)} (\mathbb{E}_{\mathcal{F}_{i/n}}[\mathbb{E}_{\mathcal{F}_{(z-1)/n}}[\eta_z^n(f_n)]] - \mathbb{E}_{\mathcal{F}_{(i-1)/n}}[\mathbb{E}_{\mathcal{F}_{(z-1)/n}}[\eta_z^n(f_n)]]]) \\ &= \eta_i^n(f_n) + \bar{m}_n f_n(X_{i/n}) - \bar{m}_n f_n(X_{(i-1)/n}) - \mathbb{E}_{\mathcal{F}_{(i-1)/n}}[\bar{m}_n f_n(X_{i/n})] \\ &\quad + \sum_{z=i+2}^{n \wedge (i+l-1)} (\mathbb{E}_{\mathcal{F}_{i/n}}[\bar{m}_n f_n(X_{(z-1)/n})] - \mathbb{E}_{\mathcal{F}_{(i-1)/n}}[\bar{m}_n f_n(X_{(z-1)/n})]). \end{aligned}$$

Since

$$\mathbb{E}_{\mathcal{F}_{i/n}}[\bar{m}_n f_n(X_{(z-1)/n})] = \int p_{(z-1-i)/n}(X_{i/n}, y) \bar{m}_n f_n(y) dy,$$

we obtain

$$\begin{aligned} \delta_i^n(f_n, l) &= \eta_i^n(f_n) + \bar{m}_n f_n(X_{i/n}) - \bar{m}_n f_n(X_{(i-1)/n}) \\ &\quad - l_1^n f_n(X_{(i-1)/n}) + \sum_{z=2}^{(n-i) \wedge (l-1)} [l_{z-1}^n f_n(X_{i/n}) - l_z^n f_n(X_{(i-1)/n})]. \end{aligned}$$

Thus,

$$\begin{aligned} \delta_i^n(f_n) &= \eta_i^n(f_n) + \bar{m}_n f_n(X_{i/n}) - \bar{m}_n f_n(X_{(i-1)/n}) \\ &\quad - l_{(n-i) \wedge (l-1)}^n f_n(X_{(i-1)/n}) \mathbb{1}_{2 \leq (n-i) \wedge (l-1)} \\ &\quad + \sum_{z=1}^{(n-i-1) \wedge (l-2)} [l_z^n f_n(X_{i/n}) - l_z^n f_n(X_{(i-1)/n})]. \end{aligned}$$

We finally get

$$\begin{aligned} V^{jk}(n, f_n) - V^{jk}(n, M_n f_n) &= \frac{2^{j/2}}{n} \sum_{i=1}^n \mathbb{1}_{jk}(i/n) \eta_i^n(f_n) \\ &= \mathcal{M}_{jk}^n(f_n, l) - H_{jk}^n(f_n, l) - K_{jk}^n(f_n, l). \end{aligned} \quad \square$$

4.3.2. Technical lemmas

We prove here some useful lemmas. In particular, they will enable us to control the different terms of the decomposition. We begin with a usual Riemann approximation.

Lemma 2 (Riemann approximation). *Let $f \in C_b^1$ and*

$$A_n = \frac{2^{j/2}}{n} \sum_{i=1}^n \mathbb{1}_{jk}(i/n) f(X_{i/n}) - 2^{j/2} \int_0^1 \mathbb{1}_{jk}(s) f(X_s) ds.$$

Then,

$$\mathbb{E}[A_n^2] \leq c 2^{-j} n^{-1}.$$

Proof. Let

$$\xi_n^i = 2^{j/2} \int_{(i-1)/n}^{i/n} [\mathbb{1}_{jk}(s) f(X_s) - \mathbb{1}_{jk}(i/n) f(X_{i/n})] ds.$$

We have

$$|\xi_n^i| \leq 2^{j/2} \int_{(i-1)/n}^{i/n} |f(X_s) - f(X_{i/n})| ds.$$

Since $f \in C_b^1$, using the Burkholder–Davis–Gundy inequality, we get $\mathbb{E}[(\xi_n^i)^2] \leq c 2^j n^{-3}$. Now, $A_n = \sum_{i=1}^n \xi_n^i$ with $n 2^{-j}$ terms in the sum. Thus,

$$\mathbb{E}[A_n^2] \leq \sum_{i=1}^n \sum_{i'=1}^n (\mathbb{E}[(\xi_n^i)^2] \mathbb{E}[(\xi_n^{i'})^2])^{1/2} \leq c 2^{-j} n^{-1}. \quad \square$$

The following lemma is a consequence of Assumption E together with Lemma 1 and inequalities (4) and (5). Details can be found in Delattre [8].

Lemma 3.

$$|m_n f_n(x, u)| \leq c(1 + \beta_n^2), \tag{6}$$

$$\int_0^1 \left| \frac{\partial^i}{\partial x^i} m_n f_n(x, u) \right| du + \left| \frac{\partial^i}{\partial x^i} M_n f_n(x) \right| \leq c, \quad 0 \leq i \leq 2, \tag{7}$$

$$|M_n f_n(x) - M f_n(x)| \leq cn^{-1/2}, \tag{8}$$

$$|l_i^n f_n(x)| \leq c\alpha_n^2(1 + n/i). \tag{9}$$

We end this section with the following bounds for H_{jk}^n and K_{jk}^n .

Lemma 4.

$$|H_{jk}^n(f_n, l)| \leq c2^{j/2}n^{-1} + c2^{j/2}\alpha_n^2[1 + n2^{-j}(l - 1)^{-1} + (\log n)\mathbb{1}_{k=2j-1}],$$

$$|K_{jk}^n(f_n, l)| \leq c2^{j/2}\alpha_n^2 \log n.$$

Proof. From inequalities (6) and (9), we get

$$n|H_{jk}^n(f_n, l)| \leq c2^{j/2}(1 + \beta_n^2) + c2^{j/2}\alpha_n^2 \sum_{i \in s_{jk}} n[(l - 1)^{-1} + (n - i)^{-1}\mathbb{1}_{2 \leq (n-i)}].$$

We also have

$$\begin{aligned} n|K_{jk}^n(f_n, l)| &= 2^{j/2} \sum_{z=1}^n \sum_{i \in s_{jk}} \mathbb{1}_{1 \leq z \leq (n-i-1) \wedge (l-2)} [l_z^n f_n(X_{i/n}) - l_z^n f_n(X_{(i-1)/n})] \\ &\leq c2^{j/2}\alpha_n^2 \sum_{z=1}^n (1 + n/z) \leq c2^{j/2}\alpha_n^2 n \log n. \end{aligned} \quad \square$$

4.3.3. *End of the proof of Proposition 1*

Until the end of Section 4.4, we take $l = n$ and omit this index in the notation. We now bound the different terms of the fundamental decomposition. By Lemma 4, since $0 \leq j \leq \lfloor (1 + \rho) \log_2 r_n^{-1} \rfloor$, we get

$$|H_{jk}^n(f_n) + K_{jk}^n(f_n)| \leq c2^{j/2}(n^{-1} + \alpha_n^2 \log n) \leq cr_n.$$

Inequality (7) together with Lemma 2 on Riemann approximation give

$$\mathbb{E} \left[\left| V^{jk}(n, M f_n) - 2^{j/2} \int_0^1 \mathbb{1}_{jk}(s) M f_n(X_s) \right|^2 \right] \leq c2^{-j}n^{-1}.$$

Inequality (8) gives

$$|V^{jk}(n, M_n f_n - M f_n)| \leq 2^{-j/2}n^{-1/2}.$$

We now turn to the approximation term $\mathcal{M}_{jk}^n(f_n)$. We have

$$\mathbb{E}[\mathcal{M}_{jk}^n(f_n)^2] = \frac{2^j}{n^2} \sum_{i \in s_{jk}} \mathbb{E}[\delta_i^n(f)^2].$$

From the results of Delattre [8], Chapters 7 and 8, we can show that

$$\mathbb{E}[\delta_i^n(f)^2] \leq c(n\alpha_n^2 + (1 + \beta_n^2)(1 + \alpha_n(n/i)^{1/2})).$$

Since $\sum_{i=1}^n i^{-1/2} \leq 2\sqrt{n}$, we have

$$\begin{aligned} \mathbb{E}[\mathcal{M}_{jk}^n(f_n)^2] &\leq c\alpha_n^2 + c(1 + \beta_n^2)(n^{-1} + 2^{j/2}\alpha_n n^{-1}) \\ &\leq c(\alpha_n^2 + n^{-1} + 2^{j/2}\alpha_n n^{-1} + 2^{j/2}\alpha_n^3). \end{aligned} \tag{10}$$

Putting all the inequalities together, we obtain Proposition 1.

4.4. Proof of Theorem 1

Using the remark on the change of probability in Section 4.1.2, the following proposition implies Theorem 1.

Proposition 3 (L^1 convergence, absolute integrated volatility). *Let $\tilde{\theta}_n$ be the estimator defined in Section 2.2. Under Assumptions A, B', C' and D,*

$$\mathbb{E}[|\tilde{\theta}_n - \theta|] \leq cr_n,$$

with c a constant not depending on n .

For expository purposes, we first treat the case $g(x) = 1$.

4.4.1. Proof of Proposition 3 in the case $g(x) = 1$

We assume here that $g(x) = 1$ and set

$$f_n(x, u, y) = (\pi/2)^{1/2} \beta_n [|u + \beta_n^{-1}y|].$$

This specification implies

$$Mf_n(X_s) = \sigma(X_s).$$

We begin with a lemma on the behavior of the wavelet coefficients. Let c_{j_0k} , d_{jk} and \hat{c}_{j_0k} be as defined in Section 2.2. Thanks to the vanishing moment of ψ , we easily get the following result:

Lemma 5.

$$c_{j_0k}^2 \leq c2^{-j_0}, \quad \mathbb{E}[d_{jk}^2] \leq c2^{-2j}.$$

Let

$$Z_j = \sum_{k=0}^{2^j-1} \mathcal{M}_{jk}^n(f_n) c_{jk}, \quad \tilde{Z}_j = \sum_{k=0}^{2^j-1} K_{jk}^n(f_n) c_{jk}.$$

We have the following lemma:

Lemma 6. *Let $0 \leq j \leq \lfloor (1 + \rho) \log_2 r_n^{-1} \rfloor$, then*

$$\mathbb{E}[|Z_j| + |\tilde{Z}_j|] \leq cr_n.$$

Proof. We have $Z_j = Z_{j,1} + Z_{j,2}$ with

$$Z_{j,1} = \sum_{k=0}^{2^j-1} 2^{j/2} \left(\int_{k/2^j}^{(k+1)/2^j} [\sigma(X_s) - \sigma(X_{k2^{-j}})] ds \right) \mathcal{M}_{jk}^n,$$

$$Z_{j,2} = \frac{1}{n} \sum_{k=0}^{2^j-1} \sigma(X_{k2^{-j}}) ds \sum_{i \in S_{jk}} \delta_i.$$

We easily get $\mathbb{E}[|Z_{j,1}|] \leq cr_n$. For $Z_{j,2}$, we have

$$\mathbb{E}[|Z_{j,2}|^2] = \frac{1}{n^2} \sum_{k=0}^{2^j-1} \sum_{k'=0}^{2^j-1} \sum_{i \in S_{jk}} \sum_{i' \in S_{j'k'}} \mathbb{E}[\sigma(X_{k2^{-j}}) \sigma(X_{k'2^{-j}}) \delta_i \delta_{i'}].$$

For $i \neq i'$, conditioning by $\mathcal{F}_{\max(i,i')-1/n}$, we get

$$\mathbb{E}[\sigma(X_{k2^{-j}}) \sigma(X_{k'2^{-j}}) \delta_i \delta_{i'}] = 0.$$

Therefore,

$$\begin{aligned} \mathbb{E}[|Z_{j,2}|^2] &= \frac{1}{n^2} \sum_{k=0}^{2^j-1} \mathbb{E} \left[\sigma(X_{k2^{-j}})^2 \mathbb{E}_{\mathcal{F}_{k2^{-j}}} \left[\sum_{i \in S_{jk}} \delta_i^2 \right] \right] \\ &= 2^{-j} \sum_{k=0}^{2^j-1} \mathbb{E}[\sigma(X_{k2^{-j}})^2 \mathbb{E}_{\mathcal{F}_{k2^{-j}}}[\mathcal{M}_{jk}^n]] \leq cr_n^2. \end{aligned}$$

For \tilde{Z}_j , recall that

$$K_{jk}^n(f_n) = \frac{2^{j/2}}{n} \sum_{i \in S_{jk}} \mathbb{1}_{jk}(i/n) \tilde{\delta}_i,$$

with

$$\tilde{\delta}_i = \sum_{z=1}^{n-i-1} [l_z^n f_n(X_{i/n}) - l_z^n f_n(X_{(i-1)/n})]$$

and that

$$l_z^n f_n(X_{i/n}) = \mathbb{E}_{\mathcal{F}_{i/n}} [m_n f_n(X_{(i+z)/n})].$$

The same method gives the result. □

We now end the proof of Proposition 3. From Proposition 1 and equation (3), we can write

$$\begin{aligned} \hat{c}_{j_0,nk} &= c_{j_0,nk} + \mathcal{M}_{j_0,nk}^n(f_n) + V^{j_0,nk}(n, M_n f_n - M f_n) \\ &\quad + V^{j_0,nk}(n, M f_n) - 2^{j_0,n/2} \int_0^1 \mathbb{1}_{j_0,nk}(s) M f_n(X_s) ds - H_{j_0,nk}^n(f_n) - K_{j_0,nk}^n(f_n) \end{aligned}$$

and

$$\mathbb{E}[|\hat{c}_{j_0,nk} - c_{j_0,nk}|^2] \leq cr_n^2.$$

We have

$$\mathbb{E}[|\tilde{\theta}_n - \theta|] \leq c \mathbb{E} \left[\sum_{j=j_0,n+1}^{+\infty} \sum_k d_{jk}^2 + \left| \sum_k c_{j_0,nk} \mathcal{R}_k^n \right| \right] + c(r_n + 2^{j_0,n} r_n^2),$$

where \mathcal{R}_k^n is equal to

$$V^{j_0,nk}(n, M_n f_n - M f_n) + V^{j_0,nk}(n, M f_n) - 2^{j_0,n/2} \int_0^1 \mathbb{1}_{j_0,nk}(s) M f_n(X_s) ds - H_{j_0,nk}^n(f_n).$$

By Lemma 5, we have

$$\mathbb{E} \left[\sum_{j=j_0,n}^{+\infty} \sum_k d_{jk}^2 \right] \leq c 2^{-j_0,n}.$$

Moreover, using the preceding computations, it is easy to see that

$$\mathbb{E} \left[\left| \sum_k c_{j_0,nk} \mathcal{R}_k^n \right| \right] \leq c(2^{j_0,n} n^{-1} + 2^{j_0,n} \alpha_n^2 + n^{-1/2} + \alpha_n^2 \log n).$$

The result follows.

4.4.2. Proof of Proposition 3 in the general case

Let

$$\hat{c}_{j_0,nk}^* = \sqrt{\frac{\pi}{2}} \frac{2^{j_0,n/2}}{\sqrt{n}} \sum_{i=1}^n \mathbb{1}_{j_0,nk}(i/n) g(X_{(i-1)/n}) |X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)}|.$$

We easily get the result in the same way as in the previous proof, remarking that

$$|\hat{c}_{j_0,nk} - \hat{c}_{j_0,nk}^*| \leq c\alpha_n \sqrt{\frac{\pi}{2}} \frac{2^{j_0,n/2}}{\sqrt{n}} \sum_{i=1}^n \mathbb{1}_{j_0,nk}(i/n) |X_{i/n}^{(\alpha_n)} - X_{(i-1)/n}^{(\alpha_n)}|.$$

Consequently, using Proposition 1, we obtain

$$\mathbb{E}[|\hat{c}_{j_0,nk} - c_{j_0,nk}|^2] \leq c\mathbb{E}[|\hat{c}_{j_0,nk} - \hat{c}_{j_0,nk}^*|^2] + c\mathbb{E}[|\hat{c}_{j_0,nk}^* - c_{j_0,nk}|^2] \leq cr_n^2$$

and

$$\sum_k c_{j_0,nk} (\hat{c}_{j_0,nk} - c_{j_0,nk}) = \sum_k c_{j_0,nk} (\hat{c}_{j_0,nk}^* - c_{j_0,nk}) + Z,$$

with $\mathbb{E}[|Z|] \leq c\alpha_n$. The result follows.

4.5. Proof of Theorem 2

In this proof, Assumptions A1, B' and D are in force for α_n and X . We also assume until the end of Section 4.5.2 that $g(x) = 1$.

4.5.1. Compensator

We have

$$\sum_k \hat{c}_{j_0k}^2 - \int_0^1 \sigma(X_s)^2 ds = \sum_k (\hat{c}_{j_0k} - c_{j_0k})^2 + 2 \sum_k c_{j_0k} (\hat{c}_{j_0k} - c_{j_0k}) - \sum_{j \geq j_0} \sum_k d_{jk}^2.$$

The central limit theorems will be derived from the double product term. If, as previously, we choose j_0 such that 2^{j_0} is of order r_n^{-1} , re-normalized by r_n^{-1} , the two other terms do not tend to zero. Hence, we can either choose $2^j > r_n^{-1}$ and compensate the first term or choose $2^j < r_n^{-1}$ and compensate the last term. The first method is classical in quadratic functionals estimation. However, it seems difficult here. Indeed, a compensator of $\sum_k (\hat{c}_{jk} - c_{jk})^2$ requires an accurate enough estimation of the function $x \rightarrow \sigma(x)$. Consequently, we compensate the last term. This is unusual, but possible in our specific setting. Of course, a one-by-one estimation of the coefficients d_{jk} is probably not suitable for building the compensator. This is simply because the error between the coefficient d_{jk}^2 and its estimation is of the same order as the error between the coefficient c_{jk}^2 and its estimate. That is why we use here the following scaling property of the wavelet coefficients.

Lemma 7. *Let*

$$Q_j = \sum_k d_{jk}^2, \quad G(u) = \int_0^u \psi(u) du \quad \text{and} \quad c(\psi) = \int_0^1 G^2(u) du.$$

We have

$$\mathbb{E} \left[\left| 2^j Q_j - c(\psi) \int_0^1 h'(W_u)^2 du \right| \right] \leq c 2^{-j/2}.$$

Proof. We briefly give the main steps of the proof. Details can be found in Rosenbaum [27]. First we define

$$d'_{jk} = \int \psi_{jk} W_t dt.$$

We easily get that the d'_{jk} are independent centered Gaussian variables such that

$$\mathbb{E}[d'^2_{jk}] = c(\psi) 2^{-2j}.$$

Let $\xi : [0, 1] \rightarrow \mathbb{R}$ be a deterministic bounded function, vanishing outside the interval $[k2^{-j_0}, k'2^{-j_0}] \subset [0, 1]$ and define

$$\Sigma_j(\xi) = 2^j \sum_{k=0}^{T(2^j-1)} (2^j d'^2_{jk} - c(\psi) 2^{-j}) \xi_{k2^{-j}}.$$

One can show that

$$\mathbb{E}[\Sigma_j(\xi)^2] \leq c \sup_t (\xi_t^2) |k' - k| 2^{j-j_0}.$$

Using a decomposition of the function h'^2 in a wavelet basis, these results enable us to prove that

$$\mathbb{E} \left[\left| 2^j \sum_k (2^j d'^2_{jk} - c(\psi) 2^{-j}) h'(W_{k2^{-j}})^2 \right| \right] \leq c 2^{j/2}. \tag{11}$$

Since ψ has a vanishing moment,

$$\int \psi_{jk}(t) h(W_t) dt \approx h'(W_{k2^{-j}}) \int \psi_{jk}(t) W_t dt$$

and so

$$d^2_{jk} \approx h'(W_{k2^{-j}})^2 d'^2_{jk}.$$

We conclude using equation (11) together with a Riemann-type approximation. □

The following lemma shows that our method enables us to estimate the remaining coefficients accurately enough.

Lemma 8. Let $S = [a, (j_{1,n}), (j_{2,n})] \in \mathcal{S}$. Then,

$$r_n^{-1} \left[\sum_k (\hat{c}_{j_{1,n}k} - c_{j_{1,n}k})^2 - \sum_{j \geq j_{1,n}} \sum_k d_{jk}^2 + R_n(S) \right] \rightarrow 0.$$

Proof. We want to compensate

$$\sum_{j=j_{1,n}}^{\lfloor (1+a) \log_2 r_n^{-1} \rfloor} Q_j.$$

We know that for big enough j and $j_{2,n} \leq j$, Q_j is close to $2^{j_{2,n}-j} Q_{j_{2,n}}$. Therefore, we estimate the preceding quantity by

$$\sum_{j=j_{1,n}}^{\lfloor (1+a) \log_2 r_n^{-1} \rfloor} 2^{j_{2,n}-j} \hat{Q}_{j_{2,n}},$$

with

$$\hat{Q}_{j_{2,n}} = \sum_k \hat{d}_{j_{2,n}k}^2$$

for appropriate $j_{1,n}$ and $j_{2,n}$. Let

$$U_n = \sum_{j=j_{1,n}}^{\lfloor (1+a) \log_2 r_n^{-1} \rfloor} Q_j - \sum_{j=j_{1,n}}^{\lfloor (1+a) \log_2 r_n^{-1} \rfloor} 2^{j_{2,n}-j} \hat{Q}_{j_{2,n}}$$

and $Y = c(\psi) \int_0^1 h'(W_u)^2 du$. We have

$$U_n = \sum_{j=j_{1,n}}^{\lfloor (1+a) \log_2 r_n^{-1} \rfloor} (Q_j - 2^{-j} Y) + 2^{-j} (Y - 2^{j_{2,n}} Q_{j_{2,n}}) + 2^{j_{2,n}-j} (Q_{j_{2,n}} - \hat{Q}_{j_{2,n}}).$$

Using the same arguments as for Proposition 1, for $j_{1,n} \leq j \leq \lfloor (1+a) \log_2 r_n^{-1} \rfloor$, we get

$$\mathbb{E}[|\hat{d}_{jk} - d_{jk}|^2] \leq cr_n^2.$$

Hence, we also obtain

$$\mathbb{E}[|d_{jk}| |\hat{d}_{jk} - d_{jk}|] \leq c2^{-j} r_n.$$

Consequently, we have

$$\mathbb{E}[|U_n|] \leq c(2^{-3j_{1,n}/2} + 2^{-(j_{1,n}+j_{2,n}/2)} + 2^{2j_{2,n}-j_{1,n}} r_n^2 + 2^{j_{2,n}-j_{1,n}} r_n).$$

As $a > 0$, it is clear that

$$r_n^{-1} \sum_{j > \lfloor (1+a) \log_2 r_n^{-1} \rfloor} Q_j \rightarrow 0.$$

□

4.5.2. Limit theorems

We prove in this section Theorem 2 in the case $g(x) = 1$. Let

$$f_n(x, u, y) = (\pi/2)^{1/2} \beta_n \lfloor u + \beta_n^{-1} y \rfloor,$$

$$q_i^n = \frac{1}{n} f_n(X_{(i-1)/n}, \{X_{(i-1)/n}/\alpha_n\}, \sqrt{n}[X_{i/n} - X_{(i-1)/n}])$$

and $z_i^n = q_i^n - \int_{(i-1)/n}^{i/n} M f_n(X_s)$. We begin with some intermediary lemmas.

Lemma 9. *Let*

$$T_1(j_n, n) = \sum_k \left[2^{j_n/2} \int \mathbb{1}_{j_n k}(s) [\sigma(X_s) - \sigma(X_{k2^{-j_n}})] ds \right] \left[2^{j_n/2} \sum_{i \in S_{j_n k}} z_i^n \right].$$

If $2^{-j_n} + r_n^{-1} 2^{j_n/2} (n^{-1} + \alpha_n^2 \log n) + 2^{j_n} r_n \rightarrow 0$, then $r_n^{-1} T_1(j_n, n) \xrightarrow{\mathbb{P}} 0$.

Proof. We write $\mathcal{M}_{jk}(l_n)$ for $\mathcal{M}_{jk}(f_n, l_n)$, T_1 for $T_1(j_n, n)$ and j for j_n . We use the decomposition $T_1 = T_{11} + T_{12}$ with

$$T_{11} = \sum_k \left[2^{j/2} \int \mathbb{1}_{jk}(s) [\sigma(X_s) - \sigma(X_{k2^{-j}})] ds \right] \mathcal{M}_{jk}(l_n).$$

We easily get

$$\mathbb{E}[|T_{12}|] \leq c 2^{-j/2} r_n + c 2^{j/2} (n^{-1} + \alpha_n^2 \log n + \alpha_n^2 n 2^{-j} / l_n).$$

We take $l_n = \lfloor n / \log n \rfloor$ and therefore $r_n^{-1} \mathbb{E}[|T_{12}|]$ tends to zero. We set $F(x) = \sigma[h(x)]$. The term T_{11} can be written as

$$T_{11} = \sum_k \sum_{i \in S_{jk}} \left[2^j \int \mathbb{1}_{jk}(s) (W_s - W_{k2^{-j}}) F'(W_{k2^{-j}}) \frac{\delta_i(l_n)}{n} ds \right] + 2^{j/2} \sum_k R_k \mathcal{M}_{jk}(l_n),$$

with $\mathbb{E}[|R_k|^2] \leq 2^{-4j}$. Following Delattre [8], Chapters 7 and 8, there exists $\tilde{\delta}_i(l_n)$ such that for $l_n = \lfloor n / \log n \rfloor$

$$\mathbb{E}[|\delta_i(l_n) - \tilde{\delta}_i(l_n)|^2] \leq c n r_n^2 / \log n,$$

$$\mathbb{E}[\tilde{\delta}_i(l_n)^2] \leq c(1 + \beta_n^2)(1 + \alpha_n \lfloor n/i \rfloor^{1/2}).$$

Hence,

$$\mathbb{E}[\mathcal{M}_{jk}(l_n)^2] = \frac{2^j}{n^2} \sum_{i \in s_{jk}} \mathbb{E}[\delta_i(l_n)^2] \leq cr_n^2.$$

Consequently, we obtain that the expectation of the second term of T_{11} is less than $2^{-j/2}r_n$. The first term can be written as $A_1 + A_2 + A_3 + A_4 + A_5$ with

$$\begin{aligned} A_1 &= \sum_k \sum_{i \in s_{jk}} 2^j \int_{k2^{-j}}^{(i-1)/n} (W_s - W_{k2^{-j}})F'(W_{k2^{-j}}) \frac{\delta_i(l_n)}{n} ds, \\ A_2 &= \sum_k \sum_{i \in s_{jk}} 2^j \int_{(i-1)/n}^{i/n} (W_s - W_{k2^{-j}})F'(W_{k2^{-j}}) \frac{\delta_i(l_n)}{n} ds, \\ A_3 &= \sum_k \sum_{i \in s_{jk}} 2^j \int_{i/n}^{(k+1)2^{-j}} (W_s - W_{i/n})F'(W_{k2^{-j}}) \frac{\delta_i(l_n)}{n} ds, \\ A_4 &= \sum_k \sum_{i \in s_{jk}} 2^j [(k+1)2^{-j} - i/n](W_{i/n} - W_{(i-1)/n})F'(W_{k2^{-j}}) \frac{\delta_i(l_n)}{n} ds, \\ A_5 &= \sum_k \sum_{i \in s_{jk}} 2^j [(k+1)2^{-j} - i/n](W_{(i-1)/n} - W_{k2^{-j}})F'(W_{k2^{-j}}) \frac{\delta_i(l_n)}{n} ds. \end{aligned}$$

We easily get that $\mathbb{E}[A_1^2 + A_5^2] \leq c2^{-j}r_n^2$. For A_2 , we have

$$\mathbb{E}[|A_2|] \leq c \frac{2^{j/2}}{n} \left(\sup_i \mathbb{E}[\delta_i(l_n)^2] \right)^{1/2} \leq c2^{j/2}(1/n + \alpha_n^2).$$

We now turn to A_3 . We write here δ_i for $\delta_i(l_n)$. We easily obtain that $\mathbb{E}[A_3^2]$ is equal to

$$2^{2j} \sum_k \sum_{\substack{i \in s_{jk} \\ i' \in s_{jk}}} \int_{i/n}^{(k+1)2^{-j}} \int_{i'/n}^{(k+1)2^{-j}} \mathbb{E} \left[F'(W_{k2^{-j}})^2 (W_s - W_{i/n})(W_{s'} - W_{i'/n}) \frac{\delta_i}{n} \frac{\delta_{i'}}{n} \right] ds ds'.$$

We consider the quantity

$$u_i = \mathbb{E}_{\mathcal{F}_{i/n}} \left[F'(W_{k2^{-j}})^2 (W_s - W_{i/n})(W_{s'} - W_{i'/n}) \frac{\delta_i}{n} \frac{\delta_{i'}}{n} \right].$$

Suppose that $i \geq i'$ and $s' > i/n$. Then

$$\begin{aligned} u_i &= F'(W_{k2^{-j}})^2 \frac{\delta_i}{n} \frac{\delta_{i'}}{n} \mathbb{E}_{\mathcal{F}_{i/n}} [(W_s - W_{i/n})(W_{s'} - W_{i'/n})] \\ &= F'(W_{k2^{-j}})^2 \frac{\delta_i}{n} \frac{\delta_{i'}}{n} \mathbb{E}[(W_s - W_{i/n})(W_{s'} - W_{i'/n})]. \end{aligned}$$

Suppose that $i \geq i'$ and $s' \leq i/n$. Then

$$u_i = F'(W_{k2^{-j}})^2 \frac{\delta_i}{n} \frac{\delta_{i'}}{n} \mathbb{E}_{\mathcal{F}_{i/n}} [(W_s - W_{i/n})(W_{s'} - W_{i'/n})] = 0.$$

Finally,

$$\mathbb{E}[A_3^2] \leq c2^j \sum_k \sum_{i \in s_{jk}} \int_{i/n}^{(k+1)2^{-j}} \int_{i/n}^{(k+1)2^{-j}} \mathbb{E} \left[F'(W_{k2^{-j}})^2 \left(\frac{\delta_i}{n} \right)^2 \right] ds ds'.$$

Hence,

$$\mathbb{E}[A_3^2] \leq c2^{-j} r_n^2.$$

For A_4 , consider the function ζ defined on $[0, 1]$ by $\zeta(t) = 1 - t$ and $\zeta_{jk}(t) = 2^{j/2} \zeta(2^j x - k)$. We have

$$\begin{aligned} A_4 &= \sum_k \sum_{i \in s_{jk}} 2^j [(k+1)2^{-j} - i/n] F'(W_{k2^{-j}}) (W_{i/n} - W_{(i-1)/n}) \frac{\delta_i(l_n) - \tilde{\delta}_i(l_n)}{n} \\ &\quad + \sum_k F'(W_{k2^{-j}}) \sum_{i \in s_{jk}} 2^{-j/2} \zeta_{jk}(i/n) (X_{i/n} - X_{(i-1)/n}) \sigma(X_{(i-1)/n})^{-1} \frac{\tilde{\delta}_i(l_n)}{n} + R, \end{aligned}$$

with $\mathbb{E}[|R|] \leq c(1/n + \alpha_n^{3/2})$. Using that the function f_n verifies in our case

$$|f_n(x, u, y)| \leq c(1 + \beta_n)(1 + |y|),$$

following Delattre [8], Chapter 6, we can show that the quantity

$$\sqrt{n}(X_{i/n} - X_{(i-1)/n}) \sigma(X_{(i-1)/n})^{-1} \tilde{\delta}_i(l_n) (1 + \beta_n)^{-1}$$

can be written as $g_n(X_{(i-1)/n}, \{X_{(i-1)/n} \alpha_n\}, \sqrt{n}[X_{i/n} - X_{(i-1)/n}])$. The function g_n satisfies Assumption E. Therefore, since $Mg_n(x) = 0$, using the same arguments as in the proof of Proposition 1, we can prove that

$$\mathbb{E} \left[\left| \frac{1}{n} \sum_{i \in s_{jk}} \zeta_{jk}(i/n) g_n(X_{(i-1)/n}, \{X_{(i-1)/n} \alpha_n\}, \sqrt{n}[X_{i/n} - X_{(i-1)/n}]) \right|^2 \right] \leq cr_n^2.$$

Consequently,

$$\mathbb{E} \left[\left| \sum_k F'(W_{k2^{-j}}) \sum_{i \in s_{jk}} 2^{-j/2} \zeta_{jk}(i/n) (X_{i/n} - X_{(i-1)/n}) \frac{\tilde{\delta}_i(l_n)}{n} ds \right|^2 \right] \leq c2^{j/2} r_n^2.$$

For the first term, we use that

$$\frac{1}{n^2} \sum_{i=1}^n \mathbb{E}[|\delta_i(l_n) - \tilde{\delta}_i(l_n)|^2] \leq cr_n^2 / \log n$$

and finally

$$\mathbb{E} \left[\left| \sum_k \sum_{i \in s_{jk}} 2^j [(k+1)2^{-j} - i/n] (W_{i/n} - W_{(i-1)/n}) F'(W_{k2^{-j}}) \frac{\delta_i(l_n) - \tilde{\delta}_i(l_n)}{n} ds \right| \right] \leq cr_n (\log n)^{-1/2}. \quad \square$$

Lemma 10. *Let*

$$T_2(n) = \sum_i \int_{(i-1)/n}^{i/n} [\sigma(X_s) - \sigma(X_{(i-1)/n})] Mf_n(X_s) ds.$$

We have $r_n^{-1} T_2(n) \xrightarrow{\mathbb{P}} 0$.

Proof. We write T_2 for $T_2(n)$ and set $c_\eta = (2/\pi)^{1/2}$. We have

$$T_2 = c_\eta \sum_i \int_{(i-1)/n}^{i/n} [\sigma(X_s) - \sigma(X_{(i-1)/n})] \sigma(X_s) ds.$$

We can write

$$T_2 = c_\eta \sum_i \int_{(i-1)/n}^{i/n} [\sigma(X_s) - \sigma(X_{(i-1)/n})]^2 ds + c_\eta \sum_i \int_{(i-1)/n}^{i/n} [\sigma(X_s) - \sigma(X_{(i-1)/n})] \sigma(X_{(i-1)/n}) ds.$$

Itô's formula gives

$$T_2 = c_\eta \sum_i \int_{(i-1)/n}^{i/n} ds \sigma(X_{(i-1)/n}) \int_{(i-1)/n}^s \sigma'(X_t) \sigma(X_t) dW_t + c_\eta \sum_i \int_{(i-1)/n}^{i/n} ds \sigma(X_{(i-1)/n}) \int_{(i-1)/n}^s \left(\frac{1}{2} \sigma'(X_t)^2 \sigma(X_t) + \frac{1}{2} \sigma(X_t)^2 \sigma''(X_t) \right) dt + R,$$

with $\mathbb{E}[|R|] \leq c/n$. Finally, we obtain

$$T_2 = R + c_\eta \sum_i \int_{(i-1)/n}^{i/n} ds \sigma(X_{(i-1)/n}) \int_{(i-1)/n}^s \sigma'(X_t) \sigma(X_t) dW_t + R',$$

with $\mathbb{E}[|\tilde{R}'|] \leq c/n$. Let

$$\eta_i = \int_{(i-1)/n}^{i/n} ds \sigma(X_{(i-1)/n}) \int_{(i-1)/n}^s \sigma'(X_t) \sigma(X_t) dW_t.$$

For $i' < i$, $\mathbb{E}_{\mathcal{F}_{i'/n}}[\eta_i] = 0$. Hence, for given n , $M_i^n = \sum_{j=1}^i \eta_j^n$ is a martingale. Consequently,

$$\mathbb{E}[(M_i^n)^2] = \sum_{i=1}^n \mathbb{E}[\eta_i^2].$$

Since

$$\eta_i^2 \leq \frac{1}{n} \int_{(i-1)/n}^{i/n} ds \left(\sigma(X_{(i-1)/n}) \int_{(i-1)/n}^s \sigma'(X_t) \sigma(X_t) dW_t \right)^2,$$

we get

$$\mathbb{E}[(M_i^n)^2] \leq c/n^2. \quad \square$$

Lemma 11. *Let*

$$T_3(j_n, n) = \sum_k \sum_{i \in \mathcal{S}_{j_n k}} [\sigma(X_{k2^{-j_n}}) - \sigma(X_{(i-1)/n})] z_i^n.$$

If $2^{-j_n} + r_n^{-1} 2^{j_n/2} (n^{-1} + \alpha_n^2 \log n) + 2^{j_n} r_n \rightarrow 0$, then $r_n^{-1} T_3(j_n, n) \xrightarrow{\mathbb{P}} 0$.

Proof. We write T_3 for $T_3(j_n, n)$ and j for j_n . We have

$$-T_3 = \sum_k \sum_{i \in \mathcal{S}_{jk}} [\sigma(X_{(i-1)/n}) - \sigma(X_{k2^{-j}})] \left(\frac{\delta_i(l_n)}{n} + r_i + R_i \right),$$

with

$$\begin{aligned} r_i &= \frac{1}{n} [\bar{m}_n f_n(X_{i/n}) - \bar{m}_n f_n(X_{(i-1)/n})] \\ &\quad - \frac{1}{n} \sum_{z=1}^{n-i-1} [l_z^n f_n(X_{i/n}) - l_z^n f_n(X_{(i-1)/n})] - \frac{1}{n} l_{n-i}^n f_n(X_{(i-1)/n}) \mathbb{1}_{i \leq n-2} \end{aligned}$$

and $|R_i| \leq cn^{-3/2}$. We easily get

$$\begin{aligned} &\mathbb{E} \left[\left| \sum_k \sum_{i \in \mathcal{S}_{jk}} [\sigma(X_{(i-1)/n}) - \sigma(X_{k2^{-j}})] \frac{\delta_i(l_n)}{n} \right|^2 \right] \\ &= \mathbb{E} \left[\left| \sum_k \sum_{i \in \mathcal{S}_{jk}} [\sigma(X_{(i-1)/n}) - \sigma(X_{k2^{-j}})]^2 \left(\frac{\delta_i(l_n)}{n} \right)^2 \right| \right] \leq c2^{-j} r_n^2. \end{aligned}$$

The second term of the decomposition can be written as

$$\begin{aligned} & \sum_k \sum_{i \in S_{jk}} [\sigma(X_{(i-1)/n}) - \sigma(X_{k2^{-j}})] r_i \\ & = B_1 + B_2 + B_3 \end{aligned}$$

with

$$\begin{aligned} B_1 &= - \sum_k \sum_{i \in S_{jk}} [\sigma(X_{i/n}) - \sigma(X_{(i-1)/n})] r_i, \\ B_2 &= \sum_k [\sigma(X_{(k+1)2^{-j}}) - \sigma(X_{k2^{-j}})] \sum_{i \in S_{jk}} r_i, \\ B_3 &= - \sum_k \sum_{i \in S_{jk}} [\sigma(X_{(k+1)2^{-j}}) - \sigma(X_i)] r_i. \end{aligned}$$

Using Lemma 4, we obtain

$$\mathbb{E}[|B_2|] \leq 2^{j/2} \left(\frac{1}{n} + \alpha_n^2 \log n \right).$$

For B_1 we consider the decomposition $B_1 = B_{11} - B_{12}$ with

$$\begin{aligned} B_{11} &= \sum_i [\sigma(X_{i/n}) - \sigma(X_{(i-1)/n})] z_i^n, \\ B_{12} &= \sum_i [\sigma(X_{i/n}) - \sigma(X_{(i-1)/n})] \left(\frac{\delta_i(I_n)}{n} + R_i \right). \end{aligned}$$

Using the same method as for A_4 , we get $\mathbb{E}[|B_{12}|] \leq cr_n(\log n)^{-1/2}$. We have for the other term

$$\begin{aligned} B_{11} &= \sum_i (X_{i/n} - X_{(i-1)/n}) \sigma'(X_{(i-1)/n}) q_i^n \\ &+ \sum_i (X_{i/n} - X_{(i-1)/n}) \sigma'(X_{(i-1)/n}) \frac{\sigma(X_{(i-1)/n})}{n} \\ &+ R \end{aligned}$$

with $\mathbb{E}[|R|] \leq c/n$. Therefore, we easily get that $\mathbb{E}[|B_{11}|] \leq cr_n n^{-1/2}$. We now treat B_3 . The quantity

$$\sum_k \sum_{i \in S_{jk}} [\sigma(X_{(k+1)2^{-j}}) - \sigma(X_{i/n})] \frac{1}{n} [\bar{m}_n f_n(X_{i/n}) - \bar{m}_n f_n(X_{(i-1)/n})]$$

can be written as

$$\begin{aligned} & \sum_k \sum_{i \in S_{jk}} [\sigma(X_{(k+1)2^{-j}}) - \sigma(X_{i/n})] \frac{1}{n} \bar{m}_n f_n(X_{i/n}) \\ & - \sum_k \sum_{i \in S_{jk}} [\sigma(X_{(k+1)2^{-j}}) - \sigma(X_{(i-1)/n})] \frac{1}{n} \bar{m}_n f_n(X_{(i-1)/n}) \\ & + \sum_k \sum_{i \in S_{jk}} [\sigma(X_{i/n}) - \sigma(X_{(i-1)/n})] \frac{1}{n} \bar{m}_n f_n(X_{(i-1)/n}). \end{aligned}$$

This is equal to

$$\begin{aligned} & - \sum_k [\sigma(X_{(k+1)2^{-j}}) - \sigma(X_{k2^{-j}})] \frac{1}{n} \bar{m}_n f_n(X_{k2^{-j}}) \\ & + \sum_k \sum_{i \in S_{jk}} (X_{i/n} - X_{(i-1)/n}) \sigma'(X_{(i-1)/n}) \frac{1}{n} \bar{m}_n f_n(X_{(i-1)/n}) \\ & + R \end{aligned}$$

with $\mathbb{E}[|R|] \leq cr_n^2$. Eventually this term is equal to

$$R' + \sum_k \sum_{i \in S_{jk}} (X_{i/n} - X_{(i-1)/n}) \sigma'(X_{(i-1)/n}) \frac{1}{n} \bar{m}_n f_n(X_{(i-1)/n}) + R,$$

with $\mathbb{E}[|R'|] \leq c2^{j/2}r_n^2$. The quantity

$$\sqrt{n}(X_{i/n} - X_{(i-1)/n}) \sigma'(X_{(i-1)/n}) \bar{m}_n f_n(X_{(i-1)/n}) (1 + \beta_n)^{-1}$$

can be written as $g_n(X_{(i-1)/n}, \{X_{(i-1)/n} \alpha_n\}, \sqrt{n}[X_{i/n} - X_{(i-1)/n}])$. This function satisfies Assumption E. Hence, since $Mg_n(x) = 0$, we obtain

$$\begin{aligned} & \mathbb{E} \left[\left| \sum_k \sum_{i \in S_{jk}} (X_{i/n} - X_{(i-1)/n}) \sigma'(X_{(i-1)/n}) \frac{1}{n} \bar{m}_n f_n(X_{(i-1)/n}) \right| \right] \\ & \leq cr_n^2. \end{aligned}$$

We now treat

$$\sum_k \sum_{i \in S_{jk}} [\sigma(X_{(k+1)2^{-j}}) - \sigma(X_{i/n})] \frac{1}{n} \sum_{z=1}^{(n-i-1) \wedge (l_n-2)} [l_z^n f_n(X_{i/n}) - l_z^n f_n(X_{(i-1)/n})].$$

This can be written as

$$\begin{aligned} & \sum_k \sum_z \sum_{i \in S_{jk}} [\sigma(X_{(k+1)2^{-j}}) - \sigma(X_{i/n})] \frac{1}{n} l_z^n f_n(X_{i/n}) \\ & - \sum_k \sum_z \sum_{i \in S_{jk}} [\sigma(X_{(k+1)2^{-j}}) - \sigma(X_{(i-1)/n})] \frac{1}{n} l_z^n f_n(X_{(i-1)/n}) \\ & + \sum_k \sum_z \sum_{i \in S_{jk}} [\sigma(X_{i/n}) - \sigma(X_{(i-1)/n})] \frac{1}{n} l_z^n f_n(X_{(i-1)/n}). \end{aligned}$$

Hence, it is equal to

$$\begin{aligned} & - \sum_k \sum_z [\sigma(X_{(k+1)2^{-j}}) - \sigma(X_{k2^{-j}})] \frac{1}{n} l_z^n f_n(X_{(i-1)/n}) \\ & + \sum_k \sum_z \sum_{i \in S_{jk}} (X_{i/n} - X_{(i-1)/n}) \sigma'(X_{(i-1)/n}) \frac{1}{n} l_z^n f_n(X_{(i-1)/n}) + R, \end{aligned}$$

with $\mathbb{E}[|R|] \leq c\alpha_n^2 \log n$. This is finally equal to

$$R' + \sum_z \sum_k \sum_{i \in S_{jk}} (X_{i/n} - X_{(i-1)/n}) \sigma'(X_{(i-1)/n}) \frac{1}{n} l_z^n f_n(X_{(i-1)/n}) + R,$$

with $\mathbb{E}[|R'|] \leq c2^{j/2} \alpha_n^2 \log n$. The quantity

$$\sqrt{n}(X_{i/n} - X_{(i-1)/n}) \sigma'(X_{(i-1)/n}) l_z^n f_n(X_{(i-1)/n}) (\alpha_n^2 [1 + n/z])^{-1}$$

can be written as $g_n(X_{(i-1)/n}, \sqrt{n}[X_{i/n} - X_{(i-1)/n}])$. This function satisfies Assumption E. Hence, since $Mg_n(x) = 0$,

$$\mathbb{E} \left[\left| \sum_k \sum_{i \in S_{jk}} (X_{i/n} - X_{(i-1)/n}) \sigma'(X_{(i-1)/n}) \frac{1}{n} l_z^n f_n(X_{(i-1)/n}) \right| \right] \leq cn^{-1} \alpha_n^2 (1 + n/z).$$

Eventually,

$$\mathbb{E} \left[\left| \sum_z \sum_k \sum_{i \in S_{jk}} (X_{i/n} - X_{(i-1)/n}) \sigma'(X_{(i-1)/n}) \frac{1}{n} l_z^n f_n(X_{(i-1)/n}) \right| \right] \leq c\alpha_n^2 (1 + \log n).$$

It is also clear that

$$\mathbb{E} \left[\left| [\sigma(X_{(k+1)2^{-j}}) - \sigma(X_{i/n})] \frac{1}{n} l_{n-i}^n f_n(X_{(i-1)/n}) \right| \right] \leq c\alpha_n^2 (1 + \log n). \quad \square$$

We finally have the following result:

Lemma 12. *Let*

$$C_{j_n} = \sum_k c_{j_n k} (\hat{c}_{j_n k} - c_{j_n k}).$$

If $2^{-j_n} + r_n^{-1} 2^{j_n/2} (n^{-1} + \alpha_n^2 \log n) + 2^{j_n} r_n \rightarrow 0$, then we have the following stable convergences in law, where B is a standard Brownian motion, independent of \mathcal{F} :

$$\text{if } \beta_n \rightarrow 0, \quad \sqrt{n} C_{j_n} \rightarrow_{\mathcal{L}_s} \frac{1}{\sqrt{2}} (\pi - 2)^{1/2} \int_0^1 \sigma(X_t)^2 dB_t,$$

$$\text{if } \beta_n \rightarrow \beta > 0, \quad \sqrt{n} C_{j_n} \rightarrow_{\mathcal{L}_s} \int_0^1 \sigma(X_t) [\Delta_\beta(X_t)]^{1/2} dB_t,$$

$$\text{if } \beta_n \rightarrow +\infty, \quad \alpha_n^{-1} C_{j_n} \rightarrow_{\mathcal{L}_s} \frac{1}{\sqrt{3}} \int_0^1 \sigma(X_t) dB_t.$$

Proof. We have

$$\begin{aligned} \sum_k c_{j_n k} (\hat{c}_{j_n k} - c_{j_n k}) &= \sum_k \left[2^{j_n/2} \int \mathbb{1}_{j_n k}(s) \sigma(X_s) ds \right] \left[2^{j_n/2} \sum_{i \in S_{j_n k}} z_i^n \right] \\ &= T_1(j_n, n) + T_2 + T_3(j_n, n) + T_4(n), \end{aligned}$$

with

$$T_4(n) = \sum_i \sigma(X_{(i-1)/n}) q_i^n - \int \sigma(X_s) M f_n(X_s) ds.$$

We get the result by applying the results of Delattre [8], Chapter 2, to the term T_4 and using that $r_n^{-1} (T_1 + T_2 + T_3)$ tends to zero in probability. \square

The proof of Theorem 2 follows using Lemma 8.

4.5.3. Proof of Theorem 2 in the general case

We give a sketch of the proof of the result in the general case. We have the following lemma:

Lemma 13. *If $2^{-j_n} + r_n^{-1} 2^{j_n/2} (n^{-1} + \alpha_n^2 \log n) + 2^{j_n} r_n \rightarrow 0$, then we have the following convergences in stable law, where B is a standard Brownian motion, independent of \mathcal{F} :*

$$\text{if } \beta_n \rightarrow 0, \quad \sqrt{n} C_{j_n} \rightarrow_{\mathcal{L}_s} \frac{1}{\sqrt{2}} (\pi - 2)^{1/2} \int_0^1 g(X_t)^2 \sigma(X_t)^2 dB_t,$$

$$\begin{aligned} \text{if } \beta_n \rightarrow \beta > 0, \quad \sqrt{n} C_{j_n} &\rightarrow_{\mathcal{L}_s} -\frac{\beta}{2} \int_0^1 g(X_s) g'(X_s) \sigma(X_s)^2 ds \\ &\quad + \int_0^1 g(X_t)^2 \sigma(X_t) [\Delta_\beta(X_t)]^{1/2} dB_t, \end{aligned}$$

$$\begin{aligned} \text{if } \beta_n \rightarrow +\infty, \quad \alpha_n^{-1} C_{j_n} &\rightarrow \mathcal{L}_S - \frac{1}{2} \int_0^1 g(X_s)g'(X_s)\sigma(X_s)^2 ds \\ &+ \frac{1}{\sqrt{3}} \int_0^1 g(X_t)^2\sigma(X_t) dB_t. \end{aligned}$$

Proof. In this case, we have by analogy with Section 4.5.2

$$f_n(x, u, y) = (\pi/2)^{1/2} g(x - \alpha_n u) \beta_n \lfloor |u + y/\beta_n| \rfloor.$$

Let

$$\tilde{f}_n(x, u, y) = (\pi/2)^{1/2} g(x) \beta_n \lfloor |u + y/\beta_n| \rfloor.$$

We have

$$\sum_k c_{j_n k} (\hat{c}_{j_n k} - c_{j_n k}) = \sum_k c_{j_n k} (\hat{c}_{j_n k}^* - c_{j_n k}) + Z,$$

with $\mathbb{E}[|Z|] \leq c\alpha_n$. Hence, we easily get the result when β_n tends to zero. We also have

$$\begin{aligned} \sum_k c_{j_n k} (\hat{c}_{j_n k} - c_{j_n k}) &= \sum_k c_{j_n k} \left(\hat{c}_{j_n k} - 2^{j_n/2} \int \mathbb{1}_{j_n k}(s) M f_n(X_s) ds \right) \\ &+ \sum_k c_{j_n k} \left(2^{j_n/2} \int \mathbb{1}_{j_n k}(s) [M f_n(X_s) - M \tilde{f}_n(X_s)] ds \right). \end{aligned}$$

A bias is induced by the second term if β_n does not tend to zero. Indeed,

$$\alpha_n^{-1} [M f_n(X_s) - M \tilde{f}_n(X_s)] \approx -\frac{1}{2} g'(X_s) \sigma(X_s).$$

Since

$$\sum_{k=0}^{2^{j_0, n} - 1} \hat{e}_{j_0, n k}^2 \xrightarrow{\mathbb{P}} \int_0^1 |g(X_s)g'(X_s)|\sigma(X_s)^2 ds,$$

we are able to compensate this bias. We conclude the proof of Theorem 2 using that if U_n tends to U in probability on Ω and Y_n tends to Y in stable law, then (U_n, Y_n) tends to (U, Y) in stable law. □

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