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Explosive Poisson shot noise processes with applications to risk reserves

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We consider explosive Poisson shot noise processes as natural extensions of the classical compound Poisson process and investigate their asymptotic properties. Our main result is a functional central limit theorem with a self-similar Gaussian limit process which, in the classical case, is Brownian motion. The theorems are derived under regularity conditions on the moment and covariance functions of the shot noise process. The crucial condition is regular variation of the covariance function which implies the self-similarity of the limit process. The model is applied to delay in claim settlement in insurance portfolios. In this context we discuss some specific models and their properties. We also use the asymptotic theory for studying the ruin time and ruin probability for a risk process which is based on the Poisson shot noise process.

Keywords: functional central limit theorem; IBNR claims; Poisson clustering point process; Poisson shot noise process; risk reserve model

1. Introduction

We consider the explosive shot noise process

$$S(t) = \sum_{n\geq 1} X_n(t - T_n), \qquad t \geq 0, \tag{1.1}$$

where $X, X_1, X_2, \ldots$ are i.i.d. random non-null measures with support on $\mathbb{R}^+$, $X_n(t) = X_n([0, t])$, $t \geq 0$, and $(T_n)_{n\in\mathbb{N}}$ are random variables such that $N(t) := \#\{n : T_n \leq t\}$ is a homogeneous Poisson process with intensity $\alpha > 0$.

Traditionally, the Poisson shot noise process as in (1.1) has been investigated for i.i.d. stochastic processes $(X_n(t))_{t\geq 0}, n \in \mathbb{N}$, whose sample paths decrease to zero (see, for example, Bondesson 1988; 1992; Parzen 1962). Shot noise processes have been proposed, for example, for bunching in traffic (Bartlett 1963), for computer failure times (Lewis 1964) and for earthquake aftershocks (Vere-Jones 1970).

A stationary version of (1.1) is defined by

$$S(A) = \sum_{n\in\mathbb{Z}} X_n(A - T_n)$$

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for any bounded Borel set $A \subset \mathbb{R}$, where $(T_n)_{n\in\mathbb{Z}}$ are the jump times of a homogeneous Poisson process on $\mathbb{R}$ as above and $(X_n)_{n\in\mathbb{Z}}$ are i.i.d. non-negative random measures with support on $\mathbb{R}^+$. Asymptotic properties of this process have been investigated by, among others, Daley (1972) and Lane (1984).

Our interest is focused on the transient process (1.1) as a natural generalization of the classical compound Poisson process. We mention that our model includes the counting process in a Poisson clustering point process and might be of some interest in that context as well. For theoretical background on random measures we refer to Kallenberg (1983) and Daley and Vere-Jones (1988).

Our paper is organized as follows. In Section 2 we derive numerical characteristics of the process $(S(t))_{t\geq 0}$ such as the moment and covariance functions. Furthermore, we recall a particular notion of regular variation in $\mathbb{R}^2$ which is a basic tool for proving our main results.

In Section 3 we describe the asymptotic properties of the explosive Poisson shot noise process $(S(t))_{t\geq 0}$ for large t. These include strong laws of large numbers (SLLN), a central limit theorem (CLT) of Berry–Esseen type and a functional central limit theorem (FCLT). Here we need regular variation of the covariance function of S which implies in turn that the limit process is self-similar and Gaussian. As mentioned above, the model contains the counting process in a Poisson clustering process; the FCLT seems to be new in this context as well.

In Section 4 we apply our results to some specific insurance problems. The explosive shot noise process can be viewed as a natural model for delay in claim settlement: the $T_n, n \in \mathbb{N}$, are considered as the claim arrival times, and the measure $X_n(\cdot - T_n)$ describes the evolution of the pay-off process for the nth claim. Since every realization of the process $X_n(t)$ is a non-decreasing function of t, the limit $\lim_{t\to\infty} X_n(t) = X_n(\infty)$ exists (possibly infinite) and is the total pay-off caused by the nth claim. Then $(S(t))_{t\geq 0}$ as defined in (1.1) is the total claim amount process.

If the random measure X degenerates at zero the process (1.1) reduces to the classical total claim amount process which is a compound Poisson process. We specify some models via the explosive Poisson shot noise process and investigate their properties. Furthermore, we propose a risk premium and a risk process and derive an approximation for the ruin probability and the distribution of the first ruin time via the FCLT.

2. Preliminaries

2.1. MOMENT AND COVARIANCE FUNCTIONS OF THE PROCESS $(S(t))_{t\geq 0}$

We consider model (1.1):

$$S(t) = \sum_{n\geq 1} X_n(t - T_n) = \sum_{n=1}^{N(t)} X_n(t - T_n), \qquad t \geq 0,$$

(as usual $\sum_1^0 = 0$) where the processes $(X_n)_{n\in\mathbb{N}}$ are i.i.d. copies of X. Throughout we assume that for $n \in \mathbb{N}$ the functions $(X_n(t))_{t\geq 0}$ are non-decreasing and cadlag. Hence the realizations of $(X_n(t))_{t\geq 0}$ are measure-defining functions. The process $(S(t))_{t\geq 0}$ is a.s. finite for every fixed t and defines a random measure S on the Borel sets. For simplicity we write

$$X_n(a,b] = X_n((a,b]) = X_n(b) - X_n(a), \qquad a < b,$$

and use analogous notation for other measures and sets.

In this section we derive numerical characteristics of the stochastic process $(S(t))_{t\geq 0}$ such as the moment and covariance functions. One way is to calculate the Laplace functional of the random measure S. This allows for extensions to shot noise processes with general counting processes $(N(t))_{t\geq 0}$. However, in the context of a Poisson shot noise process numerical characteristics of S can be derived using the special properties of the Poisson process. For example, using the well-known fact that for $N(t) = k$ the random vector $(T_1, \ldots, T_k)$ has the same distribution as the order statistics of a sample of k i.i.d. random variables with uniform distribution on $[0, t]$, we conclude that

$$\begin{aligned} \mathrm{E}\mathrm{e}^{-\lambda S(t)} &= \sum_{k\geq 0} \mathrm{e}^{-\alpha t}\frac{(\alpha t)^k}{k!}\frac{k!}{t^k}\int_0^t \mathrm{d}u_1 \int_{u_1}^t \mathrm{d}u_2 \cdots \int_{u_{k-1}}^t \mathrm{d}u_k \left(\prod_{j=1}^k \mathrm{E}(\mathrm{e}^{-\lambda X(t-u_j)})\right) \\ &= \mathrm{e}^{-\alpha t}\sum_{k\geq 0}\frac{\alpha^k}{k!}\left(\int_0^t \mathrm{E}\mathrm{e}^{-\lambda X(u)}\,\mathrm{d}u\right)^k \\ &= \exp\left\{-\alpha t\left(1 - t^{-1}\int_0^t \mathrm{E}\mathrm{e}^{-\lambda X(u)}\,\mathrm{d}u\right)\right\}. \end{aligned}$$

Then $S(t)$ has representation $S(t) \stackrel{\mathrm{d}}{=} \sum_{n=1}^{N(t)} Y_n(t)$ for i.i.d. (Y_n) independent of $(N(t))$ such that $Y_n(t) \stackrel{\mathrm{d}}{=} X(Ut)$ where U is uniformly distributed on $(0, 1)$ and independent of X. Moreover, for $0 < s < t$ the random variable $S(s, t]$ has the same distribution as the sum of two independent terms; more precisely,

$$S(s,t] \stackrel{\mathrm{d}}{=} \sum_{n=1}^{N(s)} Z_n(s,t) + S'(t-s), \tag{2.1}$$

where $Z_n(s,t) \stackrel{\mathrm{d}}{=} X(Us, Us + (t-s)]$ are i.i.d. and $S' \stackrel{\mathrm{d}}{=} S$ is independent of S. This can be seen by the following argument. Given $(N(x))_{0\leq x\leq s}$ and $(X_n)_{n\leq N(s)}$, we can use the stationarity of the Poisson process to write $S(s, t]$ in distribution as

$$\sum_{n=1}^{N(s)} X_n((s,t] - T_n) + \sum_{n=1}^{N'(t-s)} X_n'(t - s - T_n'),$$

where $(T_n'), (X_n')$ are independent copies of $(T_n), (X_n)$, and $(N'(x))$ is defined in the natural way. Then the 'order statistics property' of the Poisson process which led to the Laplace transform of $S(t)$ can be applied to obtain (2.1).

Having these formulae in mind it is not difficult to calculate the following moments and covariances which are versions of Campbell's theorem (cf. Daley and Vere-Jones 1988). We write

$$\begin{aligned} \mu(t) &= \mu(0,t] = \alpha\int_0^t \mathrm{E}\{X(u)\}\,\mathrm{d}u, \\ \sigma^2(t) &= \sigma^2(0,t] = \alpha\int_0^t \mathrm{E}\{X^2(u)\}\,\mathrm{d}u. \end{aligned} \tag{2.2}$$

Proposition 2.1 Suppose $s < t$.
(a) Assume $\mu(t) < \infty$. Then

$$\mathrm{E}(S(s,t]) = \mu(s,t], \ \mathrm{E}\{S(t)\} = \mu(t), \qquad t \geq 0.$$

(b) Assume $\sigma^2(t) < \infty$. Then

$$\begin{aligned}\operatorname{var}(S(s,t]) &= \sigma^2(t) - \sigma^2(s) - 2\alpha \int_0^s \mathrm{E}\{X(u)X(u,u+t-s]\}\,\mathrm{d}u,\\ &= \sigma^2(t-s) + \alpha \int_0^s \mathrm{E}(X^2(u,u+t-s])\,\mathrm{d}u,\\ \operatorname{var}(S(t)) &= \sigma^2(t), \qquad t \geq 0.\end{aligned}$$

(c) Assume that $\int_0^\infty \mathrm{E}\{X^4(t-u)\}\,\mathrm{d}u < \infty$. Then

$$\mathrm{E}\{S(s,t] - \mathrm{E}(S(s,t])\}^4 = 3\sigma^4(s,t] + \alpha \int_0^{t-s} \mathrm{E}\{X^4(u)\}\,\mathrm{d}u + \alpha \int_0^s \mathrm{E}(X^4(u,u+t-s])\,\mathrm{d}u.$$

(d) Assume $\sigma^2(t) < \infty$. Then

$$\begin{aligned}\operatorname{cov}\{S(s),S(t)\} &= \alpha \int_0^s \mathrm{E}\{X(u)X(u+t-s)\}\,\mathrm{d}u\\ &= \sigma^2(s) + \alpha \int_0^s \mathrm{E}\{X(u)X(u,u+t-s]\}\,\mathrm{d}u.\end{aligned}$$

Remark 2.1
The moment functions of S are not invariant under shifts except for the compound Poisson process. Hence $(S(t))_{t\geq 0}$ is not stationary of any order.

2.2. REGULAR VARIATION OF THE COVARIANCE FUNCTION

In Section 3 we will need the notion of a *regularly varying (at infinity) function* in $\mathbb{R}^2$: The measurable function $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is regularly varying if for all $x, y > 0$ the limit

$$C(x,y) = \lim_{t\to\infty} \frac{f(xt,yt)}{f(t,t)}$$

exists finite and is positive. In this case, $C(x,y)$ is homogeneous which means that

$$C(kx,ky) = k^\rho C(x,y)$$

holds for all $x, y, k > 0$ and a fixed number ρ, and $C(1,1) = 1$. The quantity ρ is called the *index of regular variation* and C is the *limit function* of f. For other notions and properties of multivariate regular variation, we refer to Bingham *et al.* (1987, Appendix 1).

We also mention the notion of univariate regular variation. The measurable function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be regularly varying if, for all $x > 0$, the limit

$$C(x) = \lim_{t\to\infty} \frac{f(xt)}{f(t)}$$

exists and is finite. In this case, $C(x) = x^{\rho}$, and ρ is called the index of regular variation. We refer to Bingham *et al.* (1987) as a standard monograph on one-dimensional regular variation and its ramifications.

Proposition 2.2 If $\mathrm{E}\{X(s)X(t)\}$ is regularly varying with index $\rho - 1$ and limit function c, then $\operatorname{cov}\{S(s), S(t)\}$ is regularly varying with index ρ and limit function C where

$$C(x,y) = \rho \int_0^x c(u, u+y-x)\,\mathrm{d}u, \qquad x < y.$$

Moreover, $\rho \geq 1$.

Proof
Let $0 < x < y$. Then

$$\frac{\operatorname{cov}\{S(xt), S(yt)\}}{\sigma^2(t)} = \frac{\int_0^{xt} \mathrm{E}\{X(u)X(u+(y-x)t)\}\,\mathrm{d}u}{\int_0^t \mathrm{E}\{X^2(u)\}\,\mathrm{d}u}$$

$$= \frac{\int_0^x \frac{\mathrm{E}\{X(ut)X((u+y-x)t)\}}{\mathrm{E}\{X^2(t)\}}\,\mathrm{d}u}{\int_0^1 \frac{\mathrm{E}\{X^2(ut)\}}{\mathrm{E}\{X^2(t)\}}\,\mathrm{d}u} \to C(x,y)$$

where the existence of the limit $C(x,y)$ is guaranteed by Lebesgue dominated convergence. In particular, this implies that $\sigma^2(t)$ varies regularly with index ρ. Since X is different from the null measure we have $\liminf_{t\to\infty} \sigma^2(t)/t > 0$. Therefore $\rho \geq 1$. □

Remark 2.2
For $x < y$ we have the estimates

$$\sigma^2(xt) \leq \operatorname{cov}\{S(xt), S(yt)\} \leq \sigma^2(yt) - \sigma^2\{(y-x)t\}.$$

Hence regular variation of σ^2 with index ρ implies that

$$x^{\rho} \leq C(x,y) \leq y^{\rho} - (y-x)^{\rho}.$$

In particular, if $\rho = 1$ then $C(x,y) = x$, i.e. it is the covariance function of Brownian motion.

3. Asymptotic theory

3.1. LAWS OF LARGE NUMBERS

We start with a law of large numbers in order to get an impression of the order of magnitude of $S(t)$.

From Chebyshev's inequality we know that

$$P\left(\left|\frac{S(t)-\mu(t)}{\mu(t)}\right|>\epsilon\right)\leq\frac{\sigma^2(t)}{\epsilon^2\mu^2(t)}.$$

Hence a sufficient condition for the weak law of large numbers $(S(t)/\mu(t))\xrightarrow{\mathrm{P}}1$ as $t\to\infty$ is

$$\xi(t)=\frac{\sigma^2(t)}{\mu^2(t)}\to 0,\qquad t\to\infty. \tag{3.1}$$

It is not difficult to prove an SLLN under simple conditions on the rate of decrease of $\xi(t)$.

Proposition 3.1 Suppose there is a sequence $t_k\uparrow\infty$ such that

$$\sum_k \xi(t_k)<\infty, \tag{3.2}$$

and

$$\frac{\mu(t_k)}{\mu(t_{k-1})}\to 1. \tag{3.3}$$

Then the SLLN $(S(t)/\mu(t))\xrightarrow{\text{a.s.}}1$ holds.

Proof
From (3.2), Chebyshev's inequality and the Borel–Cantelli lemma we conclude that $(S(t_k)/\mu(t_k))\xrightarrow{\text{a.s.}}1$. Now (3.3) and a sandwich argument applied to the inequalities

$$\frac{S(t_{k-1})}{\mu(t_{k-1})}\leq\frac{S(t)}{\mu(t)}\frac{\mu(t)}{\mu(t_{k-1})}\leq\frac{S(t_k)}{\mu(t_k)}\frac{\mu(t_k)}{\mu(t_{k-1})}$$

for $t\in[t_{k-1},t_k]$ yields the statement. □

Remark 3.1
The assumptions of Proposition 3.1 can easily be checked. For example, if $\mu^2(t)$ and $\sigma^2(t)$ vary regularly with index ρ' and ρ, respectively, and if $\rho'>\rho$ then the conditions are obviously satisfied.

Remark 3.2
The assumptions of Proposition 3.1 can be weakened as follows. Suppose that for every $d>1$ there exists a sequence $t_k\uparrow\infty$ such that

$$\lim_{k\to\infty}\frac{\mu(t_k)}{d^k}=1$$

and (3.2) is satisfied. Analogous arguments as in the proof of Proposition 3.1 and a sandwich argument yield that

$$1\leq\liminf_{t\to\infty}\frac{S(t)}{\mu(t)}\leq\limsup_{t\to\infty}\frac{S(t)}{\mu(t)}\leq d \text{ a.s.}$$

Since $d>1$ is arbitrary this implies that $\lim_{t\to\infty}(S(t)/\mu(t))=1$ a.s.

3.2. BERRY–ESSEEN THEOREM

The limit distributions of the process

$$\frac{S(t)-\mu(t)}{\sigma(t)} \qquad \text{for } t \to \infty$$

have been characterized by Lane (1984) as infinitely divisible laws. We restrict ourselves to studying the asymptotic normality. We start with a Berry–Esseen estimate under the Lyapunov-type condition (3.4) (cf. also Lane 1987).

Theorem 3.2 Suppose that

$$\lim_{t\to\infty} L(t) = \lim_{t\to\infty} \frac{\int_0^t \mathrm{E}\{X^3(u)\}\,\mathrm{d}u}{\left(\int_0^t \mathrm{E}\{X^2(u)\}\,\mathrm{d}u\right)^{3/2}} = 0. \tag{3.4}$$

Then

$$\sup_x \left| P\left(\frac{S(t)-\mu(t)}{\sigma(t)} \le x\right) - \Phi(x)\right| \le CL(t), \qquad t>0, \tag{3.5}$$

where C is a positive constant and Φ denotes the standard normal distribution function.

Proof

We follow the lines of the proof of the classical Berry–Esseen result (see Petrov 1995, Theorem 5.5). According to Esseen's inequality we have

$$\sup_x \left| P\left(\frac{S(t)-\mu(t)}{\sigma(t)} \le x\right) - \Phi(x)\right| \le \frac{1}{\pi}\int_{-1/(4L(t))}^{1/(4L(t))} \left|\frac{f_t(\lambda)-\phi(\lambda)}{\lambda}\right| \mathrm{d}\lambda + \bar{C}L(t), \tag{3.6}$$

where $\bar{C}$ is a positive constant and f_t and ϕ denote the characteristic functions of $(S(t)-\mu(t))/\sigma(t)$ and of the standard normal distribution, respectively.

For $|\lambda| \le 1/(4L(t))$ and $|\lambda| \ge 1/(2L^{1/3}(t))$ we have by Taylor series expansion that

$$\begin{aligned} |f_t(\lambda)|^2 &= \exp\left(-2\alpha\left[t - \int_0^t \mathrm{Re}\left\{\mathrm{E}\exp\left(\mathrm{i}\lambda\frac{X(u)}{\sigma(t)}\right)\mathrm{d}u\right\}\right]\right) \\ &\le \exp\{-\lambda^2 + \tfrac{4}{3}|\lambda|^3 L(t)\} \\ &\le \exp(-\tfrac{2}{3}\lambda^2). \end{aligned}$$

Now suppose that $|\lambda| \le 1/(4L(t))$ and $|\lambda| \le 1/(2L^{1/3}(t))$. Again using a Taylor series expansion we obtain

$$\begin{aligned} \log f_t(\lambda) &= -\mathrm{i}\lambda\frac{\mu(t)}{\sigma(t)} - \alpha\left(\int_0^t \left[1 - \mathrm{E}\left\{\exp\left(\mathrm{i}\lambda\frac{X(u)}{\sigma(t)}\right)\right\}\right]\mathrm{d}u\right) \\ &= -\frac{\lambda^2}{2} + \theta\,\frac{L(t)}{2}|\lambda|^3, \qquad |\theta| \le 1. \end{aligned}$$

Analogous arguments as in Petrov (1995) yield that

$$|f_t(\lambda) - \phi(\lambda)| \le 16L(t)|\lambda|^3 \exp\{-|\lambda|^2/3\}.$$

The result now follows from (3.6). □

3.3. FUNCTIONAL CENTRAL LIMIT THEOREMS

Next we prove a functional version of the CLT. To this end recall that $D[0,1]$ is the space of cadlag functions on the unit interval (see Billingsley 1968; Pollard 1984). We suppose that $D[0,1]$ is equipped with the supremum-norm topology and with the projection σ-algebra. We define

$$S_x(t) = \frac{S(xt) - \mu(xt)}{\sigma(t)}, \qquad 0 \le x \le 1, t \ge 0.$$

It is easily seen that the sample paths of $S_.(t)$ belong to $D[0,1]$ for every $t > 0$. The following result establishes the convergence of the finite-dimensional distributions of the processes $S_.(t)$.

Theorem 3.3 Suppose that $\mathrm{E}\{X(s)X(t)\}, s, t \ge 0$, is regularly varying. Then the limits

$$C(x,y) = \lim_{t\to\infty} \operatorname{cov}\{S_x(t), S_y(t)\}, \qquad x, y \in [0,1],$$

exist and are finite. Moreover, there exists a Gaussian process $(B_x)_{0\le x\le 1}$ with zero mean and covariance function $C(x,y), 0 \le x, y \le 1$, and with a.s. continuous sample paths. The finite-dimensional distributions of the process $S_.(t)$ converge to those of $B_.$ if and only if

$$\frac{1}{\sigma^2(t)} \int_{\epsilon\sigma(t)}^{\infty} y \int_0^t P\{X(u) > y\}\,du\,dy \to 0, \qquad \forall\epsilon > 0. \tag{3.7}$$

Proof

The existence of the limit function C is a consequence of Proposition 2.2. Since C is non-negative definite there exists a Gaussian process $B_.$ with C as its covariance function. Moreover, from Remark 2.2 we conclude that for ρ as in Proposition 2.2

$$\mathrm{E}(B_x - B_y)^2 = x^\rho + y^\rho - 2C(x,y) \le y^\rho - x^\rho.$$

Therefore by Kolmogorov's continuity theorem there exists a version of B with continuous sample paths.

For notational ease we restrict ourselves to the two-dimensional distributions. Let $0 < x_1 < x_2 \le 1$ and λ_1, λ_2 be fixed positive numbers. The process

$$S_{x_1,x_2}(t) = \sigma(t)\tilde\sigma(t)^{-1}\{\lambda_1 S_{x_1}(t) + \lambda_2 S_{x_2}(t)\}$$

has zero mean and unit variance where

$$\tilde\sigma^2(t) = \lambda_1^2\sigma^2(x_1 t) + \lambda_2^2\sigma^2(x_2 t) + 2\lambda_1\lambda_2\sigma^2(t)\operatorname{cov}\{S_{x_1}(t), S_{x_2}(t)\}.$$

Hence $S_{x_1,x_2}(t)$ satisfies the assumptions of Theorem 3 in Lane (1984) according to which the condition

$$\bar{\sigma}^{-2}(t)\int_{\epsilon\bar{\sigma}(t)}^{\infty} y\int_0^{\infty} P\{\lambda_1 X(x_1 t - s) + \lambda_2 X(x_2 t - s) > y\}\,\mathrm{d}s\,\mathrm{d}y \to 0, \qquad \forall\epsilon > 0,$$

is necessary and sufficient for convergence towards a standard Gaussian variable. Hence it suffices to show for normal convergence that for every fixed $\lambda > 0, \epsilon > 0, x \in (0,1]$

$$\sigma^{-2}(xt)\int_{\epsilon\sigma(xt)}^{\infty} y\int_0^{\infty} P\{\lambda X(xt - s) > y\}\,\mathrm{d}s\,\mathrm{d}y = \sigma^{-2}(xt)\int_{\epsilon\sigma(xt)}^{\infty} y\int_0^{xt} P\{\lambda X(u) > y\}\,\mathrm{d}u\,\mathrm{d}y \to 0,$$

but this follows from (3.7). Therefore $S_{x_1,x_2}(t)$ converges weakly to a standard Gaussian distribution. By the regular variation of the covariance function we obtain that

$$\begin{aligned}\lambda_1 S_{x_1}(t) + \lambda_2 S_{x_2}(t) &\xrightarrow{\mathrm{d}} N(0, \lambda_1^2 x_1^\rho + \lambda_2^2 x_2^\rho + 2\lambda_1\lambda_2 C(x_1,x_2))\\ &\stackrel{\mathrm{d}}{=} \lambda_1 B_{x_1} + \lambda_2 B_{x_2}.\end{aligned}$$

This and the Cramér–Wold device prove the convergence of the finite-dimensional distributions.

The necessity part follows from Theorem 3 in Lane (1984). □

Remark 3.3

Lane (1984) showed in a more general framework that (3.7) is necessary and sufficient for the normal convergence of $S_1(t)$.

Remark 3.4

A sufficient condition for (3.7) is the Lyapunov-type assumption

$$\sigma^{-2-\delta}(t)\int_0^t \mathrm{E}\{X^{2+\delta}(u)\}\,\mathrm{d}u \to 0$$

for some $\delta > 0$.

Next we show the tightness of the processes $S_\cdot(t)$. For the compound Poisson process it is well known that $S_\cdot(t)$ converges to Brownian motion (e.g. Gut 1988, Section V.2). But as soon as we depart from this model to the more general case (1.1) we lose the stationary and independent increments and the method of proof is no longer applicable. However, if $X(\infty) < \infty$ a.s. and if $X(t)$ approaches $X(\infty)$ sufficiently fast, then, under appropriate moment conditions, one can derive an FCLT with Brownian motion as a limit for $S_\cdot(t)$.

First we suppose that

$$X(\infty) < \infty \text{ a.s.}$$

The basic idea is to approximate the process $S_\cdot(t)$ by the compound Poisson process $\sum_{i=1}^{N(\cdot t)} X_i(\infty)$. We will frequently make use of the fact that if $0 < \mathrm{E}X^k(\infty) < \infty$ for some $k > 0$ then

$$\frac{1}{t}\int_0^t \mathrm{E}\{X^k(u)\}\,\mathrm{d}u \to \mathrm{E}\{X^k(\infty)\}.$$

Proposition 3.4 Suppose that $E\{X^2(\infty)\} < \infty$. If the conditions

$$\frac{1}{\sqrt{t}}\int_0^t [E\{X(\infty)\} - E\{X(u)\}]\,du \to 0, \tag{3.8}$$

$$\frac{1}{\sqrt{t}}\sup_{0\le s\le t}\left|\sum_{i=1}^{N(s)}\{X_i(\infty) - X_i(s-T_i)\}\right| \xrightarrow{P} 0, \tag{3.9}$$

are satisfied then

$$S_\cdot(t) \xrightarrow{d} B_\cdot, \qquad t\to\infty,$$

for a standard Brownian motion $B_\cdot$ on $[0,1]$ where the convergence holds in $D[0,1]$ equipped with the supremum norm.

Proof
Since $E\{X^2(\infty)\} < \infty$ and $E\{X(xt)X(yt)\} \to E\{X^2(\infty)\}$ for all $0 < x < y$, $C(x,y)$ is necessarily regularly varying with index $\rho = 1$. It follows that $C(x,y) = \min(x,y)$ (cf. Remark 2.2) and, that $B_\cdot$ is Brownian motion. We have for $x \in (0,1]$

$$\begin{aligned}
S_x(t) &= \sigma^{-1}(t)\left(\sum_{i=1}^{N(xt)} X_i(xt-T_i) - \mu(xt)\right)\\
&= [\alpha t E\{X^2(\infty)\}]^{-1/2}\{1+o(1)\}\left(\sum_{i=1}^{N(xt)}\{X_i(xt-T_i) - X_i(\infty)\} + \sum_{i=1}^{N(xt)}[X_i(\infty) - E\{X(\infty)\}]\right.\\
&\quad \left. + \{N(xt) - \alpha xt\}E\{X(\infty)\} + [\alpha xt E\{X(\infty)\} - \mu(xt)]\right)\\
&= I_1(x) + I_2(x) + I_3(x) + I_4(x).
\end{aligned}$$

In view of (3.9), I_1 converges to zero uniformly in probability. Moreover, for some positive constant c

$$|I_4(x)| \le \frac{c}{\sqrt{t}}\int_0^t [E\{X(\infty)\} - E\{X(u)\}]\,du.$$

Hence, and by (3.8), we conclude that I_4 uniformly converges to zero. It follows from Gut (1988) that

$$\left(\frac{1}{(\alpha t)^{1/2}}\{N(xt) - \alpha xt\}, \frac{1}{[\operatorname{var}\{X(\infty)\}t]^{1/2}}\sum_{i=1}^{[xt]}[X_i(\infty) - E\{X(\infty)\}], \frac{N(xt)}{\alpha t}\right) \xrightarrow{d} (B_x^{(1)}, B_x^{(2)}, x)$$

where $B_\cdot^{(1)}$ and $B_\cdot^{(2)}$ are two independent standard Brownian motions on $[0,1]$ and the convergence $\xrightarrow{d}$ is weak convergence in $D([0,1],\mathbb{R}^3)$ equipped with the supremum norm. (Gut actually proves the results for the Skorokhod J_1-topology but since the limit processes have continuous sample paths and are independent the convergence also holds with the supremum norm.) An application of the continuous mapping theorem implies that

$$I_2(\cdot) + I_3(\cdot) \xrightarrow{\mathrm{d}} B^{(2)}_{\cdot}\left(\frac{\operatorname{var}\{X(\infty)\}}{\mathrm{E}\{X^2(\infty)\}}\right)^{1/2} + B^{(1)}_{\cdot}\left(\frac{[\mathrm{E}\{X(\infty)\}]^2}{\mathrm{E}\{X^2(\infty)\}}\right)^{1/2} \stackrel{\mathrm{d}}{=} B_{\cdot}$$

which concludes the proof. □

In the following result we reformulate condition (3.9).

Theorem 3.5 Suppose that $\mathrm{E}X^2(\infty) < \infty$ and that (3.8) holds. Assume that there exist positive functions h and g defined on the non-negative real axis such that

$$\sqrt{t}(\mathrm{E}\{X(\infty)\} - \mathrm{E}[X\{h(t)\}]) \to 0, \qquad t \to \infty, \tag{3.10}$$

$$g(t) \sup_{h(t) \le s \le t} (N(s) - N(s - h(t))) = O_{\mathrm{P}}(1), \qquad t \to \infty, \tag{3.11}$$

and

$$tP\{X(\infty) > g(t)\sqrt{t}\} \to 0, \qquad t \to \infty. \tag{3.12}$$

Then

$$S_{\cdot}(t) \xrightarrow{\mathrm{d}} B_{\cdot}, \qquad t \to \infty,$$

for a standard Brownian motion on $[0, 1]$ where the convergence holds in $D[0, 1]$ equipped with the supremum norm.

Proof
By Proposition 3.4 it remains to show (3.9). For $\epsilon > 0$ fixed we obtain that

$$\begin{aligned}
&P\left(\sup_{0\le s\le t} \sum_{i=1}^{N(s)} \{X_i(\infty) - X_i(s - T_i)\} > \epsilon\sqrt{t}\right) \\
&\le P\left(\sup_{0\le s\le t} \sum_{i=1}^{N(s)} \{X_i(\infty) - X_i(s - T_i)\} I\{s - T_i > h(t)\} > \epsilon\sqrt{t}/2\right) \\
&\quad + P\left(\sup_{0\le s\le t} \sum_{i=1}^{N(s)} \{X_i(\infty) - X_i(s - T_i)\} I\{s - T_i \le h(t)\} > \epsilon\sqrt{t}/2\right) \\
&\le P\left(\sum_{i=1}^{N(t)} [X_i(\infty) - X_i\{h(t)\}] > \epsilon\sqrt{t}/2\right) \\
&\quad + P\left(\max_{i=1,\dots,N(t)} X_i(\infty) \sup_{0\le s\le t} \sum_{i=1}^{N(s)} I\{s - T_i \le h(t)\} > \epsilon\sqrt{t}/2\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq (\epsilon\sqrt{t}/2)^{-1}\mathrm{E}\left(\sum_{i=1}^{N(t)}[X_i(\infty) - X_i\{h(t)\}]\right) \\
&\quad + P\left(\max_{i=1,\dots,N(t)} X_i(\infty) \sup_{h(t)\leq s\leq t}\{N(s) - N(s-h(t))\} > \epsilon\sqrt{t}/2\right) \\
&= (\epsilon\sqrt{t}/2)^{-1}\alpha t(\mathrm{E}X(\infty) - \mathrm{E}X(h(t))) \\
&\quad + P\left(\max_{i=1,\dots,N(t)} X_i(\infty) \sup_{h(t)\leq s\leq t}\{N(s) - N(s-h(t))\} > \epsilon\sqrt{t}/2\right) \\
&= I_1 + I_2.
\end{aligned}
$$

From (3.10) we conclude that $I_1 \to 0$. For I_2 we get for any fixed $\delta > 0$, in view of the LLN and by (3.11) and (3.12), that

$$
\begin{aligned}
I_2 &\leq P\left(\max_{i=1,\dots,[\alpha t(1+\delta)]} X_i(\infty) \sup_{h(t)\leq s\leq t}\{N(s) - N(s-h(t))\} > \epsilon\sqrt{t}/2\right) + P(N(t) > \alpha t(1+\delta)) \\
&= P\left(\left((g(t)\sqrt{t})^{-1}\max_{i=1,\dots,[\alpha t(1+\delta)]} X_i(\infty)\right) g(t) \sup_{h(t)\leq s\leq t}\{N(s) - N(s-h(t))\} > \epsilon/2\right) + o(1) \\
&= o(1).
\end{aligned}
$$

In the last step we also made use of the fact that

$$(g(t)\sqrt{t})^{-1}\max_{i=1,\dots,[\alpha t(1+\delta)]} X_i(\infty) \xrightarrow{\mathrm{P}} 0$$

which is a consequence of (3.12) (cf. Leadbetter *et al.* 1983, Theorem 1.5.1). □

Corollary 3.6 Suppose that one of the following conditions holds:

(a) $X(x) = X(\infty)$ a.s. for $x \geq x_0$ and $\mathrm{E}\{X^{2+\delta}(\infty)\} < \infty$ for some $\delta > 0$.
(b) (3.10) holds for $h(t) = t^{\beta}$ and $\mathrm{E}\{X^{2/(1-2\beta)}(\infty)\} < \infty$ for some $\beta \in (0, 0.5)$.

Then $S_{\cdot}(t) \xrightarrow{\mathrm{d}} B_{\cdot}$ as $t \to \infty$ in $D[0,1]$ equipped with the supremum norm.

Proof
(a) If $\mathrm{E}\{X(x)\} = \mathrm{E}\{X(\infty)\}$ for $x \geq x_0$, then conditions (3.8) and (3.10) are satisfied. If we choose $g(t) = (\ln t)^{-1}$ and $h(t) = x_0$ then, in view of $\mathrm{E}\{X^{2+\delta}(\infty)\} < \infty$ and the exponential tail behaviour of a Poisson random variable, it is not difficult to check the validity of (3.11) and (3.12). Hence the conditions of Theorem 3.5 are satisfied and the statement follows.
(b) We conclude from Deheuvels and Steinebach (1989, Theorem 1), that

$$\sup_{t^{\beta}\leq s\leq t}\{N(s) - N(s-t^{\beta})\} = O_{\mathrm{P}}(t^{\beta})$$

which implies (3.11) for a function g such that $g(t) = O_{\mathrm{P}}(t^{-\beta})$, $\beta \in (0,1)$. We choose $g(t) = t^{-\beta}$ for

$\beta < 0.5$. Then $\mathrm{E}X^{2/(1-2\beta)}(\infty) < \infty$ implies condition (3.12). Moreover, (3.10) implies (3.8). An application of Theorem 3.5 yields the statement. □

Remark 3.5
Using some more sophisticated results on the increase for increments of renewal processes, the conditions of Theorem 3.5 on the functions h and g can be modified in various ways. We refer to Deheuvels and Steinebach (1989) for a recent treatment of this topic.

Notice that the results above do not apply if $X(t) \to \infty$ as $t \to \infty$. For a corresponding result we need some additional regularity assumptions:

$$\sigma^{-2}(t)\left(\sigma^2\{t(y-x)\} + \alpha\int_0^{xt} \mathrm{E}[X^2\{u, u+t(y-x)\}]\,\mathrm{d}u\right) \le \{F_1(y) - F_1(x)\}^{q_1} \tag{3.13}$$

$$\sigma^{-4}(t)\left(\int_0^{t(y-x)} \mathrm{E}\{X^4(u)\}\,\mathrm{d}u + \int_0^{xt} \mathrm{E}[X^4\{u, u+t(y-x)\}]\,\mathrm{d}u\right) \le \{F_2(y) - F_2(x)\}^{q_2} \tag{3.14}$$

for some $q_i > 1, i = 1, 2$, for non-decreasing continuous functions $F_i, i = 1, 2$, on $[0, 1]$ uniformly for $0 \le x < y \le 1$ and sufficiently large t.

Condition (3.13) requires more than a linear growth of $\sigma^2(t)$. Conditions (3.13) and (3.14) are essentially of the same type. The left-hand sides consist of two summands. The first of them describes the increase of the sample paths in time and the second one can be interpreted as an integrated modulus of continuity of the second and fourth moments of the process X. The second terms describe the dependence structure of the increments in the process X; in the case of the classical compound Poisson process these terms disappear.

Theorem 3.7 Suppose the assumptions of Theorem 3.3 are satisfied and that (3.13) or (3.14) hold. Then

$$S_{\cdot}(t) \xrightarrow{\mathrm{d}} B_{\cdot}, \qquad t \to \infty,$$

in $D[0, 1]$ equipped with the supremum norm.

Proof
By Theorem 3.3 it remains to show the tightness of the processes $S_{\cdot}(t)$. Since the limit process has a.s. sample paths in $C[0, 1]$, the space of the continuous functions on $[0, 1]$ equipped with the supremum norm, it suffices to show that for each $\epsilon > 0, \eta > 0$ there exists some $\delta > 0$ such that

$$\limsup_{t\to\infty} \sum_{j<\delta^{-1}} P\left(\sup_{x\in[j\delta,(j+1)\delta)} |S_x(t) - S_{j\delta}(t)| > \eta\right) < \epsilon$$

(see Pollard 1984, Theorem 3 in Chapter V.1). We follow the lines of the proof of Billingsley (1968, Theorem 12.3). We fix $\eta > 0, \epsilon > 0$. For the moment we also fix δ and t. Choose m large enough that

$$\sum_{j<\delta^{-1}}\left(P\left(\sup_{x\in[j\delta,(j+1)\delta)} |S_x(t) - S_{j\delta}(t)| > \eta\right) - P\left(\max_{i=1,\dots,m-1} |S_{j\delta+\delta i/m}(t) - S_{j\delta}(t)| > \eta\right)\right) < \frac{\epsilon}{2}$$

which is possible in view of the right continuity of the $S_x(t)$. Next we show that

$$\sum_{j<\delta^{-1}} P\left(\max_{i=1,\dots,m-1} |S_{j\delta+\delta i/m}(t)-S_{j\delta}(t)|>\eta\right)\le\frac{\epsilon}{2} \tag{3.15}$$

provided t and m (depending on t) are chosen large enough. First suppose that (3.13) holds. Then in view of Proposition 2.1(b)

$$\mathrm{E}\{S_{j\delta+\delta k/m}(t)-S_{j\delta+\delta i/m}(t)\}^2\le\left\{F_1\left(j\delta+\delta\frac{k}{m}\right)-F_1\left(j\delta+\delta\frac{i}{m}\right)\right\}^{q_1} \tag{3.16}$$

uniformly for $1\le i\le k\le(m-1),m,j$. Next suppose that (3.14) holds. With Proposition 2.1(b) we have

$$\begin{aligned}\sigma^{-4}(t)\sigma^4[xt,yt]&\le\sigma^{-4}(t)\left(\int_{xt}^{yt}\mathrm{E}\{X^2(u)\}\,\mathrm{d}u\right)^2\\&\le\sigma^{-4}(t)[t(y-x)\,\mathrm{E}\{X^2(t)\}]^2\\&\le c[t\,\mathrm{E}\{X^2(t)\}]^{-2}[t(y-x)\,\mathrm{E}\{X^2(t)\}]^2\\&=c(y-x)^2\end{aligned} \tag{3.17}$$

for some $c>0$ and sufficiently large t. Here we have used the regular variation of $\mathrm{E}X^2(t)$. Combining (3.17) with assumption (3.14) and recalling Proposition 2.1(c) we conclude that for $k\ge i$

$$E\{S_{j\delta+\delta k/m}(t)-S_{j\delta+\delta i/m}(t)\}^4\le\left\{F_3\left(j\delta+\delta\frac{k}{m}\right)-F_3\left(j\delta+\delta\frac{i}{m}\right)\right\}^{q_3} \tag{3.18}$$

uniformly for k,i,j,m, for sufficiently large t and for a continuous, monotone non-decreasing function F_3 on $[0,1],q_3>1$. Relations (3.18) and (3.16) and an application of Billingsley (1968, Theorem 12.2) yield that

$$P\left(\max_{i=1,\dots,m-1}|S_{j\delta+\delta i/m}(t)-S_{j\delta}(t)|>\eta\right)\le[F_4\{(j+1)\delta\}-F_4(j\delta)]^{q_4}$$

for a continuous non-decreasing function F_4 on $[0,1],q_4>1$, which implies (3.15) provided we choose δ sufficiently small. This concludes the proof. □

The restriction to the unit interval in the FCLTs above is not necessary. From the uniform convergence theorem for regularly varying functions we obtain that

$$\frac{S(xkt)-\mu(xkt)}{\sigma(t)}=\frac{\sigma(kt)}{\sigma(t)}\frac{S(xkt)-\mu(xkt)}{\sigma(kt)}\xrightarrow{\mathrm{d}}k^{\rho/2}B_x,\qquad 0\le x\le 1,$$

for any $k>0$. Hence

$$\frac{S(xt)-\mu(xt)}{\sigma(t)}\xrightarrow{\mathrm{d}}k^{\rho/2}B_{x/k},\qquad 0\le x\le k.$$

Thus $S_{\cdot}(t)$ converges weakly in $D[0,k]$ (equipped with the supremum norm) to a Gaussian process $B_{\cdot}$ on $[0,k]$ with a.s. continuous sample paths and $B_{\cdot} \stackrel{d}{=} k^{\rho/2} B_{\cdot/k}$. We define convergence in $D[0,\infty)$ as uniform convergence on compacta (see Pollard 1984, Definition 22, p. 108) where $D[0,\infty)$ is equipped with its projection σ-algebra. By Pollard (1984, Theorem 23, p. 108), we conclude the following:

Theorem 3.8 The processes $S_{\cdot}(t)$ converge weakly in $D[0,\infty)$ to a Gaussian process $B_{\cdot}$ with zero mean and covariance function $C(x,y)$ such that

$$C(kx,ky) = k^{\rho}C(x,y), \qquad x,y \geq 0, k > 0,$$

i.e. $B_{\cdot}$ is a self-similar Gaussian process.

This result extends the FCLT for the classical compound Poisson process to the shot noise process $S(t)$. In both cases the limit process is self-similar Gaussian. The crucial condition which ensures this property is regular variation in $\mathbb{R}^2$ of the covariance function of the underlying process $S(t)$. This shows that regular variation is a natural concept in this context.

The FCLT allows for sensitive estimates of the probability of the oscillations of $S(t)$ around its mean value in a given interval and also for the estimation of moments of continuous functionals of $S_x(t)$. Define

$$m_t = \min_{0 \leq x \leq 1} S_x(t) = \frac{1}{\sigma(t)} \min_{0 \leq s \leq t} (S(s) - \mu(s))$$

$$M_t = \max_{0 \leq x \leq 1} S_x(t) = \frac{1}{\sigma(t)} \max_{0 \leq s \leq t} (S(s) - \mu(s)).$$

The following is an immediate consequence of the continuous mapping theorem.

Corollary 3.9 Suppose the assumptions for the FCLT in Theorems 3.5 or 3.7 are satisfied. Then

$$(S_1(t), m_t, M_t) \xrightarrow{d} \left(B_1, \min_{0 \leq x \leq 1} B_x, \max_{0 \leq x \leq 1} B_x\right).$$

Remark 3.6

If $B_{\cdot}$ is Brownian motion, then the joint distribution of B_1 and the minimum and maximum of Brownian motion are well known (see, for example, Billingsley 1968, formula (11.10)). For other Gaussian processes the tail behaviour of suprema has been studied under entropy conditions and/or conditions on the regularity of $C(x,y)$. We refer to Adler (1990) and Samorodnitsky (1991) for some recent studies.

We continue with an application of the FCLT proved above. First define the *integrated shot noise process*

$$D_x(t) = \int_0^x S_y(t)\,\mathrm{d}y, \qquad 0 \leq x \leq 1.$$

From the FCLT and the Cramér–Wold device we conclude that for any $0 \leq x_1 < \cdots < x_n \leq 1$ the finite-dimensional distributions satisfy the relation

$$(D_{x_1}(t),\ldots,D_{x_n}(t)) \xrightarrow{\mathrm{d}} \left(\int_0^{x_1} B_y\,\mathrm{d}y,\ldots,\int_0^{x_n} B_y\,\mathrm{d}y\right).$$

We prove a functional version of this result.

Corollary 3.10 Suppose that the assumptions for the FCLT of $(S_{\cdot}(t))_{t\geq 0}$ are satisfied. Then

$$D_{\cdot}(t) \xrightarrow{\mathrm{d}} \int_0^{\cdot} B_y\,\mathrm{d}y$$

in $C[0,1]$ equipped with the supremum norm. Moreover, $\int_0^{\cdot} B_y\,\mathrm{d}y$ is a Gaussian process with mean function zero and with covariance function

$$\operatorname{cov}\left(\int_0^{x_1} B_y\,\mathrm{d}y,\int_0^{x_2} B_y\,\mathrm{d}y\right) = \int_0^{x_1}\int_0^{x_2} C(y_1,y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2.$$

Proof
For the FCLT it remains to show the tightness. According to Billingsley (1968, Theorem 12.3), it suffices to show that

$$I(x,y)(t) = \mathrm{E}\{D_y(t)-D_x(t)\}^2 \leq \{F(y)-F(x)\}^q$$

for a non-decreasing continuous function F on $[0,1], q>1, 0<x<y\leq 1$ and sufficiently large t. We have

$$I(x,y)(t) = \int_x^y\int_x^y \operatorname{cov}(S_{x_1}(t),S_{x_2}(t))\,\mathrm{d}x_1\,\mathrm{d}x_2.$$

In view of the uniform convergence theorem for regularly varying functions (Bingham *et al.* 1987, Theorem 1.5.2) and since $\operatorname{cov}(S_{x_1}(t),S_{x_2}(t)) \leq (\operatorname{var} S_{x_1}(t)\operatorname{var} S_{x_2}(t))^{0.5}$ we conclude that

$$I(x,y)(t) \leq c\int_x^y\int_x^y \mathrm{d}x_1\,\mathrm{d}x_2 \leq c(y-x)^2$$

for large t and a constant $c>0$. This proves the tightness. □

4. Some applications in insurance

The shot noise process can be considered as a natural model for delay in claim settlement. In that case,

$$S(t) = \sum_{n\geq 1} X_n(t-T_n), \qquad t\geq 0,$$

is the total claim amount process and the process X_i describes the pay-off procedure of the ith individual claim. If the random measure X degenerates at zero the process reduces to the classical total claim amount process. Here it is assumed that the claims are settled by the insurer at the time they occur. In reality this is rarely the case; very often a claim is unknown to the insurer at the time it occurs and it is reported after a certain time delay. Moreover, the future cost development is also often unknown as, for example, in rehabilitation following accidents.

In Section 4.1 we introduce some specific models and investigate their properties. In Section 4.2 we give approximations to the probability of ruin related to our model. For those interested in more details on delay in claim settlement we refer to Norberg (1993), Arjas (1989) and Neuhaus (1992) and references therein.

4.1. MODELS FOR DELAY IN CLAIM SETTLEMENT

4.1.1. Finite individual claims

Suppose that

$$X_n(\infty) = \lim_{t\to\infty} X_n(t) < \infty \qquad \text{a.s.} \tag{4.1}$$

for every n, i.e. the total size of every claim is finite with unit probability. In particular, if the X_n are a.s. uniformly bounded measures then assumption (4.1) is satisfied. From L'Hôpital's rule and Proposition 2.1 we obtain that

$$\mu(t) = \alpha \mathrm{E}\{X(\infty)\}\{1 + o(1)\}t, \qquad \sigma^2(t) = \alpha \mathrm{E}\{X^2(\infty)\}\{1 + o(1)\}t, \qquad t \to \infty,$$

provided the corresponding moments are finite. Furthermore, from Proposition 3.1 we immediately obtain that

$$\lim_{t\to\infty} \frac{S(t)}{t} = \alpha \mathrm{E}\{X(\infty)\} \qquad \text{a.s.}$$

If $\mathrm{E}X^3(\infty) < \infty$ we conclude from Theorem 3.2 that the Berry–Esseen estimate

$$\sup_x \left| P\left(\frac{S(t) - \mu(t)}{\sigma(t)} \le x\right) - \Phi(x)\right| \le Ct^{-1/2} \frac{\mathrm{E}\{X^3(\infty)\}}{[\mathrm{E}\{X^2(\infty)\}]^{3/2}}, \qquad t > 0,$$

holds. Moreover, under the moment conditions of Corollary 3.6 the FCLT applies with standard Brownian motion as limit process.

These results show that the asymptotic theory for $S(t)$ under condition (4.1) is very much like the theory for the sums

$$\sum_{n=1}^{N(t)} X_n(\infty),$$

i.e. the classical total claim amount process.

4.1.2. A simple multiplicative model

A large subclass of processes (1.1) has the representation

$$S(t) = \sum_{n\ge 1} Y_n \gamma(t - T_n) = \sum_{n=1}^{N(t)} Y_n \gamma(t - T_n) \tag{4.2}$$

for i.i.d. $Y, Y_1, Y_2, \ldots$ and a fixed non-decreasing function γ.

The mean, variance and covariance functions under suitable moment conditions are

$$\mu(t) = \alpha \mathrm{E}Y \int_0^t \gamma(u)\, \mathrm{d}u$$

$$\sigma^2(t) = \alpha \mathrm{E}Y^2 \int_0^t \gamma^2(u)\, \mathrm{d}u$$

$$\mathrm{cov}\,(S(s), S(t)) = \alpha \mathrm{E}Y^2 \int_0^s \gamma(u)\gamma(u+t-s)\, \mathrm{d}u, \qquad s < t.$$

If $\gamma(\infty) < \infty$ an application of L'Hôpital's rule yields that

$$\mu(t) = \alpha \mathrm{E}Y\gamma(\infty)\{1+o(1)\}t, \qquad \sigma^2(t) = \alpha \mathrm{E}Y^2\gamma^2(\infty)\{1+o(1)\}t, \qquad t \to \infty.$$

Hence if the appropriate moments of Y are finite the SLLN holds, and the Berry–Esseen theorem is valid with rate $t^{-1/2}$. In particular, if γ varies regularly with index $a \geq 0$, then $\mathrm{E}\{X(s)X(t)\} = \mathrm{E}Y^2\gamma(s)\gamma(t)$ is regularly varying with index $2a = \rho - 1$. Then, by Proposition 2.2, for $x < y$,

$$C(x,y) = (2a+1)\int_0^x u^a(u+y-x)^a\, \mathrm{d}u,$$

and C is then regularly varying with index ρ. For $a = 0$ the FCLT applies with standard Brownian

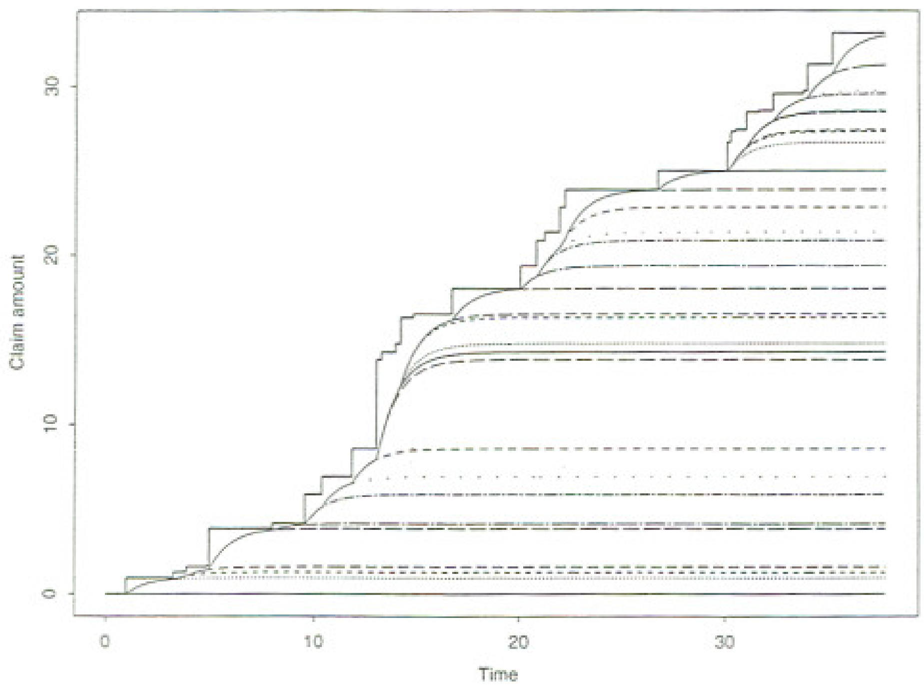

Figure 1. Sample path of the shot noise process of Example 1 and the path of the corresponding compound Poisson process

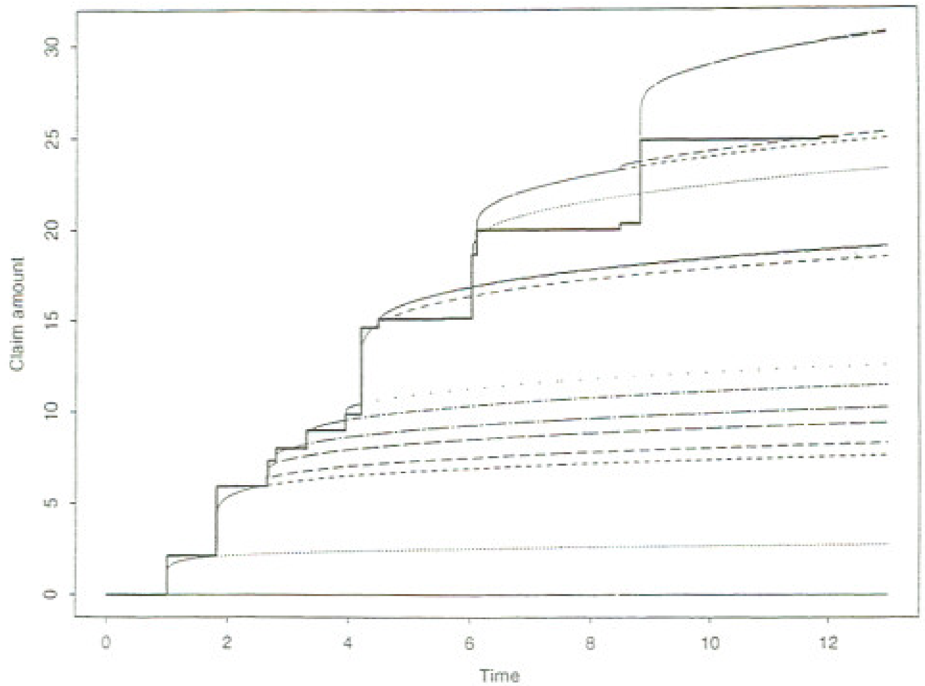

Figure 2. Sample path of the shot noise process of Example 2 and the path of the corresponding compound Poisson process

motion as limit process (see Section 3.3) and for $a > 0$ the limit process is a self-similar Gaussian process with covariance function C.

Example 1

$X(t) = Y(1 - e^{-\lambda t}), t \geq 0$, for some $\lambda > 0$. This means that the pay-off of each claim decreases exponentially fast. Notice that $\gamma(t) = 1 - e^{-\lambda t}$ is regularly varying with index 0. Hence, for Y with sufficiently high finite moments, the above asymptotic results apply for large t. Moreover, for any finite t and given Y we can calculate the Laplace transform and the moments of $(S(t))_{t\geq 0}$ explicitly; we omit the details.

Figure 1 shows a simulated sample path of $S(t)$ and the evolution of each individual claim in comparison with the corresponding classical total claim amount process (staircase). The variable Y has a standard exponential distribution.

Example 2

$X(t) = Yt^a, a \geq 0, t > 0$. For $a = 0$ this is just the compound Poisson process. For $a > 0$ the assumptions of the FCLT in Theorem 3.7 are satisfied. For example, for $a = 1$ and $x < y$ the covariance function of the limit process is given by

$$C(x, y) = \tfrac{1}{2}x^2(3y - x).$$

Figure 2 shows a simulated sample path of $(S(t))_{t\geq 0}$ and the evolution of each individual claim, in comparison with the corresponding classical total claim amount process. The variable Y is standard exponential and $\gamma(t) = t^{0.1}$. It is obvious that for small time the sample paths of $(S(t))_{t\geq 0}$ and of the classical model (staircase) are almost indistinguishable.

4.2. RUIN PROBLEMS

We define the risk process

$$Y(t) = u + P(t) - S(t), \qquad t \geq 0,$$

where $(S(t))_{t\geq 0}$ denotes the total claim amount process given by model (1.1), u is the initial capital and $P(t)$ the premium income up to time t. The *finite time ruin probability* is defined as

$$\begin{aligned}\psi(u,t) &= P(Y(s) < 0 \text{ for some } 0 \leq s \leq t)\\ &= P\Big(\inf_{0\leq s\leq t}\{P(s) - S(s)\} < -u\Big), \qquad t \geq 0.\end{aligned}$$

In our situation it is difficult to calculate ψ explicitly; instead we will apply the FCLT for an approximation. The method is related to the so-called diffusion approximation (see, for example Grandell 1992, and references therein).

In order to avoid ruin with unit probability the pure risk premium $\mu(t) = \mathrm{E}S(t)$, has to be loaded, i.e. increased by a positive function. We propose here as the premium function

$$P(t) = \mu(t) + c\sigma(t), \qquad t \geq 0,$$

for some positive constant c which corresponds to the so-called *standard deviation principle* (see, for example Gerber 1979). Then for $t_0 = t/n$ and $u_0 = u/\sigma(n)$ we obtain

$$\begin{aligned}\psi(u,t) &= P\Big(\inf_{0\leq s\leq t}\{\mu(s) + c\sigma(s) - S(s)\} < -u\Big)\\ &= P\Big\{\inf_{0\leq s\leq t/n}\Big(c\frac{\sigma(ns)}{\sigma(n)} - \frac{S(ns)-\mu(ns)}{\sigma(n)}\Big) < -\frac{u}{\sigma(n)}\Big\}\\ &= P\Big\{\inf_{0\leq s\leq t_0}\Big(c\frac{\sigma(ns)}{\sigma(n)} - \frac{S(ns)-\mu(ns)}{\sigma(n)}\Big) < -u_0\Big\}.\end{aligned}$$

Now, if u_0 and t_0 behave asymptotically as constants, i.e. if t is of order n and u is of order $\sigma(n)$, and if σ^2 is regularly varying with index ρ, then an application of the FCLT in Section 3.3 yields for u and t sufficiently large the approximation

$$\psi(u,t) \sim P\Big(\inf_{0\leq s\leq t_0}(cs^{\rho/2} - B_s) < -u_0\Big) \tag{4.3}$$

(cf. Grandell 1992). For a more thorough treatment of modelling delay in claim settlement by explosive shot noise processes we refer to Klüppelberg and Mikosch (1995).

We conclude the paper with a result which is related to (4.3). First we translate the ruin problem into a first hitting time problem: For $u, c > 0$ define

$$T_c = \inf\{t \geq 0 : B_t = u + ct^{\rho/2}\}.$$

Then

$$P(T_c \leq t_0) = P\left(\inf_{0 \leq s \leq t_0} (cs^{\rho/2} - B_s) \leq -u\right).$$

Because of the strong link between the hitting time T_c and the approximation of the ruin probability, every property of T_c also yields some insight into the ruin problem. Unfortunately, first hitting times of Gaussian processes are rather complicated objects in the general case, but also for Brownian motion (see Durbin 1985; Lerche 1986; and references therein).

In our approximation Brownian motion B corresponds to the case $\rho = \frac{1}{2}$ and hence one has to consider the first hitting time of B with a square root boundary. We present some results for this particular case.

Proposition 4.1 For every $u, c > 0$ define $T_c = \inf\{t \geq 0 : B_t = u + c\sqrt{t}\}$, then

$$T_c = L_u \exp\{2V_c\}$$

for independent L_u and V_c where

$$L_u = \inf\{t \geq 0 : B_t = u\}$$

is the first hitting time of u by Brownian motion and

$$V_c = \inf\{t \geq 0 : U_t = c\}$$

is the first hitting time of c by an Ornstein–Uhlenbeck process starting at zero.

Proof

Using the strong Markov property and the scaling property of Brownian motion we obtain for independent Brownian motions B and $\tilde{B}$ that

$$\begin{aligned}
T_c &= \inf\{t \geq 0 : B_t = u + c\sqrt{t}\} \\
&= L_u + \inf\{t \geq 0 : B_{L_u + t} - u = c\sqrt{L_u + t}\} \\
&\stackrel{d}{=} L_u + \inf\{t \geq 0 : \tilde{B}_t = c\sqrt{L_u + t}\} \\
&= L_u + \inf\{L_u v \geq 0 : \tilde{B}_{L_u v} = c\sqrt{L_u(1 + v)}\} \\
&\stackrel{d}{=} L_u(1 + \inf\{v \geq 0 : \tilde{B}_v = c\sqrt{1 + v}\}) \\
&= L_u(1 + \inf\{e^{2s} - 1 \geq 0 : e^{-s}\tilde{B}_{e^{2s}-1} = c\}).
\end{aligned}$$

Notice that

$$U_s = e^{-s}\tilde{B}_{e^{2s}-1}, \qquad s \geq 0,$$

is an Ornstein–Uhlenbeck process with covariance function $e^{-(t+s)}(e^{2s} - 1)$ for $s \leq t$ and satisfies

$U_0 = 0$ a.s. Hence

$$T_c \stackrel{d}{=} L_u \exp\left[2 \inf\{s \geq 0 : U_s = c\}\right] =: L_u \exp(2V_c).$$

□

So the ruin problem can be reduced to first hitting time problems of Brownian motion and of an Ornstein–Uhlenbeck process with constant boundaries. The distributions of the variables V_c and L_u have been studied. For L_u we refer to Revuz and Yor (1991) and for V_c to Breiman (1967) (Greenwood and Perkins 1983 give a detailed proof of the main result in Breiman 1967). We list some of these results:

Lemma 4.2 L_u is α-stable, with $\alpha = \frac{1}{2}$ and positive with density

$$f_{L_u}(t) = \frac{u}{\sqrt{2\pi t^3}} \exp\left\{-\frac{u^2}{2t}\right\}, \qquad t > 0.$$

V_c has an exponential tail, i.e. there exist constants $a \geq 0$ and $b(c) > 0$ such that

$$P(V_c > x) \sim a \exp\{-2b(c)x\}, \qquad x \to \infty.$$

Moreover, $b(c)$ is decreasing in c with $b(c) \to 0$ as $c \to \infty$, and $b(c) \to \infty$ as $c \to 0$.

We also mention that the probability $P(\inf_{0 \leq s \leq t_0}(c\sqrt{s} - B_s) < u_0)$ in (4.3) can be given as the solution to the heat equation with special boundary conditions. This allows for the numerical calculation of this probability. For more details we refer to Klüppelberg and Mikosch (1995).

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