

# Asymptotics and sharp bounds in the Poisson approximation to the Poisson-binomial distribution

BERO ROOS

*Institut für Mathematische Stochastik, Universität Hamburg, Bundesstraße 55, D-20146 Hamburg, Germany. E-mail: roos@math.uni-hamburg.de*

The Poisson-binomial distribution is approximated by a Poisson law with respect to a new multi-metric (difference metric) unifying a broad class of probability metrics between discrete distributions. The accompanying non-metric situation is also considered leading to moderate- and large-deviation results. Using the Charlier B expansion and Fourier arguments, sharp bounds and asymptotic relations are given.

*Keywords:* Charlier B expansion; large deviations; moderate deviations; Poisson approximation; Poisson-binomial distribution; probability metrics

## 1. Introduction

Let  $S_n$  be the sum of independent Bernoulli random variables  $X_1, \dots, X_n$  with success probabilities  $P(X_j = 1) = 1 - P(X_j = 0) = p_j \in [0, 1]$  for  $1 \leq j \leq n$ . We investigate the approximation of the distribution  $P^{S_n}$  of  $S_n$  by a Poisson law  $\mathcal{P}(t)$  with mean  $t \in (0, \infty)$  and also by finite signed measures derived from an expansion due to Charlier (1905). As a measure of accuracy, a new multi-metric (difference metric) is introduced (see formula (1)) unifying a broad class of probability metrics between discrete distributions. Further, the accompanying non-metric situation is investigated, leading to moderate- and large-deviation results. The task is to give sharp bounds and asymptotic relations. The method used is based on work by Shorgin (1977), Deheuvels and Pfeifer (1986a; 1986b; 1988), Deheuvels *et al.* (1989) and Roos (1995). For other publications on Poisson approximations, see, for example, Barbour (1987), Barbour *et al.* (1992; 1995), Chen and Choi (1992), Deheuvels (1992) and Prohorov (1953).

We proceed with the definition of the difference metric. Some notation is needed. Let  $\mathbb{Z}_+ = \{0, 1, \dots\}$  and  $\mathbb{R}^{\mathbb{Z}_+} = \{f | f: \mathbb{Z}_+ \rightarrow \mathbb{R}\}$ . For  $f \in \mathbb{R}^{\mathbb{Z}_+}$ , let  $\|f\|_p$  ( $p \in [1, \infty]$ ) be the  $p$ -norm of  $f$  and set  $f(m) = 0$  for  $m < 0$ . In this paper, we define the difference operator  $\Delta: \mathbb{R}^{\mathbb{Z}_+} \rightarrow \mathbb{R}^{\mathbb{Z}_+}$  by  $\Delta f(m) = f(m-1) - f(m)$  for  $f \in \mathbb{R}^{\mathbb{Z}_+}$  and  $m \in \mathbb{Z}_+$ ; for the inverse  $\Delta^{-1}$ , we have  $\Delta^{-1}f(m) = -\sum_{j=0}^m f(j)$ . The difference metric between two finite signed measures  $Q_1$  and  $Q_2$  concentrated on  $\mathbb{Z}_+$  is defined by

$$d_p^{(i,j)}(Q_1, Q_2) = \left\| \left( \sum_{u=m-j+1}^m \Delta^i(f_{Q_1} - f_{Q_2})(u) \right)_{m \in \mathbb{Z}_+} \right\|_p, \tag{1}$$

where  $i \in \mathbb{Z}$ ,  $j \in \mathbb{N}$ ,  $p \in [1, \infty]$ , and  $f_{Q_1}, f_{Q_2} \in \mathbb{R}^{\mathbb{Z}_+}$  are the counting densities of  $Q_1, Q_2$ . If we consider the restriction of  $d_p^{(i,j)}$  to the set of all probability measures concentrated on  $\mathbb{Z}_+$ , we get the total variation distance  $\frac{1}{2}d_1^{(0,1)}$ , the  $p$ -metric between distribution functions  $d_p^{(-1,1)}$ , the Kolmogorov metric  $d_\infty^{(-1,1)}$ , the Fortet–Mourier metric  $d_1^{(-1,1)}$  and the  $j$ -point metric  $d_\infty^{(0,j)}$ . Here, the  $j$ -point metric indicates the largest difference between the probabilities of half-open intervals of length  $j$ .

## 2. The bounds for the difference metric

For  $k \in \mathbb{Z}$ ,  $m \in \mathbb{Z}_+$ , and  $t \in [0, \infty)$ , let  $\pi(\cdot, t) \in \mathbb{R}^{\mathbb{Z}_+}$  with  $\pi(m, t) = \mathcal{P}(t)(\{m\}) = e^{-t} t^m / m!$  and write  $\Delta^k \pi(m, t) = (\Delta^k \pi(\cdot, t))(m)$ . Here and throughout the rest of the paper, we let  $0^0 = 1$ . Let

$$p_k(x, t) = \sum_{j=0}^k \binom{k}{j} \binom{x}{j} j! (-t)^{k-j}, \quad t, x \in \mathbb{R}, k \in \mathbb{Z}_+ \tag{2}$$

be the Charlier polynomial of degree  $k$ . The following theorem is the principal tool in the argument of this paper.

**Theorem 1.** For  $m \in \mathbb{Z}_+$  and  $t \in (0, \infty)$ ,

$$P(S_n = m) = \sum_{k=0}^\infty a_k(t) \Delta^k \pi(m, t), \tag{3}$$

where  $a_k(t) = (1/k!) \sum_{m=0}^\infty P(S_n = m) p_k(m, t)$ ,  $k \in \mathbb{Z}_+$ .

For the proof of a more general theorem, see Schmidt (1933). The series (3) is called the Charlier (B) expansion of  $P^{S_n}$ . The coefficients  $a_k(t)$  are called Charlier coefficients of  $P^{S_n}$ . For further papers on the Charlier expansion, see Boas (1949) and the references therein. We now give a review of some well-known relations for  $p_k(x, t)$  and  $\Delta^k \pi(m, t)$ .

**Lemma 1.** (a) For  $x, t \in \mathbb{R}$  and  $k \in \mathbb{N}$ ,

$$k p_{k-1}(x-1, t) = p_k(x, t) - p_k(x-1, t), \tag{4}$$

$$p_{k+1}(x, t) = (x - k - t) p_k(x, t) - t k p_{k-1}(x, t). \tag{5}$$

(b) For  $t \in (0, \infty)$ ,  $k, m \in \mathbb{Z}_+$ , and  $z \in \mathbb{C}$ ,

$$t^k \Delta^k \pi(m, t) = \pi(m, t) p_k(m, t), \tag{6}$$

$$\sum_{j=0}^\infty \Delta^k \pi(j, t) z^j = \exp(t(z-1))(z-1)^k, \tag{7}$$

$$\Delta^k \pi(m, t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-ixm + t(e^{ix} - 1))(e^{ix} - 1)^k dx. \tag{8}$$

In what follows, let  $\lambda_k = \sum_{j=1}^n p_j^k$  for  $k \in \mathbb{N}$ ,  $\lambda = \lambda_1 > 0$ , and  $\eta(t) = 2\lambda_2 + (\lambda - t)^2$  for  $t \in (0, \infty)$ . The following lemma is devoted to the Charlier coefficients of  $P^{S_n}$ .

**Lemma 2.** *Let  $k \in \mathbb{N}$  and  $t \in (0, \infty)$ . Further, let  $I_0(x) = \sum_{m=0}^{\infty} (x/2)^{2m} / (m!)^2$  be the modified Bessel function of the first kind and order 0,  $\beta(x) = e^{-x^2/4} I_0(x)$ ,  $x \in \mathbb{R}$ . Then*

$$a_k(t) = \frac{1}{k} \left( a_{k-1}(t)(\lambda - t) + \sum_{j=0}^{k-2} (-1)^{k-j-1} a_j(t) \lambda_{k-j} \right), \tag{9}$$

$$|a_k(t)| \leq \left( \frac{\eta(t)e}{2k} \right)^{k/2} \beta \left( \frac{|\lambda - t| \sqrt{2k}}{\sqrt{\eta(t)}} \right). \tag{10}$$

**Proof.** Let  $H(z) = \prod_{j=1}^n (1 + p_j(z - 1))$ ,  $z \in \mathbb{C}$ , be the probability generating function of  $S_n$ . As in Schmidt (1933, p. 141),  $\sum_{k=0}^{\infty} a_k(t) z^k = e^{-tz} H(z + 1)$  for  $z \in \mathbb{C}$ . Let  $h(z) = \ln(H(z + 1)) - tz$  and  $g(z) = \exp(h(z))$  for  $|z| < 1$ . Then (9) follows from

$$\begin{aligned} a_k(t) &= \frac{1}{k!} \frac{d^k}{dz^k} g(z) \Big|_{z=0} = \frac{1}{k!} \sum_{j=0}^{k-1} \left( \binom{k-1}{j} \frac{d^j}{dz^j} g(z) \frac{d^{k-j}}{dz^{k-j}} h(z) \right) \Big|_{z=0} \\ &= \frac{1}{k} \sum_{j=0}^{k-1} \frac{a_j(t)}{(k-1-j)!} \frac{d^{k-j}}{dz^{k-j}} h(z) \Big|_{z=0}. \end{aligned}$$

Let  $\alpha \in (0, \infty)$ . By Cauchy's theorem,

$$a_k(t) = \frac{1}{2\pi\alpha^k} \int_0^{2\pi} \exp(-ikx - \alpha t e^{ix}) \prod_{j=1}^n (1 + p_j \alpha e^{ix}) dx.$$

By  $1 + x \leq e^x$  and  $I_0(x) = (1/[2\pi]) \int_0^{2\pi} \exp(x \cos y) dy$  for  $x \in \mathbb{R}$ , this leads to

$$\begin{aligned} |a_k(t)| &\leq \frac{1}{2\pi\alpha^k} \int_0^{2\pi} \exp(-\alpha t \cos x) \prod_{j=1}^n (1 + 2\alpha p_j \cos x + R^2 p_j^2)^{1/2} dx \\ &\leq Y(\alpha) \beta(\alpha(\lambda - t)), \end{aligned}$$

where  $Y(\alpha) := \alpha^{-k} \exp(\alpha^2 \eta(t)/4)$  attains its minimum for  $\alpha = \alpha_0 := \sqrt{2k/\eta(t)}$ . Relation (10) is proved by substituting  $\alpha_0$  for  $\alpha$ . □

Note that Shorgin (1977) showed (9) and (10) in the case  $\lambda = t$ . Using (9), we derive

$$a_0(t) = 1, \quad a_1(t) = \lambda - t, \quad a_2(t) = \frac{(\lambda - t)^2 - \lambda_2}{2}. \tag{11}$$

For the rest of this paper, let  $\beta(x)$  as in Lemma 2. It is clear that  $\beta(x_1) \leq \beta(x_2)$  for  $0 \leq x_2 \leq x_1$ , and that  $0 < \beta(|x|) = \beta(x) \leq 1$  for  $x \in \mathbb{R}$ . For the bounds for the difference metric, we need the following lemma.

**Lemma 3.** For  $k \in \mathbb{N}$ ,  $t \in (0, \infty)$ , and  $p \in [1, \infty]$ ,

$$\|\Delta^k \pi(\cdot, t)\|_p \leq \frac{\sqrt{e}}{2} \left(1 + \sqrt{\frac{\pi}{2}}\right) (2k)^{1/p} e^{-1/(2p)} \left(\frac{k}{te}\right)^{(k+1)/2-1/(2p)}. \tag{12}$$

**Proof.** Shorgin (1977; see the proof of his Lemma 6) proved (12) for  $p = \infty$ . For  $p = 1$ , (12) can be shown by using (23) (which can be proved independently) and the inequality  $1 + x \leq e^x$ ,  $x \in \mathbb{R}$ . The assertion is shown by using the convexity theorem:  $\|f\|_p \leq \|f\|_q^s \|f\|_{q'}^{1-s}$  if  $1 \leq q < p < q' \leq \infty$ ,  $f \in \mathbb{R}^{\mathbb{Z}^+}$ ,  $0 < s < 1$ , and  $1/p = s/q' + (1-s)/q$ .  $\square$

For  $x \in \mathbb{R}$ , let  $[x]$ ,  $\lceil x \rceil \in \mathbb{Z}$  be defined by  $x - 1 < [x] \leq x \leq \lceil x \rceil < x + 1$ . For  $t \in (0, \infty)$ , let  $\tilde{\theta}(t) = \eta(t)/(2t)$ . We now give the main result of this section.

**Theorem 2.** Let  $t \in (0, \infty)$ ,  $k \in \mathbb{Z}_+$ ,  $s \in \{k, k + 1, k + 2, \dots\}$ ,  $i \in \{-s, -s + 1, -s + 2, \dots\}$ ,  $j \in \mathbb{N}$ ,  $p \in [1, \infty]$ ,  $r = (i + 1)/2 - 1/(2p)$ ,

$$C_1 = \frac{j(1 + \sqrt{\pi/2})}{2^{1-1/p}(s+i+1)^{\lceil r+1/p \rceil - r-1/p}}, \quad C_2 = \frac{j(2\pi)^{1/4} \exp(1/(24(s+1)))2^{(s+1)/2+i}}{(s+1)^{1/4}\sqrt{s!}},$$

and

$$\nu(t) = \beta\left(\frac{|\lambda - t\sqrt{2(s+1)}}{\sqrt{\eta(t)}}\right).$$

Let  $Q(k, t)$  denote the finite signed measure concentrated on  $\mathbb{Z}_+$  with the counting density  $f_{Q(k,t)}(m) = \sum_{v=0}^k a_v(t) \Delta^v \pi(m, t)$ ,  $m \in \mathbb{Z}_+$ . Then  $d_p^{(i,j)}(P^{S_n}, Q(k, t)) = H + R$ , where

$$H = \left\| \sum_{u=k+1}^{s+j-1} \left[ \sum_{v=k+1}^{\min\{u,s\}} a_v(t) \binom{j}{u-v+1} \right] \Delta^{u+i} \pi(\cdot, t) \right\|_p \tag{13}$$

and

$$|R| \leq \frac{C_1 \nu(t)}{\tilde{\theta}(t)^{i/2} t^r} \sum_{v=s+i+1}^{\infty} \tilde{\theta}(t)^{v/2} \nu^{\lceil r+1/p \rceil}, \tag{14}$$

$$|R| \leq C_2 \nu(t) \eta(t)^{(s+1)/2} \left(1 + \sqrt{2\eta(t)}\right) \exp(2\eta(t)). \tag{15}$$

**Proof.** By

$$\sum_{u=m-j+1}^m f(u) = \sum_{w=0}^{j-1} \binom{j}{w+1} \Delta^w f(m), \quad f \in \mathbb{R}^{\mathbb{Z}^+}, m \in \mathbb{Z}_+,$$

it is easy to see that, for  $m \in \mathbb{Z}_+$ ,

$$\sum_{u=m-j+1}^m \Delta^i(f_{P^{S_n}} - f_{Q(k,t)})(u) = \sum_{v=k+1}^{\infty} a_v(t) \sum_{w=0}^{j-1} \binom{j}{w+1} \Delta^{w+v+i} \pi(m, t) = H_m + R_m,$$

where

$$\begin{aligned} H_m &= \sum_{v=k+1}^s a_v(t) \sum_{w=0}^{j-1} \binom{j}{w+1} \Delta^{w+v+i} \pi(m, t) \\ &= \sum_{u=k+1}^{s+j-1} \left[ \sum_{v=k+1}^{\min\{u, s\}} a_v(t) \binom{j}{u-v+1} \right] \Delta^{u+i} \pi(m, t), \\ R_m &= \sum_{v=s+1}^{\infty} a_v(t) \sum_{w=0}^{j-1} \binom{j}{w+1} \Delta^{w+v+i} \pi(m, t) = \sum_{u=m-j+1}^m \Delta^i(f_{P^{S_n}} - f_{Q(s,t)})(u). \end{aligned}$$

Hence  $d_p^{(i,j)}(P^{S_n}, Q(k, t)) = \|(H_m + R_m)_{m \in \mathbb{Z}_+}\|_p$  and  $H = \|(H_m)_{m \in \mathbb{Z}_+}\|_p$ . Relation (14) can be shown by using

$$|R| \leq \|(R_m)_{m \in \mathbb{Z}_+}\|_p \leq j \left\| \sum_{v=s+1}^{\infty} a_v(t) \Delta^{v+i} \pi(\cdot, t) \right\|_p \leq j \sum_{v=s+1}^{\infty} |a_v(t)| \|\Delta^{v+i} \pi(\cdot, t)\|_p, \tag{16}$$

and the inequalities (10), (12), and  $1 + x \leq e^x$ ,  $x \in \mathbb{R}$ . Relation (15) can be proved by using (16), (10), the inequality  $\|\Delta^v \pi(\cdot, t)\|_p \leq 2^v$  ( $v \in \mathbb{Z}_+$ ), Stirling’s formula (see Feller 1968)

$$v! = \sqrt{2\pi} v^{v+1/2} \exp(\vartheta_v - v), \quad \vartheta_v \in \left[ \frac{1}{12v+1}, \frac{1}{12v} \right], \quad v \in \mathbb{N}, \tag{17}$$

and

$$\sum_{m=v}^{\infty} \frac{x^m}{\sqrt{m!}} \leq \frac{x^v}{\sqrt{v!}} \sum_{m=0}^{\infty} \frac{x^m}{\sqrt{m!}} \binom{m}{\lfloor m/2 \rfloor}^{1/2} \leq \frac{x^v}{\sqrt{v!}} (x+1) \exp(x^2) \tag{18}$$

for  $v \in \mathbb{Z}_+$ ,  $x \in [0, \infty)$ . □

Note that  $Q(0, t) = \mathcal{P}(t)$ ,  $Q(1, \lambda) = \mathcal{P}(\lambda)$ , and that Barbour (1987) used other signed measures for the total variation distance. Only the first two of his signed measures coincide with  $Q(0, \lambda)$  and  $Q(2, \lambda)$ . Observe that always  $\nu(t) \leq 1$ .

### 3. Evaluation of the norm term

In what follows, the norm term  $H$  in Theorem 2 is evaluated in the cases  $p = 1$  and  $p = \infty$ . Using these formulae, upper and lower estimates of the corresponding distances can be derived. The following two propositions are generalizations of results by Deheuvels and Pfeifer (1986a; 1986b) and Roos (1995). The proofs are easy and therefore omitted.

**Proposition 1.** Let  $q(x) = \sum_{m=0}^k c_m t^{k-m} p_{m+1}(x, t)$ , where  $t \in (0, \infty)$ ,  $k \in \mathbb{Z}_+$ , and  $c_0, \dots, c_k \in \mathbb{R}$ ,  $c_k \neq 0$ . Then  $q(x)$  has at least one zero in  $(0, \infty)$ . If  $q(x)$  has exactly  $u \in \{1, \dots, k + 1\}$  different zeros in  $[0, \infty)$ , denoted by  $x_1(t) < \dots < x_u(t)$ , then

$$\left\| \sum_{m=0}^k c_m \Delta^m \pi(\cdot, t) \right\|_{\infty} = \max_{1 \leq v \leq u} \left| \sum_{m=0}^k c_m \Delta^m \pi(\lfloor x_v(t) \rfloor, t) \right|. \tag{19}$$

**Proposition 2.** Let  $q(x) = \sum_{m=0}^k c_m t^{k-m} p_m(x, t)$ , where  $t \in (0, \infty)$ ,  $k \in \mathbb{Z}_+$ , and  $c_0, \dots, c_k \in \mathbb{R}$ ,  $c_k \neq 0$ . If, under considerations of multiplicity,  $q(x)$  has exactly  $u \in \{0, \dots, k\}$  zeros in  $[0, \infty)$ , denoted by  $x_1(t) \leq \dots \leq x_u(t)$  (if  $u \geq 1$ ), then

$$\left\| \sum_{m=0}^k c_m \Delta^m \pi(\cdot, t) \right\|_1 = \left| (-1)^u c_0 + 2 \sum_{m=0}^k c_m \sum_{v=1}^u (-1)^v \Delta^{m-1} \pi(\lfloor x_v(t) \rfloor, t) \right|. \tag{20}$$

From the theory of orthogonal polynomials it is known that the zeros of the Charlier polynomials  $p_k(x, t)$ ,  $k \in \mathbb{N}$ ,  $t \in (0, \infty)$ , are real, simple and located in the interval  $(0, \infty)$ . The preceding propositions lead to the following corollaries.

**Corollary 1.** Let  $k \in \mathbb{Z}_+$ ,  $t \in (0, \infty)$ , and  $0 < x_1(t) < \dots < x_{k+1}(t)$  be the zeros of  $p_{k+1}(x, t)$ . Then

$$\|\Delta^k \pi(\cdot, t)\|_{\infty} = \max_{1 \leq v \leq k+1} |\Delta^k \pi(\lfloor x_v(t) \rfloor, t)|. \tag{21}$$

**Corollary 2.** Let  $k \in \mathbb{N}$ ,  $t \in (0, \infty)$ , and  $0 < x_1(t) < \dots < x_k(t)$  be the zeros of  $p_k(x, t)$ . Then

$$\|\Delta^k \pi(\cdot, t)\|_1 = 2 \left| \sum_{v=1}^k (-1)^v \Delta^{k-1} \pi(\lfloor x_v(t) \rfloor, t) \right| = 2 \sum_{v=1}^k |\Delta^{k-1} \pi(\lfloor x_v(t) \rfloor, t)|. \tag{22}$$

**Proof.** The first equality follows from Proposition 2. The second equality is proved if it is shown that  $p_{k-1}(\lfloor x_v(t) \rfloor, t)$  alternates in sign as  $v$  varies from 1 through  $k$ . But this is a consequence of the following lemma. □

Note that the inequalities

$$\|\Delta^k \pi(\cdot, t)\|_1 \leq 2k \|\Delta^{k-1} \pi(\cdot, t)\|_{\infty} \leq k \|\Delta^k \pi(\cdot, t)\|_1, \quad k \in \mathbb{N}, t \in (0, \infty), \tag{23}$$

follow from (21) and (22), where equalities hold for  $k = 1$ . The first inequality of (23) is used in the proof of (12). The following lemma is needed to complete the proof of Corollary 2.

**Lemma 4.** Let  $t \in (0, \infty)$  and  $0 < x_{k,1}(t) < \dots < x_{k,k}(t)$  be the zeros of  $p_k(x, t)$  for  $k \in \mathbb{N}$ . Then  $x_{k+1,v}(t) < x_{k,v}(t) < x_{k+1,v+1}(t) - 1$  for  $k \in \mathbb{N}$  and  $v \in \{1, \dots, k\}$ .

Note that the relation above without the ‘-1’ is the well-known separation theorem for the zeros of the Charlier polynomials.

**Proof.** By induction over  $k$ , it can be shown that  $x_{k,v+1}(t) - x_{k,v}(t) > 1$  for  $k \in \{2, 3, \dots\}$  and  $v \in \{1, \dots, k - 1\}$ . Here the separation theorem and (4) are used. Now the assertion can easily be proved.  $\square$

In the following lemma, some additional properties of the norm term are given.

**Lemma 5.** Let  $k \in \mathbb{Z}_+$ ,  $c_0, \dots, c_k \in \mathbb{R}$ ,  $c_k \neq 0$ , and  $p \in [1, \infty]$ . Then the norm  $\|\sum_{m=0}^k c_m \Delta^m \pi(\cdot, t)\|_p$  is a  $(0, \infty)$ -valued, continuous function of  $t \in [0, \infty)$ .

**Proof.** The assertion can easily be shown by using

$$|p_k(m, t)| \leq \sum_{w=0}^k \binom{k}{w} m^w t^{k-w} = (m + t)^k, \quad k, m \in \mathbb{Z}_+, t \in (0, \infty), \tag{24}$$

Minkowski’s inequality, (6), (7) and (19).  $\square$

### 4. Asymptotic relations for the norm term

In what follows, let

$$H_k(x) = k! \sum_{m=0}^{\lfloor k/2 \rfloor} \frac{(-1)^m (2x)^{k-2m}}{(k - 2m)! m!} \tag{25}$$

be the Hermite polynomial of degree  $k \in \mathbb{Z}_+$ . We need the well-known relations

$$H_{k+1}(x) = 2x H_k(x) - 2k H_{k-1}(x), \quad k \in \mathbb{N}, x \in \mathbb{R}, \tag{26}$$

$$\varphi_k(x) = \frac{1}{\sqrt{2\pi}} \frac{d^k}{dx^k} e^{-x^2/2} = \frac{(-1)^k}{\sqrt{2\pi} 2^{k/2}} e^{-x^2/2} H_k(x/\sqrt{2}), \quad k \in \mathbb{Z}_+, x \in \mathbb{R}. \tag{27}$$

For  $p \in [1, \infty]$ , let  $\|\varphi_k\|_p$  be the  $p$ -norm of  $\varphi_k$ .

**Proposition 3.** Let  $k \in \mathbb{Z}_+$  and  $b: (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  be bounded. For  $t \rightarrow \infty$ ,

$$\sup_{x \in \mathbb{R}} |t^{(k+1)/2} \Delta^k \pi(\lfloor t + x\sqrt{t} + b(t, x) \rfloor, t) - (-1)^k \varphi_k(x)| = \mathcal{O}(t^{-1/2}), \tag{28}$$

$$\sup_{x \in \mathbb{R}} (1 + x^2) |t^{(k+1)/2} \Delta^k \pi(\lfloor t + x\sqrt{t} + b(t, x) \rfloor, t) - (-1)^k \varphi_k(x)| = \mathcal{O}(t^{-1/2}). \tag{29}$$

**Proof.** We use Fourier techniques as in Petrov (1975). Let  $k \in \mathbb{Z}_+$ ,  $x \in \mathbb{R}$ ,  $t \in (0, \infty)$ , and  $m = \lfloor t + x\sqrt{t} + b(t, x) \rfloor \geq 0$ . Then  $m = t + x\sqrt{t} + \tilde{b}(t, x)$ , where  $\tilde{b}: (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  is bounded. We write  $\tilde{b}$  for  $\tilde{b}(t, x)$ . Using (8) and

$$\varphi_k(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-ixy - y^2/2) (-iy)^k dy,$$

it is easy to see that

$$\begin{aligned}
 & 2\pi|t^{(k+1)/2}\Delta^k\pi(m, t) - (-1)^k\varphi_k(x)| \\
 & \leq t^{(k+1)/2}\int_{-\pi}^{\pi} e^{-ty^2/2}|\exp(t[e^{iy} - 1 - iy + y^2/2] - iy\tilde{b})(e^{iy} - 1)^k - (iy)^k| dy + I_1 \\
 & \leq t^{(k+1)/2}(I_2 + I_3 + I_4) + I_1,
 \end{aligned}$$

where

$$\begin{aligned}
 I_1 &= 2\int_{\pi\sqrt{t}}^{\infty} e^{-y^2/2}y^k dy, \\
 I_2 &= \int_{-\pi}^{\pi} e^{-ty^2/2}|\exp(t[\cos y - 1 + y^2/2]) - 1||e^{iy} - 1|^k dy, \\
 I_3 &= \int_{-\pi}^{\pi} e^{-ty^2/2}|(e^{iy} - 1)^k - (iy)^k| dy, \\
 I_4 &= \int_{-\pi}^{\pi} e^{-ty^2/2}|\exp(i[t(\sin y - y) - y\tilde{b}]) - 1||y|^k dy.
 \end{aligned}$$

Using calculus, it is possible to show that, for  $t \rightarrow \infty$ ,  $I_1 = \mathcal{O}(\exp(-\pi^2 t/2)t^{(k-1)/2})$ ,  $I_2 = \mathcal{O}(t^{-(k+3)/2})$ ,  $I_3 = \mathcal{O}(t^{-(k+2)/2})$ , and  $I_4 = \mathcal{O}(t^{-(k+2)/2})$ . For  $I_2$ , the inequality

$$I_2 \leq 2^{k+3}\left(\frac{\pi^2}{4} - 1\right)t\int_0^{\pi/2} \exp(-2t \sin^2 y)\sin^{k+4} y dy$$

and Shorgin's (1977; see the proof of his Lemma 6) estimate of the integral are used. For  $I_3$ , we use  $|(e^{iy} - 1)^k - (iy)^k| \leq k|y|^{k+1}e^{k|y|}$  and similar estimates for  $I_4$ . Hence

$$\sup_{x \in A(t)} |t^{(k+1)/2}\Delta^k\pi(\lfloor t + x\sqrt{t} + b(t, x) \rfloor, t) - (-1)^k\varphi_k(x)| = \mathcal{O}(t^{-1/2}), \quad t \rightarrow \infty,$$

where  $A(t) = \{x \in \mathbb{R} | \lfloor t + x\sqrt{t} + b(t, x) \rfloor \geq 0\}$ . The proof of (28) is easily completed. To prove (29), it suffices to estimate  $T = x^2|t^{(k+1)/2}\Delta^k\pi(m, t) - (-1)^k\varphi_k(x)|$  uniformly in  $x \in A(t)$ . Using (5), (6), (26), (27), and  $x = (m - t - \tilde{b})/\sqrt{t}$ , we obtain

$$\begin{aligned}
 x^2\Delta^k\pi(m, t) &= t\Delta^{k+2}\pi(m, t) + [2k - 2\tilde{b} + 1]\Delta^{k+1}\pi(m, t) + [2k + 1 + t^{-1}(k - \tilde{b})^2]\Delta^k\pi(m, t) \\
 &\quad + t^{-1}k[2k - 2\tilde{b} - 1]\Delta^{k-1}\pi(m, t) + k(k - 1)t^{-1}\Delta^{k-2}\pi(m, t),
 \end{aligned}$$

and  $x^2\varphi_k(x) = \varphi_{k+2}(x) + (2k + 1)\varphi_k(x) + k(k - 1)\varphi_{k-2}(x)$ , where  $\varphi_{-2}(x) = \varphi_{-1}(x) = 0$ . Hence



$$\begin{aligned}
 T \leq & |t^{(k+3)/2} \Delta^{k+2} \pi(m, t) - (-1)^{k+2} \varphi_{k+2}(x)| + [2k + 2|\tilde{b}| + 1] t^{(k+1)/2} \|\Delta^{k+1} \pi(\cdot, t)\|_\infty \\
 & + (k - \tilde{b})^2 t^{(k-1)/2} \|\Delta^k \pi(\cdot, t)\|_\infty + [2k + 1] |t^{(k+1)/2} \Delta^k \pi(m, t) - (-1)^k \varphi_k(x)| \\
 & + k[2k + 2|\tilde{b}| + 1] t^{(k-1)/2} \|\Delta^{k-1} \pi(\cdot, t)\|_\infty \\
 & + k(k - 1) |t^{(k-1)/2} \Delta^{k-2} \pi(m, t) - (-1)^{k-2} \varphi_{k-2}(x)|.
 \end{aligned}$$

Using the estimates (12), (28), and  $\|\pi(\cdot, t)\|_\infty \leq (2te)^{-1/2}$  for  $t \in (0, \infty)$  (see Deheuvels and Pfeifer 1988), the proof is easily completed.  $\square$

**Proposition 4.** Let  $p \in [1, \infty]$  and  $k \in \mathbb{Z}_+$ . Then

$$\exists M \in (0, \infty) \forall t \in (0, \infty): |t^{(k+1)/2-1/(2p)} \|\Delta^k \pi(\cdot, t)\|_p - \|\varphi_k\|_p| \leq \frac{M}{\sqrt{t}}. \tag{30}$$

**Proof.** For sufficiently large  $t$  the assertion is shown by using (28), (29) and

$$t^{-1/(2p)} \|\Delta^k \pi(\cdot, t)\|_p = \left( \int_{-\infty}^{\infty} |\Delta^k \pi(\lfloor t + x\sqrt{t} \rfloor, t)|^p dx \right)^{1/p}, \quad p \in [1, \infty).$$

For small  $t$ , Lemma 5 is used.  $\square$

### 5. Asymptotic results for the difference metric

For the following theorem, we consider a triangular scheme: we let  $n$  and  $X_1, \dots, X_n$  depend on an additional parameter  $l \in \mathbb{N}$  and assume that  $l \rightarrow \infty$ . Then the following quantities also depend on  $l$ :  $S_n, p_1, \dots, p_n, \lambda_k$  for  $k \in \mathbb{N}, \eta(t)$ , and  $\theta(t)$  for  $t \in (0, \infty)$ . Let  $\theta = \tilde{\theta}(\lambda) = \lambda_2/\lambda$ . Note that  $\theta \leq 1$ . Sometimes we write  $\theta^{(l)}$  for  $\theta$ . We now present the main result of this section.

**Theorem 3.** Let  $i \in \{-2, -1, \dots\}, j \in \mathbb{N}, p \in [1, \infty]$  be independent of  $l$ . Further, let

$$H_p^{(i,j)}(t) = \left\| \sum_{u=2}^{j+1} \binom{j}{u-1} \Delta^{u+i} \pi(\cdot, t) \right\|_p, \quad t \in [0, \infty).$$

If  $\limsup_{l \rightarrow \infty} \theta^{(l)} < 1$ , then

$$d_p^{(i,j)}(P^{S_n}, \mathcal{P}(\lambda)) = \frac{\lambda_2}{2} H_p^{(i,j)}(\lambda) \left[ 1 + \mathcal{O} \left( \min \left\{ \frac{\lambda_3}{\lambda_2 \sqrt{\lambda}} + \theta, \frac{\lambda_3}{\lambda_2} + \lambda_2 \right\} \right) \right], \tag{31}$$

and

$$d_p^{(i,j)}(P^{S_n}, \mathcal{P}(\lambda)) = \frac{j \|\varphi_{i+2}\|_p \theta}{2\lambda^{(i+1)/2-1/(2p)}} \left[ 1 + \mathcal{O} \left( \min \left\{ 1, \frac{1}{\sqrt{\lambda}} + \theta \right\} \right) \right]. \tag{32}$$

If  $a \in [0, \infty)$  is independent of  $l$  and  $\lambda_2 = \mathcal{O}(1)$ , then

$$d_p^{(i,j)}(P^{S_n}, \mathcal{P}(\lambda)) = \frac{\lambda_2}{2} H_p^{(i,j)}(a) \left[ 1 + \mathcal{O} \left( \min \left\{ 1, \frac{\lambda_3}{\lambda_2} + \lambda_2 + |\lambda - a| \right\} \right) \right]. \tag{33}$$

**Proof.** Letting  $k = 0$ ,  $s = 3$ , and  $t = \lambda$  in Theorem 2, we get  $d_p^{(i,j)}(P^{S_n}, \mathcal{P}(\lambda)) = (\lambda_2/2)H_p^{(i,j)}(\lambda) + R$ , where

$$|R| \leq \frac{\lambda_3}{3} \left\| \sum_{u=3}^{j+2} \binom{j}{u-2} \Delta^{u+i} \pi(\cdot, \lambda) \right\|_p + |R_1|$$

and the following two estimates hold:

$$|R_1| = \mathcal{O} \left( \frac{\lambda_2^2}{\lambda^{(i+5)/2-1/(2p)}} \right) \quad \text{if } \limsup_{l \rightarrow \infty} \theta^{(l)} < 1,$$

$$|R_1| = \mathcal{O}(\lambda_2^2) \quad \text{if } \lambda_2 = \mathcal{O}(1).$$

Using the triangular inequality, (12) and Lemma 5, we obtain

$$|R| = \mathcal{O} \left( \frac{\lambda_3}{\lambda^{(i+4)/2-1/(2p)}} + \frac{\lambda_2^2}{\lambda^{(i+5)/2-1/(2p)}} \right) \quad \text{if } \limsup_{l \rightarrow \infty} \theta^{(l)} < 1, \tag{34}$$

$$|R| = \mathcal{O}(\lambda_3 + \lambda_2^2) \quad \text{if } \lambda_2 = \mathcal{O}(1). \tag{35}$$

Because of (30), three constants  $M_1, M_2, M_3 \in (0, \infty)$  exist such that  $M_1 > 1$  and  $M_2 \leq t^{(i+3)/2-1/(2p)} H_p^{(i,j)}(t) \leq M_3$  for  $t \in (M_1, \infty)$ . By Lemma 5,  $0 < \inf_{t \in [0, M_1]} H_p^{(i,j)}(t) =: M_4$ . Let  $A_1 = \{l \in \mathbb{N} | \lambda > M_1\}$ ,  $A_2 = \{l \in \mathbb{N} | M_1 \geq \lambda \geq 1\}$ , and  $A_3 = \mathbb{N} \setminus (A_1 \cup A_2)$ . For (31), we may assume  $\sup_{l \in \mathbb{N}} \theta^{(l)} < 1$ . By (34) and (35), we obtain, for  $l \in A_k$  ( $k \in \{1, 2, 3\}$ ),

$$\frac{|R|}{\lambda_2 H_p^{(i,j)}(\lambda)} = \mathcal{O} \left( \min \left\{ \frac{\lambda_3}{\lambda_2 \sqrt{\lambda}} + \theta, \frac{\lambda_3}{\lambda_2} + \lambda_2 \right\} \right).$$

Relation (31) is proved. Relations (32) and (33) are easily shown by similar arguments.  $\square$

It is easy to show that (31) and (32) remain valid if  $i + 1 = 1/p$  and the condition  $\limsup_{l \rightarrow \infty} \theta^{(l)} < 1$  is dropped. Hence (32) is a generalization of results of Prohorov (1953, Theorem 2), Deheuvels and Pfeifer (1986a; 1986b; 1988), and Roos (1995) concerning the Poisson approximation of the binomial and Poisson-binomial distributions with respect to the total variation distance, the Kolmogorov metric, the Fortet–Mourier metric, and the one-point metric. It should be mentioned that, as has been observed by Barbour *et al.* (1992, p. 2), the statement of Prohorov’s Theorem 2 is inaccurate. A correct version, in our notation, is:  $d_1^{(0,1)}(\mathcal{B}(n, p), \mathcal{P}(np)) = \sqrt{2/(\pi e)} p [1 + \mathcal{O}(\min\{1, [np]^{-1/2} + p\})]$ , where  $\mathcal{B}(n, p)$  denotes the binomial distribution with parameter  $n$  and success probability  $p$ . In Prohorov’s version, the ‘+p’ is missing, which invalidates his result, for example, for  $p = 1, n \rightarrow \infty$ .

For easier  $\mathcal{O}$ -terms in (31) and (33), observe the following relations:

$$\min \left\{ \frac{\lambda_3}{\lambda_2 \sqrt{\lambda}} + \theta, \frac{\lambda_3}{\lambda_2} + \lambda_2 \right\} \leq 2 \min \{ \sqrt{\theta}, \sqrt{\lambda_2} \},$$

$$\frac{\lambda_3}{\lambda_2} + \lambda_2 + |\lambda - a| \leq (1 + \sqrt{M})(\sqrt{\lambda_2} + |\lambda - a|) \quad \text{if } \lambda_2 \leq M.$$

### 6. Non-metric considerations

In this section, we are interested in relations for  $\Delta^i(f_{P^{S_n}} - f_{\mathcal{P}(t)})(m)$ , where  $i \in \{-1, 0, \dots\}$ ,  $t \in (0, \infty)$ , and  $m \in \mathbb{Z}_+$ . The first result is a consequence of Theorem 2. Here, we consider the triangular scheme as introduced before Theorem 3. Further, let  $m \in \mathbb{Z}_+$  and  $t \in (0, \infty)$  also depend on  $l$ .

**Theorem 4.** *Let  $i \in \{-1, 0, \dots\}$ ,  $a \in \mathbb{R}$ ,  $m_0 \in \mathbb{Z}_+$  be independent of  $l$ .*

(a) *If  $\lambda \rightarrow \infty$ ,  $m = \lambda + a\sqrt{\lambda} + \mathcal{O}(1)$ ,  $\theta \rightarrow 0$ ,  $(\lambda - t)\sqrt{\lambda}/\lambda_2 \rightarrow 0$ , and  $\lambda = t + \mathcal{O}(1)$ , then*

$$\frac{\lambda^{(i+3)/2}}{\lambda_2} \Delta^i(f_{P^{S_n}} - f_{\mathcal{P}(t)})(m) \rightarrow \frac{(-1)^{i+1}}{2} \varphi_{i+2}(a).$$

(b) *If  $m = m_0$ ,  $\lambda \rightarrow a \in [0, \infty)$ ,  $t \rightarrow a$ ,  $\lambda_2/t^{i+4} \rightarrow 0$ , and  $(\lambda - t)/\lambda_2 \rightarrow 0$ , then*

$$\frac{1}{\lambda_2} \Delta^i(f_{P^{S_n}} - f_{\mathcal{P}(t)})(m) \rightarrow -\frac{1}{2} \Delta^{i+2} \pi(m_0, a).$$

**Proof.** First note that both in (a) and (b),  $\tilde{\theta}(t) \rightarrow 0$ . Hence Theorem 2 yields  $\Delta^i(f_{P^{S_n}} - f_{\mathcal{P}(t)})(m) = \tilde{H} + \tilde{R}$ , where

$$\tilde{H} = (\lambda - t) \Delta^{i+1} \pi(m, t) + \frac{(\lambda - t)^2 - \lambda_2}{2} \Delta^{i+2} \pi(m, t),$$

$$|\tilde{R}| = |\Delta^i(f_{P^{S_n}} - f_{\mathcal{Q}(2,t)})(m)| \leq d_{\infty}^{(i,1)}(P^{S_n}, \mathcal{Q}(2, t)) = \mathcal{O} \left( \frac{\tilde{\theta}(t)^{3/2}}{t^{(i+1)/2}} \right).$$

In case (a), we have  $m = t + a\sqrt{t} + \mathcal{O}(1)$  and  $t \rightarrow \infty$ . Using (28), we obtain  $t^{(k+1)/2} \Delta^k \pi(m, t) \rightarrow (-1)^k \varphi_k(a)$  for  $k \in \mathbb{Z}_+$ . Now it is easy to prove that  $\tilde{H} \lambda^{(i+3)/2} / \lambda_2 \rightarrow 2^{-1} (-1)^{i+1} \varphi_{i+2}(a)$  and  $\tilde{R} \lambda^{(i+3)/2} / \lambda_2 \rightarrow 0$ , as required.

In case (b), the relations  $\tilde{H} / \lambda_2 \rightarrow -2^{-1} \Delta^{i+2} \pi(m_0, a)$  and  $\tilde{R} / \lambda_2 \rightarrow 0$  are easily shown, completing the proof. □

**Theorem 5.** *Let  $t \in (0, \infty)$ ,  $k \in \mathbb{Z}_+$ ,  $s \in \{k, k + 1, \dots\}$ ,  $i \in \{-s - 1, -s, -s + 1, \dots\}$ ,  $m \in \mathbb{Z}_+$ ,  $V(m, t) = (m/t + 1)\sqrt{\eta(t)}/2$ ,*

$$C_3 = \frac{\exp(1/(24(s + 1)))(2\pi)^{1/4}}{(s + 1)^{1/4} \sqrt{s!}}, \quad \nu(t) = \beta \left( \frac{|\lambda - t| \sqrt{2(s + 1)}}{\sqrt{\eta(t)}} \right).$$

*Then  $\Delta^i(f_{P^{S_n}} - f_{\mathcal{Q}(k,t)})(m) = H' + R'$ , where  $H' = \sum_{u=k+1}^s a_u(t) \Delta^{u+i} \pi(m, t)$  and*

$$|R'| \leq C_3 v(t) \pi(m, t) (m/t + 1)^i V(m, t)^{s+1} (1 + V(m, t)) \exp(V(m, t)^2).$$

**Proof.** First note that, by (6) and (24),

$$|\Delta^k \pi(m, t)| \leq \pi(m, t) \left(\frac{m}{t} + 1\right)^k, \quad k, m \in \mathbb{Z}_+, t \in (0, \infty). \tag{36}$$

The assertion is easily shown by using (10), (17), (18), and (36). □

For the following result, we use the triangular scheme as considered for Theorem 4.

**Theorem 6.** Let  $i \in \{-1, 0, \dots\}$  be independent of  $l$ .

(a) If  $\lambda/m \rightarrow 0$ ,  $m \rightarrow \infty$ ,  $m\sqrt{\lambda_2}/\lambda \rightarrow 0$  and  $(\lambda - t)/\lambda_2 \rightarrow 0$ , then

$$\Delta^i(f_{P^{S_n}} - f_{\mathcal{P}(t)})(m) \sim -\frac{\lambda_2}{2} \pi(m, t) \left(\frac{m}{t}\right)^{2+i}.$$

(b) If  $m/\lambda \rightarrow a \in [0, \infty)$ ,  $\lambda \rightarrow \infty$ ,  $\lambda_2 \rightarrow 0$  and  $(\lambda - t)/\lambda_2 \rightarrow 0$ , then

$$\frac{\Delta^i(f_{P^{S_n}} - f_{\mathcal{P}(t)})(m)}{\lambda_2 \pi(m, t)} \rightarrow -\frac{1}{2} (a - 1)^{i+2}.$$

For the proof, the following lemma is needed.

**Lemma 6.** (a) Under the assumptions in Theorem 6(a), we have  $t/m \rightarrow 0$ ,  $(\lambda - t)t/(\lambda_2 m) \rightarrow 0$ ,  $p_k(m, t)/m^k \rightarrow 1$  for all  $k \in \mathbb{Z}_+$ ,  $\lambda_2 \rightarrow 0$ ,  $\lambda - t \rightarrow 0$ ,  $(\lambda - t)^2/\lambda_2 \rightarrow 0$ ,  $m\sqrt{\lambda_2}/t \rightarrow 0$  and  $(\lambda - t)m/t \rightarrow 0$ .

(b) Under the assumptions in Theorem 6(b), we have  $m/t \rightarrow a$ ,  $\lambda - t \rightarrow 0$ ,  $t \rightarrow \infty$ ,  $p_k(m, t)t^{-k} \rightarrow (a - 1)^k$  for all  $k \in \mathbb{Z}_+$ ,  $(\lambda - t)^2/\lambda_2 \rightarrow 0$ ,  $m\sqrt{\lambda_2}/t \rightarrow 0$  and  $(\lambda - t)m/t \rightarrow 0$ .

**Proof of Theorem 6.** By Theorem 5,  $\Delta^i(f_{P^{S_n}} - f_{\mathcal{P}(t)})(m) = H' + R'$ , where

$$H' = \pi(m, t) \left[ \frac{\lambda - t}{t^{i+1}} p_{i+1}(m, t) + \frac{1}{2t^{i+2}} ((\lambda - t)^2 - \lambda_2) p_{i+2}(m, t) \right],$$

$$R' = \mathcal{O}(\pi(m, t) y^{i+3} x^{3/2} (1 + y\sqrt{x}) \exp(xy^2)),$$

$x = \eta(t)/2$  and  $y = m/t + 1$ . Using Lemma 6, the assertions are easily proved. □

In what follows, let  $F$  (or  $G$ ) denote the distribution function of  $P^{S_n}$  (or  $\mathcal{P}(t)$ ). To obtain the following results on large and moderate deviations, set  $i = -1$  in Theorems 4 and 6.

**Corollary 3.** (a) Under the assumptions in Theorem 4(a),

$$\frac{1 - G(m)}{\theta} \left( \frac{1 - F(m)}{1 - G(m)} - 1 \right) \rightarrow \frac{-ae^{-a^2/2}}{2\sqrt{2\pi}}. \tag{37}$$

(b) Under the assumptions in Theorem 4(b),

$$\frac{1 - G(m_0)}{\lambda_2} \left( \frac{1 - F(m)}{1 - G(m)} - 1 \right) \rightarrow \frac{-\Delta^1 \pi(m_0, a)}{2}. \quad (38)$$

(c) Under the assumptions in Theorem 6(a),

$$\frac{1 - F(m)}{1 - G(m)} - 1 \sim -\frac{\lambda_2}{2} \left( \frac{m}{t} \right)^2. \quad (39)$$

(d) Under the assumptions in Theorem 6(b),

$$\frac{1 - G(m)}{\lambda_2 \pi(m, t)} \left( \frac{1 - F(m)}{1 - G(m)} - 1 \right) \rightarrow \frac{1 - a}{2}. \quad (40)$$

Note that Chen and Choi (1992, Corollary 2.4) obtained (39) under more restrictive assumptions in the case  $t = \lambda$ . (They overlooked the required assumption  $m \rightarrow \infty$  in their corollary. Their assertion does not hold without this condition). For  $t = \lambda$ , Barbour *et al.* (1995, Corollary 4.3) proved (39) under more general assumptions as in Corollary 3(c).

## Acknowledgement

This paper is based on parts of my Ph.D. thesis written at the University of Oldenburg in 1996 under the supervision of Professor Dietmar Pfeifer. I thank Professor Pfeifer for his guidance and a number of helpful discussions. Further, I thank the associate editor and the referees for their remarks and one of the referees for his suggestion to shorten the paper.

## References

- Barbour, A.D. (1987) Asymptotic expansions in the Poisson limit theorem. *Ann. Probab.*, **15**, 748–766.
- Barbour, A.D., Holst, L. and Janson, S. (1992) *Poisson Approximation*. Oxford: Clarendon Press.
- Barbour, A.D., Chen, L.H.Y. and Choi, K.P. (1995) Poisson approximation for unbounded functions, I: Independent summands. *Statist. Sinica*, **5**, 749–766.
- Boas, R.P. Jr (1949) Representation of probability distributions by Charlier series. *Ann. Math. Statist.*, **20**, 376–392.
- Charlier, C.V.L. (1905) Die zweite Form des Fehlergesetzes. *Ark. Mat. Astr. Fys.*, **2**, **15**, 1–8.
- Chen, L.H.Y. and Choi, K.P. (1992) Some asymptotic and large deviation results in Poisson approximation. *Ann. Probab.*, **20**, 1867–1876.
- Deheuvels, P. (1992) Large deviations by Poisson approximations. *J. Statist. Plann. Inference*, **32**, 75–88.
- Deheuvels, P. and Pfeifer, D. (1986a) A semigroup approach to Poisson approximation. *Ann. Probab.*, **14**, 663–676.
- Deheuvels, P. and Pfeifer, D. (1986b) Operator semigroups and Poisson convergence in selected metrics. *Semigroup Forum*, **34**, 203–224. Errata: *Semigroup Forum*, **35**, 251 (1987).
- Deheuvels, P. and Pfeifer, D. (1988) On a relationship between Uspensky's theorem and Poisson

- approximations. *Ann. Inst. Statist. Math.*, **40**, 671–681.
- Deheuvels, P., Pfeifer, D. and Puri, M.L. (1989) A new semigroup technique in Poisson approximation. *Semigroup Forum*, **38**, 189–201.
- Feller, W. (1968) *An Introduction to Probability Theory and Its Applications I*. New York: Wiley.
- Petrov, V.V. (1975) *Sums of Independent Random Variables*. Berlin: Springer-Verlag.
- Prohorov, Y.V. (1953) Asymptotic behaviour of the binomial distribution. *Uspekhi Mat. Nauk*, **8**, 135–142 (Russian). English translations: *Select. Transl. Math. Statist. Probab.*, **1**, 87–95 (1961).
- Roos, B. (1995) A semigroup approach to Poisson approximation with respect to the point metric. *Statist. Probab. Lett.*, **24**, 305–314.
- Schmidt, E. (1933) Über die Charlier–Jordansche Entwicklung einer willkürlichen Funktion nach der Poissonschen Funktion und ihren Ableitungen. *Z. Angew. Math. Mech.*, **13**, 139–142.
- Shorgin, S.Y. (1977) Approximation of a generalized binomial distribution. *Theory Probab. Appl.*, **22**, 846–850.

Received March 1997 and revised May 1999