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Nonparametric inference for Lévy-driven Ornstein–Uhlenbeck processes

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We consider nonparametric estimation of the Lévy measure of a hidden Lévy process driving a stationary Ornstein–Uhlenbeck process which is observed at discrete time points. This Lévy measure can be expressed in terms of the canonical function of the stationary distribution of the Ornstein–Uhlenbeck process, which is known to be self-decomposable. We propose an estimator for this canonical function based on a preliminary estimator of the characteristic function of the stationary distribution. We provide a suppport-reduction algorithm for the numerical computation of the estimator, and show that the estimator is asymptotically consistent under various sampling schemes. We also define a simple consistent estimator of the intensity parameter of the process. Along the way, a nonparametric procedure for estimating a self-decomposable density function is constructed, and it is shown that the Ornstein–Uhlenbeck process is β-mixing. Some general results on uniform convergence of random characteristic functions are included.

Keywords: Lévy process; self-decomposability; support-reduction algorithm; uniform convergence of characteristic functions

1. Introduction

For a given positive number λ and a given increasing Lévy process Z without drift component, consider the stochastic differential equation

$$\mathrm{d}X(t) = -\lambda X(t)\mathrm{d}t + \mathrm{d}Z(\lambda t), \qquad t \geqslant 0. \tag{1.1}$$

A solution X to this equation is called a Lévy-driven *Ornstein–Uhlenbeck* (OU) *process*, and the process Z is referred to as the *background driving Lévy process* (BDLP). The autocorrelation of X at lag h can be expressed in terms of the 'intensity parameter' λ as $\mathrm{e}^{-\lambda|h|}$.

By the Lévy–Khinchine representation theorem (Sato 1999, Theorem 8.1), the distribution of Z is characterized by its Lévy measure ρ. If $\int_2^\infty \log x \rho(\mathrm{d}x) < \infty$, then a unique stationary solution to (1.1) exists (Sato 1999, Theorem 17.5 and Corollary 17.9). Moreover, the stationary distribution π of $X(1)$ is *self-decomposable* with characteristic function

$$\psi(t) := \int \mathrm{e}^{\mathrm{i}tx}\pi(\mathrm{d}x) = \exp\left(\int_0^\infty (\mathrm{e}^{\mathrm{i}tx} - 1)\frac{k(x)}{x}\mathrm{d}x\right), \tag{1.2}$$

where $k(x) = \rho(x, \infty)$. This shows that π is characterized by the decreasing function k, which is called the *canonical function*. Conversely, if we presuppose that π satisfies (1.2), then there exists an increasing Lévy process Z, unique in law, such that (1.1) holds for all $\lambda > 0$. Due to the special scaling in (1.1), π does not depend on λ.

Assume that we have discrete-time observations $X_0, X_\Delta, \ldots, X_{(n-1)\Delta}$ $(\Delta > 0)$ from $(X_t, t \geqslant 0)$, as defined by (1.1), where the sampling interval Δ may depend on n. Based on these observations, we aim to estimate the parameters of the model. From the previous remarks this comes down to (i) estimating the intensity parameter λ and (ii) estimating the canonical function k. In this paper we deal with both estimation problems. Our approach to (ii) is nonparametric, although parametric submodels can be handled with our method as well (see Jongbloed and van der Meulen 2004).

One motivation for studying this problem comes from stochastic volatility models in financial mathematics. Barndorff-Nielsen and Shephard (2001a) model stock price as a geometric Brownian motion. The diffusion coefficient of this motion, referred to as the volatility, is assumed to be a Lévy-driven OU process. Based on stock prices, the objective is to estimate the Lévy measure of the BDLP and λ. Although related, this estimation problem is intrinsically harder than the one we consider, since volatility is unobservable in practice. Despite this, the present work may be extended to handle these models by the addition of a deconvolution step, and hence may provide a first step towards estimating these models nonparametrically. Another motivation comes from storage theory, where equation (1.1) is often referred to as the 'storage equation' (see, for example, Çinlar and Pinsky 1971).

Rubin and Tucker (1959) considered nonparametric estimation for general Lévy processes, based on both continuous- and discrete-time observations, and Basawa and Brockwell (1982) considered estimation for the subclass of continuously observed increasing Lévy processes. In this paper we consider indirect estimation through the observation of the OU process X at discrete time instants. Thus we deal with an inverse problem, and our estimation techniques are correspondingly quite different from the ones in these papers. Another paper on estimation for OU processes is Roberts *et al*. (2004), in which Bayesian estimation for parametric models is considered. Other papers on empirical characteristic function procedures include Knight and Satchell (1997), Feuerverger and McDunnough (1981) and, in a more general framework, Luong and Thompson (1987).

In section 2 we discuss self-decomposability via the Lévy–Khinchine representation theorem. We show that a self-decomposable distribution is characterized by the logarithm of its characteristic function, which is called the *cumulant function*. Furthermore, we state the close relationship between self-decomposability and Lévy-driven OU processes. Additional details on this can be found in Sato (1999, Section 17). We show that the process $(X_t, t \geqslant 0)$ is a Feller process (Proposition 2.1) and hence satisfies the strong Markov property. We also give some examples of self-decomposable distributions and related OU processes. In section 3 we prove that the OU process is β-mixing. In the proof, we use theory as developed in Meyn and Tweedie (1993a; 1993b) and a result from Shiga (1990).

Section 4 explains our method for estimating the canonical function. The method uses a given preliminary, consistent estimator $\tilde{\psi}_n$ for the characteristic function ψ_0 of $X(1)$, a typical example being the empirical characteristic function of the observations. Any characteristic function ψ without zeros possesses a unique (distinguished) logarithm, its associated cumulant function, which we denote by $T\psi$. Our estimator of the cumulant function $T\psi_0$ is now defined as the projection of the preliminary estimate $T\tilde{\psi}_n$ onto the class of cumulant functions of self-decomposable distributions, relative to a weighted L_2-distance. The estimates of ψ_0 and its associated canonical function are defined by inverting the respective maps. Under a 'compactness condition' on the set of canonical functions, this *cumulant M-estimator* exists and is unique (Theorem 4.5). In Section 5 we prove two uniform convergence results on random characteristic functions, which may be of independent interest. We then use these results to provide conditions under which the cumulant M-estimator is consistent (Theorem 5.3). The estimator can numerically be approximated by a support-reduction algorithm, as discussed in Groeneboom *et al*. (2003). In Section 6 we explain how this algorithm fits within our set-up.

Section 7 contains applications and examples of estimators under different observation schemes and presents some simulation results. We also consider the estimation of a self-decomposable distribution based on independent and identically distributed (i.i.d.) data. This problem is difficult to handle by standard estimation techniques, as there exists no general closed-form expression for the density of a self-decomposable distribution. The approach is to first estimate the canonical function by our cumulant M-estimator and then apply Fourier inversion.

For the intensity parameter λ, a simple explicit estimator is defined in Section 8. This estimator is shown to be asymptotically consistent, although biased upward.

The appendix contains proofs of some more technical lemmas.

2. Preliminaries

In this section we discuss self-decomposable distributions on $\mathbb{R}_+$ and Lévy-driven OU processes. Furthermore, we introduce notation that will be used throughout the rest of the paper.

2.1. Self-decomposable distributions on $\mathbb{R}_+$

A random variable X, with distribution function F, is said to be *self-decomposable* if for every $c \in (0, 1)$ there exists a random variable X_c, independent of X, such that $X \stackrel{d}{=} cX + X_c$. In particular, all degenerate random variables are self-decomposable. Since the concept of self-decomposability only involves the distribution of a random variable, we define a probability measure or a characteristic function to be self-decomposable if its corresponding random variable is self-decompsable.

The class of self-decomposable distributions is a subclass of the class of infinitely divisible distributions. For the latter type of distributions, there is a powerful characterization in terms of characteristic functions: the Lévy–Khinchine representation. A random variable Y with values in $\mathbb{R}_+$ ($= [0, \infty)$) is infinitely divisible if and only if its characteristic function has the form

$$\psi(t) = E^{\mathrm{i}tY} = \exp\left(\mathrm{i}\gamma_0 t + \int_0^\infty (\mathrm{e}^{\mathrm{i}tx} - 1)\nu(\mathrm{d}x)\right), \qquad \forall t \in \mathbb{R}, \tag{2.1}$$

where $\gamma_0 \geqslant 0$. The measure ν is called the *Lévy measure* of Y and satisfies the integrability condition $\int_0^\infty (x \wedge 1)\, \nu(\mathrm{d}x) < \infty$. The parameter γ_0 is called the *drift*.

If Y is self-decomposable, the measure ν takes a special form. It has a density with respect to Lebesgue measure (Sato 1999, Corollary 15.11) and

$$\nu(\mathrm{d}x) = \frac{k(x)}{x}\,\mathrm{d}x,$$

where k is a decreasing function on $(0, \infty)$, known as the *canonical function*. We take this function to be right-continuous. The integrability condition on ν is given by $\int_0^1 k(x)\mathrm{d}x + \int_1^\infty x^{-1}k(x)\mathrm{d}x < \infty$. By Proposition V.2.3 in van Harn and Steutel (2004), the class of self-decomposable distributions on $\mathbb{R}_+$ is closed under weak convergence. By Theorem 27.13 in Sato (1999), the distribution of Y is either absolutely continuous with respect to Lebesgue measure or degenerate.

Thus each non-degenerate positive, self-decomposable random variable is characterized by a couple (γ_0, k) consisting of a non-negative number γ_0 and decreasing function k. In the next section we shall see that the variable $X(1)$ of the process X solving (1.1) is self-decomposable. Due to our assumption that the BDLP Z in (1.1) possesses no drift, the parameter γ_0 corresponding to $X(1)$ is zero.

Next, we introduce some notation. Define a measure μ on the Lebesgue measurable sets in $(0, \infty)$ by

$$\mu(\mathrm{d}x) = \frac{1 \wedge x}{x}\,\mathrm{d}x, \qquad x \in (0, \infty).$$

Let $\mathcal{L}^1(\mu)$ be the space of μ-integrable functions on $(0, \infty)$. Define a semi-norm $\|\cdot\|_\mu$ on $\mathcal{L}^1(\mu)$ by $\|k\|_\mu = \int |k|\mathrm{d}\mu$. Note that the definition of the measure μ precisely suits the integrability condition on k, which can now be formulated as $\|k\|_\mu < \infty$.

Define a set of functions by

$$K := \{k \in \mathcal{L}^1(\mu) : k(x) \geqslant 0,\ k \text{ is decreasing and right-continuous}\}.$$

The set $K \subseteq \mathcal{L}^1(\mu)$ is a convex cone which contains precisely the canonical functions of all non-degenerate self-decomposable distributions on $\mathbb{R}_+$ and the degenerate distribution at 0.

Let Ψ be the corresponding set of characteristic functions

$$\Psi := \left\{\psi : \mathbb{R} \to \mathbb{C} | \psi(t;\, k) = \exp\left(\int_0^\infty (\mathrm{e}^{\mathrm{i}tx} - 1)\frac{k(x)}{x}\mathrm{d}x\right) \text{ for some } k \in K\right\}. \tag{2.2}$$

By the definition of Ψ the mapping $Q : K \mapsto \Psi$, assigning to each function $k \in K$ its corresponding characteristic function in Ψ, is onto. As a consequence of the Lévy–Khinchine theorem, Q is also one-to-one.

The following result from complex analysis can be found, for example, in Chung (2001, Section 7.6). Suppose $\varphi : \mathbb{R} \to \mathbb{C}$ is continuous, $\varphi(0) = 1$ and $\varphi(x) \neq 0$ for all $x \in [-T, T]$. Then there exists a unique continuous function $f : [-T, T] \to \mathbb{C}$ such that $f(0) = 0$ and $\exp(f(x)) = \varphi(x)$. The corresponding statement when $[-T, T]$ is replaced by $(-\infty, \infty)$ is also true. The function f is referred to as the *distinguished logarithm*. If φ is a characteristic function, then f is called a *cumulant function*.

Since an infinitely divisible characterstic function has no real zeros (see Sato 1999, Lemma 7.5), we can attach to each $\psi \in \Psi$ a unique continuous function g such that $e^{g(t)} = \psi(t)$ and $g(0) = 0$. Since we will switch between sets of characteristic functions and cumulant functions throughout, we define a mapping T from Ψ onto its range by

$$[T(\psi)](t) = g(t), \qquad \psi \in \Psi, \, t \in \mathbb{R},$$

By the uniqueness of the distinguished logarithm and the Lévy–Khinchine representation it follows that

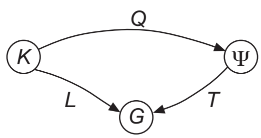
$$G := T(\Psi) = \left\{ g : \mathbb{R} \to \mathbb{C} \mid g(t) = \int_0^\infty (e^{itx} - 1) \frac{k(x)}{x} dx, \text{ for some } k \in K \right\}.$$

We have thus defined three sets, each parametrizing the class of self-decomposable distributions: (i) K, the set of canonical functions; (ii) Ψ, the set of characteristic functions; (iii) G, the set of cumulant functions. Typical members of each will be denoted by k, ψ and g respectively.

In order to switch easily between canonical functions and cumulants, we define the mapping $L : K \to G$ by $L = T \circ Q$. That is, for $k \in K$,

$$[L(k)](t) = \int_0^\infty (e^{itx} - 1) \frac{k(x)}{x} dx, \qquad t \in \mathbb{R}.$$

The following diagram may help to clarify the relations between the operators defined so far:



Next, we give a few examples of positive self-decomposable distributions.

Example 2.1. (i) Let X be Gamma(c, α) distributed with density f given by $f(x) = (\alpha^c/\Gamma(c))x^{c-1}e^{-\alpha x}\mathbf{1}_{\{x>0\}}$, $c, \alpha > 0$. The characteristic and canonical functions are given by $\psi(t) = (1 - \alpha^{-1}it)^{-c}$ and $k(x) = ce^{-\alpha x}$, respectively.

(ii) Let X be an α-stable distribution with $\alpha \in (0, 1)$. Then X has support $[0, \infty)$ if and only if its characteristic function is

$$\psi(t) = \exp\left(-|t|^\alpha \left[1 - i \tan\left(\frac{\pi\alpha}{2}\right) \operatorname{sgn}(t)\right]\right).$$

Its corresponding canonical function is given by $k(x) = c_\alpha x^{-\alpha}$, where $c_\alpha = \alpha/(\Gamma(1 - \alpha) \cos(\pi\alpha/2))$. Note that $c_{1/2} = 1/\sqrt{2\pi}$. The density function of X permits a known closed-form expression in terms of elementary functions only if $\alpha = \frac{1}{2}$. In this case $f(x) = (2\pi)^{-1/2}x^{-3/2}e^{-1/(2x)}\mathbf{1}_{\{x>0\}}$. The probability distribution with this density is called the Lévy distribution. If Z has a standard normal distribution, then W, defined by $W = 1/Z^2$ if $Z \neq 0$ and $W = 0$ otherwise, has a Lévy distribution.

(iii) The inverse Gaussian distribution with parameters δ and γ, IG(δ, γ), has probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}} \delta e^{\delta\gamma} x^{-3/2} \exp(-(\delta^2 x^{-1} + \gamma^2 x)/2) \mathbf{1}_{\{x>0\}}, \qquad \delta > 0, \gamma \geqslant 0.$$

See, for example, Barndorff-Nielsen and Shephard (2001b). Its canonical function is given by $k(x) = (2\pi)^{-1/2} \delta x^{-1/2} \exp(-\gamma^2 x/2) \mathbf{1}_{\{x>0\}}$. The case $(\delta, \gamma) = (1, 0)$ corresponds to the Lévy distribution.

2.2. Lévy-driven Ornstein–Uhlenbeck processes

In this section we discuss some properties of Lévy-driven OU processes. We can assume that the driving Lévy process $Z = (Z_t, t \geqslant 0)$ has right-continuous sample paths, with existing left-hand limits. It is easily verified that a (strong) solution $X = (X_t, t \geqslant 0)$ to (1.1) is given by

$$X_t = e^{-\lambda t} X_0 + \int_{(0,t]} e^{-\lambda(t-s)} dZ(\lambda s), \qquad t \geqslant 0. \tag{2.3}$$

Up to indistinguishability, this solution is unique (Sato 1999, Section 17). Furthermore, since X is given as a stochastic integral with respect to a cadlag semi-martingale, the OU process $(X_t, t \geqslant 0)$ can be assumed cadlag itself. The stochastic integral in (2.3) can be interpreted as a pathwise Lebesgue–Stieltjes integral, since the paths of Z are almost surely of finite variation on each interval $(0, t]$, $t \in (0, \infty)$ (Sato 1999, Theorem 21.9). Figure 1 shows a simulation of an OU process with Gamma(2, 2) marginal distribution.

Denote by $(\mathcal{F}_t^0)_{t\geqslant 0}$ the natural filtration of (X_t). That is, $(\mathcal{F}_t^0) = \sigma(X_u, u \in [0, t])$. As noted in Shiga (1990, Section 2), $(X_t, \mathcal{F}_t^0)$ is a temporally homogeneous Markov process. Denote by $(E, \mathcal{E})$ the state space of X, where $\mathcal{E}$ is the Borel σ-field on E. We take $E = [0, \infty)$. The transition kernel of (X_t), denoted by $P_t(x, B)$ $(x \in E, B \in \mathcal{E})$, has characteristic function (Sato 1999, Lemma 17.1).

$$\int e^{izy} P_t(x, dy) = \exp\left(ize^{-\lambda t} x + \lambda \int_0^t g(e^{\lambda(u-t)} z) du \right), \qquad z \in \mathbb{R}, \tag{2.4}$$

where g is the cumulant of $Z(1)$.

Let $b\mathcal{E}$ denote the space of bounded $\mathcal{E}$-measurable functions. The transition kernel induces an operator $P_t : b\mathcal{E} \to b\mathcal{E}$ by

$$P_t f(x) := \int f(y) P_t(x, dy) = \int f(e^{-\lambda t} x + y) P_t(0, dy). \tag{2.5}$$

The second equality follows directly from the explicit solution (2.3). We call P_t the transition operator. Let $C_0(E)$ denote the space of continuous functions on E vanishing at infinity (i.e. for all $\varepsilon > 0$ there exists a compact subset K of E such that $|f| \leqslant \varepsilon$ on $E \backslash K$).

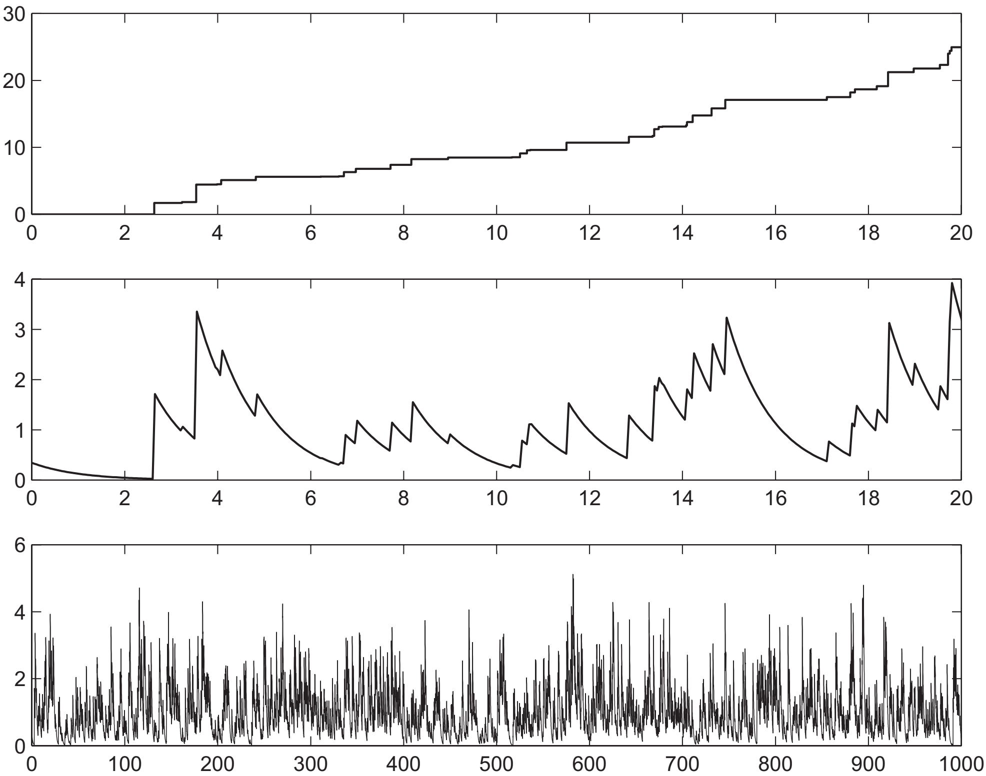


Figure 1. Top: simulation of the BDLP (compound Poisson process of intensity 2 with exponential jumps of expectation $\frac{1}{2}$). Middle: corresponding OU process with Gamma(2,2) marginal distribution. Bottom: OU process on longer time horizon.

Proposition 2.1. *The transition operator of the OU-process is of Feller type. That is,*

(i) $P_t C_0(E) \subseteq C_0(E)$ *for all* $t \geqslant 0$,

(ii) $\forall f \in C_0(E)$, $\forall x \in E$, $\lim_{t\downarrow 0} P_t f(x) = f(x)$.

For general notions concerning Markov processes of Feller type we refer to Revuz and Yor (1999, Chapter 3).

Proof. (i) Let $f \in C_0(E)$, whence f is bounded. If $x_n \to x$ in E, then $f(\mathrm{e}^{-\lambda t}x_n + y) \to f(\mathrm{e}^{-\lambda t}x + y)$ in $\mathbb{R}$, by the continuity of f, for any $y \in \mathbb{R}$. By dominated convergence, $P_t f(x_n) \to P_t f(x)$, as $n \to \infty$. Hence, $P_t f$ is continuous. Again by dominated convergence, $P_t f(x) \to 0$, as $x \to \infty$.

(ii) By dominated convergence $\int_0^t g(\mathrm{e}^{\lambda(u-t)}z)\mathrm{d}u = \int_0^t g(\mathrm{e}^{-\lambda u}z)\mathrm{d}u \to 0$, as $t \downarrow 0$. Here we use the continuity of the cumulant g and $g(0) = 0$. Then it follows from (2.4) that

$$\lim_{t\downarrow 0}\int e^{izy}P_t(x,\,dy) = e^{izx}.$$

Thus $P_t(x, \cdot)$ converges weakly to $\varepsilon_x(\cdot)$ (Dirac measure at x):

$$\lim_{t\downarrow 0}\int f(y)P_t(x,\,dy) = \int f(y)\varepsilon_x(dy) = f(x), \qquad \forall f \in C_b(E).$$

Here $C_b(E)$ denotes the class of bounded, continuous functions on E. The result follows since $C_0(E) \subseteq C_b(E)$. □

The Feller property of (X_t) implies (X_t) is a *Borel right Markov process*; see the definitions in Getoor (1975, Chapter 9). We will need this result in Section 3.

Since P_t is Feller, (X_t) satisfies the strong Markov property (Revuz and Yor 1999, Theorem III.3.1). In order to state a useful form of the latter property, we define a *canonical OU process* on the space $\Omega = D[0, \infty)$, by setting $X_t(\omega) = \omega(t)$, for $\omega \in \Omega$ (here $D[0, \infty)$ denotes the space of cadlag functions on $[0, \infty)$, equipped with its σ-algebra generated by the cylinder sets). By the Feller property, this process exists (Revuz and Yor 1999, theorem III.2.7). Let ν be a probability measure on $(E, \mathcal{E})$ and denote by P_ν the distribution of the canonical OU process on $D[0, \infty)$ with initial distribution ν. For $t \in [0, \infty)$, we define the shift maps $\theta_t : \Omega \to \Omega$ by $\theta_t(\omega(\cdot)) = \omega(\cdot + t)$.

Next, we enlarge the filtration by including certain null sets. Denote by $\mathcal{F}_\infty^\nu$ the completion of $\mathcal{F}_\infty^0 = \sigma(\mathcal{F}_t^0,\, t \geqslant 0)$ with respect to P_ν. Let $(\mathcal{F}_t^\nu)$ be the filtration obtained by adding to each $\mathcal{F}_t^0$ all the P_ν-negligible sets of $\mathcal{F}_\infty^\nu$. Finally, set $\mathcal{F}_t = \bigcap_\nu \mathcal{F}_t^\nu$ and $\mathcal{F}_\infty = \bigcap_\nu \mathcal{F}_\infty^\nu$, where the intersection is over all initial probability measures ν on $(E, \mathcal{E})$. In the special case of Feller processes, it can be shown that the filtration $(\mathcal{F}_t)$ obtained in this way is automatically right-continuous (thus, it satisfies the 'usual hypotheses'). See Proposition III.2.10 in Revuz and Yor (1999). Moreover, (X_t) is still Markov with respect to this completed filtration (Revuz and Yor 1999, Proposition III.2.14). The strong Markov property can now be formulated as follows. Let Z be an $\mathcal{F}_\infty$-measurable and positive (or bounded) random variable. Let T be an $\mathcal{F}_t$-stopping time. Then, for any initial measure ν,

$$E_\nu(Z \circ \theta_T | \mathcal{F}_T) = E_{X_T}(Z), \qquad P_\nu\text{-almost surely on } \{T < \infty\}. \tag{2.6}$$

Here $\mathcal{F}_T = \{A \in \mathcal{F} : A \bigcap \{T \leqslant t\} \in \mathcal{F}_t, \forall t \geqslant 0\}$. The expectation on the right-hand side is interpreted as $E_x Z$, evaluated at $x = X_T$.

In Section 3 we will apply the strong Markov property to random times such as $\sigma_A := \inf\{t \geqslant 0 : X_t \in A\}$ with $A \in \mathcal{E}$. By Theorem III.2.17 in Revuz and Yor (1999), σ_A is an $(\mathcal{F}_t)$-stopping time.

The following theorem gives a condition in terms of the process Z such that there exists a stationary solution to (1.1). Moreover, it shows that under this condition the marginal distribution of this stationary solution is self-decomposable with canonical function determined by the Lévy measure of the underlying process Z.

Theorem 2.2. *Suppose Z is an increasing Lévy process with Lévy measure ρ (which is by definition the Lévy measure of $Z(1)$). Suppose ρ satisfies the integrability condition*

$$\int_2^\infty \log x \rho(\mathrm{d}x) < \infty. \tag{2.7}$$

Then $P_t(x, \cdot)$ converges weakly to a limit distribution π as $t \to \infty$ for each $x \in E$ and each $\lambda > 0$. Moreover, π is self-decomposable with canonical function $k(x) = \rho(x, \infty)\mathbf{1}_{(0,\infty)}(x)$. Furthermore, π is the unique invariant probability distribution of X.

For a proof, see Sato (1999, Theorem 17.5 and Corollary 17.9). Theorem 24.10(iii) in Sato (1999) implies that π has support $[0, \infty)$.

We end this section with two examples of Lévy-driven OU processes. These examples are closely related to the ones given in Examples 2.1(i) and 2.1(iii).

Example 2.2. (i) Let $(X_t, t \geqslant 0)$ be the OU process with $\pi = \text{Gamma}(c, \alpha)$. From Theorem 2.2 and Example 2.1(i) it follows that the BDLP $(Z_t, t \geqslant 0)$ has Lévy measure ρ satisfying $\rho(\mathrm{d}x) = c\alpha \mathrm{e}^{-\alpha x}\mathrm{d}x$ (for $x > 0$). Since $\int_0^\infty \rho(\mathrm{d}x) < \infty$, Z is a compound Poisson process. By examining the characteristic function of $Z(1)$, we see that the process Z can be represented as $Z_t = \sum_{i=1}^{N_t} Y_i$, where $(N_t, t \geqslant 0)$ is a Poisson process of intensity c, and $Y_1, Y_2, \ldots$ is a sequence of independent random variables, each having an exponential distribution with parameter α. Figure 1 corresponds to the case $c = \alpha = 2$.

(ii) Let $(X_t, t \geqslant 0)$ be the OU process with $\pi = \text{IG}(\delta, \gamma)$. Similarly to (i), we obtain for the Lévy measure ρ of the BDLP Z the expression

$$\rho(\mathrm{d}x) = \left(\frac{\delta}{2\sqrt{2\pi}} \frac{1}{x\sqrt{x}} \mathrm{e}^{-\gamma^2 x/2} + \frac{\delta\gamma^2}{2\sqrt{2\pi}} \frac{1}{\sqrt{x}} \mathrm{e}^{-\gamma^2 x/2}\right)\mathrm{d}x, \qquad x > 0.$$

Write $\rho = \rho^{(1)} + \rho^{(2)}$. Then $(Z_t, t \geqslant 0)$ can be constructed as the sum of two independent Lévy processes $Z^{(1)}$ and $Z^{(2)}$, where $Z^{(i)}$ has Lévy measure $\rho^{(i)}$ $(i = 1, 2)$. It is easily seen that $Z^{(1)}(1) \sim IG(\delta/2, \gamma)$. Note that

$$\int_0^\infty \rho^{(2)}(\mathrm{d}x) = \int_0^\infty \frac{\delta\gamma^2}{2\sqrt{2\pi}} \frac{1}{\sqrt{x}} \mathrm{e}^{-\gamma^2 x/2}\mathrm{d}x < \infty,$$

so that $Z^{(2)}$ is a compound Poisson process. Some calculations show that we can construct $Z^{(2)}$ as $Z_t^{(2)} = \gamma^{-2}\sum_{i=1}^{N_t} W_i^2$, where $(N_t, t \geqslant 0)$ is a Poisson process of intensity $\delta\gamma/2$, and $W_1, W_2, \ldots$ is a sequence of independent standard normal random variables. Since $\int_0^\infty \rho(\mathrm{d}x) = \infty$, this OU process is a process of infinite activity: it has infinitely many jumps in bounded time intervals.

3. A condition for the OU process to be β-mixing

Let $(X_t, t \geqslant 0)$ be a stationary Lévy-driven OU process. The following theorem is the main result of this section.

Theorem 3.1. *If condition* (2.7) *of Theorem 2.2 holds, then the Ornstein–Uhlenbeck process* (X_t) *is β-mixing.*

This result will be used in Section 7 to obtain consistency proofs for some estimators that will be defined in the next section. For the remainder of this section we will assume that (2.7) holds. Theorem 2.2 then implies that there exists a unique invariant probability measure π_0.

By Proposition 1 in Davydov (1973), the *β-mixing coefficients* for a stationary continuous-time Markov process X are given by

$$\beta_X(t) = \int_E \pi(\mathrm{d}x) \| P_t(x, \cdot) - \pi(\cdot) \|_{\mathrm{TV}}, \qquad t > 0.$$

Here, $\| \cdot \|_{\mathrm{TV}}$ denotes the total variation norm and π the initial distribution. The process is said to be β-mixing if $\beta_X(t) \to 0$, as $t \to \infty$. The analogous definitions for the discrete-time case are obvious. Dominated convergence implies that the following condition is sufficient for (X_t) to be β-mixing:

$$\lim_{t\to\infty} \| P_t(x, \cdot) - \pi(\cdot) \|_{\mathrm{TV}} = 0, \qquad \forall\, x \in E. \tag{3.1}$$

That is, it suffices to prove that the transition probabilities converge in total variation to the invariant distribution for each initial state $x \in E$. The next theorem, taken from Meyn and Tweedie (1993b), (Theorem 6.1), can be used to verify this condition.

Theorem 3.2. *Suppose that (X_t) is positive Harris recurrent with invariant probability distribution π. Then* (3.1) *holds if and only if some skeleton chain is φ-irreducible.*

In the remainder of this section we first prove that the 1-skeleton chain, obtained from (X_t), is φ-irreducible (Corollary 3.5). Subsequently we show that (X_t) is positive Harris recurrent (Lemma 3.6). By an application of Theorem 3.2, Theorem 3.1 then follows immediately.

We start with some definitions from the general theory of stability of continuous-time Markov processes. These correspond to the ones used in Theorem 3.2. For more details, see Meyn and Tweedie (1993b). Recall from Section 2 that P_ν denotes the distribution of the OU process with initial distribution ν. We write P_x if ν is Dirac mass at x. For a measurable set A we let

$$\sigma_A = \inf\{t \geqslant 0 | X_t \in A\}, \qquad \eta_A = \int_0^\infty \mathbf{1}_{\{X_t \in A\}} \mathrm{d}t.$$

Thus, σ_A denotes the first *hitting time* of the set A and η_A denotes the *time spent* in A by the process X. A Markov process is called *φ-irreducible* if, for some non-zero σ-finite measure φ,

$$\varphi(A) > 0 \Rightarrow E_x(\eta_A) > 0, \qquad \forall x \in E,\ A \in \mathcal{E}.$$

The Markov process X is called *Harris recurrent* if, for some non-zero σ-finite measure φ,

$$\varphi(A) > 0 \Rightarrow P_x(\eta_A = \infty) = 1, \qquad \forall x \in E,\ A \in \mathcal{E}.$$

If X is a Borel right Markov process, then this condition can be shown to be equivalent to the following (see Kaspi and Mandelbaum 1994): for some non-zero σ-finite measure ψ,

$$\psi(A) > 0 \Rightarrow P_x(\sigma_A < \infty) = 1, \qquad \forall x \in E,\ A \in \mathcal{E}. \tag{3.2}$$

The latter condition is generally more easily verifiable. The process is called *positive Harris recurrent* if it is Harris recurrent and admits an invariant *probability* measure.

The Δ-*skeleton* is defined as the Markov chain obtained by sampling the original process X_t at deterministic time points $\Delta, 2\Delta, \ldots$ (observation scheme 1 in Section 7 coincides with this concept). In a slight abuse of notation, we shall henceforth denote the continuous-time process by (X_t) and its Δ-skeleton by (X_n) (thus, $X_n \equiv X_{n\Delta}$). The next proposition says that the 1-skeleton obtained from X constitutes a first-order autoregressive time series, with infinitely divisible noise terms.

Proposition 3.3. *Consider observation scheme 1 with* $\Delta = 1$ *and denote the observations by* $X_0, X_1, \ldots$. *Then the chain satisfies the first-order autoregressive relation*

$$X_n = \mathrm{e}^{-\lambda} X_{n-1} + W_n(\lambda), \qquad n \geqslant 1, \tag{3.3}$$

where $(W_n(\lambda))_n$ *is an i.i.d. sequence of random variables distributed as*

$$W_\lambda := \int_0^1 \mathrm{e}^{\lambda(u-1)} \mathrm{d}Z(\lambda u).$$

Moreover, W_λ *is infinitely divisible with Lévy measure* κ *given by*

$$\kappa(B) = \int_B w^{-1} \rho(w, \mathrm{e}^{\lambda} w] \mathrm{d}w, \qquad B \in \mathcal{E}. \tag{3.4}$$

The proof is given in the Appendix.

Remark 3.1. Since

$$\mathrm{e}^{-\lambda} Z(\lambda) \leqslant \int_0^1 \mathrm{e}^{\lambda(u-1)} \mathrm{d}Z(\lambda u) \leqslant Z(\lambda),$$

W_λ has the same tail behaviour as $Z(\lambda)$. In particular, if $Z(1)$ has infinite expectation, then so does W_λ.

We will now show that (X_n) is φ-irreducible. For the discrete-time case this means that there exists a non-zero σ-finite measure φ, such that, for all $B \in \mathcal{E}$ with $\varphi(B) > 0$, $\sum_{n=1}^{\infty} P_n(x, B) > 0$ for all $x \in E$.

Lemma 3.4. *Let* P^{W_λ} *be the distribution function of* W_λ. *Then* P^{W_λ} *has an absolutely continuous component with respect to Lebesgue measure.*

Proof. It follows from Proposition 3.3 that P^{W_λ} is infinitely divisible with Lévy measure κ. From (3.4), we see that κ is absolutely continuous with respect to Lebesgue measure.

First consider the case $\kappa[0, \infty) < \infty$. Then P^{W_λ} is compound Poisson, and hence (see equation 27.1 in Sato 1999),

$$P^{W_\lambda}(\cdot) = \mathrm{e}^{-\kappa[0,\infty)}\left(\delta_{\{0\}}(\cdot) + \sum_{k=1}^{\infty}\frac{\kappa^{*k}(\cdot)}{k!}\right), \tag{3.5}$$

where δ_0 denotes the Dirac measure at 0 and * denotes the convolution operator. Since the convolution of two non-zero finite measures σ_1 and σ_2 is absolutely continuous if either of them is absolutely continuous (Sato 1999, Lemma 27.1), it follows from the absolute continuity of κ that the second term on the right-hand side of (3.5) constitutes the absolutely continuous part of P^{W_λ}.

Next consider the case $\kappa[0,\infty) = \infty$. Define for each $n = 1, 2, \ldots, \kappa_n(B) := \kappa(B \cap (1/n, \infty)$ for Borel sets B in $(0, \infty)$. Set $c_n = \kappa_n[0, \infty)$. Then $c_n < \infty$ and κ_n is absolutely continuous. Let $P_n^{W_\lambda}$ be the distribution corresponding to κ_n. As in the previous case, we have

$$P_n^{W_\lambda}(\cdot) = \mathrm{e}^{-c_n}\left(\delta_{\{0\}}(\cdot) + \sum_{k=1}^{\infty}\frac{\kappa_n^{*k}(\cdot)}{k!}\right),$$

and $P_n^{W_\lambda}$ has an absolutely continuous component with respect to Lebesgue measure. Since P^{W_λ} contains $P_n^{W_\lambda}$ as a convolution factor, it follows that P^{W_λ} has an absolutely continuous component with respect to Lebesgue measure. □

Proposition 6.3.5 in Meyn and Tweedie (1993a) asserts that (X_n) as defined in (3.3) is φ-irreducible if the common distribution of the innovation sequence $(W_n(\lambda))$ has an absolutely continuous component with respect to Lebesgue measure. Using the previous lemma, we therefore obtain φ-irreducibility of (X_n).

Corollary 3.5 *The* 1*-skeleton chain* (X_n) *is* φ*-irreducible.*

Lemma 3.6. *Under condition* (2.7)*,* (X_t) *is positive Harris recurrent.*

Proof. Let $\sigma_a = \inf\{t \geqslant 0 : X_t = a\}$. We will prove $P_x(\sigma_a < \infty) = 1$, for all $x, a \in E$. Then condition (3.2) is satisfied for any non-zero measure ψ on E.

First, we consider the case $x \geqslant a$. Since we assume condition (2.7), Lemma A.1 in the Appendix applies:

$$\int_0^1 \frac{\mathrm{d}z}{z}\exp\left(-\int_z^1 \frac{\lambda_\rho(y)}{y}\,\mathrm{d}y\right) = +\infty. \tag{3.6}$$

Here λ_ρ is given as in (A.4) in the Appendix. Theorem 3.3 in Shiga (1990) now asserts that $P_x(\sigma_a < \infty) = 1$ for every $x \geqslant a > 0$.

Next, suppose $x < a$. Let (X_n) be the skeleton chain obtained from (X_t). Define $\tau_a = \inf\{n \geqslant 0 : X_n \geqslant a\}$. Then, for each $m \in \mathbb{N}$,

$$P_x(\tau_a > m) = P_x(X_1 < a, \ldots, X_m < a)$$
$$= P_0(X_1 + \mathrm{e}^{-\lambda}x < a, \ldots, X_m + \mathrm{e}^{-\lambda m}x < a)$$
$$\leqslant P_0(X_1 < a, \ldots, X_m < a)$$
$$= P_0(W_1 < a, \ldots, \mathrm{e}^{-\lambda}X_{m-1} + W_m < a)$$
$$\leqslant P(W_1 < a, \ldots, W_m < a) = [P(W_\lambda < a)]^m \in [0, 1).$$

The last assertion holds since the support of any non-degenerate infinitely divisible random variable is unbounded (Sato 1999, Theorem 24.3). From this, it follows that

$$P_x(\tau_a < \infty) \geqslant \lim_{m\to\infty}(1 - [P(W_\lambda < a)]^m) = 1.$$

It is easy to see that $\{\tau_a + \sigma_a \circ \theta_{\tau_a} < \infty\} \subseteq \{\sigma_a < \infty\}$. Hence,

$$P_x(\sigma_a < \infty) \geqslant P_x(\tau_a + \sigma_a \circ \theta_{\tau_a} < \infty) = E_x\{E_x(\mathbf{1}_{\{\tau_a + \sigma_a \circ \theta_{\tau_a} < \infty\}} | \mathcal{F}_{\tau_a})\}$$
$$= E_x\{\mathbf{1}_{\{\tau_a < \infty\}} E_x(\mathbf{1}_{\{\sigma_a \circ \theta_{\tau_a} < \infty\}} | \mathcal{F}_{\tau_a})\} = E_x\{E_{X_{\tau_a}} \mathbf{1}_{\{\sigma_a < \infty\}}\} = 1.$$

The second inequality holds since $\{\tau_a + \sigma_a \circ \theta_{\tau_a} < \infty\} = \{\tau_a < \infty\} \bigcap \{\sigma_a \circ \theta_{\tau_a} < \infty\}$. The third equality follows from the strong Markov property, as formulated in (2.6). The last equality follows from the case $x \geqslant a$.

Hence, for all $x \in E$, we have proved that $P_x(\sigma_a < \infty) = 1$. Thus (X_t) is Harris recurrent.

By Theorem 2.2, the invariant measure of a Lévy-driven OU process is a probability measure, which shows that (X_t) is *positive* Harris recurrent. □

Remark 3.2. The β-mixing property of general (multidimensional) OU processes is also treated in Masuda (2004, Section 4). There it is assumed that the OU process is strictly stationary, and moreover that $\int |x|^\alpha \pi(\mathrm{d}x) < \infty$, for some $\alpha > 0$. The latter assumption is stronger than our assumption (2.7), but also yields the stronger conclusion that $\beta_X(t) = O(\mathrm{e}^{-at})$, as $t \to \infty$, for some $a > 0$ (i.e. the process (X_t) is *geometrically ergodic*). It seems hard to extend the argument in Masuda (2004) under assumption (2.7).

4. Definition of a cumulant M-estimator

Let π_0 be the unique invariant probability distribution of X. Any reference to the true underlying distribution will be denoted by a subscript 0. For example, F_0 denotes the true underlying distribution function of $X(1)$ and k_0 the true underlying canonical function.

To estimate k_0, based on discrete-time observations from X, we first define a preliminary estimator $\tilde{\psi}_n$ for ψ_0. In what follows, we choose $\tilde{\psi}_n$ such that either

$$\text{for each } n, \tilde{\psi}_n \text{ is a characteristic function and, } \forall t \in \mathbb{R}, \tilde{\psi}_n(t) \xrightarrow{\text{a.s.}} \psi_0(t), \text{ as } n \to \infty, \quad (4.1)$$

or

$$\text{for each } n, \tilde{\psi}_n \text{ is a characteristic function and, } \forall t \in \mathbb{R}, \tilde{\psi}_n(t) \xrightarrow{\text{p}} \psi_0(t), \text{ as } n \to \infty. \quad (4.2)$$

We will show in Section 5 that any preliminary estimator satisfying this condition will yield a consistent estimator for k_0. A natural preliminary estimator is the *empirical characteristic function*. We will return to possible choices for $\tilde{\psi}_n$ in Section 7.

Given any preliminary estimator $\tilde{\psi}_n$ for ψ_0, a first idea for constructing an estimator for k_0 would be to minimize some distance between $Q(k)$ and $\tilde{\psi}_n$ over all canonical functions $k \in K$. For example, if we let w be a positive (Lebesgue) integrable compactly supported weight function, we could take a weighted L^2-distance and define an estimator by

$$\hat{k}_n = \operatorname*{argmin}_{k \in K} \int |[Q(k)](t) - \tilde{\psi}_n(t)|^2 w(t)\mathrm{d}t.$$

Apart from the issue of whether this estimator is well defined, one disadvantage of this estimation method is that the objective function is non-convex (convexity being desirable from a computational point of view). This problem can be avoided by comparing cumulants. We will see below that $\tilde{\psi}_n$ is non-vanishing on S_w for sufficiently large n and thus admits a distinguished logarithm there. Then the idea is to define an estimator $\hat{k}_n$ as

$$\hat{k}_n = \operatorname*{argmin}_{k \in K} \int |[L(k)](t) - \tilde{g}_n(t)|^2 w(t)\mathrm{d}t.$$

We call this estimator a *cumulant M-estimator*. Next, we will make this idea more precise.

Let w be a non-negative integrable weight function with compact support, denoted by S_w. Assume w is non-zero in a neighbourhood of the origin and even. Define the space of square-integrable functions with respect to $w(t)\mathrm{d}t$ by

$$L^2(w) := \left\{ f : \mathbb{R} \to \mathbb{C} | f \text{ is (Lebesgue) measurable and } \int |f(t)|^2 w(t)\mathrm{d}t < \infty \right\},$$

where we identify functions which are equal almost everywhere with respect to $w(t)\mathrm{d}t$. We define an inner-product $\langle \cdot, \cdot \rangle_w$ on $L^2(w)$ by

$$\langle f, g \rangle_w = \Re \int f(t)\overline{g(t)} w(t)\mathrm{d}t,$$

where the bar over g denotes complex conjugation and $\Re$ the operation of taking the real part of an element of $\mathbb{C}$. For $g \in L^2(w)$ define a norm by $\|g\|_w = \sqrt{\langle g, g \rangle_w}$. The space $(L^2(w), \langle \cdot, \cdot \rangle_w)$ is a Hilbert space. For the rest of this paper, we assume n is large enough such that $\tilde{g}_n$ exists on S_w.

Next, we define an estimator for $g_0 = T(\psi_0)$ as the minimizer of

$$\Gamma_n(g) := \|g - T\tilde{\psi}_n\|_w^2 = \int |g(t) - T\tilde{\psi}_n(t)|^2 w(t)\mathrm{d}t$$

over an appropriate subset of G, which we consider as a subspace of $L^2(w)$. It is a standard fact from Hilbert space theory that every non-empty, closed, convex set in $L^2(w)$ contains a unique element of smallest norm. We will use this result to establish the existence and uniqueness of our estimator.

Since Γ_n is a squared norm in a Hilbert space, we only need to specify an appropriate subset of G. For this purpose, we first derive some properties of the mapping L, as defined in Section 2.

Lemma 4.1. *The mapping $L : K \to G$ is continuous, onto and one-to-one.*

Proof. Let $\{k_n\}$ be a sequence in K converging to $k_0 \in K$, that is $\|k_n - k_0\|_\mu \to 0$ as $n \to \infty$.

For $t \in S_w$,

$$\begin{aligned}|L(k_n)(t) - L(k_0)(t)| &= \left|\int_0^\infty (\mathrm{e}^{\mathrm{i}tx} - 1)\frac{k_n(x) - k_0(x)}{x}\,\mathrm{d}x\right| \\ &\leqslant |t|\int_0^1 |k_n(x) - k_0(x)|\mathrm{d}x + 2\int_1^\infty x^{-1}|k_n(x) - k_0(x)|\mathrm{d}x \\ &\leqslant \max\{|t|, 2\}\|k_n - k_0\|_\mu,\end{aligned}$$

where we use the inequality $|\mathrm{e}^{\mathrm{i}x} - 1| \leqslant \min\{|x|, 2\}$. Thus $L(k_n) \to L(k_0)$ uniformly on S_w which implies $\|L(k_n) - L(k_0)\|_w \to 0$ $(n \to \infty)$. Hence, L is continuous.

The surjectivity is trivial by the definition of G. If $g_1, g_2 \in G$ and $\|g_1 - g_2\|_w = 0$, then (by continuity of elements in G), $g_1 = g_2$ on S_w. Then also $\psi_1 := \mathrm{e}^{g_1} = \mathrm{e}^{g_2} := \psi_2$ on S_w. Lemma 4.2 below implies $\psi_1 = \psi_2$ on $\mathbb{R}$. Since Q is one-to-one, we must have $k_1 = k_2$. □

The following lemma extends the uniqueness theorem for characteristic functions. A proof can be found in Loève (1977, Chapter 4).

Lemma 4.2. *Let X be a positive random variable with characteristic function ψ. If ψ_M is the restriction of ψ to an interval $(-M, M)$, then ψ_M determines ψ.*

The set

$$G' := \left\{g : \mathbb{R} \to \mathbb{C} : g(t) = \beta_0\mathrm{i}t + \int_0^\infty \frac{\mathrm{e}^{\mathrm{i}tx} - 1}{x}k(x)\mathrm{d}x,\, \beta_0 \geqslant 0,\, k \in K\right\},$$

is closed under uniform convergence on compact sets containing the origin. To see this, let S be such a compact set. If $\{g_n\}_n \in G'$ is such that $\sup_{t\in S}|g_n(t) - g(t)| \to 0$ for some g, then $\sup_{t\in S}|\psi_n(t) - \psi(t)| \to 0$ and then (by the same argument as in the proof of Lévy's continuity theorem) the random variables corresponding to $\{\psi_n\}$ are uniformly tight. Denote these random variables by $\{X_n\}$. By Prohorov's theorem, there exists a subsequence n_l such that X_{n_l} converges weakly to a random variable X^*. Since X_n is a positive self-decomposable random variable, and the class of positive self-decomposable random variables is closed under weak convergence, X^* is positive self-decomposable. Let g^* be the cumulant of X^*. Then $g^* \in G'$ and $\sup_{t\in S}|g_{n_l}(t) - g^*(t)| \to 0$. Together with the continuity of g and g^* on S, this implies $g^* = g$ on S. Hence $g = g^* \in G'$.

However, the set G is not closed under uniform convergence on compact sets containing the origin. Let S again be such a set and define a sequence $\{k_n\}_{n\geq 1} \in K$ by $k_n(x) = n\mathbf{1}_{[0,1/n)}(x)$. Then, for each $t \in \mathbb{R}$,

$$g_n(t) = [L(k_n)](t) = n\int_0^{1/n} \frac{e^{itx} - 1}{x}\,dx \to it, \qquad \text{as } n \to \infty.$$

Let $g(t) = it$. Then, since each g_n and g are uniformly continuous on the compact set S, we have $\sup_{t\in S}|g_n(t) - g(t)| \to 0$. However, $g \notin G$, since g can only correspond to a point mass at one. Returning to the set G', we see that this example is the canonical example that can preclude closedness of G.

In view of dominated convergence, this counterexample also shows why the set G is not closed in $L^2(w)$. To obtain an appropriate *closed* subset of G, we first define a set of envelope functions in K. Pick for each $R > 0$ a function $k_R \in K$ such that $\|k_R\|_\mu \leq R$ (for example $k_R(x) = R/(4\sqrt{x})$). The collection $\{k_R,\ R > 0\}$ defines a set of envelope functions. Now let

$$K_R := \{k \in K | k(x) \leq k_R(x) \text{ for } x \in (0, \infty)\}.$$

and put $G_R = L(K_R)$, that is, G_R is the image of K_R under L.

Lemma 4.3. *Let $R > 0$. Then*

(i) *K_R is a compact, convex subset of $\mathcal{L}^1(\mu)$,*
(ii) *G_R is a compact, convex subset of $L^2(w)$.*

Proof. (i) Convexity of K_R is obvious.

Let $\{k_n\}$ be a sequence in K_R. Since each k_n is bounded on all strictly positive rational points, we can use a diagonalization argument to extract a subsequence n_j from n such that the sequence k_{n_j} converges to some function $\bar{k}$ on all strictly positive rationals. For $x \in (0, \infty)$ define

$$\tilde{k}(x) = \sup\{\bar{k}(q),\ x < q,\ q \in \mathbb{Q}\}.$$

This function is (by its definition) decreasing and right-continuous and satisfies $\tilde{k} \leq k_R$ on $(0, \infty)$. Thus $\tilde{k} \in K_R$. Furthermore, k_{n_j} converges pointwise to $\tilde{k}$ at all continuity points of $\tilde{k}$. Since the number of discontinuity points of $\tilde{k}$ is at most countable, k_{n_j} converges to $\tilde{k}$ almost everywhere on $(0, \infty)$. Moreover, since $k_{n_j} \leq k_R$ on $(0, \infty)$ and $k_R \in \mathcal{L}^1(\mu)$, Lebesgue's dominated convergence theorem applies: $\|k_{n_j} - \tilde{k}\|_\mu \to 0$, as $n_j \to \infty$. Hence, K_R is sequentially compact.

(ii) G_R is compact since it is the image of the compact set K_R under the continuous mapping L. Convexity of G_R follows from convexity of K_R. □

Corollary 4.4. *The inverse operator of L, $L^{-1} : G_R \to K_R$, is continuous.*

Proof. This is a standard result from topology; see Corollary 9.12 in Jameson (1974). □

Since we want to define our objective function in terms of the canonical function, one last step is necessary. Since Γ_n has a unique minimizer over G_R and to each G_R there belongs a unique member of K_R, there exists a unique minimizer of $\Gamma_n \circ L$ (which we will henceforth write as $\Gamma_n L$) over K_R. More precisely:

Theorem 4.5. *Let* $\hat{g}_n = \operatorname{argmin}_{g \in G_R} \Gamma_n(g)$ *(known to exist and to be unique). Then* $\hat{k}_n = \operatorname{argmin}_{k \in K_R} [\Gamma_n L](k)$ *exists. Moreover,* $\hat{k}_n = L^{-1}(\hat{g}_n)$ *and* $\hat{k}_n$ *is unique.*

Proof. Since $L : K_R \to G_R$ is onto and one-to-one, to each $g \in G_R$ there corresponds a unique $k \in K_R$ such that $L(k) = g$. Thus

$$\beta := \min_{g \in G_R} \Gamma_n(g) = \min_{k \in K_R} [\Gamma_n L](k).$$

Now define $\hat{k}_n = L^{-1}(\hat{g}_n)$ and choose an arbitrary $k \in K_R$ (but $k \neq \hat{k}_n$). Then $\hat{k}_n \in K_R$ and

$$[\Gamma_n L](\hat{k}_n) = \Gamma_n(\hat{g}_n) = \beta < [\Gamma_n L](k),$$

which shows that $\hat{k}_n$ is the unique minimizer of $\Gamma_n L$ over K_R. □

5. Consistency

In this section we discuss the consistency of the cumulant M-estimator. We start with two results, which strengthen the pointwise convergence in (4.1) and (4.2) to uniform convergence.

Lemma 5.1. *Let* $(\Omega, \mathcal{U}, \mathbb{P})$ *be a probability space. For each* $\omega \in \Omega$, *suppose* $\varphi_n(\cdot, \omega)$ $(n = 1, 2, \ldots)$ *and* φ *are characteristic functions such that for each* $t \in \mathbb{R}$, $\varphi_n(t, \cdot) \xrightarrow{\text{p}} \varphi(t)$, *as* $n \to \infty$. *Then, as* $n \to \infty$

$$\sup_{t \in K} |\varphi_n(t, \cdot) - \varphi(t)| \xrightarrow{\text{p}} 0, \qquad \text{for every compact set } K \subseteq \mathbb{R}.$$

Proof. Denote the distribution functions corresponding to $\varphi_n(\cdot, \omega)$ and φ by $F_n(\cdot, \omega)$ and F. The functions $x \mapsto \mathrm{e}^{\mathrm{i}tx}$ for $t \in K$ are uniformly bounded and equicontinuous. Therefore (by the Arzelà–Ascoli theorem), if $F_n(\cdot, \omega) \xrightarrow{\text{w}} F$ for some ω along some subsequence, then $\sup_{t \in K} |\varphi_n(t, \cdot) - \varphi(t)| \to 0$ for this ω and subsequence. It follows that it suffices to show that for every subsequence of $\{n\}$ there exists a further subsequence $\{n'\}$ and a set $A \in \mathcal{U}$ with $\mathbb{P}(A) = 1$ such that $F_{n'}(\cdot, \omega) \xrightarrow{\text{w}} F$, for all $\omega \in A$, along the subsequence.

By assumption, for every t there exists a subsequence $\{n\}$ such that $\varphi_n(t, \omega) \xrightarrow{\text{a.s.}} \varphi(t)$. Denote $\mathbb{Q} = \{q_1, q_2, \ldots\}$. There exists a subsequence $\{n^{(1)}\}$ of $\{n\}$ and a set $A^{(1)} \in \mathcal{U}$ with $\mathbb{P}(A^{(1)}) = 1$ such that $\varphi_{n^{(1)}}(q_1, \omega) \to \varphi(q_1)$, for all $\omega \in A^{(1)}$. There exists a subsequence $\{n^{(2)}\}$ of $\{n^{(1)}\}$ and a set $A^{(2)} \in \mathcal{U}$ with $\mathbb{P}(A^{(2)}) = 1$ such that $\varphi_{n^{(2)}}(q_2, \omega) \to \varphi(q_2)$, for all

$\omega \in A^{(2)}$. Proceed iteratively in this way. Consider the diagonal sequence, obtained by $n_i := n_i^{(i)}$, and set $A = \cap_{i=1}^{\infty} A^{(i)}$, then $\mathbb{P}(A) = 1$ and

$$\varphi_n(q, \omega) \to \varphi(q), \qquad \forall\, q \in \mathbb{Q}, \forall \omega \in A. \tag{5.1}$$

For every $\delta > 0$,

$$\int_{|x|>2/\delta} F_n(\mathrm{d}x, \omega) \leqslant \frac{1}{2\delta} \int_{-\delta}^{\delta} |1 - \varphi_n(t, \omega)| \mathrm{d}t =: a_n(\delta, \omega), \tag{5.2}$$

by a well-known inequality (see, for instance, Chung 2001, Section 6.3). Furthermore, with $a(\delta) := (2\delta)^{-1} \int_{-\delta}^{\delta} |1 - \varphi(t)| \mathrm{d}t$, by Fubini's theorem

$$\mathbb{E}|a_n(\delta, \cdot) - a(\delta)| \leqslant \frac{1}{2\delta} \int_{-\delta}^{\delta} \mathbb{E} \Big| |1 - \varphi_n(t)| - |1 - \varphi(t)| \Big| \mathrm{d}t \to 0, \qquad n \to \infty,$$

by dominated convergence and the assumed convergence in probability. Thus, for every $\delta > 0$ there exists a further subsequence $\{n\}$ and a set $B \in \mathcal{U}$ with $\mathbb{P}(B) = 1$ such that $a_n(\delta, \omega) \to a(\delta)$ for all $\omega \in B$. By a diagonalization scheme we can find a further subsequence $\{n\}$ and a set $C \in \mathcal{U}$ with $\mathbb{P}(C) = 1$ such that

$$\lim_{n\to\infty} |a_n(\delta, \omega) - a(\delta)| = 0, \qquad \forall\, \delta \in \mathbb{Q} \cap (0, \infty), \forall \omega \in C.$$

Combined with (5.2), this shows that

$$\limsup_{n\to\infty} \int_{\{|x|>2/\delta\}} F_n(\mathrm{d}x, \omega) \leqslant a(\delta), \qquad \forall \delta \in \mathbb{Q} \cap (0, \infty), \forall\, \omega \in C,$$

taking the limsup over the subsequence. Because $a(\delta)\downarrow 0$ as $\delta\downarrow 0$, we see that $\{F_n(\cdot, \omega)\}_{n=1}^{\infty}$ is tight for all $\omega \in C$.

If G is a limit point of $F_n(\cdot, \omega)$, then by (5.1),

$$\int \mathrm{e}^{\mathrm{i}tx} \mathrm{d}G(x) = \lim_{n\to\infty} \int \mathrm{e}^{\mathrm{i}tx} F_n(\mathrm{d}x, \omega) = \int \mathrm{e}^{\mathrm{i}tx} dF(x), \qquad \forall t \in \mathbb{Q}, \forall \omega \in A.$$

Hence $F = G$, and it follows that $\{F_n(\cdot, \omega)\}_n$ has only one limit point, whence $F_n(\cdot, \omega) \xrightarrow{\mathrm{w}} F$, for all $\omega \in A \cap C$, along the subsequence. □

Lemma 5.2. *Let $(\Omega, \mathcal{U}, \mathbb{P})$ be a probability space. For each $\omega \in \Omega$, suppose that $\varphi_n(\cdot, \omega)$ $(n = 1, 2, \ldots)$ and φ are characteristic functions such that for each $t \in \mathbb{R}$, $\varphi_n(t, \cdot) \xrightarrow{\text{a.s.}} \varphi(t)$, as $n \to \infty$. Then, as $n \to \infty$,*

$$\sup_{t\in K} |\varphi_n(t, \cdot) - \varphi(t)| \xrightarrow{\text{a.s.}} 0, \qquad \textit{for every compact set } K \subseteq \mathbb{R}.$$

Proof. It suffices to show that there exists an $A \in \mathcal{U}$ with $\mathbb{P}(A) = 1$ such that $F_n(\cdot, \omega) \xrightarrow{\mathrm{w}} F$, for all $\omega \in A$.

With a_n and a as in the proof of the previous lemma,

$$\mathbb{E}\sup_{m\geqslant n}|a_m(\delta,\cdot)-a(\delta)|\leqslant\frac{1}{2\delta}\int_{-\delta}^{\delta}\mathbb{E}\left(\sup_{m\geqslant n}\Big||1-\varphi_m(t,\cdot)|-|1-\varphi(t)|\Big|\right)\mathrm{d}t\to 0,\qquad n\to\infty.$$

This implies that $|a_n(\delta,\cdot)-a(\delta)|\xrightarrow{\text{a.s.}}0$ ($n\to\infty$), for all $\delta>0$. Combined with (5.2), we see that

$$\limsup_{n\to\infty}\int_{\{|x|>2/\delta\}}F_n(\mathrm{d}x,\omega)\leqslant a(\delta),\qquad\forall\delta\in\mathbb{Q},\,\omega\in A_1,$$

for some set $A_1\in\mathcal{U}$ with $\mathbb{P}(A_1)=1$. Thus, for $\omega\in A_1$ the whole sequence $\{F_n(\cdot,\omega)\}$ is tight.

Let $A_2\in\mathcal{U}$ be a set of probability one such that $\varphi_n(t,\omega)\to\varphi(t)$, for all $t\in\mathbb{Q}$ and for all $\omega\in A_2$. Let $\omega\in A_2$. Then (as at the end of the proof of Lemma 5.1), $F_n(\cdot,\omega)$ has only F as a limit point.

Hence, for all $\omega\in A:=A_1\cap A_2$, $F_n(\cdot,\omega)\xrightarrow{\text{w}}F$. □

Remark 5.1. If φ_n is the empirical characteristic function of independent random variables with common distribution F, there is a large literature on results as in Lemma 5.2. We mention the final result of Csörgő and Totik (1983): if $\lim_{n\to\infty}(\log T_n)/n=0$, then $\sup_{|t|\leqslant T_n}|\varphi_n(t)-\varphi(t)|\xrightarrow{\text{a.s.}}0$, as $n\to\infty$, where $\varphi(t)=\int e^{itx}F(\mathrm{d}x)$, $t\in\mathbb{R}$. The rate $T_n=\exp(o(n))$ is the best possible in general for almost sure convergence.

We now come to the consistency result which will be applied in Section 7.

Theorem 5.3. *Assume that the sequence of preliminary estimators* $\tilde\psi_n$ *satisfies* (4.1). *If* $k_0\in K_R$ *for some* $R>0$*, then the cumulant M-estimator is consistent. That is,*

$$\|\hat g_n-g_0\|_w\to 0\ \textit{almost surely},\qquad \textit{as }n\to\infty,\tag{5.3}$$

$$\|\hat k_n-k_0\|_\mu\to 0\ \textit{almost surely},\qquad \textit{as }n\to\infty.\tag{5.4}$$

The same results hold in probability if we only assume (4.2).

Proof. We first prove the statement in the case where $\tilde\psi_n$ converges almost surely to ψ_0. By Lemma 5.2, $\sup_{t\in S_w}|\tilde\psi_n(t,\cdot)-\psi_0(t)|\xrightarrow{\text{a.s.}}0$. Let $A\subseteq\Omega$ be the set of probability one on which the convergence occurs. Fix $\omega\in A$. Since ψ_0 has no zeros, there exists an $\varepsilon>0$ such that $\inf_{t\in S_w}|\psi_0(t)|>2\varepsilon$. For this ε there exists an $N=N(\varepsilon,\omega)\in\mathbb{N}$ such that $\sup_{t\in S_w}|\tilde\psi_n(t,\omega)-\psi_0(t)|<\varepsilon$ for all $n\geqslant N$. Hence, for all $n\geqslant N$ and for all $t\in S_w$, $|\tilde\psi_n(t,\omega)|\geqslant|\psi_0(t)|-|\tilde\psi(t,\omega)-\psi_0(t)|\geqslant\varepsilon>0$.

For $n\geqslant N$ we can define $\tilde g_n(\omega)=T\tilde\psi_n(\omega)$ on S_w. Theorem 7.6.3 in Chung (2001) implies that the uniform convergence of $\tilde\psi_n(\omega)$ to ψ_0 on S_w carries over to uniform convergence of $\tilde g_n(\omega)$ to g_0 on S_w. By dominated convergence, $\lim_{n\to\infty}\|\tilde g_n(\omega)-g_0\|_w=0$.

Since $\hat g_n(\cdot,\omega)$ minimizes Γ_n over G_R, we have

$$\|\hat g_n(\cdot,\omega)-g_0\|_w\leqslant\|\hat g_n(\cdot,\omega)-\tilde g_n(\cdot,\omega)\|_w+\|g_0-\tilde g_n(\cdot,\omega)\|_w\leqslant 2\|g_0-\tilde g_n(\cdot,\omega)\|_w\to 0,$$

as n tends to infinity. By Corollary 4.4 this implies

$$\|\hat{k}_n(\cdot, \omega) - k_0\|_\mu = \|L^{-1}(\hat{g}_n(\cdot, \omega)) - L^{-1}(g_0)\|_\mu \to 0, \qquad n \to \infty.$$

Hence, for all ω in a set with probability one, $\lim_{n\to\infty}\|\hat{k}_n(\cdot, \omega) - k_0\|_\mu \to 0$.

Next, we prove the corresponding statement for convergence in probability. By Lemma 5.1, $Y_n := \sup_{t\in S_w}|\tilde{\psi}_n(t, \cdot) - \psi_0(t)| \xrightarrow{\text{p}} 0$, as $n \to \infty$.

The following characterization of convergence in probability holds: $Y_n \xrightarrow{\text{p}} Y$ if and only if each subsequence of (Y_n) possesses a further subsequence that converges almost surely to Y.

Let (n_k) be an arbitrary increasing sequence of natural numbers. Then $Y_{n_k} \xrightarrow{\text{p}} 0$. Then there exists a subsequence (n_m) of (n_k) such that $Y_{n_m} \xrightarrow{\text{a.s.}} 0$. Now we can apply the statement of the theorem for almost sure convergence; this gives $\|\hat{k}_{n_m} - k_0\|_\mu \xrightarrow{\text{a.s.}} 0$. This in turn shows that $\|\hat{k}_n - k_0\|_\mu \xrightarrow{\text{p}} 0$. □

Corollary 5.4. *Assume* (4.1) *and* $k_0 \in k_r$ *for some* $R > 0$. *Denote the distribution function corresponding to* $\hat{\psi}_n(\cdot, \omega)$ *by* $\hat{F}_n(\cdot, \omega)$. *Then, for all* ω *in a set of probability one,*

$$\|\hat{F}_n(\cdot, \omega) - F_0(\cdot)\|_\infty \to 0, \qquad n \to \infty.$$

Here $\|\cdot\|_\infty$ denotes the supremum norm. If we only assume (4.2), then

$$\|\hat{F}_n - F_0\|_\infty \xrightarrow{\text{p}} 0, \qquad n \to \infty.$$

Proof. First assume (4.1). Theorem 5.3 implies $\|\hat{k}_n - k_0\|_\mu \xrightarrow{\text{a.s.}} 0$, as $n \to \infty$. Fix an arbitrary ω of the set on which the convergence takes place. From the proof of Lemma 4.1, we obtain that $\hat{g}_n(\cdot, \omega)$ converges uniformly on compacta to g_0. Then $\hat{\psi}_n(\cdot, \omega)$ also converges uniformly on compacta to ψ_0. By the continuity theorem (Chung 2001, Section 6.3), $\hat{F}_n(\cdot, \omega) \xrightarrow{\text{w}} F_0(\cdot)$. Since F_0 is continuous, this is equivalent to $\|\hat{F}_n(\cdot, \omega) - F_0(\cdot)\|_\infty \to 0$, as $n \to \infty$.

The statement for convergence in probability follows by arguing along subsequences, as in the proof of Theorem 5.3. □

Theorem 5.3 involves only functional analytic properties of various operators and sets. To fulfil the probabilistic assumption that the sequence of preliminary estimators satisfies a law of large numbers, we can use the β-mixing result from Section 3.

6. Computing the cumulant M-estimator

For numerical purposes we will approximate the convex cone K by a finite-dimensional subset. For $N \geqslant 1$, let $0 < \theta_1 < \theta_2 < \ldots < \theta_N$ be a fixed set of positive numbers and set $\Theta = \{\theta_1, \ldots, \theta_N\}$. For example, we can take an equidistant grid with grid points $\theta_j = jh$ $(1 \leqslant j \leqslant N)$, where h is the mesh width. Define 'basis functions' by

$$u_\theta(x) := \mathbf{1}_{[0,\theta)}(x), \qquad x \geqslant 0,$$

$$z_\theta(t) = [Lu_\theta](t) = \int_0^{\theta t} \frac{e^{iu} - 1}{u} du, \qquad t \in \mathbb{R},$$

and set $\mathcal{U}_\Theta := \{u_\theta, \theta \in \Theta\}$. Let K_Θ be the convex cone generated by $\mathcal{U}_\Theta$,

$$K_\Theta := \left\{ k \in K | k = \sum_{i=1}^N \alpha_i u_{\theta_i}, \alpha_i \in [0, \infty), 1 \leq i \leq N \right\}.$$

Define a *sieved estimator* by

$$\check{k}_n = \operatorname*{argmin}_{k \in K_\Theta} \Gamma_n L(k) = \operatorname*{argmin}_{\alpha_1 \geq 0, \ldots, \alpha_N \geq 0} \left\| \sum_{i=1}^N \alpha_i z_{\theta_i} - \tilde{g}_n \right\|_w^2. \tag{6.1}$$

Since the set $\{\mathbf{x} : \mathbf{x} = (x_1, \ldots, x_N), x_i \geq 0 \text{ for all } 1 \leq i \leq N\}$ is a closed convex subset of $\mathbb{R}^N$ and $\Gamma_n L$ is a continuous mapping, we have:

Theorem 6.1. *The sieved estimator $\check{k}_n$ is uniquely defined.*

Note that in this case we do not need conditions in terms of envelope functions, as in Section 4.

Next, we study the problem of computing $\check{k}_n$ numerically. Since each $k \in K_\Theta$ is a finite positive mixture of basis functions $u_\theta \in \mathcal{U}_\Theta$, our minimization problem fits precisely in the set-up of Groeneboom *et al*. (2003). We will follow the approach adopted there to solve (6.1).

6.1. The support-reduction algorithm

Define the directional derivative of $\Gamma_n L$ at $k_1 \in K$ in the direction of $k_2 \in K$ by

$$D_{\Gamma_n L}(k_2; k_1) := \lim_{\varepsilon \to 0} \varepsilon^{-1}([\Gamma_n L](k_1 + \varepsilon k_2) - [\Gamma_n L](k_1)).$$

This quantity exists (it may be infinite), since $\Gamma_n L$ is a convex functional on K (Γ_n, as an L^2-distance on a Hilbert space, is a strictly convex functional on G, and L satisfies $L(k_1 + k_2) = L(k_1) + L(k_2)$).

Groeneboom *et al*. (2003) show that under conditions that are satisfied here the following characterization of $\check{k}_n$ holds. Write $\check{k}_n = \sum_{j \in J} \alpha_j u_{\theta_j}$, where $J := \{j \in \{1, \ldots, N\} | \alpha_j > 0\}$. Then

$$\check{k}_n \text{ minimizes } \Gamma_n L \text{ over } K_\Theta \iff D_{\Gamma_n L}(u_{\theta_j}; \check{k}_n) \begin{cases} \geq 0 & \forall j \in \{1, \ldots, N\}, \\ = 0 & \forall j \in J. \end{cases} \tag{6.2}$$

This result forms the basis for the *support-reduction algorithm*, which is an iterative algorithm for solving (6.1). We discuss this algorithm briefly. For additional details we refer to Section 3 of Groeneboom *et al*. (2003).

Suppose at each iteration we are given a 'current iterate' $k^J \in K_\Theta$ which can be written as

$$k^J = \sum_{j \in J} \alpha_j u_{\theta_j}$$

(J refers to the index set of positive α-weights). Relation (6.2) gives a criterion for checking whether k^J is optimal. As we will shortly see, each iterate k^J will satisfy the equality part of (6.2): $D_{\Gamma_n L}(u_{\theta_j};\, k^J) = 0$, for all $j \in J$. This fact, together with (6.2), implies that if k^J is not optimal, then there is an $i \in \{1, \ldots, N\} \backslash J$ with $D_{\Gamma_n L}(u_{\theta_i},\, k^J) < 0$. Thus u_{θ_i} provides a direction of descent for $\Gamma_n L$. In that case the algorithm prescribes two steps that have to be carried out:

Step 1. Determine a direction of descent for $\Gamma_n L$. Let

$$\Theta_< := \{\theta \in \Theta : D_{\Gamma_n L}(u_\theta,\, k^J) < 0\};$$

then $\Theta_<$ is non-empty. From $\Theta_<$ we choose a direction of descent. Suppose θ_{j^*} is this direction. (A particular choice is the direction of steepest descent, in which case $\theta_{j^*} := \operatorname{argmin}_{\theta \in \Theta_<} D_{\Gamma_n L}(u_\theta,\, k^J)$. This boils down to finding a minimum element in a vector of length at most N. We give an alternative choice below.)

Step 2. Let the new iterate be given by

$$k^{J^*} = \sum_{j \in J^*} \beta_j u_{\theta_j}, \qquad J^* := J \cup \{j^*\},$$

where $\{\beta_j,\, j \in J^*\}$ are (as yet unknown) weights. We first minimize $\Gamma_n L(k^{J^*})$ with respect to $\{\beta_j,\, j \in J^*\}$, without positivity constraints. In our situation this is a (usually low-dimensional) quadratic unconstrained optimization problem.

If $\min\{\beta_j,\, j \in J^*\} \geqslant 0$, then $k^{J^*} \in K_\Theta$ and k^{J^*} satisfies the equality part of (6.2). In that case, we check the inequality part of (6.2) and possibly return to step 1. Otherwise, we perform a support-reduction step. Since it can be shown that β_{j^*} is always positive, we can make a move from k^J towards k^{J^*} and stay within the cone K_Θ initially. As a next iterate, we take $k := k^J + \hat{c}(k^{J^*} - k^J)$, where

$$\begin{aligned}
\hat{c} &= \max\{c \in [0, 1] : k^J + c(k^{J^*} - k^J) \in K_\Theta\} \\
&= \max\left\{ c \in [0, 1] : \sum_{j \in J} \left[c\beta_j + (1 - c)\alpha_j \right] u_{\theta_j} + c\beta_{j^*} u_{\theta_{j^*}} \in K_\Theta \right\} \\
&= \max\{c \in [0, 1] : c\beta_j + (1 - c)\alpha_j \geqslant 0\,,\ \text{for all } \beta_j (j \in J) \text{ with } \beta_j < 0\} \\
&= \min\{\alpha_j/(\alpha_j - \beta_j),\ j \in J,\ \text{for which } \beta_j < 0\}. \qquad (6.3)
\end{aligned}$$

Then $k \in K_\Theta$. Let j^{**} be the index for which the minimum in (6.3) is attained, that is, for which $\hat{c}\beta_{j^{**}} + (1 - \hat{c})\alpha_{j^{**}} = 0$. Define $J^{**} := J^* \backslash \{j^{**}\}$. Then k is supported on $\{\theta_j,\, j \in J^{**}\}$. That is, in the new iterate, the support point $\theta_{j^{**}}$ is removed. Next, set $k^{J^{**}} = \sum_{j \in J^{**}} \gamma_j u_{\theta_j}$ and compute optimal weights γ_j. If all weights γ_j are non-negative, the equality part of (6.2) is satisfied and we can check the inequality part of (6.2) and possibly return to step 1. Otherwise, a new support-reduction step can be carried out, since all weights of k are positive. In the end, our iterate k will satisfy the equality part of (6.2).

To start the algorithm, we fix a starting value $\theta^{(0)} \in \Theta$. Then we determine the function $cu_{\theta^{(0)}}$ minimizing $\Gamma_n L$ as a function of $c > 0$. Once the algorithm has been initialized it starts iteratively adding and removing support points, while in between computing optimal weights.

Theorem 3.1 in Groeneboom *et al*. (2003) gives conditions to guarantee that the sequence of iterates $\{k^{(i)}\}_i$ (generated by the support-reduction algorithm) indeed converges to the solution of our minimization problem. Since these conditions are met in our case, we have

$$(\Gamma_n L)(k^{(i)}) \downarrow (\Gamma_n L)(\check{k}_n), \qquad \text{as } i \to \infty.$$

6.2. Implementation details

We now work out the actual computations involved when implementing the algorithm. Suppose $k = \sum_{j=1}^m \alpha_j u_{\theta_j}$.

Step 1. Given the 'current iterate' k, we aim to add a function u_θ which provides a direction of descent for $\Gamma_n L$. By linearity of L,

$$\begin{aligned}[\Gamma_n L](k + \varepsilon u_\theta) - [\Gamma_n L](k) &= \|L(k + u_\theta) - \tilde{g}_n\|_w^2 - \|Lk - \tilde{g}_n\|_w^2 \\ &= \varepsilon c_1(\theta, k) + \frac{1}{2}\varepsilon^2 c_2(\theta), \end{aligned} \tag{6.4}$$

where $c_2(\theta) = 2\|Lu_\theta\|_w^2 = 2\|z_\theta\|_w^2 > 0$ and

$$c_1(\theta, k) = 2\langle Lk - \tilde{g}_n, Lu_\theta\rangle_w = 2\left\langle \sum_{j=1}^m \alpha_j z_{\theta_j} - \tilde{g}_n, z_\theta \right\rangle_w.$$

In order to find a direction of descent, we can pick any $\theta \in \Theta$ for which $c_1(\theta, k) < 0$. However, since the right-hand side of (6.4) is quadratic in ε, it can be minimized explicitly (and we choose to do so). If $c_1(\theta, k) < 0$, then

$$\operatorname*{argmin}_{\varepsilon > 0} (\varepsilon c_1(\theta, k) + \frac{1}{2}\varepsilon^2 c_2(\theta)) = -\frac{c_1(\theta, k)}{c_2(\theta)} =: \hat{\varepsilon}_\theta.$$

Minimizing $[\Gamma_n L](k + \hat{\varepsilon}_\theta u_\theta)$ over all points $\theta \in \Theta$ with $c_1(\theta, k) < 0$ gives

$$\hat{\theta} = \operatorname*{argmin}_{\{\theta \in \Theta : c_1(\theta, k) < 0\}} -\frac{c_1(\theta, k)^2}{2c_2(\theta)} = \operatorname*{argmin}_{\theta \in \Theta} \frac{c_1(\theta, k)}{\sqrt{c_2(\theta)}}.$$

Step 2. Given a set of support points, we compute optimal weights. This is a standard least-squares problem, that is solved by the normal equations. In our set-up, these are obtained by differentiating $[\Gamma_n L](k)$ with respect to α_j ($j \in \{1, \ldots, m\}$) and setting the partial derivatives equal to zero. This gives the system $A\boldsymbol{\alpha} = \mathbf{b}$, where

$$A_{i,j} = \langle z_{\theta_i}, z_{\theta_j}\rangle_w, \qquad i, j = 1, \ldots, m, \tag{6.5}$$

and

$$b_i = \langle z_{\theta_i}, \tilde{g}_n\rangle_w, \qquad i = 1, \ldots, m.$$

The matrix A is easily seen to be symmetric. By the next lemma, A is non-singular, whence the system $A\boldsymbol{\alpha} = \mathbf{b}$ has a unique solution.

Lemma 6.2. *The matrix A, as defined in* (6.5)*, is non-singular.*

Proof. Denote by $\mathbf{a}_j$ the jth column of A. Let $h_1, \ldots, h_m \in \mathbb{R}$. We aim to show that if $\sum_{i=1}^m h_i \mathbf{a}_i = \mathbf{0}$, then all h_j are zero. Now $\sum_{i=1}^m h_i \mathbf{a}_i = (\langle z_{\theta_1}, \varphi\rangle_w, \ldots, \langle z_{\theta_m}, \varphi\rangle_w)^{\mathrm{T}}$, where $\varphi \in L^2(w)$ is given by $\varphi := \sum_{i=1}^m h_j z_{\theta_j}$. Thus if $\sum_{i=1}^m h_i \mathbf{a}_i = \mathbf{0}$, then $\varphi \perp \operatorname{span}(z_{\theta_1}, \ldots, z_{\theta_m})$ in $L^2(w)$. Since $\varphi \in \operatorname{span}(z_{\theta_1}, \ldots, z_{\theta_m})$, we must have $\varphi = 0$ almost everywhere with respect to Lebesgue measure on S_w. By continuity of $t \mapsto z_\theta(t)$, $\varphi = 0$ on S_w.

Now $\varphi = \sum_{i=1}^m h_i z_{\theta_i} = L(\sum_{i=1}^m h_i u_{\theta_i}) = 0$. If, for $k \in K$, $L(k) = 0$, then $k \equiv 0$. Therefore, $\sum_{i=1}^m h_i u_{\theta_i} \equiv 0$, which can only be true if all h_i are equal to zero. □

Tucker (1967, Section 4.3) gives an explicit way to calculate the imaginary part of $\tilde{g}_n$. An estimator $\check{f}_n$ of the density function can be obtained by inverting the characteristic function $Q(\check{k}_n)$. For the density plots in Figures 2 and 3, we used the method of Schorr (1975).

7. Applications and examples

We consider two observation schemes for (X_t):

1. Observe (X_t) on a regularly spaced grid with fixed mesh width Δ. Write $X_{k\Delta}$ for the observations $(k = 0, 1, \ldots)$.
2. As observation scheme 1, but now suppose that the mesh width Δ_n decreases as n increases. This gives, for each n, observations $(X_0, X_{\Delta_n}, X_{2\Delta_n}, \ldots)$.

7.1. Data from the OU process: observation scheme 1

Suppose $X_0, X_\Delta, \ldots, X_{(n-1)\Delta}$ are n observations from the stationary OU process (X_t). Let F_0 denote the marginal distribution of $X_{i\Delta}$ $(0 \leqslant i \leqslant n-1)$. By Theorem 2.2, F_0 is positive, self-decomposable and characterized by a function k_0 in K. As a preliminary estimator for $\psi_0 = Q(k_0)$ we propose the empirical characteristic function defined by

$$\tilde{\psi}_n(t) := \int \mathrm{e}^{\mathrm{i}tx} \mathrm{d}\mathbb{F}_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} \mathrm{e}^{\mathrm{i}tX_{j\Delta}}, \qquad t \in \mathbb{R}.$$

Here $\mathbb{F}_n$ denotes the empirical distribution function of $X_0, \ldots, X_{(n-1)\Delta}$. By Theorem 3.1 the process (X_t) is β-mixing. This implies that (X_n) is β-mixing. This in turn implies that (X_n) is *ergodic* (Genon-Catalot *et al.* 2000). An application of Birkhoff's ergodic theorem (Krengel 1985, pp. 9–10) gives, for $t \in \mathbb{R}$,

$$\tilde{\psi}_n(t) \xrightarrow{\text{a.s.}} \int \mathrm{e}^{\mathrm{i}tx} \mathrm{d}F_0(x) = \psi_0(t),$$

as n tends to infinity. Consistency of $\hat{k}_n$ now follows directly upon an application of Theorem 5.3.

So far, the weight function has been fixed in advance of the estimation procedure. The choice of this function has been more or less arbitrary. Roughly, the larger the number of observations, the larger one would like to take S_w. This suggests a data-adaptive choice for w (or at least for its support). Numerical experiments indicate that such a choice can improve the numerical results obtained so far. Therefore, we have implemented a bootstrap procedure to determine the right-hand end-point of S_w, which we denote by t_n^*. This procedure runs as follows. Take a large number (say, N) of samples of n observations from the empirical distribution function $\hat{F}_n$. For the ith set of observations, let $g_n^{(i)}$ denote its corresponding empirical cumulant function. Then approximate $E|\tilde{g}_n(t) - g_0(t)|$ by the average $U_n(t) := N^{-1}\sum_{i=1}^N |g_n^{(i)}(t) - \tilde{g}_n(t)|$ for each t in a (large) interval $[0, M]$. Finally, take some threshold $\eta > 0$ and define $t_n^* := \inf\{t \geqslant 0 : U_n(t) > \eta\}$.

Next, we apply the support-reduction algorithm of Section 6 with $w(\cdot) = \mathbf{1}_{[-t_n^*, t_n^*]}(\cdot)$ ($\eta = 0.1$). Figure 2 shows some simulation results in the case where π_0 is Gamma(3, 2). We simulated the OU process on the interval [0, 1000] and took observations at time instants 0, 1, ... (i.e. we observe the 1-skeleton).

Although the estimate for k_0 is quite inaccurate, the estimate for the density function shows a much better fit. The density plot is obtained by inversion of the characteristic function, corresponding to the estimated canonical function. See the final remark of Section 6.

7.2. Data from the OU process: observation scheme 2

For each $n \geqslant 1$, denote the observations by $X_0, X_{\Delta_n}, X_{2\Delta_n}, \ldots, X_{(n-1)\Delta_n}$. We will now show that the empirical characteristic function based on these observations converges pointwise in probability to ψ_0.

Define for each fixed $u \in \mathbb{R}$ the continuous process $(Y_t^u,\ t \geqslant 0)$ by

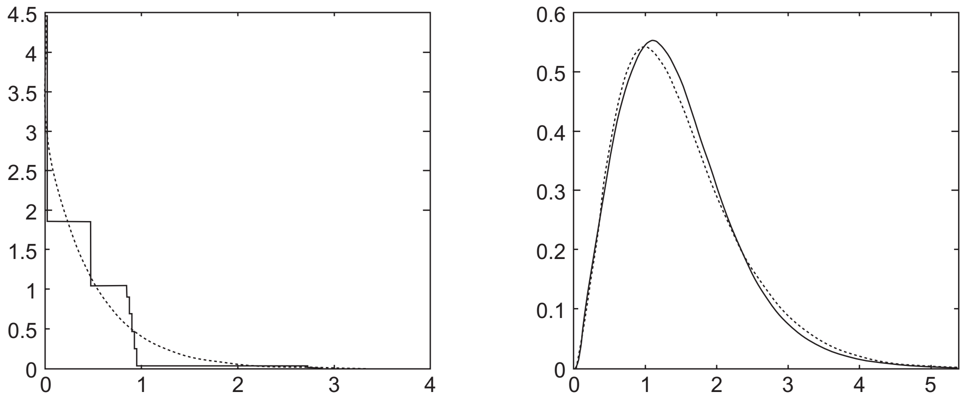


Figure 2. Gamma(3,2) distribution, $n = 1000$. Left: estimated (solid) and true (dotted) canonical function. Right: estimated (solid) and true (dotted) density function.

$$Y_t^u = \mathrm{e}^{\mathrm{i}uX_t} - E^{\mathrm{i}uX_t}.$$

Denote by $(Y^u_{k\Delta_n})_k$ the discretely sampled process, obtained from $(Y_t^u,\ t \geqslant 0)$ by observation scheme 2. Thus,

$$Y^u_{k\Delta_n} = \mathrm{e}^{\mathrm{i}uX_{k\Delta_n}} - E^{\mathrm{i}uX_{k\Delta_n}}, \qquad k = 0, \ldots, n-1.$$

For a certain stochastic process $(U_s,\ s \geqslant 0)$ the α-mixing 'numbers' are defined by

$$\alpha_U(h) = 2\sup_t \sup_{\substack{A\in\sigma(U_s, s\leqslant t)\\ B\in\sigma(U_s, s\geqslant t+h)}} |P(A\cap B) - P(A)P(B)|, \qquad h > 0.$$

The process $(U_s,\ s \geqslant 0)$ is called α-mixing if $\alpha_U(h) \to 0$ as $h \to \infty$. As shown in Genon-Catalot *et al*. (2000), β-mixing is a stronger property than α-mixing. In fact, for any process $(U_s,\ s \geqslant 0)$ we have $\alpha_U(t) \leqslant \beta_U(t)$ $(t > 0)$.

Lemma 7.1. *Suppose that* $\Delta_n \to 0$ *and* $n\Delta_n \to \infty$, *as* $n \to \infty$. *Then, for* $u \in \mathbb{R}$,

$$\frac{1}{n}\sum_{k=0}^{n-1} Y^u_{k\Delta_n} \xrightarrow{p} 0, \qquad n \to \infty.$$

Proof. Let $u \in \mathbb{R}$ arbitrary. Denote the α-mixing numbers of $(|Y_t^u|,\ t \geqslant 0)$ by $\alpha_{|Y^u|}$, and similarly for $(|Y^u_{k\Delta_n}|)_k$ by $\alpha_{|Y^{u,n}|}$. Since $\sigma(|Y_t^u|,\ t \in T) \subseteq \sigma(X_t,\ t \in T)$ for any interval $T \subseteq [0, \infty)$, the definition of the α-mixing numbers implies that for any $h > 0$, $\alpha_{|Y^u|}(h) \leqslant \alpha_X(h)$. In the same way one can verify that for $j \in \mathbb{N}$, $\alpha_{|Y^{u,n}|}(j) \leqslant \alpha_{|Y^u|}(j\Delta_n)$. Combining these inequalities gives, for $j \in \mathbb{N}$,

$$\alpha_{|Y^{u,n}|}(j) \leqslant \alpha_{|Y^u|}(j\Delta_n) \leqslant \alpha_X(j\Delta_n) \leqslant \beta_X(j\Delta_n), \tag{7.1}$$

where the last inequality follows from the remark just before the lemma.

Lemma A.2 implies that the following inequality holds: for each $h \in \mathbb{N}$,

$$P\left(\frac{1}{n}\sum_{k=0}^{n-1} |Y^u_{k\Delta_n}| \geqslant 2\varepsilon\right) \leqslant \frac{2h}{n\varepsilon^2}\int_0^1 Q^2(1-w)\mathrm{d}w + \frac{2}{\varepsilon}\int_0^{\alpha_{|Y^{u,n}|}(h)} Q(1-w)\mathrm{d}w, \tag{7.2}$$

where $Q = F^{-1}_{|Y_1|}$. Since $P(|Y_1| \leqslant y) = 1$ if $y \geqslant 2$, we have $Q(u) \leqslant 2$ for all $u \in (0, 1)$. Hence, for all $\varepsilon > 0$,

$$\begin{aligned} P\left(\left|\frac{1}{n}\sum_{k=0}^{n-1} Y^u_{k\Delta_n}\right| \geqslant 2\varepsilon\right) &\leqslant P\left(\frac{1}{n}\sum_{k=0}^{n-1} |Y^u_{k\Delta_n}| \geqslant 2\varepsilon\right) \\ &\leqslant \frac{8h}{n\varepsilon^2} + \frac{4}{\varepsilon}\alpha_{|Y^{u,n}|}(h) \leqslant \frac{8h}{n\varepsilon^2} + \frac{4}{\varepsilon}\beta_X(h\Delta_n), \end{aligned} \tag{7.3}$$

where the last inequality follows from (7.1).

Take $h = h_n = \sqrt{n/\Delta_n}$, then $h_n/n \to 0$ and $h_n\Delta_n \to \infty$ $(n \to \infty)$. Hence both terms in (7.3) can be made arbitrarily small by letting $n \to \infty$. □

If we define $\tilde{\psi}_n(u) = n^{-1}\sum_{k=0}^{n-1} e^{iuX_{k\Delta_n}}$, then the above lemma shows that $\tilde{\psi}_n(u) \xrightarrow{p} \psi_0(u)$ for each $u \in \mathbb{R}$. An application of Theorem 5.3 gives $\|\hat{k}_n - k_0\|_\mu \xrightarrow{p} 0$ as $n \to \infty$, proving consistency.

7.3. Estimating a positive self-decomposable density from i.i.d. data

We momentarily digress to consider the problem of estimating a positive self-decomposable density from i.i.d. data. Let $X_1, \ldots, X_n$ be independent random variables with common distribution function F_0. As before, F_0 is characterized by k_0 in K. As a preliminary estimator for ψ_0 we take again the empirical characteristic function. Since $\mathbb{F}_n$ converges weakly to F_0 almost surely, it follows that $\tilde{\psi}_n$ converges pointwise almost surely to ψ_0, as n tends to infinity. Consistency of $\hat{k}_n$ now follows from Theorem 5.3.

Let f_0 denote the density of F_0. We remark that a general closed-form expression for the density function f_0 in terms of k_0 is not known, This hampers the use of maximum likelihood techniques for estimating a self-decomposable density based on i.i.d. data. However, given $\hat{k}_n$, we can calculate $\hat{\psi}_n = Q(\hat{k}_n)$, and then numerically invert this function to obtain a nonparametric estimator $\hat{F}_n$ for F_0. In contrast to the empirical distribution function, our estimator $\hat{F}_n$ is guaranteed to be of the correct type (i.e. self-decomposable). Figure 3 shows plots for the canonical function and the density function, in the case where π_0 follows an IG(2, 1) distribution.

Alternative preliminary estimators are also possible. For example, suppose we know, in addition to the assumptions already made, that the density of F_0 is decreasing. Then we can take as a preliminary estimator the Grenander estimator $F_{n,\mathrm{Gren}}$, which is defined as the least concave majorant of the empirical distribution function $\mathbb{F}_n$. Using similar arguments to the foregoing, we can show that the estimator for k based on $F_{n,\mathrm{Gren}}$ is consistent. As another example, one could also take the maximum likelihood estimator for a unimodal

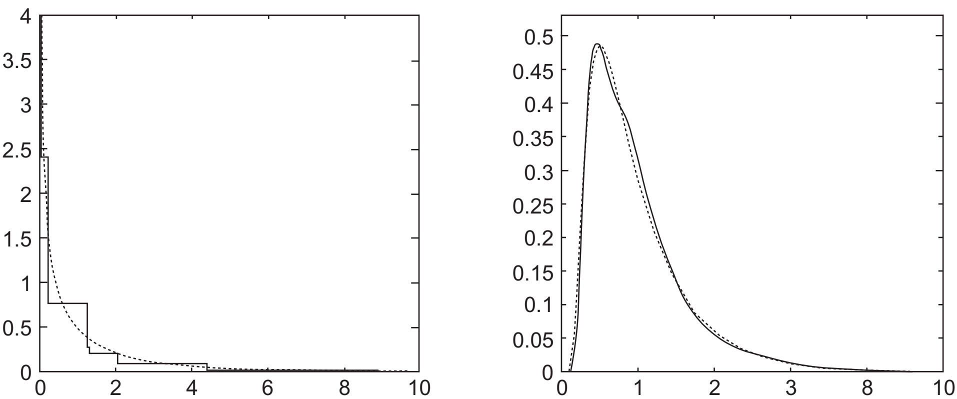


Figure 3. IG(2,1) distribution, $n = 1000$. Left: estimated (solid) and true (dotted) canonical function. Right: estimated (solid) and true (dotted) density function.

density as a preliminary estimator for f_0. This makes sense, since every self-decomposable density is unimodal (Sato 1999, Theorem 53.1).

8. Estimation of the intensity parameter λ

Suppose $X_0, X_\Delta, \ldots, X_{n\Delta}$ are discrete-time observations from the stationary OU process according to observation scheme 1, for some fixed $\Delta > 0$. In this section, we define an estimator for λ. For ease of notation we write $X_i = X_{i\Delta}$ $(i = 0, \ldots, n)$. From the proof of Proposition 3.3 we know that for each $n \geqslant 1$, $X_n = \mathrm{e}^{-\lambda} X_{n-1} + W_n(\lambda)$. Here $\{W_n(\lambda)\}_{n \geqslant 1}$ is a sequence of independent random variables with infinitely divisible distribution function Γ_λ. Since $(X_n,\ n \geqslant 0)$ is stationary, $X_0 \sim \pi_0$, where π_0 has Lévy density $x \mapsto \rho(x, \infty)/x$.

Let $\theta = \mathrm{e}^{-\lambda}$ and denote the true parameter by θ_0. Since $W_n(\lambda) \geqslant 0$ for each $n \geqslant 1$, we easily obtain the bound $\theta_0 \leqslant \min_{n \geqslant 1}(X_n / X_{n-1})$. Define the estimator

$$\hat{\theta}_n = \min_{1 \leqslant k \leqslant n} \frac{X_k}{X_{k-1}}.$$

Then $\hat{\theta}_n(\omega) \geqslant \theta_0$, for each ω. Hence $\hat{\theta}_n$ is always biased upwards. However, we have:

Lemma 8.1. *The estimator $\hat{\theta}_n$ is consistent: $\hat{\theta}_n \xrightarrow{\mathrm{p}} \theta_0$, as n tends to infinity.*

Proof. Let $\varepsilon > 0$. Since

$$\begin{aligned}
\{\hat{\theta}_n \leqslant \theta_0 + \varepsilon\} &= \left\{\exists\, k \in \{1, \ldots, n\} \text{ such that } \frac{X_k}{X_{k-1}} \leqslant \theta_0 + \varepsilon\right\} \\
&= \left\{\exists\, k \in \{1, \ldots, n\} \text{ such that } \frac{\theta_0 X_{k-1} + W_k(\lambda)}{X_{k-1}} \leqslant \theta_0 + \varepsilon\right\} \\
&= \{\exists\, k \in \{1, \ldots, n\} \text{ such that } W_k(\lambda) \leqslant \varepsilon X_{k-1}\} := A_{n,\varepsilon},
\end{aligned}$$

we have $P(|\hat{\theta}_n - \theta_0| \leqslant \varepsilon) = P(A_{n,\varepsilon})$. We aim to show that for each $\varepsilon > 0$, $\lim_{n \to \infty} P(A^c_{n,\varepsilon}) = 0$. Define $N_n := \sum_{k=1}^n \mathbf{1}_{\{X_{k-1} > 1\}}$. Then

$$\begin{aligned}
P(A^c_{n,\varepsilon}) &= P(W_k(\lambda) > \varepsilon X_{k-1}\,,\ \forall\, k \in \{1, \ldots, n\}) \\
&= \sum_{j=0}^n P(W_k(\lambda) > \varepsilon X_{k-1},\ \forall\, k \in \{1, \ldots, n\} | N_n = j) P(N_n = j) \\
&\leqslant \sum_{j=0}^n (P(W_1(\lambda) > \varepsilon))^j\, P(N_n = j),
\end{aligned}$$

where the last inequality holds since $\{W_k(\lambda)\}_{k \geqslant 1}$ is an i.i.d. sequence. Since $W_1(\lambda)$ has support $[0, \infty)$ (Sato 1999, Corollary 24.8), $P(W_1(\lambda) > \varepsilon) := \alpha_\varepsilon \in [0, 1)$. This gives

$$P(A_{n,\varepsilon}^c) \leqslant \sum_{j=0}^{\infty} \alpha_\varepsilon^j P(N_n = j).$$

By dominated convergence, $\lim_{n\to\infty} P(A_{n,\varepsilon}^c) \leqslant \sum_{j=0}^{\infty} \alpha_\varepsilon^j [\lim_{n\to\infty} P(N_n = j)]$. We are done, once we have proved that $\lim_{n\to\infty} P(N_n = j) = 0$.

We claim $N_n \xrightarrow{\text{a.s.}} \infty$, as $n \to \infty$. From Section 3 we know that $\lim_{n\to\infty} \|P^n(x, \cdot) - \pi_0\|_{\text{TV}} = 0$, for all $x \in E$. By Proposition 6.3 in Nummelin (1984), this implies that the chain $(X_n)_n$ is positive Harris recurrent and aperiodic. Now Harris recurrence implies that the set $(1, \infty)$ is visited infinitely many times by $(X_n)_n$, almost surely. Therefore, the claim holds and we conclude $P(A_{n,\varepsilon}^c) \to 0$. □

By the continuous mapping theorem we have:

Corollary 8.2. *Define* $\hat{\lambda}_n = -\log \hat{\theta}_n$. *Then* $\hat{\lambda}_n \xrightarrow{\text{p}} \lambda_0$, *as* $n \to \infty$, *where* λ_0 *denotes the true value of* λ.

If all innovations $W_n(\lambda)$ are exponentially distributed, $\hat{\theta}_n$ equals the maximum likelihood estimator for the model. A detailed asymptotic analysis for this model is given in Nielsen and Shephard (2003).

Appendix

Proof of Proposition 3.3. The solution of the OU equation is given in (2.3). If we discretize the expression for this solution, we obtain

$$X_n = \mathrm{e}^{-\lambda} X_{n-1} + \int_0^1 \mathrm{e}^{\lambda(u-1)} \mathrm{d}Z(\lambda(u + n - 1)), \qquad n \geqslant 1.$$

Since Z has stationary and independent increments, we can write

$$X_n = \mathrm{e}^{-\lambda} X_{n-1} + W_n(\lambda), \tag{A.1}$$

where $(W_n(\lambda))_n$ is an i.i.d. sequence of random variables distributed as W_λ.

Next, we show that the distribution of $(\tilde{X}_t)$, defined by

$$\tilde{X}_t := \int_0^t \mathrm{e}^{-\lambda(t-s)} \mathrm{d}Z(\lambda s), \tag{A.2}$$

is infinitely divisible for each $t \geqslant 0$. Since $W_\lambda \stackrel{d}{=} \tilde{X}_1$ we then obtain infinite divisibility for the noise variables. Note that $(\tilde{X}_t)$ is simply the OU process with initial condition $X(0) = 0$.

Similarly to equation (17.3) in Sato (1999), we have the following relation between the characteristic function of $\tilde{X}$ and $T(\psi_{Z(1)})$ (the cumulant of $Z(1)$):

$$E^{\mathrm{i}z\tilde{X}_t} = \exp\left(\lambda \int_0^t T(\psi_{Z(1)})(\mathrm{e}^{-\lambda(t-u)} z) \mathrm{d}u\right).$$

Since we assume Z has Lévy measure ρ (i.e. $T(\psi_{Z(1)})(u) = \int_0^\infty (\mathrm{e}^{\mathrm{i}ux} - 1)\rho(\mathrm{d}x)$), we have

$$\begin{aligned}\log E^{\mathrm{i}z\tilde{X}_t} &= \lambda\int_0^t\int_0^\infty \left(\mathrm{e}^{\mathrm{i}e^{-\lambda(t-u)}zx} - 1\right)\rho(\mathrm{d}x)\mathrm{d}u \\ &= \lambda\int_0^\infty\int_0^t \left(\mathrm{e}^{\mathrm{i}e^{-\lambda(t-u)}zx} - 1\right)\mathrm{d}u\rho(\mathrm{d}x) = \int_0^\infty\int_{\mathrm{e}^{-\lambda t}x}^x (\mathrm{e}^{\mathrm{i}wz} - 1)w^{-1}\mathrm{d}w\rho(\mathrm{d}x) \\ &= \int_0^\infty (\mathrm{e}^{\mathrm{i}wz} - 1)w^{-1}\mathrm{d}w\int_w^{\mathrm{e}^{\lambda t}w}\rho(\mathrm{d}x) = \int_0^\infty (\mathrm{e}^{\mathrm{i}wz} - 1)\kappa_t(\mathrm{d}w).\end{aligned}$$

Here

$$\kappa_t(B) = \int_B w^{-1}\rho(w, \mathrm{e}^{\lambda t}w]\mathrm{d}w, \qquad B \in \mathcal{E}.$$

Hence if we let $\kappa := \kappa_1$, the Lévy measure has the form as given in (3.4).

It remains to be shown that κ_t satisfies $\int_0^\infty (1 \wedge x)\kappa_t(\mathrm{d}x) < \infty$ for each $t > 0$. This follows from

$$\int_0^1 x\kappa_t(\mathrm{d}x) = \int_0^1 \rho(x, \mathrm{e}^{\lambda t}x]\mathrm{d}x \leqslant \int_0^{\mathrm{e}^{\lambda t}} y\rho(\mathrm{d}y) < \infty$$

and

$$\begin{aligned}\int_1^\infty \kappa_t(\mathrm{d}x) = \kappa_t(1, \infty) &= \int_1^\infty\int_{(1\vee \mathrm{e}^{-\lambda t}y)}^y \frac{1}{w}\mathrm{d}w\rho(\mathrm{d}y) \\ &= \int_1^\infty \log\left(\frac{y}{(1\vee \mathrm{e}^{-\lambda t}y)}\right)\rho(\mathrm{d}y) < \infty \\ &= \int_1^{\mathrm{e}^{\lambda t}} \log y\rho(\mathrm{d}y) + \int_{\mathrm{e}^{\lambda t}}^\infty \lambda t\rho(\mathrm{d}y) < \infty.\end{aligned}$$

□

Lemma A.1. *Under condition* (2.7),

$$I := \int_0^1 \frac{\mathrm{d}z}{z}\exp\left(-\int_z^1 \frac{\lambda_\rho(y)}{y}\mathrm{d}y\right) = +\infty, \tag{A.3}$$

where

$$\lambda_\rho(y) = \int_0^\infty (1 - \mathrm{e}^{-yx})\rho(\mathrm{d}x). \tag{A.4}$$

Proof. Let $y \in (0, 1)$. Since $1 - \mathrm{e}^{-u} \leqslant \min(u, 1)$ for $u > 0$, we obtain

$$\begin{aligned}\lambda_\rho(y) &= \int_0^1 \ldots + \int_1^{1/\sqrt{y}} \ldots + \int_{1/\sqrt{y}}^\infty (1-\mathrm{e}^{-yx})\rho(\mathrm{d}x)\\ &\leqslant y\int_0^1 x\rho(\mathrm{d}x) + \int_1^{1/\sqrt{y}} \frac{y}{\sqrt{y}}\rho(\mathrm{d}x) + \int_{1/\sqrt{y}}^\infty \frac{1-\mathrm{e}^{-yx}}{\log x}\log x\rho(\mathrm{d}x)\\ &\leqslant c_1 y + c_2\sqrt{y} - \frac{2}{\log y}\int_{1/\sqrt{y}}^\infty \log x\rho(\mathrm{d}x),\end{aligned}$$

where $c_1 = \int_0^1 x\rho(\mathrm{d}x)$ and $c_2 = \rho(1, \infty)$.

Choose $\alpha \in (0, 1)$ such that $c_3 := 2\int_{1/\sqrt{\alpha}}^\infty \log x\rho(\mathrm{d}x) < 1$, which is possible by (2.7). Since $y \mapsto \int_{1/\sqrt{y}}^\infty \log x\rho(\mathrm{d}x)$ is increasing on (0, 1), we have

$$\lambda_\rho(y) \leqslant c_1 y + c_2\sqrt{y} - c_3/\log y, \qquad \text{if } y \in (0, \alpha).$$

For $y \in (\alpha, 1)$, we have the simple estimate $\lambda_\rho(y) \leqslant c_1 y + c_2$. If $z \in (0, \alpha)$, then

$$\begin{aligned}\int_z^1 \frac{\lambda_\rho(y)}{y}\mathrm{d}y &= \int_z^\alpha \frac{\lambda_\rho(y)}{y}\mathrm{d}y + \int_\alpha^1 \frac{\lambda_\rho(y)}{y}\mathrm{d}y\\ &\leqslant c_1(\alpha - z) + 2c_2(\sqrt{\alpha} - \sqrt{z}) - c_3\int_z^\alpha \frac{1}{y\log y}\mathrm{d}y + c_1(1-\alpha) - c_2\log\alpha\\ &= K_\alpha - c_1 z - 2c_2\sqrt{z} + c_3\log(-\log z),\end{aligned}$$

where

$$K_\alpha = c_1 + c_2(2\sqrt{\alpha} - \log(\alpha)) - c_3\log(-\log\alpha) \in \mathbb{R}.$$

Using this inequality, we obtain

$$I \geqslant \int_0^\alpha \frac{\mathrm{d}z}{z}\exp\left(-\int_z^1 \frac{\lambda_\rho(y)}{y}\mathrm{d}y\right) \geqslant \mathrm{e}^{-K_\alpha}\int_0^\alpha \mathrm{e}^{c_1 z + 2c_2\sqrt{z}}(-\log z)^{-c_3}\frac{\mathrm{d}z}{z}.$$

The last integral exceeds

$$\int_0^\alpha \frac{1}{z(-\log z)^{c_3}}dz = \int_{-\log\alpha}^\infty \frac{1}{u^{c_3}}\mathrm{d}u = \infty,$$

since α was chosen such that $c_3 < 1$. □

The statement and proof of the following lemma are similar to Theorem 3.2 in Rio (2000).

Lemma A.2. *For any mean-zero time series X_t with α-mixing numbers $\alpha(h)$, every $x > 0$ and every $h, n \in \mathbb{N}$, with $Q_t = F_{|X_t|}^{-1}$,*

$$P(\overline{X}_n \geqslant 2x) \leqslant \frac{2}{nx^2}\int_0^1 \frac{h}{n}\sum_{t=1}^n Q_t^2(1-u)\,\mathrm{d}u + \frac{2}{x}\int_0^{\alpha(h)} \frac{1}{n}\sum_{t=1}^n Q_t(1-u)\,\mathrm{d}u. \quad \text{(A.5)}$$

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References

Barndorff-Nielsen, O.E. and Shephard, N. (2001a) Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics. *J. Roy. Statist. Soc. Ser. B*, **63**, 167–241.

Barndorff-Nielsen, O.E. and Shephard, N. (2001b) Modelling by Lévy processes for financial econometrics. In O.E. Barndorff-Nielsen, T. Mikosch and S. Resnick (eds), *Lévy Processes: Theory and Applications*, pp. 238–318. Boston: Birkhäuser.

Basawa, I.V. and Brockwell, P.J. (1982) Nonparametric estimation for nondecreasing Lévy processes. *J. Roy. Statist. Soc. Ser. B*, **44**, 262–269.

Chung, K.L. (2001) *A Course in Probability Theory* (3rd edition). San Diego, CA: Academic Press.

Çinlar, E. and Pinsky, M. (1971) A stochastic integral in storage theory. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, **17**, 227–240.

Csörgő, S. and Totik, V. (1983) On how long interval is the empirical characteristic function uniformly consistent? *Acta Sci. Math. (Szeged)*, **45**, 141–149.

Davydov, Yu.A. (1973) Mixing conditions for Markov chains. *Theory Probab. Appl.*, **18**, 312–328.

Feuerverger, A. and McDunnough, P. (1981) On the efficiency of empirical characteristic function procedures. *J. Roy. Statist. Soc. Ser. B*, **43**, 20–27.

Genon-Catalot, V., Jeantheau, T. and Laredo, C. (2000) Stochastic volatility models as hidden Markov models and statistical applications. *Bernoulli*, **6**, 1051–1079.

Getoor, R.K. (1975) *Markov Processes: Ray Processes and Right Processes*, Lecture Notes in Math. 440. Berlin: Springer-Verlag.

Groeneboom, P., Jongbloed, G. and Wellner, J.A. (2003) The support reduction algorithm for computing nonparametric function estimates in mixture models. Under revision.

Jameson, G.J.O. (1974) *Topology and Normed Spaces*. London: Chapman & Hall.

Jongbloed, G. and van der Meulen, F.H. (2004) Parametric estimation for subordinators and induced OU-processes. Preprint (http://www.math.vu.nl/~meulen).

Kaspi, H. and Mandelbaum, A. (1994) On Harris recurrence in continuous time. *Math. Oper. Res.*, **19**(1).

Knight, J.L. and Satchell, S.E. (1997) The cumulant generating function method: implementation and asymptotic efficiency. *Econometric Theory*, **13**, 170–184.

Krengel, U. (1985) *Ergodic Theorems*. Berlin: de Gruyter.

Loève, M. (1977) *Probability Theory I* (4th edition). New York: Springer-Verlag.

Luong, A. and Thompson, M.E. (1987) Minimum-distance methods based on quadratic distances for transforms. *Canad. J. Statist.*, **15**, 239–251.

Masuda, H. (2004) On multidimensional Ornstein–Uhlenbeck processes driven by a general Lévy process. *Bernoulli*, **10**, 97–120.

Meyn, S.P. and Tweedie, R.L. (1993a) *Markov Chains and Stochastic Stability.* London: Springer-Verlag.

Meyn, S.P. and Tweedie, R.L. (1993b) Stability of Markovian processes II: Continuous-time processes and sampled chains. *Adv. Appl. Probab.*, **25**, 487–517.

Nielsen, B. and Shephard, N. (2003) Likelihood analysis of a first order autoregressive model with exponential innovations. *J. Time Series Anal.*, **24**, 337–344.

Nummelin, E. (1984) *General Irreducible Markov Chains and Non-negative Operators*. Cambridge: Cambridge University Press.

Revuz, D. and Yor, M. (1999) *Continuous Martingales and Brownian Motion* (3rd edition). Berlin: Springer-Verlag.

Rio, E. (2000) *Théorie Asymptotique des Processus Aléatoires Faiblement Dependants*, Math. Appl. 31. Berlin: Springer-Verlag.

Roberts, G.O., Papaspiliopoulos, O. and Dellaportas, P. (2004) Bayesian inference for non-Gaussian Ornstein–Uhlenbeck stochastic volatility processes. *J. Roy. Statist. Soc. Ser. B*, **66**, 369–393.

Rubin, H. and Tucker, H.G. (1959) Estimating the parameters of a differential process. *Ann. Math. Statist.*, **30**, 641–658.

Sato, K.I. (1999) *Lévy Processes and Infinitely Divisible Distributions*. Cambridge: Cambridge University Press.

Schorr, B. (1975) Numerical inversion of a class of characteristic functions. *BIT*, **15**, 94–102.

Shiga, T. (1990) A recurrence criterion for Markov processes of Ornstein–Uhlenbeck type. *Probab. Theory Related Fields*, **85**, 425–447.

Tucker, H.G. (1967) *A Graduate Course in Probability.* New York: Academic Press.

van Harn, K. and Steutel, F.W. (2004) *Infinite Divisibility of Probability Distributions on the Real Line*. New York: Marcel Dekker.

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