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# Extreme value theory for moving average processes with light-tailed innovations

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We consider stationary infinite moving average processes of the form

$$Y_n = \sum_{i=-\infty}^{\infty} c_i Z_{n+i}, \qquad n \in \mathbb{Z},$$

where $(Z_i)_{i\in\mathbb{Z}}$ is a sequence of independent and identically distributed (i.i.d.) random variables with light tails and $(c_i)_{i\in\mathbb{Z}}$ is a sequence of positive and summable coefficients. By 'light tails' we mean that $Z_0$ has a bounded density $f(t) \sim \nu(t)\exp(-\psi(t))$, where $\nu(t)$ behaves roughly like a constant as $t \to \infty$ and $\psi$ is strictly convex satisfying certain asymptotic regularity conditions. We show that the i.i.d. sequence associated with $Y_0$ is in the maximum domain of attraction of the Gumbel distribution. Under additional regular variation conditions on $\psi$, it is shown that the stationary sequence $(Y_n)_{n\in\mathbb{N}}$ has the same extremal behaviour as its associated i.i.d. sequence. This generalizes Rootzen's results where $f(t) \sim ct^{\alpha}\exp(-t^p)$ for $c > 0$, $\alpha \in \mathbb{R}$ and $p > 1$.

*Keywords:* domain of attraction; extreme value theory; generalized linear model; light-tailed innovations; moving average process

## 1. Introduction

The goal of this paper is to study extreme value theory of strictly stationary moving average processes of the form

$$Y_n = \sum_{i=-\infty}^{\infty} c_i Z_{n+i}, \qquad n \in \mathbb{Z}, \tag{1.1}$$

where $(Z_i)_{i\in\mathbb{Z}}$ is a sequence of independent and identically distributed (i.i.d.) random variables with $\mathrm{E}|Z_0| < \infty$ and $(c_i)_{i\in\mathbb{Z}}$ is a sequence of non-negative real coefficients satisfying $\sum_{i=-\infty}^{\infty} c_i < \infty$. The extremal behaviour of such processes can be classified according to the tail behaviour of the innovation sequence $(Z_i)_{i\in\mathbb{Z}}$ and the manner in which the coefficient sequence $(c_i)_{i\in\mathbb{Z}}$ decreases. Davis and Resnick (1985) investigated the extremes of such moving average processes for innovations whose distributions have regularly varying tails. In that case $Y$ belongs to the maximum domain of attraction of the Fréchet distribution and the point processes of exceedances of $(Y_n)_{n\in\mathbb{Z}}$ converge to a compound Poisson process; that is, extremes appear in clusters. Davis and Resnick (1988) also considered innovations in the domain of attraction of the Gumbel distribution, which are

1350–7265 

convolution equivalent. Here only the multiplicity of the maximum of the coefficients $(c_i)_{i\in\mathbb{Z}}$ determines the cluster size of the limiting compound Poisson process. A summary of results for innovations with subexponential tails can be found in Embrechts *et al.* (1997, Section 5.5). All such innovations have tails which are heavier than exponential.

A different regime was considered in Rootzén (1986; 1987), who investigated innovations whose tails are lighter than exponential. More precisely, he considered innovations with densities of the form $f(t) \sim Kt^\alpha \exp(-t^p)$ as $t \to \infty$, with $p > 1$. Here $a(t) \sim b(t)$ as $t \to \infty$ means that the quotient of the left-hand side and right-hand side converges to 1 as $t \to \infty$. The present paper can be seen as a generalization of Rootzén's results.

We work under the following conditions on the innovations. Let $Z$ be a generic random variable with the same distribution as $Z_0$. We assume that $Z$ has a bounded probability density and that it satisfies

$$f(t) \sim \nu(t)\exp(-\psi(t)) \qquad t \to \infty. \tag{1.2}$$

Here $\psi$ is convex, $C^2$, with $\psi'' > 0$ and $\psi'(\infty) = \infty$, and the function $\phi = 1/\sqrt{\psi''}$ is *self-neglecting*, that is,

$$\lim_{t\to\infty} \frac{\phi(t + x\phi(t))}{\phi(t)} = 1, \qquad \text{uniformly on bounded } x\text{-intervals.} \tag{1.3}$$

The function $\nu$ is measurable and is *flat for* $\phi$, that is,

$$\lim_{t\to\infty} \frac{\nu(t + x\phi(t))}{\nu(t)} = 1, \qquad \text{uniformly on bounded } x\text{-intervals,} \tag{1.4}$$

which guarantees that it is more or less flat on intervals of the appropriate length determined by $\phi$. Such densities are closed with respect to finite convolutions, which applies to a finite moving average process; see Balkema *et al.* (1993). This is a basic property needed to analyse such light-tailed linear models. As the assumptions in Balkema *et al.* (1993) are minimal, our framework is to our knowledge the most general framework possible.

Our paper is organized as follows. In Section 2 we introduce the necessary assumptions, state the main results and conclude with some examples. Assumption 2.1 redefines any density (1.2) satisfying (1.3) and (1.4) such that it satisfies certain conditions which do not constitute a restriction, but make calculations easier. Assumption 2.2 allows for a generalization of results from the finite moving average to the general model (1.1). Assumption 2.2 will suffice to determine the tail behaviour of $Y_0$ up to a certain order (Theorem 2.1) and to show that $Y_0$ belongs to the domain of attraction of the Gumbel distribution (Theorem 2.2). To investigate the extremal behaviour of the stationary sequence $(Y_n)_{n\in\mathbb{Z}}$, we have to impose certain regularity conditions on the function $\psi$. As is natural in extreme value theory, we require regular variation or rapid variation of $\psi$, as given in Assumptions 2.3 and 2.4. Theorem 2.3 then shows that the extremal behaviour of the moving average process $(Y_n)_{n\in\mathbb{Z}}$ is exactly that of its associated i.i.d. sequence; that is, $(Y_n)_{n\in\mathbb{Z}}$ belongs to the domain of attraction of the Gumbel distribution with the same norming constants as the associated i.i.d. sequence.

In Section 3 we state some auxiliary results and discuss our assumptions. Section 4 is devoted to the proof of the tail behaviour and domain of attraction of $Y_0$ as stated in

Theorems 2.1 and 2.2, while the extremal behaviour of the stationary sequence $(Y_n)_{n\in\mathbb{Z}}$ as stated in Theorem 2.3 is proved in Section 5. Applications of the results to financial time series such as stochastic volatility models and the exponential generalized autoregressive conditional heteroscedastic model are considered in Section 6. Finally, in Section 7 we give some extensions of our results, treating for example the case of positive and negative coefficients.

## 2. Assumptions and main results

We make the general assumptions of the Introduction more precise, introduce the necessary notation, state our main results and give some examples. Throughout the paper we shall assume the following condition (such a representation can always be found for the class of densities introduced in Section 1).

***Assumption 2.1.*** *The random variable $Z$ has finite expectation and a bounded density $f$, which satisfies*

$$f(t) = \nu(t)\exp(-\psi(t)), \qquad t \geqslant t_0, \tag{2.1}$$

*for some $t_0 \in \mathbb{R}$ and functions $\nu$, $\psi : [t_0, \infty) \to \mathbb{R}$, where $\psi$ is $C^2$, $\psi'(t_0) = 0$, $\psi'(\infty) = \infty$, $\psi''$ is strictly positive on $[t_0, \infty)$ and $1/\sqrt{\psi''}$ is self-neglecting. The function $\nu$ is measurable and flat for $1/\sqrt{\psi''}$.*

The function $\psi'$ is continuous and strictly increasing on $[t_0, \infty)$ with range $[0, \infty)$. Therefore, for any $\tau \in [0, \infty)$ and the non-negative summable sequence $(c_i)_{i\in\mathbb{Z}}$, we can define

$$\begin{aligned}
q(\tau) &:= \psi'^{\leftarrow}(\tau),\\
S^2(\tau) &:= q'(\tau) = 1/\psi''(q(\tau)),\\
q_i(\tau) &:= c_i q(c_i\tau),\\
\sigma_i^2(\tau) &:= q_i'(\tau) = c_i^2 S^2(c_i\tau),
\end{aligned}$$

where $\psi'^{\leftarrow}$ denotes the inverse of $\psi'$. Note that $q(0) = t_0$, and that $q$ is $C^1$ on $[t_0, \infty)$ and strictly increasing with $q(\infty) = \infty$. Furthermore, on any compact interval of the form $[t_0, s]$ for $s \in [t_0, \infty)$, $S^2 = q'$ is bounded above and bounded away from zero.

Then, by the previous considerations,

$$Q(\tau) := \sum_{i=-\infty}^{\infty} q_i(\tau) \quad \text{and} \quad \sigma_\infty^2(\tau) := \sum_{i=-\infty}^{\infty} \sigma_i^2(\tau)$$

can be defined pointwise for any $\tau \geqslant 0$. The sum defining $\sigma_\infty^2$ converges uniformly on any compact interval $[0, s]$ $(s > 0)$, which then implies that the sum defining $Q$ converges uniformly on compacts, and that $Q$ is $C^1$ satisfying

$$Q'(\tau) = \sigma_\infty^2(\tau) = \sum_{i=-\infty}^{\infty} q_i'(\tau), \qquad \tau \geqslant 0. \tag{2.2}$$

Furthermore, $Q$ is strictly increasing and maps $[0, \infty)$ onto $[t_0 \sum_{i=-\infty}^{\infty} c_i, \infty)$. Set $S := \sqrt{S^2}$, $\sigma_i := \sqrt{\sigma_i^2}$, $\sigma_\infty := \sqrt{\sigma_\infty^2}$. To describe the tail behaviour of $Y_0$, we will need further conditions on the speed of convergence of the sum defining $\sigma_\infty^2$. More precisely, we will impose the following assumption:

***Assumption 2.2.*** *$(c_i)_{i\in\mathbb{Z}}$ is a summable sequence of non-negative real numbers, not all zero, and the following two conditions hold:*

$$\lim_{m\to\infty} \limsup_{\tau\to\infty} \frac{\sum_{|j|>m} \sigma_j^2(\tau)}{\sigma_\infty^2(\tau)} = 0, \tag{2.3}$$

$$\lim_{m\to\infty} \limsup_{\tau\to\infty} \frac{\sum_{|j|>m} \sigma_j(\tau)}{\sigma_\infty(\tau)} = 0. \tag{2.4}$$

Clearly, Assumption 2.2 is satisfied if all but finitely many of the $c_i$ are zero. Assumptions 2.1 and 2.2 allow us to obtain the tail behaviour of $Y_0$. Denote by $\Phi$ the moment generating function of $Y_0$, which in Lemma 4.1 will be shown to exist under Assumptions 2.1 and 2.2. Then with the aid of $\Phi$ we can express the exact tail behaviour of $Y_0$, and without using $\Phi$ we obtain the tail behaviour of $Y_0$ up to a certain order:

**Theorem 2.1.** *Suppose that Assumptions 2.1 and 2.2 hold. Then*

$$P\left(\sum_{i=-\infty}^{\infty} c_i Z_i > Q(\tau)\right) \sim \frac{1}{\sqrt{2\pi}\tau\sigma_\infty(\tau)} \mathrm{e}^{-\tau Q(\tau)} \Phi(\tau), \qquad \tau \to \infty. \tag{2.5}$$

*Furthermore, there is a function $\rho(\tau) = o(1/\sigma_\infty(\tau))$, $\tau \to \infty$, such that*

$$P\left(\sum_{i=-\infty}^{\infty} c_i Z_i > t\right) \sim \frac{1/\sqrt{2\pi}}{Q^{\leftarrow}(t)\sigma_\infty(Q^{\leftarrow}(t))} \exp\left(\int_{t\sum c_i}^{t} (Q^{\leftarrow}(v) + \rho(Q^{\leftarrow}(v)))\mathrm{d}v\right), \qquad t \to \infty, \tag{2.6}$$

*and $1/\sigma_\infty(\tau) = o(\tau)$, $\tau \to \infty$, so the first term in the integral is the leading term.*

As $Y_0$ is light-tailed, it is no surprise that $Y_0$ belongs to the domain of attraction of the Gumbel distribution; we write $Y_0 \in \mathrm{MDA}(\Lambda)$. We also say that the associated i.i.d. sequence to $(Y_n)_{\in\mathbb{Z}}$ belongs to MDA($\Lambda$); this is a sequence $(\tilde{Y}_n)_{n\in\mathbb{Z}}$ of i.i.d. random variables all with the stationary distribution. Then $Y_0 \in \mathrm{MDA}(\Lambda)$ means that there exist norming constants $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that $a_n > 0$, $b_n \in \mathbb{R}$, and

$$\lim_{n\in\infty} P\left(a_n\left(\max_{j=1,\dots,n} \tilde{Y}_j - b_n\right) \leqslant x\right) = \Lambda(x) = \exp(-\mathrm{e}^{-x}), \qquad x \in \mathbb{R}.$$

For more details on classical extreme value theory we refer to Embrechts *et al.* (1997), Leadbetter *et al.* (1983) or Resnick (1987).

**Theorem 2.2.** *Suppose that Assumptions 2.1 and 2.2 hold. Then*

$$\lim_{t\to\infty} \frac{P(Y_0 > t + x/Q^{\leftarrow}(t))}{P(Y_0 > t)} = \mathrm{e}^{-x}, \qquad x \in \mathbb{R}. \tag{2.7}$$

*The i.i.d. sequence associated with $(Y_n)_{n\in\mathbb{Z}}$ belongs to* $\mathrm{MDA}(\Lambda)$, *with norming constants $a_n$ and $b_n$ given by the equations*

$$\lim_{n\in\infty} nP(Y_0 > b_n) = 1 \quad \textit{and} \quad a_n := Q^{\leftarrow}(b_n). \tag{2.8}$$

It does not seem to be too restrictive to impose further regular variation conditions on $\psi$. We shall denote the class of functions regularly varying in infinity with index $\beta$ by $\mathrm{RV}_\beta$; for definitions and results we refer to the monograph by Bingham *et al.* (1987).

***Assumption 2.3.*** *Suppose that $\psi'' \in \mathrm{RV}_\beta$ for $\beta \in [-1, \infty]$. For $\beta = \infty$, which corresponds to the class of rapidly varying functions, we require additionally that $\psi''$ is ultimately absolutely continuous on compacts (i.e. there exists $T$ such that $\psi''$ is absolutely continuous on $[T, T + x]$ for any $x > 0$) and that*

$$\lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\psi'(t)}{\psi''(t)} = 0.$$

*Define $\beta'$ such that $1 + \beta' = 1/(1 + \beta)$ with the convention that the left-hand side is equal to $0$ for $\beta = \infty$ and equal to $\infty$* if *$\beta = -1$.*

*Furthermore, suppose there exists $\theta \in [0, 2)$ such that $\theta + \beta' > 0$ and $\sum_{i=-\infty}^{\infty} c_i^{1-\theta/2} < \infty$, where $(c_i)_{i\in\mathbb{Z}}$ is a sequence of non-negative real numbers, not all zero.*

In Proposition 3.2 it will be shown that Assumptions 2.3 and 2.1 together imply Assumption 2.2. Under the following slightly stronger assumption we will show that the extremal behaviour of the moving average process $(Y_n)_{n\in\mathbb{Z}}$ is the same as the extremal behaviour of its associated i.i.d. sequence: the dependence vanishes in the extremes.

***Assumption 2.4.*** *Suppose that $\psi$, $\beta$ and $\beta'$ are as in Assumption 2.3. Furthermore, suppose there is some constant $\vartheta > \max\{1, 2/(2 + \beta')\}$ such that $c_i = O(|i|^{-\vartheta})$, $i \to \infty$, where $(c_i)_{i\in\mathbb{Z}}$ is a sequence of non-negative real numbers, not all zero. Finally, suppose that $Z$ has finite variance.*

Assumption 2.4 implies Assumption 2.3: if we choose $\theta \in [0, 2 - 2/\vartheta)$ such that $\theta + \beta' > 0$, then Assumption 2.3 follows, since $(1 - \theta/2)\vartheta > 1$. The extremal behaviour of the stationary $(Y_n)_{n\in\mathbb{Z}}$ can now be described as follows:

**Theorem 2.3.** *Suppose that Assumptions 2.1 and 2.4 hold. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, as given* *in* (2.8)*, be norming constants of the i.i.d. sequence associated with* $Y_0$*. Then* $(Y_n)_{n\in\mathbb{N}}$ *belongs to* $\mathrm{MDA}(\Lambda)$ *with the same norming constants, that is,*

$$\lim_{n\in\infty} P\left(a_n\left(\max_{j=1,\dots,n} Y_j - b_n\right) \leqslant x\right) = \exp(-\mathrm{e}^{-x}), \qquad x \in \mathbb{R}.$$

In the course of proving our results, we will use the following notation. For any summable sequence $(c_i)_{i\in\mathbb{Z}}$ of non-negative real numbers let $i_0$ be an index such that $c_{i_0} = \max\{c_i : i \in \mathbb{Z}\}$. Let $c$ and $d$ be strictly positive real numbers, and let $0 \leqslant \theta < 2$. Denote by $\mathcal{G}_{c,d,\theta}$ the set of all non-negative sequences $(c_i)_{i\in\mathbb{Z}}$ such that $\sum_{i=-\infty}^{\infty} c_i \leqslant d$, $\sum_{i=-\infty}^{\infty} c_i^{2-\theta} \leqslant d$, $\sum_{i=-\infty}^{\infty} c_i^{1-\theta/2} \leqslant d$, and $c/2 \leqslant c_{i_0} \leqslant c$. If in the following limits of summation are missing, then it is understood that summation is over $\mathbb{Z}$. Convergence in distribution will be denoted by $\xrightarrow{d}$, and convergence in probability by $\xrightarrow{P}$.

We conclude this section with some examples.

***Example 2.1.*** (a) Let $\psi(t) := (\beta + 2)^{-1} t^{\beta+2}$, where $\beta \in (-1, \infty)$. Then $\psi'' \in \mathrm{RV}_\beta$ and $\psi$ satisfies Assumption 2.1 with $t_0 = 0$. An example for a flat function $\nu$ for $1/\sqrt{\psi''}$ would be any function behaving asymptotically like a rational function, or also $\nu(t) = \mathrm{e}^t$ if $\beta > 0$. Put $\beta' := (1+\beta)^{-1} - 1$ and suppose that $c_i = O(|i|^{-\vartheta})$ for some $\vartheta > \max(1, 2/(2+\beta'))$. If $Z$ is then such that it has finite variance and bounded density $f$ as in (2.1), then Assumptions 2.1 and 2.4 hold and Theorems 2.1–2.3 can be applied. In particular, since $Q^{\leftarrow}(t) = (t/\sum c_i^{2+\beta'})^{1+\beta}$ and $Q'(Q^{\leftarrow}(t)) = ct^{-\beta}$ for some constant $c$, (2.6) gives

$$P\left(\sum_{i=-\infty}^{\infty} c_i Z_i > t\right) = \exp\left(-(2+\beta)^{-1}\left(\sum c_i^{2+\beta'}\right)^{-1-\beta} t^{2+\beta} + o(t^{1+\beta/2})\right), \qquad t \to \infty.$$

This agrees with Theorem 6.1 in Rootzén (1987); however, focusing on this example and under an additional smoothness condition, Rootzén obtains the estimate $O(t^{(1+\beta)/\vartheta})$ for the remaining term (as $t \to \infty$), which can be seen to be slightly better than our estimate, since $\vartheta > 2/(2+\beta')$ implies $(1+\beta)/\vartheta < 1 + \beta/2$.

(b) Let $\psi$: $[1, \infty) \to \mathbb{R}$ be given by $\psi(t) = t\log t - t$. Then $\psi''(t) = 1/\mathrm{t} \in \mathrm{RV}_{-1}$ and $\psi$ satisfies Assumption 2.1 with $t_0 = 1$. Any rational function would then be flat for $1/\sqrt{\psi''}$. Let $c_i = O(|i|^{-\vartheta})$ for some $\vartheta > 1$. For simplicity, assume that $c_{i_0} = 1$, and that this maximum $c_{i_0}$ is taken with multiplicity $N$. Let $c' := \max\{c_i : i \in \mathbb{Z},\, c_i \neq 1\} < 1$. Assume that $Z$ also satisfies all other properties of Assumptions 2.1 and 2.4. Then Theorems 2.1–2.3 are applicable. For the tail, note that $q(\tau) = \mathrm{e}^\tau$, $Q(\tau) = N\mathrm{e}^\tau + O(\mathrm{e}^{c'\tau})$, $\tau \to \infty$, and approximate inversion shows

$$Q^{\leftarrow}(t) = \log t - \log N + O(t^{c'-1}), \qquad t \to \infty.$$

Since $Q'(\tau) \sim N\mathrm{e}^\tau$, $\tau \to \infty$, it follows that $\sigma_\infty^{-1}(Q^{\leftarrow}(\tau)) \sim t^{-1/2}$, so that (2.6) gives

$$P\left(\sum_{i=-\infty}^{\infty} c_i Z_i > t\right) = \exp(-t\log t + t(1+\log N) + O(t^{\max\{c',1/2\}})), \qquad t \to \infty.$$

(c) Examples where $\psi''$ is in $\mathrm{RV}_\infty$ and satisfies the additional condition in Assumption 2.3 are $\psi(t) = \mathrm{e}^t$ or $\psi(t) = \exp(\mathrm{e}^t)$ for large $t$. If then $c_i = O(|i|^{-\vartheta})$ for some $\vartheta > 2$ and the additional conditions in Assumptions 2.1 and 2.4 are satisfied (a flat function could be a rational function, or also $\nu(t) = \mathrm{e}^t$), then Theorems 2.1–2.3 can be applied. We consider one example in more detail. Let $\psi : [0, \infty) \to [0, \infty)$ be given by $\psi(t) = \mathrm{e}t^2/2$ for $t \in [0, 1]$ and $\psi(t) = \mathrm{e}^t - \mathrm{e}/2$ for $t > 1$. Let $\theta \in (1, 2)$ such that $\sum c_i^{1-\theta/2} < \infty$. For simplicity, assume that $\sum c_i = 1$. Then $q(\tau) = \tau/\mathrm{e}$ for $0 \leqslant \tau \leqslant \mathrm{e}$ and $q(\tau) = \log \tau$ for $\tau \geqslant \mathrm{e}$. This shows that

$$Q(\tau) = \sum_{i=-\infty}^{\infty} c_i \log c_i + \log \tau + \sum_{i:c_i\tau<\mathrm{e}} \left( \frac{c_i^2 \tau}{\mathrm{e}} - c_i \log(c_i\tau) \right),$$

where

$$\sum_{i:c_i\tau<\mathrm{e}} \left( \frac{c_i^2 \tau}{\mathrm{e}} - c_i \log(c_i\tau) \right) = \tau^{-\theta/2} \sum_{i:c_i\tau<\mathrm{e}} c_i^{1-\theta/2} \left( \frac{(c_i\tau)^{1+\theta/2}}{\mathrm{e}} - (c_i\tau)^{\theta/2} \log(c_i\tau) \right) = o(\tau^{-\theta/2}),$$

as $\tau \to \infty$. Approximate inversion yields

$$Q^{\leftarrow}(t) = \mathrm{e}^{t - \sum c_i \log c_i} + o(\mathrm{e}^{t(1-\theta/2)}), \qquad t \to \infty.$$

Furthermore,

$$Q'(\tau) = \frac{1}{\tau} \left( \sum_{i:c_i\tau\geqslant\mathrm{e}} c_i + \sum_{c_i\tau<\mathrm{e}} (c_i\tau) \frac{c_i}{\mathrm{e}} \right) \sim \frac{1}{\tau}, \qquad \tau \to \infty,$$

so that $\sigma_\infty^{-1}(Q^{\leftarrow}(t)) = O(\mathrm{e}^{t/2})$, $t \to \infty$. An application of (2.6) then shows that

$$P\left( \sum_{i=-\infty}^{\infty} c_i Z_i > t \right) = \exp(-\mathrm{e}^{t - \sum c_i \log c_i} + O(\mathrm{e}^{t/2})), \qquad t \to \infty.$$

# 3. Auxiliary results

## 3.1. Exponential families

A basic role in our proofs will be played by exponential families. Let $X$ be a random variable whose moment generating function $\mathrm{E}\mathrm{e}^{\tau X}$ exists for all $\tau \in [0, \infty)$. Then the *exponential family* $(\overline{X}_\tau)_{\tau\geqslant 0}$ is defined to be a family of random variables such that

$$F_{\overline{X}_\tau}(\mathrm{d}z) = \frac{\mathrm{e}^{\tau z} F_X(\mathrm{d}z)}{\mathrm{E}\mathrm{e}^{\tau X}}, \qquad \tau \geqslant 0,$$

where $F_X$ and $F_{\overline{X}_\tau}$ denote the distribution function of $X$ and $\overline{X}_\tau$, respectively. Exponential families have the following useful properties, which follow by standard calculations (see, for example, Rootzén 1987, Section 3):

$$P(X \in A) = \mathrm{E}(\mathrm{e}^{-\tau\overline{X}_\tau} 1_{\overline{X}_\tau \in A})\mathrm{E}\mathrm{e}^{\tau X}, \qquad \tau \geqslant 0,\ A \text{ a Borel set}, \tag{3.1}$$

$$\overline{(cX)}_\tau \stackrel{d}{=} c\overline{X}_{c\tau}, \qquad c,\ \tau \geqslant 0. \tag{3.2}$$

We will consider the exponential families of the random variables $X_i := c_i Z_i$. Denote by $\Phi_i$ the moment generating function of $X_i$, which by Assumption 2.1 exists and is finite for all $\tau \geqslant 0$, as shown in Balkema *et al.* (1993, Proposition 5.11). Denote the density of $X_i$ by $f_i$, and the exponential family associated with $X_i$ by $(\overline{X}_{i,\tau})_{\tau\geqslant 0}$. Assume throughout that the exponential families are taken such that $(\overline{X}_{i,\tau})_{i\in\mathbb{Z}}$ are mutually independent for any $\tau \geqslant 0$. The exponential family associated with the generic random variable $Z$ will be denoted by $(\overline{Z}_\tau)_{\tau\geqslant 0}$. In Lemma 4.1 it will be shown that the moment generating function $\Phi$ of $\sum X_i$ exists and is finite for every argument $\tau \geqslant 0$, and that $\sum_{i=-\infty}^{\infty}\overline{X}_{i,\tau}$ converges almost surely for any $\tau \geqslant 0$. In particular, the exponential family of $\sum X_i$ exists, and since taking exponential families commutes with taking convolution (see, for example, Rootzén 1987, equation (3.4)), this exponential family is given by $(\sum_{i=-\infty}^{\infty}\overline{X}_{i,\tau})_{\tau\geqslant 0}$.

## 3.2. ANET convergence

A family $(W_\tau)_{\tau\geqslant 0}$ of random variables with densities $w_\tau$ is called *asymptotically normal with exponential tails* (ANET) if $w_\tau(x)$ converges locally uniformly in $x$ to the density $\varphi(x) = \mathrm{e}^{-x^2/2}/\sqrt{2\pi}$ of the standard normal distribution as $\tau \to \infty$, and if for any $\varepsilon > 0$ there exist $\tau_\varepsilon$ and a constant $M_\varepsilon > 1$ such that

$$w_\tau(x) \leqslant \mathrm{e}^{-|x|/\varepsilon}, \qquad \forall |x| \geqslant M_\varepsilon,\ \tau \geqslant \tau_\varepsilon.$$

If a sequence is ANET, it is known that the moment generating functions and the (absolute) moments of all orders converge to the corresponding moment generating function and (absolute) moments of the standard normal distribution, and that $W_\tau$ converges in distribution to $N(0, 1)$; see Balkema *et al.* (1993, Proposition 6.3).

In Balkema *et al.* (1993, Theorem 6.6) it is shown that under Assumption 2.1 a suitable centring and normalization transforms the exponential family associated with $Z$ into an ANET sequence. More precisely, the sequence $((\overline{Z}_\tau - q(\tau))/\ S(\tau))_{\tau\geqslant 0}$ is ANET. Since the set of random variables satisfying Assumption 2.1 is closed under finite convolution, as shown in Balkema *et al.* (1993, Theorem 1.1), it follows that for any $m \in \mathbb{N}_0$ such that at least one of the $c_i$ for $|i| \leqslant m$ is non-zero, the exponential family associated with $\sum_{i=-m}^{m} X_i$ can be transformed into an ANET sequence. More precisely, the sequence $(\sum_{i=-m}^{m} (\overline{X}_{i,\tau} - q_i(\tau))/\sqrt{\sum_{i=-m}^{m}\sigma_i^2(\tau)})_{\tau\geqslant 0}$ is ANET; see Balkema *et al.* (1993, p. 586). See also Barndorff-Nielsen and Klüppelberg (1992) for further calculations.

## 3.3. Discussion of the assumptions

Recall that a function $g : [0, \infty) \to \mathbb{R}$ is in $\mathrm{RV}_\beta$ ($\beta \in \mathbb{R}$) if and only if there are constants

$a$, $c > 0$, a measurable function $c(\cdot)$ and a locally Lebesgue integrable function $\varepsilon$ on $[a, \infty)$ such that $\lim_{x\to\infty} c(x) = c$, $\lim_{x\to\infty} \varepsilon(x) = 0$, and

$$g(x) = x^\beta c(x) \exp\left(\int_a^x \frac{\varepsilon(u)}{u}\, \mathrm{d}u\right), \qquad x \geqslant a. \tag{3.3}$$

If the function $c(\cdot)$ in (3.3) can be taken as a constant, then $g$ is said to be *normalized regularly varying with index* $\beta$; we write $g \in \mathrm{NRV}_\beta$.

The following lemma clarifies Assumption 2.3. In particular,

$$\lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\psi'(t)}{\psi''(t)} = 0$$

means nothing more than $q' \in \mathrm{NRV}_{-1}$, which already implies that $\psi'' \in \mathrm{RV}_\infty$.

**Lemma 3.1.** *Suppose that $\psi : [t_0, \infty) \to \mathbb{R}$ is $C^2$, $\psi'(\infty) = \infty$, and $\psi'' > 0$. Let $q = \psi'^{\leftarrow}$, and for $\beta \in [-1, \infty]$ define $\beta'$ through $1 + \beta' = (1 + \beta)^{-1}$.*

(a) *For all $\beta \in [-1, \infty]$, we have $\psi' \in \mathrm{RV}_{1+\beta}$ if and only if $q \in \mathrm{RV}_{1+\beta'}$.*
(b) *If $\psi'' \in \mathrm{RV}_\beta$ where $\beta \in \mathbb{R}$, then $\beta \geqslant -1$, $\psi' \in \mathrm{RV}_{1+\beta}$, $1/\sqrt{\psi''}$ is self-neglecting, and $q' \in \mathrm{RV}_{\beta'}$. If $\beta \in (-1, \infty)$, then $\psi'' \in \mathrm{RV}_\beta$ if and only if $q' \in \mathrm{RV}_{\beta'}$.*
(c) *Let $\beta' \in [-1, \infty)$. Then $\psi''$ is ultimately absolutely continuous on compacts and satisfies*

$$\lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\psi'(t)}{\psi''(t)} = 1 + \beta'$$

*if and only if $q' \in \mathrm{NRV}_{\beta'}$.*
(d) *If $q' \in \mathrm{RV}_{-1}$, then $1/\sqrt{\psi''}$ is self-neglecting and $\psi'' \in \mathrm{RV}_\infty$.*
(e) *$1/\sqrt{\psi''}$ is self-neglecting if and only if $1/\sqrt{q'}$ is self-neglecting.*

***Proof.*** (a) This follows from Proposition 1.5.15 and Theorem 2.4.7 of Bingham *et al.* (1987).

(b) Since $\psi'(\infty) = \infty$ and $\psi'' \in \mathrm{RV}_\beta$, it follows from l'Hôpital's rule that $\psi' \in \mathrm{RV}_{1+\beta}$ and further that $1 + \beta \geqslant 0$. Since $q'(\tau) = 1/\psi''(q(\tau))$, it follows by composition that $q' \in \mathrm{RV}_{\beta'}$ if $\beta \neq -1$, and the converse follows similarly. If $\beta = -1$, then $\psi' \in \mathrm{RV}_0$, hence $q \in \mathrm{RV}_\infty$. By the monotone equivalence theorem (Bingham *et al.* 1987, Theorem 1.5.3), $\psi''$ is asymptotically equivalent to a decreasing function $h$, say. Then if $c \in (0, 1)$, for any $\varepsilon > 0$ there exists $\tau_\varepsilon$ such that $q(c\tau) < \varepsilon q(\tau)$ for $\tau \geqslant \tau_\varepsilon$, since $q \in \mathrm{RV}_\infty$. This then implies

$$\frac{q'(c\tau)}{q'(\tau)} \sim \frac{h(q(\tau))}{h(q(c\tau))} \leqslant \frac{h(q(\tau))}{h(\varepsilon q(\tau))} \to \varepsilon, \qquad \tau \to \infty,$$

showing that $q' \in \mathrm{RV}_\infty$. To show that $1/\sqrt{\psi''}$ is self-neglecting, note that

$$\lim_{t\to\infty} \frac{t + x/\sqrt{\psi''(t)}}{t} = 1 + \lim_{t\to\infty} \frac{x}{t\sqrt{\psi''(t)}} = 1$$

uniformly in $x \in \mathbb{R}$, since $t \mapsto t\sqrt{\psi''(t)}$ is in $\mathrm{RV}_{1+\beta/2}$.

(c) Note that $\psi''$ is ultimately absolutely continuous on compacts and satisfies the relation

$$\lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\psi'(t)}{\psi''(t)} = 1 + \beta'$$

if and only if $q'$ is ultimately absolutely continuous on compacts and satisfies

$$\lim_{\tau\to\infty} \frac{\tau q''(\tau)}{q'(\tau)} = \lim_{\tau\to\infty} \frac{-\psi'(q(\tau))\psi'''(q(\tau))}{\psi''(q(\tau))^2} = \lim_{t\to\infty} \frac{-\psi'(t)\psi'''(t)}{\psi''(t)^2} = \beta'.$$

But this is equivalent to $q'$ being ultimately absolutely continuous on compacts and satisfying

$$\lim_{\tau\to\infty} \frac{\tau\, \mathrm{d}(\tau^{-\beta'} q'(\tau))/\mathrm{d}\tau}{\tau^{-\beta'} q'(\tau)} = 0,$$

which is equivalent to $q' \in \mathrm{NRV}_{-1}$; see Bingham *et al.* (1987, p. 15).

The proof of (d) is similar to the proof of (b), using (e) to show that $1/\sqrt{\psi''}$ is self-neglecting.

The proof of (e) itself is given in Balkema *et al.* (1993, Theorem 5.3). □

Next we show that Assumptions 2.1 and 2.3 imply Assumption 2.2.

**Proposition 3.2.** *Suppose that Assumptions 2.1 and 2.3 are satisfied. Then Assumption 2.2 holds. Furthermore, there exists a positive constant $D$, depending only on $\psi$ and on $\theta$, such that for every constant $c$ bounding $(c_i)_{i\in\mathbb{Z}}$ from above,*

$$\sigma_\infty^2(\tau) \leqslant D \sum_{i=-\infty}^{\infty} \left(\frac{c_i}{c}\right)^{2-\theta} c^2 q'(c\tau), \qquad \tau \geqslant 0. \tag{3.4}$$

***Proof.*** Note that $q' \in \mathrm{RV}_{\beta'}$ by Lemma 3.1. Define $p_1(\tau) := \tau^\theta q'(\tau)$ for $\tau \geqslant 0$. Then there exists an increasing function $p_2 : [0, \infty) \to \mathbb{R}$ such that $p_1(\tau) \leqslant p_2(\tau)$ for any $\tau \geqslant 0$, and $p_1(\tau) \sim p_2(\tau)$ as $\tau \to \infty$. For $\beta' \neq \infty$, this follows from the monotone equivalence theorem (Bingham *et al.* 1987, Theorem 1.5.3), and for $\beta' = \infty$ from $q'(\tau) = 1/\psi''(q(\tau))$, the monotonicity of $q$ and an application of the monotone equivalence theorem to $1/\psi'' \in \mathrm{RV}_1$. We conclude that there exists a positive constant $d_1$ such that $p_2(\tau) \leqslant d_1 p_1(\tau)$ for all $\tau \geqslant 1$. Let $c \geqslant \max\{c_i : i \in \mathbb{Z}\}$. Then if $c\tau \geqslant 1$, we have

$$p_1(c_i\tau) \leqslant p_2(c_i\tau) \leqslant p_2(c\tau) \leqslant d_1 p_1(c\tau).$$

Since $q'$ is continuous and strictly positive on $[0, 1]$, there exists some $d_2 > 0$ such that $q'(x) \leqslant d_2 q'(y)$ for every $x, y \in [0, 1]$. In particular, for $c\tau \leqslant 1$, $q'(c_i\tau) \leqslant d_2 q'(c\tau)$. Then, with $D := \max(d_1, d_2)$, it follows that

$$c_i^\theta q'(c_i\tau) \leqslant D c^\theta q'(c\tau), \qquad \tau \geqslant 0, \tag{3.5}$$

giving (3.4). Since $\sum c_i^{1-\theta/2} < \infty$, it follows from (3.5), the dominated convergence theorem and the fact that $p_1 \in \mathrm{RV}_{\beta'+\theta}$, that

$$\lim_{\tau\to\infty} \frac{\sum_{i=-\infty}^{\infty} c_i \sqrt{q'(c_i\tau)}}{c\sqrt{q'(c\tau)}} = \sum_{i=-\infty}^{\infty} \left(\frac{c_i}{c}\right)^{1-\theta/2} \lim_{\tau\to\infty} \sqrt{\frac{c_i^\theta \tau^\theta q'(c_i\tau)}{c^\theta \tau^\theta q'(c\tau)}} = \sum_{i=-\infty}^{\infty} \left(\frac{c_i}{c}\right)^{1+\beta'/2},$$

where the right-hand side has to be interpreted as $\operatorname{card}\{i : c_i = c\}$ if $\beta' = \infty$. Similarly, for any $m > 0$,

$$\lim_{\tau\to\infty} \frac{\sum_{|i|>m} c_i \sqrt{q'(c_i\tau)}}{c\sqrt{q'(c\tau)}} = \sum_{|i|>m} \left(\frac{c_i}{c}\right)^{1+\beta'/2},$$

and (2.4) follows. The limit relation (2.3) follows similarly. □

***Remark 3.1.*** The proof shows that the condition

$$\lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\psi'(t)}{\psi''(t)} = 0$$

(for the case $\psi'' \in \mathrm{RV}_\infty$), which by Lemma 3.1 is equivalent to $q' \in \mathrm{NRV}_{-1}$, can be slightly relaxed to $q' \in \mathrm{RV}_{-1}$, and Assumption 2.2 still follows.

There are also many examples where Assumptions 2.1 and 2.2 hold, but 2.3 does not:

***Example 3.1.*** Let $\psi : [0, \infty) \to (0, \infty)$ such that $\psi'(0) = 0$ and $\psi''(t) = (2 + \cos(\pi\sqrt{t}))^{-2}$. Then the derivative of $1/\sqrt{\psi''(t)}$ tends to 0 as $t \to \infty$, and the mean value theorem implies that $1/\sqrt{\psi''}$ is self-neglecting. A flat function $\nu$ would be any rational function or $\nu(t) = \exp(t^\alpha)$ for $\alpha \in [0, 1)$. If then $Z$ has finite expectation and bounded density $f$ satisfying (2.1), then Assumption 2.1 holds. If, furthermore, $(c_i)_{i\in\mathbb{Z}}$ is a summable sequence of non-negative numbers, then it is easy to see that Assumption 2.2 holds, too. Note, however, that Assumption 2.3 is not satisfied for this example.

# 4. Proof of Theorems 2.1 and 2.2

In this section we shall prove the tail behaviour of $Y_0$ as stated in Theorem 2.1 and then use this result to prove Theorem 2.2, that is, that the associated i.i.d. sequence is in $\mathrm{MDA}(\Lambda)$. The proofs will be split up into several lemmas, and exponential families will play an important role. We will also give some uniform estimates under the extra condition of Assumption 2.3 and for coefficient sequences in $\mathcal{G}_{c,d,\theta}$. These will be used in Section 5 when proving Theorem 2.3. Recall the notation of Section 3.1.

**Lemma 4.1.** *Under Assumptions 2.1 and 2.2, the moment generating function $\Phi$ of $\sum X_i = \sum c_i Z_i$ exists and is finite for all $\tau \geqslant 0$, and we have*

$$\Phi(\tau) = \prod_{i=-\infty}^{\infty} \Phi_i(\tau), \qquad \tau \geqslant 0,$$

*as well as*

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \log \Phi(\tau) = \sum_{i=-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}\tau} \log \Phi_i(\tau) = \sum_{i=-\infty}^{\infty} \mathrm{E}\overline{X}_{i,\tau}, \qquad \tau \geqslant 0, \tag{4.1}$$

*where the sum and the product converge uniformly on compact subsets of* $[0, \infty)$. *The exponential family associated with* $\sum X_i$ *is* $(\sum_{i=-\infty}^{\infty} \overline{X}_{i,\tau})_{\tau \geqslant 0}$, *where the sum converges absolutely almost surely.*

***Proof.*** By the definition of the exponential family,

$$\mathrm{E}\overline{X}_{i,\tau} = \frac{\mathrm{E}X_i \mathrm{e}^{\tau X_i}}{\Phi_i(\tau)} = \frac{\int_{-\infty}^{\infty} f_i(t) t \mathrm{e}^{\tau t}\,\mathrm{d}t}{\Phi_i(\tau)} = \frac{\mathrm{d}\Phi_i(\tau)/\mathrm{d}\tau}{\Phi_i(\tau)} = \frac{\mathrm{d}}{\mathrm{d}\tau} \log \Phi_i(\tau),$$

where we have used the differentiation lemma for the third equality. Furthermore, we see (since $\mathrm{E}|X_i| < \infty$) that $[0, \infty) \to \mathbb{R}$, $\tau \mapsto \mathrm{E}|\overline{X}_{i,\tau}|$ is continuous. Since $(\overline{Z}_\tau - q(\tau))/S(\tau))_{\tau \geqslant 0}$ is ANET as noted in Section 3.2, it follows that the absolute moment $\mathrm{E}|(\overline{Z}_\tau - q(\tau))/S(\tau)|$ converges to the absolute moment of $N(0, 1)$ as $\tau \to \infty$. Furthermore, $q(\tau)$, $1/S(\tau)$ and $\mathrm{E}|\overline{Z}_\tau|$ are bounded on compact subintervals of $[0, \infty)$. This shows that there is a constant $C$ such that $\mathrm{E}|\overline{Z}_\tau - q(\tau)| \leqslant CS(\tau)$ for all $\tau \geqslant 0$. Using (3.2), this implies that

$$\mathrm{E}|\overline{X}_{i,\tau} - q_i(\tau)| \leqslant C\sigma_i(\tau), \qquad \forall \tau \geqslant 0, \forall i \in \mathbb{Z}. \tag{4.2}$$

In particular, it follows that for any $s > 0$,

$$\sup_{0 \leqslant \tau \leqslant s} \mathrm{E}|\overline{X}_{i,\tau}| \leqslant C \sup_{0 \leqslant \tau \leqslant s} \sigma_i(\tau) + \sup_{0 \leqslant \tau \leqslant s} |q_i(\tau)|,$$

implying absolute and uniform convergence on compacts of $\sum_{i=-\infty}^{\infty} \mathrm{E}\overline{X}_{i,\tau}$. The convergence of $\sum_{i=-\infty}^{\infty} \mathrm{E}|\overline{X}_{i,\tau}|$ gives almost sure convergence of $\sum_{i=-\infty}^{\infty} \overline{X}_{i,\tau}$. Note that uniform convergence on compacts of $\sum \mathrm{d} \log \Phi_i(\tau)/\mathrm{d}\tau$ implies uniform convergence on compacts of $\sum \log \Phi_i(\tau)$ and hence of $\prod_{i=-\infty}^{\infty} \Phi_i(\tau)$. That the limit is in fact $\Phi(\tau)$ follows from the dominated convergence theorem. For application of the latter, construct a random variable $\tilde{Z}$ such that $\tilde{Z} = Z$ if $Z \geqslant 0$, and $\tilde{Z} \in [0, 1]$ if $Z < 0$, and such that $\tilde{Z}$ has a bounded density. Then if $(\tilde{Z}_i)_{i \in \mathbb{Z}}$ is an i.i.d. sequence with distribution $\tilde{Z}$, the same calculations as before show that $\prod_{i=-\infty}^{\infty} \mathrm{e}^{c_i \tilde{Z}_i}$ is an integrable majorant. That the exponential family associated with $\sum X_i$ is indeed $(\sum_{i=-\infty}^{\infty} \overline{X}_{i,\tau})_{\tau \geqslant 0}$ has already been noted in Section 3.1. □

**Lemma 4.2.** *Under Assumptions 2.1 and 2.2,*

$$\frac{1}{\sigma_\infty(\tau)} \sum_{i=-\infty}^{\infty} (\overline{X}_{i,\tau} - q_i(\tau)) \xrightarrow{d} N(0, 1), \qquad \tau \to \infty. \tag{4.3}$$

***Proof.*** For $\tau \geqslant 0$ and $m \in \mathbb{N}$ such that not all of the $(c_i)_{|i| \leqslant m}$ are zero, define

$$A_{m\tau} := \sum_{i=-m}^{m} (\overline{X}_{i,\tau} - q_i(\tau)) \left( \frac{1}{\sigma_\infty(\tau)} - \frac{1}{\left(\sum_{j=-m}^{m} \sigma_j^2(\tau)\right)^{1/2}} \right),$$

$$B_{m\tau} := \frac{\sum_{|i|>m} (\overline{X}_{i,\tau} - q_i(\tau))}{\sigma_\infty(\tau)}.$$

Then

$$\frac{\sum_{i=-\infty}^{\infty}(\overline{X}_{i,\tau}-q_i(\tau))}{\left(\sum_{i=-\infty}^{\infty}\sigma_i^2(\tau)\right)^{1/2}}-\frac{\sum_{i=-m}^{m}(\overline{X}_{i,\tau}-q_i(\tau))}{\left(\sum_{i=-m}^{m}\sigma_i^2(\tau)\right)^{1/2}}=A_{m\tau}+B_{m\tau}.$$

By the ANET property,

$$\frac{\sum_{|i|\leqslant m}(\overline{X}_{i,\tau}-q_i(\tau))}{\left(\sum_{|i|\leqslant m}\sigma_i^2(\tau)\right)^{1/2}}\xrightarrow{d}N(0,\,1),\qquad \tau\to\infty.$$

Then (4.3) follows from a variant of Slutsky's theorem (see Billingsley 1999, Theorem 3.2), provided that for any $\varepsilon>0$,

$$\lim_{m\to\infty}\limsup_{\tau\to\infty}P(|A_{m\tau}|>\varepsilon)=0=\lim_{m\to\infty}\limsup_{\tau\to\infty}P(|B_{m\tau}|>\varepsilon). \tag{4.4}$$

To show (4.4), write

$$A_{m\tau}=\frac{\sum_{i=-m}^{m}(\overline{X}_{i,\tau}-q_i(\tau))}{\left(\sum_{i=-m}^{m}\sigma_i^2(\tau)\right)^{1/2}}\left(\left(\frac{\sum_{j=-m}^{m}\sigma_j^2(\tau)}{\sigma_\infty^2(\tau)}\right)^{1/2}-1\right).$$

Since $\lim_{\tau\to\infty}\mathrm{E}|\sum_{i=-m}^{m}(\overline{X}_{i,\tau}-q_i(\tau))/(\sum_{i=-m}^{m}\sigma_i^2(\tau))^{1/2}|=\sqrt{2/\pi}$, it follows from (2.3) that

$$\limsup_{m\to\infty}\limsup_{\tau\to\infty}\mathrm{E}(|A_{m\tau}|)\leqslant\sqrt{\frac{2}{\pi}}\limsup_{m\to\infty}\limsup_{\tau\to\infty}\left(1-\left(\frac{\sum_{j=-m}^{m}\sigma_j^2(\tau)}{\sigma_\infty^2(\tau)}\right)^{1/2}\right)=0,$$

implying the left-hand equality of (4.4) by Markov's inequality. The right-hand side of (4.4) follows similarly from (2.4), noting that

$$\mathrm{E}|B_{m\tau}|\leqslant\frac{\sum_{|i|>m}\mathrm{E}|\overline{X}_{i,\tau}-q_i(\tau)|}{\sigma_\infty(\tau)}\leqslant\frac{C\sum_{|i|>m}\sigma_i(\tau)}{\sigma_\infty(\tau)}$$

by (4.2). □

**Lemma 4.3.** (a) *Suppose that Assumptions 2.1 and 2.2 hold. Then* $\sigma_\infty(\tau)^{-1}\sum(\overline{X}_{i,\tau}-q_i(\tau))$ *has a density, denoted by* $r_\tau(x)$*, which converges locally uniformly to the density* $\varphi(x)$ *of the standard normal distribution, as* $\tau\to\infty$*. Furthermore, the densities* $r_\tau$ *are uniformly bounded by the same constant for sufficiently large* $\tau$*.*

(b) *Suppose that Assumption 2.1 holds and that* $\psi$ *and* $\theta$ *are as in Assumption 2.3. Let* $c$*,* $d$ *be positive constants. Then there are positive constants* $\tau_0$*,* $D_0$*, such that for any coefficient sequence in* $\mathcal{G}_{c,d,\theta}$ *the density* $r_\tau$ *is bounded by* $D_0$ *for any* $\tau\geqslant\tau_0$*.*

***Proof.*** (a) By (2.3), there is some $m\in\mathbb{N}_0$ such that

$$\frac{1}{2}\leqslant\frac{1}{\sigma_\infty(\tau)}\sqrt{\sum_{|i|\leqslant m}\sigma_i^2(\tau)}\leqslant 1\qquad\text{for large }\tau. \tag{4.5}$$

Denote by $g_\tau$ the density of $\sum_{|i|\leqslant m}(\overline{X}_{i,\tau}-q_i(\tau))/\sqrt{\sum_{|i|\leqslant m}\sigma_i^2(\tau)}$. By the ANET property,

$g_\tau(x)$ converges locally uniformly to $\varphi(x)$ as $\tau \to \infty$, and $|g_\tau(x)| \leqslant \mathrm{e}^{-|x|}$ for large $x$ and $\tau$. This implies that for any $\varepsilon > 0$ there exist $\delta_{1,\varepsilon} > 0$ and $\tau_{1,\varepsilon}$ such that

$$|g_\tau(x) - g_\tau(y)| \leqslant \varepsilon, \qquad \forall \tau \geqslant \tau_{1,\varepsilon}, \forall x, y \in \mathbb{R} : |x - y| \leqslant \delta_{1,\varepsilon}.$$

The density of $\sum_{|i|\leqslant m}(\overline{X}_{i,\tau} - q_i(\tau))/\sigma_\infty(\tau)$ is given by

$$x \mapsto g_\tau\left(\frac{\sigma_\infty(\tau)}{\sqrt{\sum_{|i|\leqslant m}\sigma_i^2(\tau)}}x\right)\frac{\sigma_\infty(\tau)}{\sqrt{\sum_{|i|\leqslant m}\sigma_i^2(\tau)}} =: h_\tau(x).$$

By (4.5) there exist $\delta_{2,\varepsilon} > 0$ and $\tau_{2,\varepsilon}$ such that

$$|h_\tau(x) - h_\tau(y)| \leqslant \varepsilon, \qquad \forall \tau \geqslant \tau_{2,\varepsilon}, \forall x, y \in \mathbb{R} : |x - y| \leqslant \delta_{2,\varepsilon}.$$

Denote by $H_\tau$ the distribution function of $\sum_{|i|>m}(\overline{X}_{i,\tau} - q_i(\tau))/\sigma_\infty(\tau)$. Then

$$\frac{\sum_{i=-\infty}^{\infty}(\overline{X}_{i,\tau} - q_i(\tau))}{\sigma_\infty(\tau)} = \frac{\sum_{|i|\leqslant m}(\overline{X}_{i,\tau} - q_i(\tau))}{\sigma_\infty(\tau)} + \frac{\sum_{|i|>m}(\overline{X}_{i,\tau} - q_i(\tau))}{\sigma_\infty(\tau)}$$

has a density, say $r_\tau(x)$ (since the first summand has a density), which satisfies

$$|r_\tau(x) - r_\tau(y)| = \left|\int_{-\infty}^{\infty}(h_\tau(x-t) - h_\tau(y-t))\mathrm{d}H_\tau(t)\right| \leqslant \int_{-\infty}^{\infty}\varepsilon\,\mathrm{d}H_\tau(t) = \varepsilon \tag{4.6}$$

for all $\tau \geqslant \tau_{2,\varepsilon}$ and $x, y \in \mathbb{R}$ such that $|x - y| \leqslant \delta_{2,\varepsilon}$. Similarly, one obtains that the $r_\tau$ are uniformly bounded for large $\tau$. Now assume that $r_\tau(x)$ does not converge to $\varphi(x)$ as $\tau \to \infty$ for all $x \in \mathbb{R}$. Without loss of generality assume that

$$\varphi(x_0) + 3\varepsilon \leqslant \limsup_{\tau\to\infty} r_\tau(x_0)$$

in some $x_0$ and for sufficiently small $\varepsilon > 0$. Then there is a subsequence $(\tau_n)_{n\in\mathbb{N}}$ tending to $\infty$ such that $\lim_{n\in\infty} r_{\tau_n}(x_0) = \limsup_{\tau\to\infty} r_\tau(x_0)$. By (4.6) this implies that there is some $\delta > 0$ such that for sufficiently large $n$,

$$r_{\tau_n}(y) \geqslant \varphi(y) + \varepsilon, \qquad \forall y \in [x_0 - \delta, x_0 + \delta].$$

It follows that

$$\lim_{n\to\infty}\int_{x_0-\delta}^{x_0+\delta} r_{\tau_n}(y)\mathrm{d}y \geqslant \int_{x_0-\delta}^{x_0+\delta}(\varphi(y) + \varepsilon)\mathrm{d}y,$$

contradicting Lemma 4.2. This shows that $r_\tau(x)$ converges to $\varphi(x)$ in any $x \in \mathbb{R}$ as $\tau \to \infty$, and by (4.6) we see that this convergence is locally uniform.

(b) By Proposition 3.2, there is a constant $D_1 > 0$ such that for any $(c_i)_{i\in\mathbb{Z}} \in \mathcal{G}_{c,d,\theta}$, $D_1 \leqslant \sigma_{i_0}(\tau)/\sigma_\infty(\tau) \leqslant 1$ for $\tau \geqslant 0$. Denote by $g_\tau$ the density of $(\overline{X}_{i_0,\tau} - q_{i_0}(\tau))/\sigma_{i_0}(\tau) \stackrel{d}{=} (\overline{Z}_{c_{i_0}\tau} - q(c_{i_0}\tau))/S(c_{i_0}\tau)$. Since $c/2 \leqslant c_{i_0}$, it follows from the ANET property of $((\overline{Z}_\tau - q(\tau))/\mathrm{S}(\tau))_{\tau\geqslant 0}$ that there exist $\tau_0$, $D_2$, depending only on $f$, $\psi$ and $c$, such that $g_\tau$ is bounded by $D_2$ for $\tau \geqslant \tau_0$. The density $h_\tau$ of $(\overline{X}_{i_0,\tau} - q_{i_0}(\tau))/\sigma_\infty(\tau)$ is then bounded by $D_0 := D_2/D_1$ for $\tau \geqslant \tau_0$. Similarly to (4.6), this then implies that $r_\tau$ is bounded by $D_0$ for $\tau \geqslant \tau_0$. $\square$

We are now able to prove the first part of Theorem 2.1.

***Proof of* (2.5) *in Theorem 2.1.*** Using (3.1), it follows that

$$P\left(\sum_{i=-\infty}^{\infty} c_i Z_i > Q(\tau)\right)$$

$$= \mathrm{E}(\mathrm{e}^{-\tau\sum \overline{X}_{i,\tau}} 1_{\sum \overline{X}_{i,\tau} > Q(\tau)})\Phi(\tau)$$

$$= \mathrm{E}(\mathrm{e}^{-\tau\sigma_\infty(\tau)\sum(\overline{X}_{i,\tau}-q_i(\tau))/\sigma_\infty(\tau)} 1_{\sum(\overline{X}_{i,\tau}-q_i(\tau))/\sigma_\infty(\tau)>0})\mathrm{e}^{-\tau Q(\tau)}\Phi(\tau)$$

$$= \mathrm{e}^{-\tau Q(\tau)}\Phi(\tau)\int_0^\infty \mathrm{e}^{-\tau\sigma_\infty(\tau)x} r_\tau(x)\mathrm{d}x.$$

Noting that

$$\lim_{\tau\to\infty} \tau^2 q'(\tau) = \lim_{\tau\to\infty} \frac{\tau^2}{\psi''((\psi')^{\leftarrow}(\tau))} = \lim_{t\to\infty} \frac{\psi'(t)^2}{\psi''(t)},$$

where the last limit was shown to equal $\infty$ in Balkema *et al.* (1993, Proposition 5.8), it follows that

$$\lim_{\tau\to\infty} \tau\sigma_\infty(\tau) = \infty. \tag{4.7}$$

Then using dominated convergence and Lemma 4.3(a) gives

$$\tau\sigma_\infty(\tau)\int_0^\infty \mathrm{e}^{-\tau\sigma_\infty(\tau)x} r_\tau(x)\mathrm{d}x = \int_0^\infty \mathrm{e}^{-z} r_\tau(z/(\tau\sigma_\infty(\tau))\mathrm{d}z$$

$$\to \int_0^\infty \mathrm{e}^{-z}\frac{1}{\sqrt{2\pi}}\mathrm{d}z = \frac{1}{\sqrt{2\pi}}, \qquad \tau\to\infty,$$

implying (2.5). □

With exactly the same proof, but now using part (b) of Lemma 4.3 instead of part (a), we obtain the following uniform estimate, which will be used in Lemma 4.6:

**Lemma 4.4.** *Suppose that Assumption 2.1 holds and that $\psi$ and $\theta$ are as in Assumption 2.3. Let c, d be positive constants. Then there exist positive constants $\tau_0$, $D_0$, such that for any coefficient sequence $(c_i)_{i\in\mathbb{Z}}$* in *$\mathcal{G}_{c,d,\theta}$,*

$$P\left(\sum_{i=-\infty}^{\infty} c_i Z_i > Q(\tau)\right) \leqslant \frac{D_0}{\tau\sigma_\infty(\tau)}\mathrm{e}^{-\tau Q(\tau)}\Phi(\tau), \qquad \tau \geqslant \tau_0. \tag{4.8}$$

In order to derive the approximation for the tail behaviour of $Y_0$ as stated in the second part of Theorem 2.1, we need estimates for $\Phi$, which are derived in the following lemma:

**Lemma 4.5.** (a) *Suppose that Assumptions 2.1 and 2.2 hold. Then, for $\tau \geqslant 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\log(\mathrm{e}^{-\tau Q(\tau)}\Phi(\tau)) = -\tau\sigma_\infty^2(\tau) + \sum_{i=-\infty}^{\infty}(\mathrm{E}\overline{X}_{i,\tau} - q_i(\tau)) = -\tau\sigma_\infty^2(\tau) + o(\sigma_\infty(\tau)), \qquad \tau\to\infty.$$

(b) *Suppose that Assumption 2.1 holds and that $\psi$ and $\theta$ are as in Assumption 2.3. Let $c$, $d$ be positive constants. Then there exists a positive constant $D$, such that, for any coefficient sequence $(c_i)_{i\in\mathbb{Z}}$ in $\mathcal{G}_{c,d,\theta}$,*

$$\sum_{i=-\infty}^{\infty}|\mathrm{E}\overline{X}_{i,\tau} - q_i(\tau)| \leqslant D\sigma_\infty(\tau), \qquad \tau \geqslant 0. \tag{4.9}$$

***Proof.*** (a) From Lemma 4.1 and (2.2) follows that, for any $\tau \geqslant 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(-\tau Q(\tau) + \log\Phi(\tau)) = -\tau Q'(\tau) - Q(\tau) + \sum_{i=-\infty}^{\infty}\mathrm{E}\overline{X}_{i,\tau} = -\tau\sigma_\infty^2(\tau) + \sum_{i=-\infty}^{\infty}(\mathrm{E}\overline{X}_{i,\tau} - q_i(\tau)).$$

Let $\varepsilon > 0$. By (4.2) and (2.4), there exists an $m_\varepsilon \in \mathbb{N}$ such that

$$\limsup_{\tau\to\infty}\mathrm{E}\sum_{|i|>m_\varepsilon}\left|\frac{\overline{X}_{i,\tau} - q_i(\tau)}{\sigma_\infty(\tau)}\right| \leqslant \varepsilon.$$

Furthermore, from the ANET property of $\sum_{i=-m_\varepsilon}^{m_\varepsilon}(\overline{X}_{i,\tau} - q_i(\tau))/\sqrt{\sum_{i=-m_\varepsilon}^{m_\varepsilon}\sigma_i^2(\tau)}$ it follows that

$$\limsup_{\tau\to\infty}\left|\frac{\mathrm{E}\sum_{i=-m_\varepsilon}^{m_\varepsilon}(\overline{X}_{i,\tau} - q_i(\tau))}{\sigma_\infty(\tau)}\right| \leqslant \limsup_{\tau\to\infty}\left|\frac{\mathrm{E}\sum_{i=-m_\varepsilon}^{m_\varepsilon}(\overline{X}_{i,\tau} - q_i(\tau))}{\sqrt{\sum_{i=-m_\varepsilon}^{m_\varepsilon}\sigma_i^2(\tau)}}\right| = 0.$$

Since $\varepsilon > 0$ was arbitrary, the assertion follows.

(b) From (4.2) it follows that there is a positive constant $C$, depending only on the density $f$ and $\psi$, such that $|\mathrm{E}\overline{X}_{i,\tau} - q_i(\tau)| \leqslant C\sigma_i(\tau)$ for $\tau \geqslant 0$. By (3.5), there exists a constant $C_1$, depending only on $\psi$ and $\theta$, such that for any coefficient sequence in $\mathcal{G}_{c,d,\theta}$,

$$\sum_{i=-\infty}^{\infty}\sigma_i(\tau) \leqslant \sqrt{C_1}\sum_{i=-\infty}^{\infty}c_i^{1-\theta/2}c_{i_0}^{\theta/2-1}c_{i_0}\sqrt{q'(c_{i_0}\tau)} \leqslant \sqrt{C_1}d(c/2)^{\theta/2-1}\sigma_{i_0}(\tau), \qquad \tau \geqslant 0,$$

giving (4.9). □

We are now able to complete the proof of Theorem 2.1.

***Proof of* (2.6) *in Theorem 2.1.*** By (2.5) and Lemma 4.5(a), there is a function $\zeta(\tau) = o(\sigma_\infty(\tau))$, $\tau\to\infty$, such that

$$P\left(\sum_{i=-\infty}^{\infty}c_iZ_i > Q(\tau)\right) \sim \frac{1}{2\pi\tau\sigma_\infty(\tau)}\exp\left(-\int_0^\tau(uQ'(u) + \zeta(u))\mathrm{d}u\right), \qquad \tau\to\infty. \tag{4.10}$$

Setting $t = Q(\tau)$ and $\rho(\tau) := \zeta(\tau)/\sigma_\infty^2(\tau) = o(1/\sigma_\infty(\tau))$, $\tau\to\infty$, (2.6) follows from

$$\int_0^{Q^{\leftarrow}(t)} \left( uQ'(u) + \frac{\zeta(u)}{Q'(u)} Q'(u) \right) \mathrm{d}u = \int_{t_0 \sum c_i}^{t} (Q^{\leftarrow}(v) + \rho(Q^{\leftarrow}(v)))\mathrm{d}v.$$

That $1/\sigma_\infty(\tau) = o(\tau)$, $\tau \to \infty$, follows from (4.7). □

In Section 5 we will need uniform estimates for the tail behaviour, which are derived in the following lemma:

**Lemma 4.6.** *Suppose that Assumption 2.1 holds and that $\psi$ and $\theta$ are as in Assumption 2.3. Let $c$, $d$ be positive constants. Then there are positive constants $D_1$, $D_2$, $t_1$ such that for any coefficient sequence $(c_i)_{i\in\mathbb{Z}}$ in $\mathcal{G}_{c,d,\theta}$,*

$$P\left( \sum_{i=-\infty}^{\infty} c_i Z_i > t \right) \leqslant D_1 \exp\left( -\int_{t_0 \sum c_i}^{t} \left( Q^{\leftarrow}(v) - \frac{D_2}{\sigma_\infty(Q^{\leftarrow}(v))} \right) \mathrm{d}v \right), \qquad t \geqslant t_1. \quad (4.11)$$

*Furthermore, for any fixed sequence $(c_i)_{i\in\mathbb{Z}}$ in $\mathcal{G}_{c,d,\theta}$, there exist positive constants $D_3$, $D_4$, $t_2$ such that*

$$P\left( \sum_{i=-\infty}^{\infty} c_i Z_i > t \right) \geqslant D_3 \exp\left( -\int_{t_0 \sum c_i}^{t} \left( Q^{\leftarrow}(v) + \frac{D_4}{\sigma_\infty(Q^{\leftarrow}(v))} \right) \mathrm{d}v \right), \qquad t \geqslant t_2. \quad (4.12)$$

***Proof.*** Similarly to (4.10), but now using Lemma 4.4 and Lemma 4.5(b), there exist $\tau_0$, $D_0 > 0$ such that

$$P\left( \sum_{i=-\infty}^{\infty} c_i Z_i > Q(\tau) \right) \leqslant \frac{D_0}{\tau\sigma_\infty(\tau)} \exp\left( -\int_0^{\tau} (uQ'(u) + \zeta(u))\mathrm{d}u \right), \quad (4.13)$$

for $\tau \geqslant \tau_0$ and any coefficient sequence $(c_i)_{i\in\mathbb{Z}}$ in $\mathcal{G}_{c,d,\theta}$. Further, $|\zeta(\tau)| \leqslant D\sigma_\infty(\tau)$ for $\tau \geqslant 0$, with $D$ from Lemma 4.5. Choosing $\tau_1 \geqslant \tau_0$ such that $q(c\tau_1) \geqslant 0$ and using the monotonicity of $q$, it follows that for $t \geqslant t_1 := dq(c\tau_1)$,

$$t \geqslant dq(c\tau_1) \geqslant \sum_{i=-\infty}^{\infty} c_i q(c\tau_1) \geqslant \sum_{i=-\infty}^{\infty} c_i q(c_i\tau_1) = Q(\tau_1). \quad (4.14)$$

This shows that (4.13) holds for any $t = Q(\tau) \geqslant t_1$, and $t_1$ is independent of the specific coefficient sequence in $\mathcal{G}_{c,d,\theta}$. Since $\tau^2\sigma_\infty^2(\tau) \geqslant \tau^2 c_{i_0}^2 q'(c_{i_0}\tau)$, it follows as in the proof of (2.5) that (4.7) holds uniformly for the sequences in $\mathcal{G}_{c,d,\theta}$, hence $D_0/(\tau\sigma_\infty(\tau))$ in (4.13) can be replaced by some $D_1$. Then (4.11) follows as in the proof of (2.6).

For the proof of (4.12), for a fixed coefficient sequence, note that (4.10) implies that the inequality in (4.13) can be reversed, by replacing $D_0$ by $1/3 < 1/\sqrt{2\pi}$. Once it is shown that for large $\tau$,

$$\tau\sigma_\infty(\tau) \leqslant \exp\left( \int_0^{\tau} \sigma_\infty(v)\mathrm{d}v \right), \quad (4.15)$$

relation (4.12) follows similarly to (4.11). From (3.4) and the dominated convergence theorem it follows that there is a $C > 0$ such that $\sigma_\infty(\tau) \sim C\sqrt{q'(c_{i_0}\tau)}$, $\tau \to \infty$. Now if $\beta \in (-1, \infty]$, that is, $q' \in \mathrm{RV}_{\beta'}$ with $\beta' \in [-1, \infty)$, then $\tau\sigma_\infty(\tau)/\int_0^\tau \sigma_\infty(u)\mathrm{d}u \to 1 + \beta'/2$, $\tau \to \infty$, by Karamata's theorem (see, for example, Bingham *et al.* 1987, Theorem 1.5.11), clearly implying (4.15) for large $\tau$. If $\psi'' \in \mathrm{RV}_{-1}$, then $q' \in \mathrm{RV}_\infty$, and by Proposition 3.2, $\tau\sigma_\infty(\tau) \leqslant (q'(c_{i_0}\tau))^{2/3}$ for large $\tau$. For simplicity, assume that $c_{i_0} = 1$. With $s := q(\tau)$ it follows for large $s$ that $q^{\leftarrow}(s)\sigma_\infty(q^{\leftarrow}(s)) \leqslant (q'(q^{\leftarrow}(s)))^{2/3} = (1/\psi''(s))^{2/3}$, and the latter function is in $\mathrm{RV}_{2/3}$. On the other hand,

$$\int_0^{q^{\leftarrow}(s)} \sigma_\infty(v)\mathrm{d}v \geqslant \int_0^{q^{\leftarrow}(s)} \sqrt{q'(v)}\,\mathrm{d}v = \int_{t_0}^{s} \frac{1}{\sqrt{q'(q^{\leftarrow}(u))}}\,\mathrm{d}u = \int_{t_0}^{s} \sqrt{\psi''(u)}\,\mathrm{d}u,$$

which (as a function in $s$) is in $\mathrm{RV}_{1/2}$. But this then clearly implies (4.15) for large $s = q(\tau)$. □

We can now show that the i.i.d. sequence associated with $Y_0$ is in MDA($\wedge$).

***Proof of Theorem 2.2.*** Once (2.7) has been shown, it follows readily that

$$\lim_{n\to\infty} nP\left(Y_0 > b_n + \frac{x}{a_n}\right) = \lim_{n\to\infty} \frac{P(Y_0 > b_n + x/Q^{\leftarrow}(b_n))}{P(Y_0 > b_n)} = \mathrm{e}^{-x}, \qquad x \in \mathbb{R},$$

showing that the associated i.i.d. sequence is in MDA($\wedge$) with norming constants $a_n$ and $b_n$, (see, for example, Embrechts *et al.* 1997, Proposition 3.3.2). Thus, it only remains to show (2.7). Let

$$\tau := Q^{\leftarrow}(t) \quad \text{and} \quad \tau^* := Q^{\leftarrow}\left(t + \frac{x}{Q^{\leftarrow}(t)}\right).$$

Then by (2.5),

$$\lim_{t\to\infty} \frac{P(Y_0 > t + x/Q^{\leftarrow}(t))}{P(Y_0 > t)} = \lim_{t\to\infty} \frac{P(Y_0 > Q(\tau^*))}{P(Y_0 > Q(\tau))} = \lim_{t\to\infty} \frac{\tau\sigma_\infty(\tau)}{\tau^*\sigma_\infty(\tau^*)} \frac{\mathrm{e}^{-\tau^* Q(\tau^*)}\Phi(\tau^*)}{\mathrm{e}^{-\tau Q(\tau)}\Phi(\tau)}.$$

Thus (2.7) will follow once we have shown that

$$\lim_{t\to\infty} \frac{Q^{\leftarrow}(t)}{Q^{\leftarrow}(t + x/Q^{\leftarrow}(t))} = 1 = \lim_{t\to\infty} \frac{Q'(Q^{\leftarrow}(t))}{Q'(Q^{\leftarrow}(t + x/Q^{\leftarrow}(t)))} \tag{4.16}$$

and

$$\lim_{t\to\infty} \int_\tau^{\tau^*} \frac{\mathrm{d}}{\mathrm{d}u} \log(\mathrm{e}^{-uQ(u)}\Phi(u))\mathrm{d}u = -x.$$

By (2.3), for any $\varepsilon > 0$ there exist $m = m_\varepsilon$ in $\mathbb{N}$ and $u_\varepsilon \in \mathbb{R}$ such that

$$P_m'(u) \leqslant Q'(u) \leqslant (1+\varepsilon)P_m'(u), \qquad \forall u \geqslant u_\varepsilon,$$

where $P_m(u) := \sum_{|i|\leqslant m} c_i q(c_i u)$. But in Balkema *et al.* (1993, Theorem 1.1) it is shown that $\sqrt{P_m'(P_m^{\leftarrow})}$ is self-neglecting. By Lemma 3.1(e) this implies that $1/\sqrt{P_m'}$ is self-neglecting. In particular,

$$\lim_{u\to\infty} \frac{P'_m(u + x/\sqrt{P'_m(u)})}{P'_m(u)} = 1,$$

uniformly on bounded $x$-intervals. But

$$\frac{1}{1+\varepsilon}\frac{P'_m(u + x/\sqrt{Q'(u)})}{P'_m(u)} \leqslant \frac{Q'(u + x/\sqrt{Q'(u)})}{Q'(u)} \leqslant (1+\varepsilon)\frac{P'_m(u + x/\sqrt{Q'(u)})}{P'_m(u)}$$

uniformly in bounded $x$ for large $u$ by (4.18) and (4.7). Since $P'_m \leqslant Q'$ and $1/\sqrt{P'_m}$ is self-neglecting, we estimate

$$\frac{1}{1+\varepsilon} \leqslant \liminf_{u\to\infty} \frac{Q'(u + x/\sqrt{Q'(u)})}{Q'(u)} \leqslant \limsup_{u\to\infty} \frac{Q'(u + x/\sqrt{Q'(u)})}{Q'(u)} \leqslant 1+\varepsilon$$

uniformly in bounded $x$-intervals, showing that $1/\sqrt{Q'}$ is self-neglecting and hence so is $\sigma_\infty(Q^{\leftarrow})$ by Lemma 3.1(e). But this then implies the right-hand side of (4.16), since $1/Q^{\leftarrow}(t)$ is smaller than $\sigma_\infty(Q^{\leftarrow}(t))$ for large $t$ by (4.7). The left-hand side of (4.16) follows from Resnick (1987, Lemma 1.3), noting that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{Q^{\leftarrow}(t)} = -(Q^{\leftarrow}(t))^{-2}\sigma_\infty^{-2}(Q^{\leftarrow}(t)) \to 0, \qquad t\to\infty,$$

by (4.7). For the proof of (4.17), note that by Lemma 4.5 and (4.7),

$$\frac{\mathrm{d}}{\mathrm{d}u}\log(\mathrm{e}^{-uQ(u)}\Phi(u)) = -u\sigma_\infty^2(u) + o(u\sigma^2{}_\infty(u)), \qquad u\to\infty.$$

Now

$$\int_\tau^{\tau^*} u\sigma^2{}_\infty(u)\mathrm{d}u = \int_{Q^{\leftarrow}(t}^{Q^{\leftarrow}(t+x/Q^{\leftarrow}(t))} uQ'(u)\mathrm{d}u = \int_t^{t+x/Q^{\leftarrow}(t)} Q^{\leftarrow}(v)\mathrm{d}v = \frac{x}{Q^{\leftarrow}(t)}Q^{\leftarrow}(\xi)$$

with some $\xi$ between $t$ and $t + x/Q^{\leftarrow}(t)$. As $t\to\infty$, the last expression converges to $x$ since $\tau^*/\tau \to 1$ and by monotonicity of $Q$. This implies (4.17), completing the proof. □

# 5. Proof of Theorem 2.3

In this section we prove that the extremal behaviour of the moving average process is the same as the behaviour of the associated i.i.d. sequence. This will be achieved by verifying Leadbetter's $D(u_n)$ and $D'(u_n)$ conditions. For definitions and results we refer to Embrechts *et al.* (1997, Section 4.4) or Leadbetter *et al.* (1983, Chapter 3). $D(u_n)$ is a mixing condition, and $D'(u_n)$ can be interpreted as an anti-clustering condition. We shall show that both conditions hold for $(Y_n)_{n\in\mathbb{N}}$, which implies then that its extremal behaviour is exactly as for the associated i.i.d. sequence. We need the following result of Rootzén (1986, Lemmas 3.1 and 3.2):

**Proposition 5.1.** *Suppose that the i.i.d. sequence associated with* $(Y_n)_{n\in\mathbb{N}}$*, given by* (1.1)*, is in* MDA($\Lambda$) *with norming constants* $a_n$ *and* $b_n$*, and that* $u_n := x/a_n + b_n$.

(a) *If* $\mathrm{E}Z^2 < \infty$, $|c_i| = O(|i|^{-\vartheta})$ *for some* $\vartheta > 1$ *as* $|i| \to \infty$, *and* $a_n = O((\log n)^\alpha)$ *for some* $\alpha > 0$ *as* $n \to \infty$, *then* $D(u_n)$ *holds.*

(b) *If, in addition to the conditions of (a), for some constant* $\gamma_0 \in (0, 1]$ *for* $n' := \lfloor n^{\gamma_0} \rfloor$ *as* $n \to \infty$,

$$n \sum_{m=1}^{2n'} P(Y_0 + Y_m > 2u_n) \to 0, \tag{5.1}$$

$$n^2 P\left( a_n \sum_{i=n'+1}^{\infty} c_i Z_i > 1 \right) \to 0, \qquad n^2 P\left( a_n \sum_{i=-\infty}^{-n'-1} c_i Z_i > 1 \right) \to 0, \tag{5.2}$$

$$a_n \sum_{i=n'+1}^{\infty} c_i Z_i \xrightarrow{P} 0, \qquad a_n \sum_{i=-\infty}^{-n'-1} c_i Z_i \xrightarrow{P} 0, \tag{5.3}$$

*then* $D'(u_n)$ *holds.*

In order to verify (5.1) under Assumption 2.1 and 2.4 we shall need Lemma 5.3. We shall see that we have to consider two different regimes, one corresponding to the case $\beta = \infty$, that is, $\psi'' \in \mathrm{RV}_\infty$, which implies $\psi \in \mathrm{RV}_\infty$, the other case being $\beta \in [-1, \infty)$, that is, $\psi \in \mathrm{RV}_\alpha$ for some $\alpha \in [1, \infty)$. We split up the proof into the cases $\beta \in [-1, \infty)$ and $\beta = \infty$, and for the latter case we need some preparation:

**Lemma 5.2.** *Suppose that Assumption 2.1 holds, that* $\psi''$ *is ultimately absolutely continuous on compacts and that*

$$\lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\psi'(t)}{\psi''(t)} = 0.$$

*Then there exist a constant* $\tau_1 \geqslant 0$ *and a* $C^1$ *function* $p : [0, \infty) \to (0, \infty)$ *which is (almost everywhere) twice differentiable, satisfies*

$$p(\tau) = q(\tau), \qquad \tau \geqslant \tau_1,$$

$p'(\tau) > 0$ *for all* $\tau \geqslant 0$, $p''(\tau) \leqslant 0$ *for* $\tau \geqslant 0$ *(almost everywhere), and, for any constants* $c_2 \geqslant c_1 \geqslant 0$,

$$c_1 p(c_1\tau) + c_2 p(c_2\tau) - (c_1 + c_2) p\Big(\frac{c_1 + c_2}{2}\tau\Big) \geqslant \frac{3(c_2 - c_1)^2}{32} \tau p'\left( \Big( \frac{c_1}{4} + \frac{3c_2}{4} \Big) \tau \right) \geqslant 0, \qquad \tau \geqslant 0. \tag{5.4}$$

***Proof.*** From Lemma 3.1(c) and its proof it follows that $q'$ is in $\mathrm{NRV}_{-1}$ and that $q''(\tau) \sim -q'(\tau)/\tau$ as $\tau \to \infty$ (where $q''$ exists almost everywhere). In particular, there exists $\tau_1$ such that $q''(\tau_1)$ exists and that

$$-\tfrac{3}{4} q'(\tau) \geqslant \tau q''(\tau) \geqslant -\tfrac{5}{4} q'(\tau), \qquad \tau \geqslant \tau_1 (\text{almost everywhere}).$$

Set $\mu := -\tau_1 q''(\tau_1)/q'(\tau_1)$. Then $\frac{3}{4} \leqslant \mu \leqslant \frac{5}{4}$. Define the function $p$ through

$$p(\tau) := \begin{cases} q(\tau), & \text{for } \tau \geqslant \tau_1, \\ q(\tau_1) - q'(\tau_1)\mathrm{e}^{\mu} \int_\tau^{\tau_1} \mathrm{e}^{-\mu t/\tau_1}\,\mathrm{d}t, & \text{for } 0 \leqslant \tau < \tau_1. \end{cases}$$

Then $p$ is $C^1$ and (almost everywhere) twice differentiable, and for $0 \leqslant \tau \leqslant \tau_1$,

$$p'(\tau) = q'(\tau_1)\mathrm{e}^{\mu}\mathrm{e}^{-\mu\tau/\tau_1}, \qquad p''(\tau) = -\mu p'(\tau)/\tau_1,$$

hence for $0 \leqslant \tau \leqslant \tau_1$,

$$\tau p''(\tau) = -\mu \frac{\tau}{\tau_1} p'(\tau) \geqslant -\mu p'(\tau) \geqslant -\tfrac{5}{4}p'(\tau).$$

Thus $p$ satisfies $p'(\tau) > 0$ for $\tau \geqslant 0$, and $p''(\tau) < 0$ as well as $\tau p''(\tau) \geqslant -\frac{5}{4}p'(\tau)$ for $\tau \geqslant 0$ (almost everywhere). For the positivity of $p$, note that $p(0) \geqslant q(\tau_1) - \mathrm{e}^{\mu}\tau_1 q'(\tau_1)$, which is positive for large enough $\tau_1$, since $\lim_{\tau\to\infty}\tau q'(\tau)/q(\tau) = 0$ by Karamata's theorem (see, Bingham *et al.* 1987, p. 26).

Now let $0 \leqslant c_1 < c_2$, set $c := c_1 + c_2$ and $c_0 := \frac{3}{4}c_1 + \frac{1}{4}c_2$. For fixed $\tau > 0$, define the function

$$k : [0, c] \to \mathbb{R}, \qquad a \mapsto k(a) := ap(a\tau) + (c-a)p((c-a)\tau).$$

Then

$$k'(a) = a\tau p'(a\tau) + p(a\tau) - p((c-a)\tau) - (c-a)\tau p'((c-a)\tau),$$

$$k''(a) = \tau[a\tau p''(a\tau) + 2p'(a\tau) + (c-a)\tau p''((c-a)\tau) + 2p'((c-a)\tau)]$$

$$\geqslant \tfrac{3}{4}\tau[p'(a\tau) + p'((c-a)\tau)] > 0 \qquad \text{almost everywhere.}$$

This shows that $k'$ is strictly increasing on $[0, c]$. Since $k'(c/2) = 0$, it follows that $k$ has an absolute minimum at $a = \frac{c}{2}$. To estimate $k(c_1) - k(c/2)$, note that $c_1 < c_0 < \frac{1}{2}c < \frac{1}{4}c_1 + \frac{3}{4}c_2 < c$. Using the mean value theorem, we see that

$$k(c_1) - k\left(\tfrac{c}{2}\right) \geqslant k(c_1) - k(c_0) = (c_0 - c_1)|k'(\xi)| \geqslant \frac{c_2 - c_1}{4}|k'(c_0)|,$$

where $\xi$ is between $c_1$ and $c_0$. Using $k'(c/2) = 0$, we obtain

$$|k'(c_0)| = \int_{c_0}^{c/2} k''(a)\mathrm{d}a$$

$$\geqslant \tfrac{3}{4}\tau\int_{c_0}^{c/2} (p'(a\tau) + p'((c-a)\tau))\mathrm{d}a$$

$$= \tfrac{3}{4}\left(p\left(\frac{c}{2}\tau\right) - p(c_0\tau) - p\left(\frac{c}{2}\tau\right) + p((c-c_0)\tau)\right).$$

Using the mean value theorem and the fact that $p'$ decreases, it then follows that

$$k(c_1) - k\left(\frac{c}{2}\right) \geqslant \frac{3(c_2 - c_1)}{16}[p((c-c_0)\tau) - p(c_0\tau)] \geqslant \frac{3(c_2-c_1)^2\tau}{32}p'((c-c_0)\tau),$$

which proves the assertion. □

We now come to the crucial step in showing (5.1). If in the following $m_0$ can be chosen to be equal to 1, then (5.6) is redundant and the stronger assertion (5.5) holds for all positive $m$:

**Lemma 5.3.** *Suppose that Assumptions 2.1 and 2.4 hold. Then there exist a constant $\gamma_0 \in (0, 1]$, a positive integer $m_0$, a constant $t_3 \geqslant t_0$ and a family $(B_t)_{t \geqslant t_3}$ of non-negative real numbers, tending to zero as $t \to \infty$, such that*

$$\frac{P(\sum_{i=-\infty}^{\infty} \frac{1}{2}(c_i + c_{i-m}) Z_i > t)}{(P(\sum_{i=-\infty}^{\infty} c_i Z_i > t))^{1+\gamma_0}} \leqslant B_t, \qquad \forall t \geqslant t_3, \forall m \geqslant m_0, \tag{5.5}$$

$$\lim_{t \to \infty} \frac{P(\sum_{i=-\infty}^{\infty} \frac{1}{2}(c_i + c_{i-m}) Z_i > t)}{P(\sum_{i=-\infty}^{\infty} c_i Z_i > t)} = 0, \qquad \forall m \in \{1, \ldots, m_0 - 1\}. \tag{5.6}$$

***Proof.*** Define $c := c_{i_0} = \max\{c_i : i \in \mathbb{Z}\}$. Choose $\theta \in [0, 2 - 2/\vartheta)$ such that $\theta + \beta' > 0$. For any $m \in \mathbb{N}_0$, define the sequence $(c_{i,m})_{i \in \mathbb{Z}}$ by $c_{i,m} := (c_i + c_{i-m})/2$. Then $c_{i,0} = c_i$ for all $i$. The corresponding quantities associated with the sequence $(c_{i,m})_{i \in \mathbb{Z}}$ will be denoted by $Q_m$ and $\sigma_{\infty,m}$, respectively. In particular,

$$Q_m(\tau) = \sum_{i=-\infty}^{\infty} \frac{c_i + c_{i-m}}{2} q\left(\frac{c_i + c_{i-m}}{2} \tau\right).$$

If the index $m = 0$ we usually omit it, so that $Q_0 = Q$ and $\sigma_{\infty,0} = \sigma_\infty$.

By assumption, it follows that there exists $d > 0$ such that $(c_{i,m})_{i \in \mathbb{Z}} \in \mathcal{G}_{c,d,\theta}$ for all $m \in \mathbb{N}_0$. Then it follows from (4.11) and (4.12) that there are positive constants $t_3$, $D_1, \ldots, D_4$ such that, for every $m \in \mathbb{N}_0$, $\gamma \geqslant 0$ and $t \geqslant t_3$,

$$\frac{P(\sum_{i=-\infty}^{\infty} c_{i,m} Z_i > t)}{(P(\sum_{i=-\infty}^{\infty} c_i Z_i > t))^{1+\gamma}}$$
$$\leqslant \frac{D_1}{D_3^{1+\gamma}} \exp\left( - \int_{t_0 \sum c_i}^{t} \left( Q_m^{\leftarrow}(v) - (1+\gamma) Q^{\leftarrow}(v) - \frac{D_2}{\sigma_{\infty,m}(Q_m^{\leftarrow}(v))} - \frac{D_4(1+\gamma)}{\sigma_\infty(Q^{\leftarrow}(v))} \right) \mathrm{d}v \right).$$

The assertion will then follow once we have shown that there exist $m_0 \in \mathbb{N}$ and $\gamma_0 \in (0, 1]$ such that

$$\lim_{t\to\infty} \inf_{m\geqslant m_0} \int_{t_0\sum c_i}^{t} (Q_m{}^{\leftarrow}(v) - (1+\gamma_0)Q^{\leftarrow}(v))\mathrm{d}v = \infty, \tag{5.7}$$

$$\lim_{v\to\infty} \sup_{m\geqslant m_0} \frac{\sigma_{\infty,m}^{-1}(Q_m{}^{\leftarrow}(v)) + \sigma_{\infty}^{-1}(Q^{\leftarrow}(v))}{Q_m{}^{\leftarrow}(v) - (1+\gamma_0)Q^{\leftarrow}(v)} = 0, \tag{5.8}$$

$$\lim_{t\to\infty} \int_{t_0\sum c_i}^{t} (Q_m{}^{\leftarrow}(v) - Q^{\leftarrow}(v))\mathrm{d}v = 0, \qquad \forall m \in \{1, \ldots, m_0 - 1\}, \tag{5.9}$$

$$\lim_{v\to\infty} \frac{\sigma_{\infty,m}^{-1}(Q_m{}^{\leftarrow}(v)) + \sigma_{\infty}^{-1}(Q^{\leftarrow}(v))}{Q_m{}^{\leftarrow}(v) - Q^{\leftarrow}(v)} = 0, \qquad \forall m \in \{1, \ldots, m_0 - 1\}. \tag{5.10}$$

For the purpose of showing (5.7)–(5.10), we will distinguish between the cases where $\beta = \infty$ and $\beta \in [-1, \infty)$. Note that (5.9) and (5.10) are redundant if $m_0$ can be chosen to be 1.

Suppose, firstly, that $\beta = \infty$, that is, $\beta' = -1$. Set $m_0 := 1$. Since modifications of $q$ on bounded intervals can be compensated by the function $\nu$ appearing in Assumption 2.1, we can assume that $q$ already has the properties of $p$ as stated in Lemma 5.2. In particular, $q$ is strictly positive on $[0, \infty)$, and from the definitions of $Q$ and $Q_m$ we see that $Q(\tau) \leqslant Q_m(2\tau)$ for $\tau \geqslant 0$ and $m \in \mathbb{N}$. Furthermore, it is easy to see that for any $m \in \mathbb{N}$ there exists $j = j(m) \in \mathbb{Z}$ such that $\inf_{m\in\mathbb{N}}(c_{j(m)} - c_{j(m)-m}) > 0$. It then follows from (5.4) that there are positive constants $b_1$, $b_2$, such that

$$Q(\tau) - Q_m(\tau) \geqslant b_1\tau q'(b_2\tau), \qquad \forall \tau \geqslant 0, \forall m \in \mathbb{N}.$$

Thus we have

$$Q^{\leftarrow}(t) \leqslant Q_m^{\leftarrow}(t) \leqslant 2Q^{\leftarrow}(t), \qquad \forall t \geqslant t_0 \sum c_i, \forall m \in \mathbb{N}. \tag{5.11}$$

Using the mean value theorem, for fixed $t$ we find some $\xi_m \in [t, Q(Q_m^{\leftarrow}(t))]$ such that

$$\begin{aligned} Q_m^{\leftarrow}(t) - Q^{\leftarrow}(t) &= Q^{\leftarrow}(Q(Q_m^{\leftarrow}(t))) - Q^{\leftarrow}(t) \\ &= \frac{Q(Q_m{}^{\leftarrow}(t)) - Q_m(Q_m{}^{\leftarrow}(t))}{Q'(Q^{\leftarrow}(\xi_m))} \geqslant \frac{b_1 Q_m^{\leftarrow}(t) q'(b_2 Q_m^{\leftarrow}(t))}{Q'(Q^{\leftarrow}(\xi_m))}. \end{aligned}$$

Since $Q^{\leftarrow}(\xi) \in [Q^{\leftarrow}(t), Q_m^{\leftarrow}(t)]$, it follows from (3.4) and the fact that $q'$ is decreasing that there exist $b_3$, $b_4 > 0$ such that

$$Q'(Q^{\leftarrow}(\xi_m)) \leqslant b_3 q'(b_4 Q^{\leftarrow}(\xi_m)) \leqslant b_3 q'(b_4 Q^{\leftarrow}(t)).$$

Since $q' \in \mathrm{RV}_{-1}$ it follows from (5.11) that there exist $d_1$, $d_2$, $t_4 > 0$ such that

$$d_1 \leqslant \frac{q'(b_2 Q_m^{\leftarrow}(t))}{q'(b_4 Q^{\leftarrow}(t))} \leqslant d_2, \qquad \forall t \geqslant t_4, \forall m \in \mathbb{N}.$$

Then it follows from the previous estimates and (5.11) that there exists $d_3 > 0$ such that

$$Q_m^{\leftarrow}(t) - Q^{\leftarrow}(t) \geqslant d_3 Q^{\leftarrow}(t), \qquad \forall t \geqslant t_4, \forall m \in \mathbb{N}.$$

This then clearly implies (5.7) with $\gamma_0 := \min\{d_3/2, 1\}$. For the proof of (5.8), observe that with the same arguments as above, there exist constants $t_5 > 0$, $b_5 > 0$ such that for any $m \in \mathbb{N}_0$ and $v \geqslant t_5$,

$$(Q^{\leftarrow}(v))^2 \sigma^2_{\infty,m}(Q_m{}^{\leftarrow}(v)) \geqslant c^2_{i_0,m}(Q^{\leftarrow}(v))^2 q'(c_{i_0,m} Q_m{}^{\leftarrow}(v)) \geqslant b_5 (Q^{\leftarrow}(v))^2 q'(Q^{\leftarrow}(v)),$$

and the latter tends to $\infty$ by (4.7).

Now suppose that $\beta \in [-1, \infty)$, that is, $\beta' \in (-1, \infty]$. Again, modifying $q$ such that $q(0) = t_0 > 0$ does not constitute a restriction. Firstly, we show that there are constants $0 < A_1 < A_2$ and $\tau_2 > 0$ such that

$$Q_m(\tau) \leqslant A_1 q(c_{i_0}\tau) < A_2 q(c_{i_0}\tau) \leqslant Q(\tau), \qquad \forall \tau \geqslant \tau_2, \forall m \geqslant 1, \tag{5.12}$$

and, if $\beta' = \infty$, that additionally there exist $m_0 \geqslant 1$, $\tau_3 \geqslant 0$ and a constant $c' < c = c_{i_0}$ such that

$$Q_m(\tau) \leqslant A_1 q(c'\tau), \qquad \forall \tau \geqslant \tau_3, \forall m \geqslant m_0. \tag{5.13}$$

To show (5.12), note that

$$Q(\tau) = \sum_{i=-\infty}^{\infty} c_i q(c_i \tau) \sim \sum_{i=-\infty}^{\infty} \left(\frac{c_i}{c_{i_0}}\right)^{2+\beta'} c_{i_0} q(c_{i_0}\tau), \qquad \tau \to \infty,$$

by dominated convergence. Here, $\sum (c_i/c_{i_0})^{2+\beta'}$ has to be interpreted as card $\{i \in \mathbb{Z} : c_i = c_{i_0}\}$ if $\beta' = \infty$. Similarly,

$$Q_m(\tau) \sim \sum_{i=-\infty}^{\infty} \left(\frac{c_{i,m}}{c_{i_0}}\right)^{2+\beta'} c_{i_0} q(c_{i_0}\tau), \qquad \tau \to \infty,$$

if $\beta' \neq \infty$, or if $\beta' = \infty$ and $c_{i_m,m} = c_{i_0}$, where $i_m$ is defined to be an index such that $c_{i_m,m} = \max\{c_{i,m} : i \in \mathbb{Z}\}$. It is easy to check (e.g. with methods similar to those used in the proof of Lemma 5.2) that

$$A_3 := c_{i_0} \sup_{m \in \mathbb{N}} \sum_{i \in \mathbb{Z}} \left(\frac{c_{i,m}}{c_{i_0}}\right)^{2+\beta'} < c_{i_0} \sum_{i \in \mathbb{Z}} \left(\frac{c_i}{c_{i_0}}\right)^{2+\beta'} =: A_4.$$

Let $\mathrm{M} \subset \mathbb{Z}$ be a finite subset such that $\sum_{i \notin M} c_i \leqslant (A_4 - A_3)/4$, and put $M_m := M \cup (M + m)$. Then $\sum_{i \notin M_m} c_{i,m} q(c_{i,m}\tau) \leqslant (A_4 - A_3) q(c_{i_0}\tau)/4$. Furthermore, since $M$ is finite, it follows from the uniform convergence theorem for RV functions (see Bingham *et al.* 1987, Theorems 1.5.2 and 2.4.1) that

$$\lim_{\tau \to \infty} \sum_{i \in \mathrm{M}_m} \left( \frac{c_{i,m} q(c_{i,m}\tau)}{c_{i_0} q(c_{i_0}\tau)} - \left(\frac{c_{i,m}}{c_{i_0}}\right)^{2+\beta'} \right) = 0,$$

uniformly in $m \in \mathbb{N}$. Thus there exists $\tau_2$, such that for any $m \in \mathbb{N}$ and any $\tau \geqslant \tau_2$,

$$Q_m(\tau) \leqslant \frac{A_4 - A_3}{4} q(c_{i_0}\tau) + \left(A_3 + \frac{A_4 - A_3}{4}\right) q(c_{i_0}\tau) = \frac{A_4 + A_3}{2} q(c_{i_0}\tau).$$

Inequality (5.12) then follows with $A_1 := (A_4 + A_3)/2$ and $A_2 := \frac{1}{4}A_3 + \frac{3}{4}A_4$. The proof of (5.13) is similar, choosing $m_0$ and $c'$ such that

$$\sup_{m \geqslant m_0} c_{i_m, m} < c' < c_{i_0}. \tag{5.14}$$

Since $Q_m(\tau) \leqslant \sum c_i q(c_{i_0}\tau)$ for any $\tau \geqslant 0$, we have $Q_m{}^{\leftarrow}(t) \geqslant 1/c_{i_0} q^{\leftarrow}(t/\sum c_i)$, which as $t \to \infty$ converges uniformly in $m$ to $\infty$. Thus we can invert (5.12) uniformly in $m$ and obtain a constant $t_6 > 0$ such that

$$Q^{\leftarrow}(t) \leqslant \frac{1}{c_{i_0}} q^{\leftarrow}\left(\frac{t}{A_2}\right) < \frac{1}{c_{i_0}} q^{\leftarrow}\left(\frac{t}{A_1}\right) \leqslant Q_m{}^{\leftarrow}(t), \qquad \forall t \geqslant t_6, \forall m \geqslant 1.$$

If $\beta' \neq \infty$, that is, $\beta \neq -1$, set $m_0 := 1$ and choose $\gamma_0 \in (0, 1]$ such that there exists $A_5 \in (A_1, A_2)$ such that $(1+\gamma_0)\psi'(t/A_2) \leqslant \psi'(t/A_5)$ for $t \geqslant t_6$. Then for $t \geqslant t_6$ and $m \in \mathbb{N}$,

$$Q_m{}^{\leftarrow}(t) - (1+\gamma_0)Q^{\leftarrow}(t) \geqslant \frac{1}{c_{i_0}}\left(\psi'\left(\frac{t}{A_1}\right) - \psi'\left(\frac{t}{A_5}\right)\right) = \frac{1}{c_{i_0}}\left(\frac{1}{A_1} - \frac{1}{A_5}\right) t\psi''(\xi), \tag{5.15}$$

where $\xi \in [t/A_5, t/A_1]$. If $\beta' = \infty$, set $m_0$ as in (5.14), and $A_5 := A_2$. Then there is a constant $t_7$ such that

$$Q_m{}^{\leftarrow}(t) - Q^{\leftarrow}(t) \geqslant \frac{1}{c_{i_0}}\left(\frac{1}{A_1} - \frac{1}{A_5}\right) t\psi''(\xi), \qquad \forall t \geqslant t_7, \forall m \in \{1, \ldots, m_0 - 1\}, \tag{5.16}$$

with $\xi \in [t/A_5, t/A_1]$; choosing $0 < \gamma_0 < \min\{c_{i_0}/c' - 1, 1\}$, it follows by inversion of (5.13) that there is a constant $t_8$ such that for $t \geqslant t_8$ and $m \geqslant m_0$,

$$Q_m{}^{\leftarrow}(t) - (1+\gamma_0)Q^{\leftarrow}(t) \geqslant \frac{1}{c'}\psi'\left(\frac{t}{A_1}\right) - \frac{1+\gamma_0}{c_{i_0}}\psi'\left(\frac{t}{A_5}\right) \geqslant \frac{1+\gamma_0}{c_{i_0}}\left(\frac{1}{A_1} - \frac{1}{A_5}\right) t\psi''(\xi), \tag{5.17}$$

$\xi \in [t/A_5, t/A_1]$. Since $\psi'' \in \mathrm{RV}_\beta$ where $\beta \geqslant -1$, we have $\lim_{t\to\infty} t^2\psi''(t) = \infty$, and (5.7) and (5.9) are then implied by (5.15)–(5.17). To show (5.8) and (5.10), note that for $m \geqslant 0$,

$$Q'_m(Q_m{}^{\leftarrow}(t)) \geqslant c_{i_m,m}^2 q'(c_{i_m,m} Q_m{}^{\leftarrow}(t)), \qquad t \geqslant t_0.$$

Since

$$\frac{c_{i_0}}{2} q(c_{i_m,m}\tau) \leqslant Q_m(\tau) \leqslant \sum_{i=-\infty}^{\infty} c_i q(c_{i_m,m}\tau), \qquad \tau \geqslant 0,$$

it follows that

$$\frac{1}{c_{i_m,m}} q^{\leftarrow}\left(\frac{t}{\sum c_i}\right) \leqslant Q_m^{\leftarrow}(t) \leqslant \frac{1}{c_{i_m,m}} q^{\leftarrow}\left(\frac{2t}{c_{i_0}}\right), \qquad t \geqslant t_0.$$

Thus, there exists $\eta_m \in [t/\sum c_i, (2/c_{i_0})t]$ such that $c_{i_m,m}Q_m{}^{\leftarrow}(t) = q^{\leftarrow}(\eta_m)$, implying

$$Q'_m(Q_m{}^{\leftarrow}(t)) \geqslant \left(\frac{c_{i_0}}{2}\right)^2 q'(q^{\leftarrow}(\eta_m)) = \left(\frac{c_{i_0}}{2}\right)^2 \frac{1}{\psi''(\eta_m)}.$$

Then (5.15)–(5.17) imply (5.8) and (5.10), since $\lim_{t\to\infty} t^2(\psi''(\xi))^2/\psi''(\eta_m) = \infty$ uniformly in $m$, using regular variation of $\psi''$. □

Now we can use Proposition 5.1 to show Theorem 2.3.

***Proof of Theorem 2.3.*** Set $u_n := x/a_n + b_n$. By (4.7), (4.11) and (4.12),

$$P\left(\sum_{i=-\infty}^{\infty} c_i Z_i > t\right) = \exp\left(-\int_{t_0\sum c_i}^{t} Q^{\leftarrow}(v)\mathrm{d}v + o\left(\int_{t_0\sum c_i}^{t} Q^{\leftarrow}(v)\mathrm{d}v\right)\right), \qquad t \to \infty.$$

Since $b_n$ is such that $P(\sum c_i Z_i > b_n) \sim n^{-1}$ as $n \to \infty$, this implies

$$\log n = \int_{t_0\sum c_i}^{b_n} Q^{\leftarrow}(v)\mathrm{d}v + o\left(\int_{t_0\sum c_i}^{b_n} Q^{\leftarrow}(v)\mathrm{d}v\right), \qquad n \to \infty.$$

Dividing by $\int_{t_0\sum c_i}^{b_n} Q^{\leftarrow}(v)\mathrm{d}v$ gives $(\int_{t_0\sum c_i}^{b_n} Q^{\leftarrow}(v)\mathrm{d}v)/(\log n) \to 1$ as $n \to \infty$. Since $a_n = Q^{\leftarrow}(b_n)$, that is, $b_n = \mathrm{Q}(a_n)$, there exist $\tau_2 > 0$ and $C_1 > 0$ such that for large $n$,

$$\begin{aligned}\int_{t_0\sum c_i}^{b_n} Q^{\leftarrow}(v)\mathrm{d}v &= \int_0^{a_n} uQ'(u)\mathrm{d}u \\ &\geqslant \int_0^{a_n} c_{i_0}^2 u^{3/2} q'(c_{i_0}u)u^{-1/2}\,\mathrm{d}u \\ &\geqslant C_1\int_{\tau_2}^{a_n} u^{-1/2}\,\mathrm{d}u = 2C_1(\sqrt{a_n} - \sqrt{\tau_2}),\end{aligned}$$

since $\lim_{u\to\infty} u^{3/2}q'(c_{i_0}u) = \infty$ since $\beta' \geqslant -1$. But this shows that $a_n/(\log n)^2$ is bounded as $n \to \infty$, showing that $D(u_n)$ holds by Proposition 5.1.

For the proof of $D'(u_n)$, we will verify conditions (5.1)–(5.3). Let $\gamma_0$, $m_0$ and $(B_t)_{t\geqslant t_3}$ be as in Lemma 5.3 and set $n' := \lfloor n^{\gamma_0} \rfloor$. Since $\lim_{n\in\infty} nP(Y_0 > u_n) = \mathrm{e}^{-x}$, it follows from (5.6) that

$$n\sum_{m=1}^{m_0-1} P(Y_0 + Y_m > 2u_n) \sim \sum_{m=1}^{m_0-1} \frac{P(Y_0 + Y_m > 2u_n)}{P(Y_0 > u_n)}\mathrm{e}^{-x} \to 0, \qquad n \to \infty.$$

On the other hand, (5.5) gives, for large $n$,

$$n\sum_{m=m_0}^{2n'} P(Y_0 + Y_m > 2u_n) \leqslant \frac{(\mathrm{e}^{-x}+1)^{1+\gamma_0}}{n^{\gamma_0}} \sum_{m=m_0}^{2n'} \frac{P(Y_0 + Y_m > 2u_n)}{P(Y_0 > u_n)^{1+\gamma_0}} \leqslant (\mathrm{e}^{-x}+1)^{1+\gamma_0} 2B_{u_n},$$

and the latter converges to 0 as $n \to \infty$, showing (5.1).

Consider the exponential families $(\overline{Z}_\tau)_{\tau\geqslant 0}$ and $(\overline{X}_{i,\tau})_{\tau\geqslant 0}$ as defined in Section 3.1. By (3.2), $\mathrm{E}\overline{X}_{i,\tau} = c_i\mathrm{E}\overline{Z}_{c_i\tau}$. Since $|c_i| \leqslant C_2|i|^{-\vartheta}$ for $i \neq 0$, for some constant $C_2$, it follows that for any $n \in \mathbb{N}$,

$$|c_i\tau| \leqslant C_2, \qquad \text{for } \tau \leqslant n^{\vartheta} \text{ and } |i| \geqslant n.$$

Since $[0, C_2] \to \mathbb{R}$, $s \mapsto \mathrm{E}\overline{Z}_s$ is a continuous function, it follows that there is some constant $C_3 > 0$ such that

$$|\mathrm{E}\overline{X}_{i,\tau}| \leqslant c_i C_3, \qquad \text{for all } \tau \leqslant n^{\vartheta} \text{ and } |i| \geqslant n.$$

This implies, for any $\tau \leqslant n^{\vartheta}$,

$$\sum_{i=n+1}^{\infty} |\mathrm{E}\overline{X}_{i,\tau}| \leqslant C_2 C_3 \sum_{i=n+1}^{\infty} |i|^{-\vartheta} \leqslant C_4 n^{1-\vartheta} \tag{5.18}$$

for some constant $C_4 > 0$. Let $\overline{\Phi}_n$ be the moment generating function of $\sum_{i=n+1}^{\infty} c_i Z_i$. As in the proof of Lemma 4.1, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \log \overline{\Phi}_n(\tau) = \sum_{i=n+1}^{\infty} \mathrm{E}\overline{X}_{i,\tau}, \qquad \tau \geqslant 0,$$

implying

$$\overline{\Phi}_n(\tau) = \exp\left( \int_0^{\tau} \sum_{i=n+1}^{\infty} \mathrm{E}\overline{X}_{i,v}\, \mathrm{d}v \right),$$

since $\overline{\Phi}_n(0) = 1$. Using (5.18), we have

$$\overline{\Phi}_n(\tau) \leqslant \exp(C_4 n^{1-\vartheta}\tau), \qquad \text{for } \tau \leqslant n^{\vartheta}.$$

Using Markov's inequality, replacing $n$ by $n'$ and setting $\tau := (n')^{\vartheta}$, we obtain

$$\begin{aligned} P\left( \sum_{i=n'+1}^{\infty} c_i Z_i > 1/a_n \right) &\leqslant \overline{\Phi}_{n'}((n')^{\vartheta}) \exp(-(n')^{\vartheta}/a_n) \\ &\leqslant \exp(C_4 n' - (n')^{\vartheta}/a_n) = o(n^{-2}), \qquad n \to \infty, \end{aligned}$$

since $a_n = O((\log n)^2)$. This is the left-hand side of (5.2). A similar procedure leads to the right-hand side of (5.2), as well as to (5.3). Thus it follows that $D'(u_n)$ holds, giving the assertion; see Embrechts *et al.* (1997, Theorem 4.4.6) or Leadbetter *et al.* (1983, Theorem 3.5.2). □

## 6. Applications to financial time series

Financial variables such as stock returns are often modelled using a stochastic volatility process. Prominent models are autoregressive conditional heteroscedastic (ARCH) and generalized ARCH (GARCH) models as introduced by Engle (1982) and Bollerslev (1986), stochastic volatility models as in Taylor (1986) and the exponential GARCH (EGARCH) model of Nelson (1991). GARCH models have generally heavy tails, so we shall concentrate on stochastic volatility and EGARCH models.

An example of a (discrete time) stochastic volatility model $(\xi_n)_{n\in\mathbb{Z}}$ with volatility process $(\sigma_n)_{n\in\mathbb{Z}}$ is given by

$$\xi_n = \sigma_n \eta_n, \qquad n \in \mathbb{Z}, \tag{6.1}$$

$$\log \sigma_n^2 = \sum_{i=1}^{\infty} \alpha_i Z_{n-i}, \qquad n \in \mathbb{Z}. \tag{6.2}$$

Here, $(Z_i)_{i\in\mathbb{Z}}$ is a sequence of i.i.d. random variables, the coefficient sequence $(\alpha_i)_{i\in\mathbb{N}}$ is such that the sum in (6.2) converges absolutely almost surely, and $(\eta_n)_{n\in\mathbb{Z}}$ is independent of $(Z_i)_{i\in\mathbb{Z}}$, hence of $(\sigma_n)_{n\in\mathbb{Z}}$. Typically, $\eta_0$ is Gaussian and $Z_0$ has light left and right tails, or is assumed to be Gaussian. Extreme value theory for such stochastic volatility models $(\xi_n)_{n\in\mathbb{Z}}$ with Gaussian noise has been provided by Breidt and Davis (1998). Much information is already contained in the volatility process $(\sigma_n)_{n\in\mathbb{Z}}$, and Theorems 2.1–2.3 provide extreme value theory for the process $(\log \sigma_n^2)_{n\in\mathbb{Z}}$ under mild conditions on $Z_0$ and non-negative coefficient sequences. A simple monotone transformation then yields extremal results for the volatility process $(\sigma_n)_{n\in\mathbb{Z}}$. In particular, from Theorem 2.2 it follows that $\log \sigma_0^2$ and hence $\sigma_0$ are in MDA($\wedge$), and Theorem 2.3 shows that extremes of the log-volatility process and hence of the volatility process do not cluster. The restriction of the coefficients being non-negative can be relaxed to a great extent, as follows from Theorems 7.1 and 7.2 and their discussion in the next section.

The EGARCH model $(\xi_n)_{n\in\mathbb{Z}}$ has a similar structure, given by

$$\xi_n = \sigma_n Z_n, \qquad n \in \mathbb{Z}, \tag{6.3}$$

$$\log \sigma_n^2 = \mu + \sum_{i=1}^{\infty} \alpha_i g(Z_{n-i}), \qquad n \in \mathbb{Z}.$$

Here, $\mu$ is a real constant, the coefficient sequence $(a_i)_{i\in\mathbb{N}}$ is as before, $g$ is typically a deterministic piecewise affine linear function (allowing for asymmetry in negative and positive innovations), and $(Z_n)_{n\in\mathbb{Z}}$ is an i.i.d. innovation sequence, typically Gaussian. The main difference from the stochastic volatility model considered before is that $\xi_n$ is defined in terms of the innovation sequence $(Z_n)_{n\in\mathbb{Z}}$ only, while the stochastic volatility model is defined in terms of a second independent driving noise sequence $(\eta_n)_{n\in\mathbb{Z}}$. For the extreme value theory of $(\log \sigma_n^2)_{n\in\mathbb{Z}}$ and hence $(\sigma_n)_{n\in\mathbb{Z}}$, however, this is irrelevant, and Theorems 2.1–2.3 can be applied for fairly general light-noise terms, similar to the stochastic volatility model discussed before. The extreme value behaviour of the price process $(\xi_n)_{n\in\mathbb{Z}}$ itself for Gaussian innovations and a finite coefficient sequence $(\alpha_i)_{i=1,\ldots,N}$ has been investigated in Lindner and Meyer (2002).

# 7. Extensions

The proofs of Theorems 2.1 and 2.2 can easily be generalized to cover independent finite sums of infinite moving average processes. Let $K \in \mathbb{N}$. For $k = 1, \ldots, K$, let $Z^{(k)}$ be a generic random variable which satisfies Assumption 2.1 with $\nu^{(k)}$, $\psi^{(k)}$ and $t_0^{(k)}$. Suppose that for each $k$, $(Z_i^{(k)})_{i\in\mathbb{Z}}$ is i.i.d. with the distribution of $Z^{(k)}$, and that $(Z_i^{(k)})_{i\in\mathbb{Z},k=1,\ldots,K}$ is independent. Let $(c_i^{(k)})_{i\in\mathbb{Z},k=1,\ldots,K}$ be a summable sequence of non-negative coefficients and define

$$Y_0 := \sum_{k=1}^{K} \sum_{i=-\infty}^{\infty} c_i^{(k)} Z_{n+i}^{(k)}. \tag{7.1}$$

Set

$$\begin{aligned}
q^{(k)}(\tau) &:= (\psi^{(k)})'^{\leftarrow}(\tau),\\
(\sigma_i^{(k)})^2(\tau) &:= (c_i^{(k)})^2 (q^{(k)})'(c_i^{(k)}\tau),\\
Q(\tau) &:= \sum_{k=1}^{K} \sum_{i=-\infty}^{\infty} c_i^{(k)} q^{(k)}(c_i^{(k)}\tau),\\
\sigma_\infty^2(\tau) &:= Q'(\tau).
\end{aligned}$$

Instead of Asumption 2.2, suppose that

$$\lim_{m\to\infty} \limsup_{\tau\to\infty} \frac{\sum_{k=1}^{K}\sum_{|j|>m}(\sigma_j{}^{(k)})^2(\tau)}{\sigma_\infty^2(\tau)} = 0,$$

$$\lim_{m\to\infty} \limsup_{\tau\to\infty} \frac{\sum_{k=1}^{K}\sum_{|j|>m}\sigma_j{}^{(k)}(\tau)}{\sigma_\infty(\tau)} = 0.$$

Denote by $\Phi$ the moment generating function of $Y_0$. Then we have the following extension of Theorems 2.1 and 2.2:

**Theorem 7.1.** *Under the assumptions and with the notation above, the assertions of Theorems 2.1 and 2.2 hold, with $Y_0$ as in* (7.1) *replacing $\sum_{i=-\infty}^{\infty} c_i Z_i$ in* (2.5) *and* (2.6)*, and $\sum_{k=1}^{K} t_0^{(k)} \sum_{i=-\infty}^{\infty} c_i^{(k)}$ replacing the lower integration limit $t_0 \sum c_i$ in* (2.6).

Theorem 7.1 can be used to cover infinite moving average processes with negative and positive coefficients. This can be achieved by splitting the sum in (1.1) into $Y_n = \sum_{c_i\geqslant 0} c_i Z_{n+i} + \sum_{c_i<0}(-c_i)(-Z_{n+i})$. If Assumptions 2.1 and 2.2 are then valid for each of the two sums (posing conditions on the left- as well on the right-tail behaviour of the density $f$ of $Z$), then Theorems 2.1 and 2.2 hold.

Theorem 7.1 can also be used to derive further results for the stochastic volatility model and EGARCH model of the previous section. Not only does it allow for positive and negative terms in the coefficient sequence, but also it follows from (6.1) and (6.3) that $\log \xi_n^2 = \log \sigma_n^2 + \log \eta_n^2$ and $\log \xi_n^2 = \log \sigma_n^2 + \log Z_n^2$, respectively. Then $\log \xi_0^2$ has the general form (7.1), and Theorem 7.1 allows us to derive the tail behaviour of $\log \xi_0^2$ (and hence of $|\xi_0|$) and to show that $\log \xi_0^2 \in \mathrm{MDA}(\wedge)$, under mild conditions on the light-tail behaviour of the noise sequences.

There is also an extension of Theorem 2.3 to moving average processes with negative and positive coefficients; its proof follows by slight modifications of the proof of Theorem 2.3.

**Theorem 7.2.** *Suppose that $Z$ as well as $-Z$ satisfy Assumptions 2.1 and 2.4 with functions $\psi_+$ and $\psi_-$ and regular (rapid) variation indices $\beta_+$ and $\beta_-$, respectively. Define $\beta'_+$ and $\beta'_-$ as in Assumption 2.3, and suppose that the real coefficient sequence $(c_i)_{i\in\mathbb{Z}}$ satisfies $|c_i| = O(|i|^{-\vartheta})$ as $|i| \to \infty$, for some $\vartheta > \max\{1, 2/(2+\beta'_+), 2/(2+\beta'_-)\}$. Suppose that $\beta_+ \neq \beta_-$, or that $\psi_+ = \psi_-$. Then the assertion of Theorem 2.3 holds for $(Y_n)_{n\in\mathbb{Z}}$ as defined in* (1.1).

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