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# Filtered Brownian motions as weak limit of filtered Poisson processes

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The main result of this paper is a limit theorem which shows the convergence in law, on a Hölderian space, of filtered Poisson processes (a class of processes which contains shot noise process) to filtered Brownian motion (a class of processes which contains fractional Brownian motion) when the intensity of the underlying Poisson process is increasing. We apply the theory of convergence of Hilbert space valued semimartingales and use a radonification result.

*Keywords:* filtered Poisson process; fractional Brownian motion; Hilbert-valued martingales; weak convergence

## 1. Introduction

There are already a number of articles, among them Pipiras and Taqqu (2000) and Taqqu *et al.* (1997), in which the fractional Brownian motion is shown to be the weak limit of a sequence of (simpler) processes. The present work was inspired by a paper by Szabados (2001) in which a strong approximation of the fractional Brownian motion is obtained by moving averages of a strong approximation of an ordinary Brownian motion. We retain here the principle of moving averages, but we only have a weak convergence since we approximate a Brownian motion by a sequence of renormalized Poisson processes.

More precisely, the Lévy fractional Brownian motion of Hurst index $H \in (0, 1)$, denoted by $B^H$, is defined by the following moving-average representation

$$B_t^H = \frac{1}{\Gamma(H+\frac{1}{2})}\int_0^t (t-s)^{H-1/2}\,\mathrm{d}B_s,$$

where $B$ is a one-dimensional standard Brownian motion. Since $\hat{N}^\lambda := \{\lambda^{-1/2}(N^\lambda(s) - \lambda.s),\ s \geqslant 0\}$, where $N^\lambda$ is a Poisson process of intensity $\lambda$, converges weakly to $B$, as $\lambda$ goes to infinity, it is natural to hope that $\{(\Gamma(H+1/2))^{-1}\int_0^t (t-s)^{H-1/2}\,\mathrm{d}\hat{N}_s^\lambda,\ t \geqslant 0\}$ will converge to $B^H$. Convergence is understood here as weak convergence in law on $\mathcal{C}([0, 1], \mathbb{R})$. We then have to distinguish between two situations. When $H$ is greater than $\frac{1}{2}$, the problem can be treated by Kolmogorov's tightness criterion and the answer is positive. On the other hand, when $H < \frac{1}{2}$, the latter result is no longer usable and it is necessary to have another method. Actually, we will prove, in a unified way, that in situations similar to the case $H > \frac{1}{2}$, the

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weak convergence mentioned above holds. We will also prove that we have weak convergence in law on some Hölderian space, a result which cannot be proved with Kolmogorov's criterion. In situations similar to the case $H < \frac{1}{2}$, we have a similar but weaker result (see Corollary 4) because of the potential singularity of the process $\int_0^t (t-s)^{H-1/2}\,\mathrm{d}\hat{N}_s^\lambda$ (Remark 3). The techniques, which seem new and interesting in themselves, involve a fine result on radonification (see Jakubowski *et al.* 2002; Badrikian and Üstünel 1996; Schwartz 1994), that is, conditions under which a cylindric semimartingale on a space $V_1$ is in fact a Hilbert-valued semimartingale on a space $V_2$.

Consider a kernel $K$ satisfying some hypothesis developed below. We can define the family of processes indexed by $\lambda \in \mathbb{R}^+$ :

$$\left\{ Y_t^\lambda = \int_0^t K(t, s)\mathrm{d}\hat{N}_s^\lambda,\ t \geqslant 0 \right\}, \tag{1}$$

where

$$\hat{N}_s^\lambda = \frac{\tilde{N}_s^n}{\sqrt{\lambda}} = \frac{N_s^\lambda - \lambda s}{\sqrt{\lambda}},$$

$N^\lambda$ being a Poisson process of constant intensity $\lambda$.

Lane (1984) shows the convergence of finite-dimensional laws of $Y^\lambda$ to a normal distribution when $\lambda$ increases to infinity. Here, we aim to establish the convergence in law in terms of processes. The usual techniques of martingale convergence seem at first glance unusable since $Y^\lambda$ is neither a martingale nor a semimartingale. However, if we freeze one of the $t$, that is, if we consider $\mathcal{X}_t^{\hat{N}^n}(r) = \int_0^t K(r, s)\mathrm{d}\hat{N}_s^\lambda$ for $r$ fixed, we obtain a process which is a martingale with respect to $t$ and $y_t^\lambda$ is nothing but $\mathcal{X}_t^{\hat{N}^n}(t)$. This remark (already used in Coutin and Decreusefond 1999, eqn. (19)) is the basis of our strategy. We will transform the original problem into a Hilbert-valued martingale convergence problem and then derive the convergence of $Y_4^\lambda$ by a contraction property. A key problem is to prove that $\mathcal{X}^{\hat{N}^\lambda}$ is a cadlag semimartingale in a convenient Hilbert space, and that is achieved using a radonification result.

This paper was in fact originally written with the above-mentioned application in mind. During the refereeing process, one referee kindly pointed out to us that the radonification result from Badrikian and Ustünel (1996) and Schwartz (1994) we were using, had been just extended from martingales to semimartingales (see Jakubowski *et al.* 2002). We then decided to modify our proofs to encompass a wider class of approximation schemes, but the main motivation remains the same.

In Section 2, we introduce our notation and main tools. In Section 3, we show the convergence of the Hilbert-valued semimartingales and then apply this result to our original problem.

## 2. Preliminary results

For $f \in \mathcal{L}^1([0, 1])$, the left and right fractional integrals of $f$ are defined by

$$(I^{\alpha}_{0^+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x f(t)(x-t)^{\alpha-1}\,\mathrm{d}t, \qquad x \geqslant 0,$$

$$(I^{\alpha}_{b^+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b f(t)(t-x)^{\alpha-1}\,\mathrm{d}t, \qquad x \leqslant b,$$

where $\alpha > 0$ and $I^0 = Id$. For any $\alpha \geqslant 0$, any $f \in \mathcal{L}^p([0, 1])$ and $g \in \mathcal{L}^q([0, 1])$ where $p^{-1} + q^{-1} \leqslant \alpha$, we have

$$\int_0^1 f(s)(I^{\alpha}_{0^+}g)(s)\mathrm{d}s = \int_0^1 (I^{\alpha}_{1^-}f)(s)g(s)\mathrm{d}s. \tag{2}$$

The Besov space $I^{\alpha}_{0^+}(\mathcal{L}^p)$, which for convenience we write as $\mathcal{I}_{\alpha,p}$, is usually equipped with the norm

$$\|f\|_{\mathcal{I}_{\alpha,p}} = \|g\|_{\mathcal{L}^p},$$

where $g$ is the unique element of $\mathcal{L}^p$ such that $f \equiv I^{\alpha}_{0^+}g$. In particular, $\mathcal{I}_{\alpha,2}$ is a (separable) Hilbert space and we have the following results (see Feyel and de La Pradelle 1999; Samko *et al.* 1993):

**Proposition 1.**

(a) *If* $\alpha - 1/p < 0$, *then* $\mathcal{I}_{\alpha,p}$ *is isomorphic to* $I^{\alpha}_{1^-}(\mathcal{L}^p)$.
(b) *For any* $0 < \alpha$ *and any* $p \geqslant 1$, $\mathcal{I}_{\alpha,p}$ *is continuously embedded in* $\mathrm{Hol}(\alpha - 1/p)$ *provided that* $\alpha - 1/p > 0$. *For* $0 < \nu \leqslant 1$, $\mathrm{Hol}(\nu)$ *denotes the space of Hölder-continuous functions, null at time* 0, *equipped with the usual norm*

$$\|f\|_{\mathrm{Hol}(\nu)} = \sup_{t\neq s}\frac{|f(t)-f(s)|}{|t-s|^{\nu}}.$$

Our main references for Hilbert-valued martingales are Métivier (1988) and Walsh (1986). We quote here the main results we need. Let $(\Omega, \mathcal{F} = (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ be a filtered probability space. Let $V$ be a separable Hilbert space; a $V$-valued process $X$ is an $\mathcal{F}$-martingale if and only if $\mathbb{E}[\|X_t\|_V]$ is finite for any $t$ and if, for any $s \geqslant t$,

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s, \quad \mathbb{P}\text{-almost surely.}$$

The analogue of the square bracket is here defined as $\lessdot X \gtrdot$ the unique predictable process with finite variation and with values in the space of positive symmetric nuclear operators from $V$ into $V$, such that, for $u, v \in V$,

$$\{\langle X_t, u\rangle_V\langle X_t, v\rangle_V - \langle \lessdot X \gtrdot_t u, v\rangle_V,\ t \geqslant 0\},$$

is a martingale. Since $\lessdot X \gtrdot_t$ is also a Hilbert–Schmidt operator, we can take its square root, denoted by $\lessdot X \gtrdot_t^{1/2}$, which is Hilbert–Schmidt because we are dealing with a non-negative definite operator of trace class. We denote by $\mathcal{L}_2(V; V)$, the space of Hilbert–Schmidt maps from $V$ into $V$. The most important result for us is Theorem 6.8 of Walsh (1986, p. 354) which is as follows:

**Proposition 2.** *Let* $(X^n)$ *be a sequence of cadlag V-valued processes. Then the laws of the processes* $(X^n,\ n \geqslant 1)$ *form a tight sequence of probalilities on* $\mathcal{D}(\mathbb{R}^+, V)$ *if the following hypotheses are satisfied:*

(a) *For each rational* $t \in (0, 1)$ *the family of random variables* $(X^n_t)$ *is tight.*
(b) *There exist* $p > 0$ *and processes* $(A^n(\delta), 0 < \delta < 1)$ *such that:*

$$\mathbb{E}\|X^n(t+\delta) - X^n(t)\|_V^p \,|\, \mathcal{F}_t] \leqslant \mathbb{E}[A^n(\delta) \,|\, \mathcal{F}_t],$$

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{E}[A^n(\delta)] = 0.$$

Beyond the trivial examples of $V$-valued Brownian motion or diffusions, it is rather hard to determine whether a $V$-valued process is a $V$-valued semimartingale. One the other hand, it is very easy to see if it is a cylindrical semimartingale, that is, if $\{\langle X_t, u\rangle_V,\ t \geqslant 0\}$ is a real-valued semimartingale for any $u \in V$. The following 'radonification' result is thus of paramount interest:

**Theorem 1.** *Let E and F be two Hilbert spaces and consider* $u : E \to F$ *a Hilbert–Schmidt operator. Let* $\mathcal{M}([0, 1], \mathbb{R})$ *be the space of cadlag square-integrable real semimartingales equipped with the norm*

$$\|M\|^2_{\mathcal{M}([0,1],\mathbb{R})} = \mathbb{E}\left[\sup_{t\in[0,1]} |M_s|^2\right].$$

*If L is in* $\mathcal{L}(E^*; \mathcal{M}([0, 1], \mathbb{R}))$*, the set of linear continuous maps from the dual of E, denoted by* $E^*$*, into* $\mathcal{M}([0, 1], \mathbb{R}))$*, then* $u \circ L$ *is an F-valued cadlag semimartingale.*

See Jakubowski *et al.* (2002) for this very statement, and Badrikian and Üstünel (1996) and Schwartz (1994) for the original statement restricted to martingales.

Assume that we are given a Hilbert–Schmidt map from $\mathcal{L}^2$ into itself, denoted by $K$, such that the following hypothesis is satisfied:

**Hypothesis 1.** *There exists* $\alpha > 0$ *such that K is a continuous one-to-one linear map from* $\mathcal{L}^2$ *into* $\mathcal{I}_{\alpha+1/2,2}$.

***Remark 1.*** Since the embedding from $\mathcal{I}_{\alpha+1/2,2}$ into $\mathcal{L}^2$ is Hilbert–Schmidt, it guarantees that $K$ is a Hilbert–Schmidt map from $\mathcal{L}^2$ into itself. Thus there is a kernel, still denoted by $K$, such that the operator $K$ takes the form

$$(Kf)(t) = \int_0^1 K(t, s) f(s) \mathrm{d}s \qquad \text{with} \int_0^1\int_0^1 K(t, s)^2 \,\mathrm{d}t\,\mathrm{d}s < \infty.$$

**Hypothesis 2.**

(a) *K is triangular, i.e.,* $K(t, s) = 0$ *for any* $s \geqslant t \geqslant 0$.
(b) *There exists* $\gamma > 0$ *such that for any* $(s, t) \in [0, 1]^2$,

$$\int_s^t \int_s^t K(u,\, r)^2 \, \mathrm{d}u \, \mathrm{d}r \leqslant c|t-s|^\gamma.$$

***Remark 2.*** Note that these two hypotheses are satisfied for any $\alpha$, by the kernel

$$K(t,\, s) = \frac{1}{\Gamma(\alpha + 1/2)} (t-s)^{\alpha - 1/2} \mathbf{1}_{[0,t)}(s),$$

which corresponds to $B^\alpha$ since in this case $K$, as a map, coincides with $I_{0^+}^{\alpha+1/2}$. The process usually called fractional Brownian motion admits the representation $\int_0^t J_\alpha(t,\, s) \mathrm{d}B_s$, with $J_\alpha$ an $(H - \frac{1}{2})$-homogeneous function of the form

$$J_\alpha(t,\, s) = L_\alpha(t,\, s)(t-s)^{\alpha-1/2} s^{-|\alpha - 1/2|},$$

where $L_\alpha$ is a bicontinuous function (see Coutin and Decreusefond 1999). Moreover, following Samko *et al.* (1993), we know that $J_\alpha$ is an isomorphism from $\mathcal{L}^2([0,\, 1])$ onto $I_{0^+}^{\alpha+1/2}(\mathcal{L}^2([0,\, 1]))$. It follows that $J_\alpha$ satisfies the Hypotheses 1 and 2 for any $\alpha \in (0,\, 1)$ with $\gamma = 2\alpha + 1$.

We denote by $K^*$, the adjoint of $K$ in $\mathcal{L}^2$.

**Lemma 1.** *Let $X = M + A$ be a cadlag semimartingale: $M$ denotes the martingale part and $A$ the finite-variation process. Assume that $\langle M \rangle_t = \int_0^t V(s) \mathrm{d}s$ and $A_t = \int_0^t \dot{A}(s) \mathrm{d}s$. Consider the following hypotheses:*

(a) *$V$ is bounded $\mathbb{P}$-p.s. by a constant $c > 0$.*
(b) $\mathbb{E}[\sup_{s \leqslant t} |\Delta X_s|] < \infty$.
(c) $\mathbb{E}[\int_0^1 |\dot{A}(s)|^2 \, \mathrm{d}s] < \infty$.

*Let $K$ satisfy hypotheses 1 and 2. Then, for any $\Phi \in (\mathcal{I}_{\alpha+1/2,2})^*$,*

$$\left\{ \mathcal{Z}_t^X(\Phi) := \int_0^t K^* \Phi(s) \mathrm{d}X_s,\ t \in [0,\, 1] \right\}$$

*is a cadlag semimartingale. Moreover, for any $\varepsilon \in (0,\, \alpha]$, there is a cadlag, $\mathcal{I}_{\alpha-\varepsilon,2}$-valued semimartingale $\mathcal{X}^X$, such that, for all $\Phi \in (\mathcal{I}_{\alpha-\varepsilon,2})^*$, we have*

$$\mathcal{Z}_t^X(\Phi) = \langle \Phi,\, \mathcal{X}_t^X \rangle_{(\mathcal{I}_{\alpha-\varepsilon,2})^*, \mathcal{I}_{\alpha-\varepsilon,2}}.$$

***Proof.*** Fix $\varepsilon \in (0,\, \alpha]$. Consider the linear map

$$\begin{aligned} L : (\mathcal{I}_{\alpha+1/2,2})^* &\to \mathcal{M}([0,\, 1],\, \mathbb{R}) \\ \Phi &\to \{\mathcal{Z}_t^X(\Phi),\ t \in [0,\, 1]\}. \end{aligned}$$

According to Hypotheses 1 and 2, there exists a constant $m$ such that

$$
\begin{aligned}
L(\Phi) &= \mathbb{E}\left[\sup_{t\leqslant 1} |\mathcal{Z}_t^X(\Phi)|^2\right] \\
&\leqslant \frac{1}{2}\left(\mathbb{E}\left[\sup_{t\leqslant 1}\left|\int_0^t K^*\Phi(s)\mathrm{d}M_s\right|^2\right] + \mathbb{E}\left[\sup_{t\leqslant 1}\left|\int_0^t K^*\Phi(s)\mathrm{d}A_s\right|^2\right]\right) \\
&\leqslant \frac{1}{2}\left(\mathbb{E}\left[\int_0^1 (K^*\Phi(s))^2 |V(s)|\mathrm{d}s\right] + \mathbb{E}\left[\left(\int_0^1 |K^*\Phi(s)||\dot{A}_s|\mathrm{d}s\right)^2\right]\right) \\
&\leqslant \frac{1}{2}\left(c\|K^*\Phi\|_{\mathcal{L}^2}^2 + \|K^*\Phi\|_{\mathcal{L}^2}^2 \mathbb{E}\left[\int_0^1 |\dot{A}_s|^2\,\mathrm{d}s)\right]\right) \\
&\leqslant m\|K^*\Phi\|_{\mathcal{L}^2}^2 \\
&\leqslant m\|\Phi\|_{(\mathcal{I}_{\alpha+1.2,2})^*}^2.
\end{aligned}
$$

Thus $L$ belongs to $\mathcal{L}((\mathcal{I}_{\alpha+1/2,2})^*, \mathcal{M}([0, 1], \mathbb{R}))$. Since the embedding of $\mathcal{I}_{\alpha+1/2,2}$ into $\mathcal{I}_{\beta+1/2,2}$ is Hilbert–Schmidt for $\beta < \alpha - \frac{1}{2}$, the result follows by Theorem 1. □

***Remark 3.*** We denote by $\epsilon_t$, the Dirac mass at time $t$. When $\alpha > \frac{1}{2}$, for $\varepsilon$ sufficiently small, $\alpha - \frac{1}{2} - \varepsilon > 0$, $\epsilon_t$ belongs to $(\mathcal{I}_{\alpha-\varepsilon,2})^*$ and *a fortiori* to $(\mathcal{I}_{\alpha+1/2-\varepsilon,2})^*$. Hence, $\mathcal{Z}_t^X(\epsilon_t)$ is well defined, is equal to $\int_0^t K(t, s)\mathrm{d}X_s$ by definition and is equal to $\langle \epsilon_t, \mathcal{X}_t^X\rangle$ by Lemma 1.

When $\alpha \leqslant \frac{1}{2}$, $\epsilon_t$ does not belong to $(\mathcal{I}_{\alpha-\varepsilon,2})^*$ and we cannot give a sense to $\mathcal{Z}_t^X(\epsilon_t)$. By the way, when $K(t, s) = (t - s)^{\alpha-1/2}$ and $X$ is a Poisson process, when $\alpha < \frac{1}{2}$, $\int_0^{\cdot} K(\cdot, s)\mathrm{d}X_s$ is a process which is positively infinite after each jump time and then takes finite values everywhere else. On the other hand, $\varepsilon^{-1}\int_{t-\varepsilon}^{t+\varepsilon} \mathcal{Z}_t^X(s)\mathrm{d}s$ is well defined and may serve, for small $\varepsilon$, as a substitute for $\int_0^t K(t, s)\mathrm{d}X_s$.

# 3. Convergence

Consider a sequence of semimartingales $X^n = M^n + A^n$ with

$$\langle M^n\rangle_t = \int_0^t V^n(s)\mathrm{d}s \quad \text{and} \quad A_t^n = \int_0^t \dot{A}^n(s)\mathrm{d}s.$$

**Hypothesis 3.**

(a) $\sup_{n\geqslant 1} V^n$ *is bounded* $\mathbb{P}$*-p.s. by a constant* $c > 0$;
(b) $\sup_{n\geqslant 1} \mathbb{E}[\sup_{s\leqslant t} |\Delta X_s^n|] < \infty$;
(c) $\sup_{n\geqslant 1} \mathbb{E}[\int_0^1 |\dot{A}^n(s)|^2\,\mathrm{d}s] < \infty$.

Suppose that $X^n$ converges to $X = M + A$ in $\mathcal{D}([0, 1]; \mathbb{R})$. From Lemma 1, we define two $\mathcal{I}_{\alpha-\varepsilon,2}$-valued processes $\mathcal{X}^{X^n}$ and $\mathcal{X}^X$ with respect to the semimartingales $X^n$ and $X$.

Our key result is the following.

**Theorem 2.** *For any* $\varepsilon > 0$ *sufficiently small, as* $n$ *goes to infinity, the laws of* $\mathcal{X}^{X^n}$ *in* $\mathcal{D}([0, 1]; \mathcal{I}_{\alpha-\varepsilon,2})$ *converge to the law of* $\mathcal{X}^X$.

***Proof.*** $K$ is supposed to be continuous from $\mathcal{L}^2$ into $\mathcal{I}_{\alpha+1/2,2}$, thus $K^*$ is continuous from $(\mathcal{I}_{\alpha+1/2,2})^*$ into $\mathcal{L}^2$. Denote by $\|K^*\|$ the corresponding operator norm. Since the embedding of $\mathcal{I}_{\alpha+1/2,2}$ into $\mathcal{I}_{\alpha-\varepsilon,2}$ is Hilbert–Schmidt and thus radonifying, it follows from Schwartz (1994, Theorem I) and Hypothesis 3 that

$$\mathbb{E}\left[\|\mathcal{X}_t^{X^n}\|^2_{\mathcal{I}_{\alpha-\varepsilon,2}}\right] \leqslant c \sup_{\|f\|_{(\mathcal{I}_{\alpha+1/2,2})^*}=1} \mathbb{E}\left[\left(\int_0^1 K^* f(s)\mathrm{d}X_s^n\right)^2\right]$$

$$\leqslant c\|K^*\|^2.$$

It then follows that, for any $\eta > 0$, there exists $M$ such that

$$\sup_n \mathbb{P}\left[\|\mathcal{X}_t^{X^n}\|_{\alpha-\varepsilon,2} > M\right] \leqslant \eta$$

and that, for any $N > 0$,

$$\lim_{r\to+\infty} \sup_n \sum_{k=r}^{\infty} \mathbb{E}\left[\langle \mathcal{X}_t^{X^n}, f_k\rangle^2 \mathbf{1}_{\|\mathcal{X}_t^{X^n}\|_{\mathcal{I}_{\alpha-\varepsilon,2}}\leqslant N}\right] = 0,$$

where $(f_k, k \geqslant 1)$ is a complete orthonormal basis of $(\mathcal{I}_{\alpha-\varepsilon,2})^*$. According to Gihman and Skorohod (1980, Theorem 2, p. 377), this implies that, for each $t \in [0, 1]$, $(\mathcal{X}_t^{X^n}, n \geqslant 1)$ is a tight sequence in $\mathcal{I}_{\alpha-\varepsilon,2}$.

On the other hand, we have,

$$\|\mathcal{X}_{t+s}^{X^n} - \mathcal{X}_t^{X^n}\|^2_{\mathcal{I}_{\alpha-\varepsilon,2}} = \sum_{k=1}^{\infty} |\langle \mathcal{X}_{t+s}^{X^n} - \mathcal{X}_t^{X^n}, f_k\rangle|^2$$

$$= \sum_{k=1}^{\infty} \left|\int_t^{t+s} K^* f_k(r)\mathrm{d}X_r^n\right|^2.$$

According to Hypothesis 2, we have:

$$\mathbb{E}\left[\sum_{k=1}^{\infty} \left|\int_t^{t+s} K^* f_k(r)\mathrm{d}X_r^n\right|^2\right] \leqslant m \sum_{k=1}^{\infty} \int_t^{t+s} |K^* f_k(r)|^2 \,\mathrm{d}r$$

$$\leqslant m\|\mathbb{1}_{[t,t+s]}K^*\|^2_{\mathrm{HS}}$$

$$\leqslant m|t-s|^{\gamma}.$$

This relation obviously implies hypothesis (b) of Proposition 2 and the sequence $\{\mathcal{X}^{X^n} : n \geqslant 1\}$ is thus tight in $\mathcal{D}([0, 1], \mathcal{I}_{\alpha-\varepsilon,2})$.

Let $\{\mathcal{X}^{X^{n_k}} : k \geqslant 1\}$ be a subsequence which converges to a limit denoted by $L$. We have, for any $u \in (\mathcal{I}_{\alpha-\varepsilon,2})^*$, $\langle u, L\rangle = \langle u, \mathcal{X}^X\rangle$. That is to say, all convergent subsequences converge to the same limit. It follows that the laws of $\mathcal{X}^{X^n}$ in $\mathcal{D}([0, 1]; \mathcal{I}_{\alpha-\varepsilon,2})$ converge to the law of $\mathcal{X}^X$. $\square$

**Corollary 1.** *Under Hypotheses 1 and 2 with* $\alpha > \frac{1}{2}$, *the laws of the processes* $\{\int_0^t K(t, s)\mathrm{d}X_s^n,\ t \in [0, 1]\}$ *in* $\mathrm{Hol}(\alpha - 1/2 - \varepsilon)$, *converge to the law of* $\{\int_0^t K(t, s)\mathrm{d}X_s,\ t \in [0, 1]\}$.

***Proof.*** For $\varepsilon$ sufficiently small, $\alpha - \frac{1}{2} - \epsilon > 0$ and for any $f \in \mathcal{I}_{\alpha-\varepsilon,2}$, $|f(s) - f(t)| \leqslant c\|f\|_{\mathcal{I}_{\alpha-\varepsilon,2}} |t - s|^{\alpha-1/2-\varepsilon}$. Thus, the map

$$B : \mathcal{I}_{\alpha-\varepsilon,2} \to \mathrm{Hol}(\alpha - \tfrac{1}{2} - \varepsilon)$$

$$f \to (s \mapsto f(s) = \langle \epsilon_s, f\rangle_{(\mathcal{I}_{\alpha-\varepsilon,2})^*, \mathcal{I}_{\alpha-\varepsilon,2})},$$

is well defined and continuous. Hence for $F$ bounded and continuous from $\mathrm{Hol}(\alpha - \frac{1}{2} - \varepsilon)$ into $\mathbb{R}$, $F \circ B$ is continuous from $\mathcal{I}_{\alpha-\varepsilon,2}$ into $\mathbb{R}$. By Theorem 2, we have

$$\mathbb{E}[F \circ B(\mathcal{X}^{X^n})] \underset{n\to\infty}{\longrightarrow} \mathbb{E}[F \circ B(\mathcal{X}^X)],$$

which amounts to saying that

$$\mathbb{E}\left[F\left(\int_0 K(t, s)\mathrm{d}X_s^n\right)\right] \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[F\left(\int_0 K(t, s)\mathrm{d}X_s\right)\right].$$

The proof is thus complete. $\square$

# 4. Application

The space of simple, integer-valued measures, locally finite on [0, 1], is denoted $\Omega$. We define the probability $\mathbb{P}$ as the unique measure on $\Omega$ such that the canonical measure $\omega$ is a Poisson random measure of compensator $\lambda\,\mathrm{d}s$. The canonical filtration $\mathcal{F}$ is defined by:

$$\mathcal{F}_0 = \{\varnothing, \Omega\},$$

$$\mathcal{F}_t = \sigma\left\{\int_0^s \omega(\mathrm{d}s),\ s \leqslant t\right\}, \qquad \text{for all } t \in [0, 1].$$

We set $N_s^\lambda = \omega([0, 1])$. Our basic object is the process $Y^\lambda$, defined by

$$Y_t^\lambda = \lambda^{-1/2}\int_0^t K(t, s)(\mathrm{d}N_s^\lambda - \lambda\,\mathrm{d}s)$$

$$= \frac{1}{\sqrt{\lambda}}\sum_{n\geqslant 1} K(t, T_n)\mathbb{1}_{[T_n\leqslant t]} - \int_0^t K(t, s)\sqrt{\lambda}\,\mathrm{d}s,$$

where $K$ satisfies Hyptheses 1 and 2.

From Lemma 1, we define two $\mathcal{I}_{\alpha-\varepsilon,2}$-valued processes $\mathcal{X}^{\hat{N}^n}$ and $\mathcal{X}^B$ defined with respect to the martingales $\hat{N}^n$ and $B$, a standard Brownian motion. It is clear that Hypothesis 3 is satisfied by $\mathcal{X}^{\hat{N}^n}$. We now have to distinguish two cases according to the position of $\alpha$ with respect to $\frac{1}{2}$. Actually, when $\alpha > \frac{1}{2}$, $\mathcal{I}_{\alpha-\varepsilon,2}$ is a subset of the set of continuous functions and thus its dual contains Dirac measures. On the other hand, when $\alpha < \frac{1}{2}$, the map $s \mapsto f(s) = \langle \epsilon_s, f \rangle_{(\mathcal{I}_{\alpha-\varepsilon,2})^*, \mathcal{I}_{\alpha-\varepsilon,2}}$ is not defined for $f \in \mathcal{I}_{\alpha-\varepsilon,2}$.

**Proposition 3.** *Under Hypotheses 1 and 2 with* $\alpha > \frac{1}{2}$, *the laws of the processes* $\{Y_t^n = \int_0^t K(t, s) \mathrm{d}\hat{N}_s^n,\ t \in [0, 1]\}$ *in* $\mathrm{Hol}(\alpha - \frac{1}{2} - \varepsilon)$ *converge to the law of* $\{Y_t = \int_0^t K(t, s) \mathrm{d}B_s,\ t \in [0, 1]\}$.

***Remark 4.*** As a consequence, we have the convergence in law on $\mathcal{C}([0, 1], \mathbb{R})$. We now show how Hypothesis 1 and Kolmogorov's criterion are sufficient prove this result. Since $K(t, s) = K^*(\epsilon_t)$, we have

$$
\begin{aligned}
\mathbb{E}[|Y_t^n - Y_s^n|^2] &= \int_0^1 |K(t, r) - K(s, r)|^2 \, \mathrm{d}r \\
&\leqslant c \|K^*(\epsilon_t - \epsilon_s)\|_{\mathcal{L}^2}^2 \\
&\leqslant c \|\epsilon_t - \epsilon_s\|_{(\mathcal{I}_{\alpha+1/2,2})'}^2 \\
&= c|t - s|^{2\alpha}.
\end{aligned}
$$

It is sufficient, according to Kolmogorov's criterion, to show that $Y^n$ converges in law to $Y$, on $\mathcal{C}([0, 1], \mathbb{R})$.

Following the same lines, we have:

**Proposition 4.** *Let* $\alpha \in (0, \frac{1}{2})$ *and let* $\eta$ *be continuous from* $[0, 1]$ *into* $T_{\alpha-\varepsilon,2}^*$. *Assume that the Hypotheses 1 and 2 hold. Then the laws of the processes* $\{\langle \eta_t, \mathcal{X}_t^n \rangle_{(\mathbb{I}_{\alpha-\varepsilon,2})^*, \mathcal{I}_{\alpha-\varepsilon,2}},\ t \in [0, 1]\}$ *in* $\mathcal{C}([0, 1]; \mathbb{R})$ *converge to the law of* $\{\langle \eta_t, \mathcal{X}_t \rangle_{(\mathcal{I}_{\alpha-\varepsilon,2})^*, \mathcal{I}_{\alpha-\varepsilon,2}},\ t \in [0, 1]\}$.

For instance, we can choose $\eta$ as

$$
\begin{aligned}
\langle \eta_t, f \rangle_{(\mathcal{I}_{\alpha-\varepsilon,2})^*, \mathcal{I}_{\alpha-\varepsilon,2}} &= \varepsilon^{-1} \int_{(t)\varepsilon)\vee 0}^{(t+\varepsilon)\wedge 1} f(s) \mathrm{d}s \\
&= \epsilon^{-1} (I_{0^+}^1 f((t + \varepsilon) \wedge 1) - 1_{0^+}^1 f((t - \varepsilon) \vee 0)).
\end{aligned}
$$

Since $f \in \mathcal{I}_{\alpha-\varepsilon,2}$. $I_{0^-}^1 f$ belongs to $\mathcal{I}_{1+\alpha-\epsilon}$ which is a subset of $\mathrm{Hol}(\frac{1}{2} + \alpha + \epsilon)$. It is then clear that $\eta$ is continuous from $[0, 1]$ into $\mathcal{I}_{\alpha-\varepsilon,2}^*$. As a consequence, the law of the process $\{\varepsilon^{-1} \int_{t-\varepsilon}^{t+\varepsilon} \mathcal{X}_t^n(s) \mathrm{d}s,\ t \in [0, 1]\}$ in $\mathcal{C}([0, 1]; \mathbb{R})$ converges to the law of the process $\{\varepsilon^{-1} \int_{t-\varepsilon}^{t+\varepsilon} \mathcal{X}_t(s) \mathrm{d}s,\ t \in [0, 1]\}$.

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