Bernoulli **6**(6), 2000, 1001–1006

Comparison of stochastic Volterra equations

GUILLERMO FERREYRA* and PADAMANBHAN SUNDAR**

Department of Mathematics, Louisiana State University, Baton Rouge LA 70803, USA.
E-mail: *ferreyra@math.lsu.edu; **sundar@math.lsu.edu

A pathwise comparison theorem for a class of one-dimensional stochastic Volterra equations driven by continuous semimartingales is proved under suitable conditions. The result is applied to equations appearing in applications.

Keywords: semimartingale; stochastic differential equation; stochastic Volterra equation

1. Introduction

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t : t \geqslant 0\}, P)$ be a probability space with the filtration satisfying the usual hypotheses. For each $j = 1, \ldots, n$, let $\{M_j(t)\}$ be a real-valued continuous local martingale adapted to $(\mathscr{F}_t)$, and $\{V_j(t)\}$ be a continuous $(\mathscr{F}_t)$-adapted process, each with paths of bounded variation on compacts. Let $X_i(t)$, $i = 1, 2$, be the pathwise unique strong solutions of the one-dimensional stochastic differential equations (SDEs)

$$X_i(t) = \xi_i + \sum_{j=1}^{n} \int_0^t \sigma_j(t, s, X_i(s))\, \mathrm{d}M_j(s) + \sum_{j=1}^{n} \int_0^t b_j^i(t, s, X_i(s))\, \mathrm{d}V_j(s). \tag{1.1}$$

Such equations are known as stochastic Volterra equations and have been studied by several authors (see, for example, Berger and Mizel 1980; Cochran *et al.* 1995; Kolodii 1983; Protter 1985). The existence and uniqueness of solutions of stochastic Volterra equations driven by right-continuous semimartingales have been established by Protter (1985). A comparison theorem for solutions of stochastic equations with Volterra-type drifts was proved by Tudor (1989). We prove a comparison theorem for stochastic equations (1.1) where the diffusion coefficient is also of the Volterra type, namely when $\sigma_j(t, s, x)$ assumes the form $H(t)\sigma_j(s, x)$. It is important to note that a comparison of solutions of stochastic Volterra equations with general Volterra-type 'diffusion' coefficients is not possible. A counter-example is given in Tudor (1989). Our result is motivated by typical Volterra models that arise in practice. The method of proof employed in this paper is the variation of parameters which is akin to that in Protter (1990) for classical SDEs.

1350–7265

2. The comparison theorem

We consider the following class of one-dimensional Volterra equations:

$$X_i(t) = \xi_i + \sum_{j=1}^{n} H(t)\int_0^t \sigma_j(s,\ X_i(s))\,\mathrm{d}M_j(s) + \sum_{j=1}^{n}\int_0^t b_j^i(t,\ s,\ X_i(s))\,\mathrm{d}V_j(s), \tag{2.1}$$

for $i = 1, 2$. We assume that $\{H(t)\}$ is a continuous, adapted, positive, strictly decreasing process. The coefficients $\sigma_j : \Omega \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $b_j^i : \Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ are assumed to be jointly continuous for each fixed $\omega \in \Omega$, and satisfy a global Lipschitz condition in the space variable: $|\sigma_j(\omega, s, y) - \sigma_j(\omega, s, x)| \leqslant L|y - x|$ and $|b_j^i(\omega, t, s, y) - b_j^i(\omega, t, s, x)| \leqslant L|y - x|$. We will suppress the first argument in the coefficients in the rest of the paper. The coefficients $\sigma_j(s, x)$ and $b_j^i(t, s, x)$ are $\mathscr{F}_s$ measurable for each $s \geqslant 0$. Moreover, we assume that the drift coefficients b_j^i are differentiable in the first variable and $|D_1 b_j^i(t, s, y) - D_1 b_j^i(t, s, x)| \leqslant L\,|y - x|$, where D_1 denotes this first derivative. Then we can divide (2.1) by $H(t)$, and, letting $Y_i(t)$ denote $X_i(t)/H(t)$, obtain the following equation for $Y_i(t)$:

$$Y_i(t) = \frac{\xi_i}{H(t)} + \sum_{j=1}^{n}\int_0^t \sigma_j(s,\ Y_i(s)H(s))\,\mathrm{d}M_j(s) + \sum_{j=1}^{n}\int_0^t \frac{b_j^i(t,\ s,\ Y_i(s)H(s))}{H(t)}\,\mathrm{d}V_j(s). \tag{2.2}$$

Using integration by parts in the third term on the right-hand side, we obtain

$$\begin{aligned} Y_i(t) = {} & \frac{\xi_i}{H(t)} + \sum_{j=1}^{n}\int_0^t \sigma_j(s,\ Y_i(s)H(s))\,\mathrm{d}M_j(s) + \sum_{j=1}^{n}\int_0^t \frac{b_j^i(s,\ s,\ Y_i(s)H(s))}{H(s)}\,\mathrm{d}V_j(s) \\ & + \sum_{j=1}^{n}\int_0^t \frac{1}{H(s)}\int_0^s D_1 b_j^i(s,\ r,\ H(r)Y_i(r))\,\mathrm{d}V_j(r)\,\mathrm{d}s \\ & - \int_0^t \frac{1}{H^2(s)}\sum_{j=1}^{n}\int_0^s b_j^i(s,\ r,\ H(r)Y_i(r))\,\mathrm{d}V_j(r)\,\mathrm{d}H(s). \end{aligned} \tag{2.3}$$

Using the Picard iteration and the Doob inequality, it is a routine matter to establish the existence of a strong solution and pathwise uniqueness of solutions to (2.3). Hence we obtain the existence and uniqueness of $X_i(t)$. In order to prove a comparison theorem, we need the following additional hypotheses on the coefficients.

Hypotheses H. *Let $b_j^i : \Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be such that, for each $\omega \in \Omega$, i and j,*

1. *$b_j^2(s, r, y) \geqslant b_j^1(s, r, x)$ for all $0 \leqslant r \leqslant s$ and $y \geqslant x$;*
2. *$D_1 b_j^2(s, r, y) \geqslant D_1 b_j^1(s, r, x)$ for all $0 \leqslant r \leqslant s$ and $y \geqslant x$, where D_1 denotes the first derivative in the first variable.*

Before we proceed further, the following lemma is needed and is therefore stated without proof. This is the 'variation of parameters' formula in the stochastic set-up. A proof of it can be found in Protter (1990, pp. 266–267).

Lemma 2.1. *Let $\{M_t\}$ be a continuous semimartingale starting at 0, and $\{A_t\}$ be an adapted continuous process with paths of finite variation. If*

$$X_t = A_t + \int_0^t X_s \, \mathrm{d}M_s$$

then $X_t = \mathscr{E}(M)_t\{A_0 + \int_0^t \mathscr{E}(M)_s^{-1}\, \mathrm{d}A_s\}$, where $\mathscr{E}(M)_t$ is the stochastic exponential of M given by $\mathrm{e}^{M_t - 1/2\langle M\rangle_t}$.

Theorem 2.2. *Assume Hypotheses H. If V_j are increasing adapted processes for each j, and $\xi_1 \leqslant \xi_2$, then $P(X_1(t) \leqslant X_2(t)\ \forall\ t \in \mathbb{R}^+) = 1$.*

Proof. Let $Y_i(t)$ denote $X_i(t)/H(t)$ as before. Let $Z_t = Y_2(t) - Y_1(t)$. From (2.2), Z_t solves the equation

$$\begin{aligned} Z_t = {} & \frac{\xi_2 - \xi_1}{H(t)} + \sum_{j=1}^{n} \int_0^t Z_s \, \mathrm{d}N_j(s) \\ & + \sum_{j=1}^{n} \int_0^t \frac{1}{H(t)} (b_j^2(t, s, Y_2(s)H(s)) - b_j^1(t, s, Y_1(s)H(s)))\, \mathrm{d}V_j(s), \end{aligned} \tag{2.4}$$

where

$$N_j(t) = \int_0^t \frac{\sigma_j(s, Y_2(s)H(s)) - \sigma_j(s, Y_1(s)H(s))}{Z_s} I_{\{Z_s \neq 0\}}\, \mathrm{d}M_j(s)$$

are local martingales by the Lipschitz hypothesis on σ_j. Let $N_t = \sum_{j=1}^n N_j(t)$. Let $\mathscr{E}(N)_t$ denote the stochastic exponential of the local martingale N. Using Lemma 2.1, we can write (2.4) as

$$\begin{aligned} Z_t = \mathscr{E}(N)_t \Bigg[& \frac{\xi_2 - \xi_1}{H_0} - \int_0^t \frac{\mathscr{E}(N)_s^{-1}}{H(s)^2} (\xi_2 - \xi_1)\, \mathrm{d}H(s) \\ & + \sum_{j=1}^{n} \int_0^t \frac{\mathscr{E}(N)_s^{-1}}{H(s)} (b_j^2(s, s, Y_2(s)H(s)) - b_j^1(s, s, Y_1(s)H(s)))\, \mathrm{d}V_j(s) \\ & - \frac{1}{H(s)} \int_0^s (b_j^2(s, r, Y_2(r)H(r)) - b_j^1(s, r, Y_1(r)H(r)))\, \mathrm{d}V_j(r)\, \mathrm{d}H(s) \\ & + \int_0^s (D_1 b_j^2(s, r, Y_2(r)H(r)) - D_1 b_j^1(s, r, Y_1(r)H(r)))\, \mathrm{d}V_j(r)\, \mathrm{d}s) \Bigg] \end{aligned} \tag{2.5}$$

Let $J_s := Z_s/\mathscr{E}(N)_s$. For any $u \leqslant s$, we can write

$$
\begin{aligned}
&b_j^2(s, r, Y_2(r)H(r)) - b_j^1(s, r, Y_1(r)H(r)) \\
&\quad = b_j^2(s, r, J_r\,\mathscr{E}(N)_r\, H(r) + Y_1(r)\, H(r)) - b_j^1(s, r, Y_1(r)H(r)).
\end{aligned} \tag{2.6}
$$

The right-hand side of (2.6) will be denoted by $c_j(s, r, J_r)$. Let $D_1 c_j$ denote the first derivative of c_j with respect to the first variable. Equation (2.5) yields

$$
\begin{aligned}
J_t = {} & \frac{\xi_2 - \xi_1}{H_0} + \int_0^t \frac{\mathscr{E}(N)_s^{-1}}{H(s)^2}(\xi_2 - \xi_1)\,\mathrm{d}(-H(s)) \\
& + \sum_{j=1}^n \int_0^t \frac{\mathscr{E}(N)_s^{-1}}{H(s)}\left(c_j(s, s, J_s)\,\mathrm{d}V_j(s) + \frac{1}{H(s)}\int_0^s c_j(s, r, J_r)\,\mathrm{d}V_j(r)\,\mathrm{d}(-H(s)\right) \\
& + \int_0^s D_1 c_j(s, r, J_r)\,\mathrm{d}V_j(r)\,\mathrm{d}s).
\end{aligned} \tag{2.7}
$$

We have thus written J_t as the solution of a random integral equation with continuous increasing integrators. The following lemma finishes the proof. □

Lemma 2.3. *For each $i = 1, \ldots, N$, assume that;*

(i) A_i *are continuous increasing functions on* $\mathbb{R}^+$.
(ii) $F_i: C[0, \infty) \to C[0, \infty)$ *with* $|F_i(u)_t - F_i(v)_t| \leqslant K_i(T)\sup_{0 \leqslant s \leqslant t}|u(s) - v(s)|$, *for all* $t \leqslant T$;
(iii) for any given $t \geqslant 0$, $u(s) = v(s)\ \forall\ 0 \leqslant s \leqslant t$ *implies that* $F_i(u)(s) = F_i(v)(s)\ \forall 0 \leqslant s \leqslant t$, *where* $u, v \in C[0, \infty)$;
(iv) $u(t) = 0 \Rightarrow F_i(u)_t \geqslant 0$.

Consider the integral equation

$$
U(t) = \xi + \sum_{i=1}^N \int_0^t F_i(U)_s\,\mathrm{d}A_i(s),
$$

where $\xi \geqslant 0$. *Then* $U(t) \geqslant 0$ *for all* t.

Proof. Let U_ε be the solution of $U_\varepsilon(t) = \xi + \sum_{i=1}^N \int_0^t G_i(U_\varepsilon)_s\,\mathrm{d}A_i(s)$, where $G_i(u)_s = F_i(u)_s + \varepsilon$. Then let $\tau = \inf\{t: U_\varepsilon(t) < 0\}$. If $\tau < \infty$, then by continuity of U_ε, $U_\varepsilon(\tau) = 0$. By condition (iv), $G_i(U_\varepsilon)_\tau > 0$. $G_i(U_\varepsilon) \in C[0, \infty)$ so that there exists a δ such that $G_i(U_\varepsilon)_t > 0$ for all $t \in [\tau, \tau + \delta]$. Therefore $U_\varepsilon(t) \geqslant 0$ for all $t \in [\tau, \tau + \delta]$. This contradicts the finiteness of τ. Therefore $\tau = \infty$. Thus $U_\varepsilon(t) \geqslant 0$ for all $t \geqslant 0$. Allowing ε to tend to 0, we obtain $U_\varepsilon \to U$ uniformly on $[0, T]$ for any $T \geqslant 0$. Therefore $U(t) \geqslant 0$ for all t. □

Corollary 2.4. *Let* V_j *be increasing adapted processes for each* j, *and* $\xi_1 \leqslant \xi_2$. *Let* $b_j^i(t, s, x) = H(t) f_j^i(s, x)$ *for all* $0 \leqslant s \leqslant t$, $x \in \mathbb{R}$, $i = 1, 2$ *and* $j = 1, \ldots, n$. *If*

$f_j^1(s, x) \leqslant f_j^2(s, x)$ *for all* $s \geqslant 0$, $x \in \mathbb{R}$, *and* $j = 1, \ldots, n$, *then* $P(X_1(t) \leqslant X_2(t)\ \forall t \in \mathbb{R}^+) = 1$.

Proof. For the particular form of the drift coefficients, equation (2.2) becomes

$$Y_i(t) = \frac{\xi_i}{H(t)} + \sum_{j=1}^{n} \int_0^t \sigma_j(s,\ Y_i(s)H(s))\,\mathrm{d}M_j(s) + \sum_{j=1}^{n} \int_0^t f_j^i(s,\ Y_i(s)H(s))\,\mathrm{d}V_j(s) \qquad (2.8)$$

This is an SDE driven by continuous processes. The proof of the above comparison theorem yields the desired conclusion. □

Example. Consider the following linear Volterra equation which arises in applications (for details, see Berger and Mizel 1980, Example 6B, p. 321; Miller 1971, p. 67):

$$X_i(t) = \mathrm{e}^{-at} \int_0^t \mathrm{e}^{as}\{E_i(s) - f(s)X_i(s)\}\,\mathrm{d}s - \mathrm{e}^{-at} \int_0^t \mathrm{e}^{as} c(s)X_i(s)\,\mathrm{d}W(s), \qquad (2.9)$$

where $i = 1, 2$. Here $a > 0$ and E_i, f and c are continuous functions of s. The above corollary implies that if $E_2(t) \geqslant E_1(t)$ for all $t > 0$, then $X_2(t) \geqslant X_1(t)$ for all $t > 0$ almost surely.

The above conclusion can also be obtained by considering the explicit solution of a linear stochastic equation. However, if the coefficients are nonlinear in the above equations, explicit solutions are seldom found. In addition, if the exponent $-at$ is replaced by an adapted, decreasing and continuous process $A(t)$, our results seem to be the best possible.

Thus far we have assumed the differentiability of the drift coefficient in the first variable which allowed us to write the Volterra equation as an SDE with functional coefficients. A true Volterra drift is considered in the following theorem.

Hypotheses H′. *Let* $b_j^i : \Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *be such that, for each* $\omega \in \Omega$, *i and j,*

1. $b_j^2(s, r, y) \geqslant b_j^1(s, r, x)$ *for all* $0 \leqslant r \leqslant s$ *and* $x \leqslant y$;
2. $b_j^2(s', r, y) - b_j^1(s', r, x) \geqslant b_j^2(s, r, y) - b_j^1(s, r, x)$ *for all* $0 \leqslant r \leqslant s \leqslant s'$ *and* $x \leqslant y$.

Theorem 2.5. *Assume Hypotheses* H′. *If* V_j *are increasing adapted processes for each j, and* $\xi_1 \leqslant \xi_2$, *then* $P(X_1(t) \leqslant X_2(t)\ \forall\ t \in \mathbb{R}^+) = 1$.

Proof. Let $\varepsilon > 0$ be fixed. Define $\overline{b}_j^i(t, s, x) = (1/\varepsilon)\int_t^{t+\varepsilon} b_j^i(u, s, x)\,\mathrm{d}u$. Then $\overline{b}_j^i$ and $D_1\overline{b}_j^i$ are Lipschitz continuous in the space variable x by using the Lipschitz property of b_j^i. Therefore, by the discussion at the beginning of this section,

$$\overline{X}_i(t) = \xi_i + \sum_{j=1}^{n} H(t)\int_0^t \sigma_j(s,\ \overline{X}_i(s))\,\mathrm{d}M_j(s) + \sum_{j=1}^{n} \int_0^t \overline{b}_j^i(t,\ s, \overline{X}_i(s))\,\mathrm{d}V_j(s) \qquad (2.10)$$

admits a pathwise unique strong solution for $i = 1, 2$. Hypotheses H′ enable us to apply Theorem 2.2 to the solutions $\overline{X}_i$ of (2.10), so that $P(\overline{X}_2(t) \geqslant \overline{X}_1(t)\ \forall t \in \mathbb{R}^+) = 1$.

Note that $\overline{X}_i(t)$ depends on ε, and we write it as $\overline{X}_i^{\varepsilon}(t)$ from now on:

$$\mathrm{E}\sup_{t\in[0,T]}(\overline{X}_i^{\varepsilon}(t)-\overline{X}_i^{\varepsilon'}(t))^2 \leqslant 8CK^2\sum_{j=1}^{n}\mathrm{E}\int_0^T(\overline{X}_i^{\varepsilon}(t)-\overline{X}_i^{\varepsilon'}(t))^2\,\mathrm{d}\langle N_j\rangle\,(t)$$

As ε_n decreases to 0, $\mathrm{E}(\sup_{t\in[0,T]}|\overline{X}_i(t)-X_i(t)|)^2$ converges to 0 for any fixed T. Therefore, there is a subsequence of $\{\varepsilon_n\}$ along which the corresponding solutions $\{\overline{X}_i(t)\}$ converge to $\{X_i(t)\}$ uniformly on compacts almost surely. Therefore, $P(X_1(t)\leqslant X_2(t)\ \forall\ t\in\mathbb{R}^+)=1$. $\square$

Acknowledgements

The second author would like to thank the US Army Research Office for its support for this research through grant DAAH04-94-G-0249.

References

Berger, M. and Mizel, V. (1980) Volterra equations with Itô integrals. *J. Integral Equations*, **2**, 187–245, 319–337.

Cochran, W.G., Lee, J.-S. and Potthoff, J. (1995) Stochastic Volterra equations with singular kernels, *Stochastic Process. Appl.*, **56**, 337–349.

Kolodii, A.M. (1983) On the existence of solutions of stochastic Volterra integral equations. *Teor. Sluchajnykh Protsessov*, **11**, 51–57 (in Russian).

Miller, R.K. (1971) *Nonlinear Volterra Integral Equations*. Menlo Park, CA: W.A. Benjamin Inc.

Protter, P. (1985) Volterra equations driven by semimartingales. *Ann. Probab.*, **13**, 519–530.

Protter, P. (1990) *Stochastic Integration and Differential Equations*. Berlin: Springer-Verlag.

Tudor, C. (1989) A comparison theorem for stochastic equations with Volterra drifts. *Ann. Probab.*, **17**, 1541–1545.

Received September 1998 and revised October 1999