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On the existence or non-existence of solutions for certain backward stochastic differential equations

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We investigate the existence of (local) solutions and explosions for backward stochastic differential equations with generator $|f(t, \omega, y, z)| \leqslant G(y) + F(y)R(z)$, where G, F, R are continuous, G is increasing in $\mathbb{R}_+$ (decreasing in $\mathbb{R}_-$) and R is subquadratic. We study in detail the case $f(t, \omega, y, z) = G(y) + A|z|^2$.

Keywords: backward stochastic differential equations; explosion time; ordinary differential equations

1. Introduction

In this paper we consider backward stochastic differential equations (BSDEs) of the form

$$Y_t = \xi + \int_t^1 f(s, Y_s, Z_s)\mathrm{d}s - \int_t^1 Z_s \mathrm{d}W_s, \qquad 0 \leqslant t \leqslant 1,$$

where (W_t) is a standard d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)_{0\leqslant t\leqslant 1}$ is the standard Brownian filtration. The random function $f: [0, 1] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is called the coefficient or generator and the $\mathbb{R}$-valued, $\mathcal{F}_1$-adapted variable ξ is called the terminal condition. This equation is denoted by $Eq(\xi, f)$.

Pardoux and Peng (1990) gave the first existence and uniqueness result in the case when f is Lipschitz continuous in (y, z) uniformly in (t, ω), and ξ is square-integrable. Lepeltier and San Martín (1997) extended the existence result to the case where the coefficient is only continuous with linear growth (in (y, z)). Kobylanski (1997) obtained an existence result in the case where ξ is bounded, and f is continuous with linear growth in y and quadratic growth in z. Lepeltier and San Martín (1998) extended her result to the case $|f(t, \omega, y, z)| \leqslant l(y) + c|z|^2$, with $l > 0$ such that $\int_0^\infty \mathrm{d}y/l(y) = \int_{-\infty}^0 \mathrm{d}y/l(y) = \infty$. The question is then what can be said when the coefficient is more than superlinear in the above sense.

We begin by considering the case $|f(t, \omega, y, z)| \leqslant G(y) + F(y)R(z)$, with G, F, R continuous functions such that R is subquadratic, that is,

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$$\limsup_{|z|\to\infty} \frac{R(z)}{|z|^2} < \infty,$$

and vanishes at 0 and the function G is increasing on $\mathbb{R}_+$ and decreasing on $\mathbb{R}_-$. Our main result is Theorem 4, where we prove that if $b \leqslant \xi \leqslant a$ for some a and b with $b \leqslant 0 \leqslant a$, then there exists a maximal solution (Y, Z) on $(t^*, 1]$ with $t^* = t_1 \vee t_2$ and

$$t_1 = 1 - \int_{-\infty}^{b} \frac{\mathrm{d}y}{G(y)}, \qquad t_2 = 1 - \int_a^{\infty} \frac{\mathrm{d}y}{G(y)}.$$

In Section 3 we first consider the case $f(t, \omega, y, z) = G(y)$. If $\xi \geqslant 0$, $\|\xi\|_\infty = a$, with $\int_a^\infty \mathrm{d}y/G(y) > 1$, we obtain, from Section 2, the existence of a solution on [0, 1]. In the critical case ($\int_a^\infty \mathrm{d}y/G(y) = 1$, which corresponds to $t^* = 0$) we obtain a sufficient condition for no explosion which is also necessary when G is convex. In the case $\int_a^\infty \mathrm{d}y/G(y) < 1$ (still wide open) we give some sufficient conditions for no explosion.

We then investigate the case $f(t, \omega, y, z) = G(y) + A|z|^2$, $A > 0$, and obtain results by using an exponential Itô formula.

Finally, using a comparison theorem, we obtain some sufficient conditions for no explosion in the case

$$|f(t, \omega, y, z)| \leqslant G(y) + A|z|^2.$$

2. The general case

Throughout this paper we shall deal with BSDEs of the form

$$Y_t = \xi + \int_t^1 f(s, \omega, Y_s, Z_s)\mathrm{d}s - \int_t^1 Z_s\,\mathrm{d}W_s, \tag{1}$$

where we assume that ξ is $\mathcal{F}_1$-measurable and bounded, and that $f : [0, 1] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfies the following hypothesis:

Hypothesis H.

(*i*) *f is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable*
(*ii*) *$f(s, \omega, \cdot, \cdot)$ is continuous for all (s, ω).*
(*iii*) *$|f(s, \omega, y, z)| \leqslant G(y) + F(y)R(z)$ where G, F, R are continuous non-negative functions satisfying $R(0) = 0$, R is subquadratic, and G is assumed to be increasing (decreasing) on $\mathbb{R}_+$ ($\mathbb{R}_-$).*

We say that (Y, Z) is a (bounded) local solution of (1) if there exists $\bar{t} < 1$ such that (Y, Z) is defined on $[\bar{t}, 1]$, Y is adapted (and bounded), Z is predictable and, for all $t \in [\bar{t}, 1]$, we have

$$Y_t = \xi + \int_t^1 f(s, \omega, Y_s, Z_s)\mathrm{d}s - \int_t^1 Z_s\,\mathrm{d}W_s.$$

A maximal (bounded) local solution of (1) is a (bounded) local solution (Y, Z) such that for any other (bounded) local solution (X, Λ), in any interval $[t_0, 1]$ where both solutions are defined we have $X_t \leqslant Y_t$, $t \in [t_0, 1]$. We observe that if (Y, Z) and (Y, U) are local solutions of (1) then in the common interval of definition $Z = U$. This implies immediately that if maximal solutions exist they are unique.

The study of (1) is carried out by comparison with ordinary differential equations (ODEs) of the form

$$u(t) = a + \int_t^1 G(u(s))\mathrm{d}s. \tag{2}$$

For this reason we shall first deduce some properties for this type of ODE. Given $a \in \mathbb{R}_+$, we write $F_a(x) = \int_a^x \mathrm{d}y/G(y)$ and $\int_{a+} \mathrm{d}y/G(y) = F_a(a+) \in \{0, \infty\}$. We shall see that (2) has a unique maximal solution up to an explosion time $t^* \in [-\infty, 1)$. t^* is characterized as $1 - t^* = \int_a^\infty \mathrm{d}y/G(y)$. We write $t^* = t^*(a, G)$. We say that a is the upper critical value for (2) if $\int_a^\infty \mathrm{d}y/G(y) = 1$, which amounts to saying that $t^* = 0$.

Example. If $G(y) = y^2$ then $a = 1$ is the upper critical value for (2) and, in general, $t^*(a, G) = 1 - 1/a$. One can verify in this case that the unique solution of (2) is

$$u(t) = \frac{a}{a(t-1)+1}, \qquad t \in]t^*, 1].$$

We can prove the following result relative to the solutions of (2).

Lemma 1. *Let $t_0 > t^*(a, G)$. For every $\epsilon > 0$, $\delta > 0$ sufficiently small, the equation*

$$u^{\epsilon,\delta}(t) = a + \epsilon + \int_t^1 (G(u^{\epsilon,\delta}(s)) + \delta)\mathrm{d}s \tag{3}$$

has a unique solution on $[t_0, 1]$. Moreover, $u^{\epsilon,\delta}$ is monotonously increasing in (ϵ, δ). The limit $u(t) = \lim_{(\epsilon,\delta)\to 0} u^{\epsilon,\delta}(t)$ exists on $[t_0, 1]$, and is the maximal solution of (2). If $\int_{a+} \mathrm{d}y/G(y) = \infty$, the unique solution of (2) is $u \equiv a$.

Proof. Let

$$F^{\epsilon,\delta}(x) = \int_{a+\epsilon}^x \frac{\mathrm{d}y}{G(y) + \delta}.$$

Using the monotone convergence theorem, we have that

$$F^{\epsilon,\delta}(\infty) \nearrow F_a(\infty) = 1 - t^*(a, G)$$

when $(\epsilon, \delta) \searrow 0$. Therefore, for (ϵ, δ) small enough, we have $F^{\epsilon,\delta}(\infty) > 1 - t_0$. A simple calculation shows that the unique solution of (3) is given implicitly by $F^{\epsilon,\delta}(u^{\epsilon,\delta}(t)) = 1 - t$, $t \geqslant t_0$. That is, $u^{\epsilon,\delta}(t) = (F^{\epsilon,\delta})^{-1}(1 - t)$.

Since

$$1-t=\int_{a+\epsilon}^{u^{\epsilon,\delta}(t)}\frac{\mathrm{d}y}{G(y)+\delta}\leqslant\int_{a+\epsilon'}^{u^{\epsilon,\delta}(t)}\frac{\mathrm{d}y}{G(y)+\delta'}$$

if $\epsilon'\leqslant\epsilon$, $\delta'\leqslant\delta$, we deduce that $u^{\epsilon,\delta}$ is monotonously increasing in (ϵ,δ), and is clearly bounded below by a. Therefore, the limit

$$u(t):=\lim_{(\epsilon,\delta)\searrow 0}u^{\epsilon,\delta}(t)$$

exists. It is easy to verify that $u(t)$ is a solution of (2) on $[t_0, 1]$.

Let $v(t)$ be another continuous solution of (2). Then, for (ϵ,δ) small, we find that $v(1)=a<u^{\epsilon,\delta}(1)=a+\epsilon$. If there exists $\tau<1$ such that $v(\tau)=u^{\epsilon,\delta}(\tau)$, we can find another time $\bar\tau<1$ such that, for all $s\in(\bar\tau,1]$, $v(s)<u^{\epsilon,\delta}(s)$ and $v(\bar\tau)=u^{\epsilon,\delta}(\bar\tau)$. Then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}v(\bar\tau)=-G(v(\bar\tau))>-G(u^{\epsilon,\delta}(\bar\tau))-\delta=\frac{\mathrm{d}}{\mathrm{d}t}u^{\epsilon,\delta}(\bar\tau),$$

which implies that $v(s)>u^{\epsilon,\delta}(s)$ in some small interval to the right of $\bar\tau$, which is a contradiction. Therefore, in the common interval where v and $u^{\epsilon,\delta}$ are defined we have $v\leqslant u^{\epsilon,\delta}$. Passing to the limit (in (ϵ,δ)), we conclude that $v\leqslant u$.

Finally, assume that $\int_{a+}\mathrm{d}y/G(y)=\infty$. Then $u^{\epsilon,\delta}$ is well defined on [0, 1] for (ϵ,δ) small enough, and u is also defined on [0, 1]. If $u(t)>a$ for some $t\in[0,1]$, we obtain, for small (ϵ,δ), that

$$\int_{a+\epsilon}^{u(t)}\frac{\mathrm{d}y}{G(y)+\delta}\leqslant\int_{a+\epsilon}^{u^{\epsilon,\delta}(t)}\frac{\mathrm{d}y}{G(y)+\delta}=1-t.$$

Then

$$\lim_{(\epsilon,\delta)\searrow 0}\int_{a+\epsilon}^{u(t)}\frac{\mathrm{d}y}{G(y)+\delta}=\int_{a}^{u(t)}\frac{\mathrm{d}y}{G(y)}=\infty,$$

which is a contradiction. Therefore $u(t)=a$ for all $t\in[0,1]$ and the result follows. $\square$

Remark. Take the ODE $u(t)=\int_t^1\sqrt{|u(s)|}\,\mathrm{d}s$. Then $u\equiv 0$ is a possible solution, but is not the maximal one. In fact $u(t)=(1-t)^2/4$ is the maximal solution. Notice that in this case $\int_{0+}\mathrm{d}y/\sqrt{y}=0$. On the other hand, the equation $u(t)=\int_t^1 u^2(s)\mathrm{d}s$ has a unique solution $u\equiv 0$, since $\int_{0+}\mathrm{d}y/y^2=\infty$.

We have in the following result a comparison theorem between solutions and subsolutions of ODEs of the type given by (2).

Lemma 2. *Let v be a non-negative continuous function such that $v(t)\leqslant a+\int_t^1 G_1(v(s))\mathrm{d}s$, where G_1 is a non-negative continuous function such that $G_1\leqslant G$. If u is the maximal solution of* (2)*, then $v\leqslant u$, in the common interval where v and u are defined.*

Proof. We have $v(1)\leqslant a<u^{\epsilon,\delta}(1)=a+\epsilon$. As before, suppose that there exists $\bar\tau$ such that

$v(\bar{\tau}) = u^{\epsilon,\delta}(\bar{\tau})$ and $v(s) < u^{\epsilon,\delta}(s)$ for all $s \in (\bar{\tau}, 1]$. Since $G_1 \leqslant G$ and G is increasing on $\mathbb{R}_+$, we obtain

$$v(\bar{\tau}) \leqslant a + \int_{\bar{\tau}}^1 G_1(v(s))\mathrm{d}s \leqslant a + \int_{\bar{\tau}}^1 G(v(s))\mathrm{d}s \leqslant a + \int_{\bar{\tau}}^1 G(u^{\epsilon,\delta}(s))\mathrm{d}s < u^{\epsilon,\delta}(\bar{\tau}),$$

which is a contradiction. □

Remark. In the same way as before, for $b \leqslant 0$ we can obtain on $(t^*, 1]$ a minimal solution for $l(t) = b - \int_t^1 G(l(s))\mathrm{d}s$, with $1 - t^* = \int_{-\infty}^b \mathrm{d}y/G(y)$.

For all $C > 0$, we consider the following truncation of the identity

$$\varphi_C(y) = \begin{cases} -C & y < -C, \\ y & -C \leqslant y \leqslant C, \\ C & y > C. \end{cases}$$

Our assumption on G implies that $G \circ \varphi_C \leqslant G$, and then from Lemma 2 we obtain that if v is a solution of $v(t) = a + \int_t^1 G(\varphi_C(v(s))\mathrm{d}s$, then $v \leqslant u$. From this fact we deduce the following comparison lemma between the solutions of (1) and (2).

Lemma 3 (A priori estimation). *If* (Y, Z) *is a bounded solution of* (1) *and* $\xi \leqslant a$ *then* $Y_t \leqslant u(t)$ *for all* $t \in (t^*, 1] \cap [0, 1]$, *where* $u(t)$ *is the maximal solution of* (2).

Proof. We take C large enough such that $|Y| \leqslant C$. Clearly (Y, Z) is also a bounded solution of the equation

$$Y_t = \xi + \int_t^1 f_C(s, \omega, Y_s, Z_s)\mathrm{d}s - \int_t^1 Z_s\,\mathrm{d}W_s$$

with $f_C(s, \omega, y, z) = f(s, \omega, \varphi_C(y), z)$. Also we have

$$|f_C(s, \omega, y, z)| \leqslant G(\varphi_C(y)) + AR(z),$$

for a suitable constant $A \geqslant 0$. Using the fact that $R(0) = 0$ and the quadratic bound on R, we find that, for all $\delta > 0$, there exists $D = D(\delta) < \infty$ such that

$$|f_C(s, \omega, y, z)| \leqslant G(\varphi_C(y)) + \delta + D(\delta)|z|^2.$$

Since,

$$\int_{-\infty}^0 \frac{\mathrm{d}y}{G(\varphi_C(y)) + \delta} = \int_0^\infty \frac{\mathrm{d}y}{G(\varphi_C(y)) + \delta} = \infty,$$

we conclude, using the results of Lepeltier and San Martín (1998), that

$$Y_t \leqslant v^{\epsilon,\delta}(t), \qquad \text{for all } t \in [0, 1],$$

where

$$v^{\epsilon,\delta}(t) = a + \epsilon + \int_t^1 (G(\varphi_C(v^{\epsilon,\delta}(s))) + \delta)\mathrm{d}s.$$

Taking any $t_0 > t^*(a, G)$, by Lemma 2 we have that

$$v^{\epsilon,\delta}(t) \leqslant u^{\epsilon,\delta}(t) \qquad \text{on } t \geqslant t_0.$$

Then we deduce that P-almost surely

$$Y_t \leqslant u(t), \qquad \text{for all } t \in (t^*, 1] \cap [0, 1],$$

proving the result. □

Assume we have a bounded local solution (Y, Z) which is defined on an interval that contains $[\bar{t}, 1]$. Consider

$$f^{\bar{t}}(s, \omega, y, z) = \begin{cases} f(s, \omega, y, z) & \bar{t} \leqslant s \leqslant 1, \\ 0 & 0 \leqslant s < \bar{t}, \end{cases}$$

and define $X_t = \mathrm{E}(Y_{\bar{t}} | \mathcal{F}_t)$, $t \leqslant \bar{t}$. Then $X_t = Y_{\bar{t}} - \int_t^{\bar{t}} \Lambda_s \, \mathrm{d}W_s$ for some predictable process Λ. In this way the process $(\overline{Y}, \overline{Z})$ defined by

$$\overline{Y}_t = \begin{cases} X_t & 0 \leqslant t < \bar{t}, \\ Y_t & \bar{t} \leqslant t \leqslant 1, \end{cases} \qquad \overline{Z}_t = \begin{cases} \Lambda_t & 0 \leqslant t < \bar{t}, \\ Z_t & \bar{t} \leqslant t \leqslant 1, \end{cases}$$

is a bounded solution of the BSDE

$$\overline{Y}_t = \xi + \int_t^1 f^{\bar{t}}(s, \omega, \overline{Y}_s, \overline{Z}_s) \mathrm{d}s - \int_t^1 \overline{Z}_s \, \mathrm{d}W_s, \qquad 0 \leqslant t \leqslant 1.$$

Since $|f^{\bar{t}}(s, \omega, y, z)| \leqslant |f(s, \omega, y, z)| \leqslant G(y) + F(y)R(z)$, we conclude that on $(t^*, 1] \cap [0, 1]$, $\overline{Y}_t \leqslant u(t)$, but then on $(t^*, 1] \cap [0, 1] \cap [\bar{t}, 1]$ we have $Y_t \leqslant u(t)$ P-a.s.

Henceforth, for each bounded local solution (Y, Z) of (1) defined on $[\bar{t}, 1]$, where $\bar{t}$ will be clear from the context, we shall denote the extension to $[0, 1]$ constructed above by $(\overline{Y}, \overline{Z})$. We can state now an existence theorem for a local solution of (1).

Theorem 4. *Let ξ be a bounded $\mathcal{F}_1$ random variable. Assume $b \leqslant \xi \leqslant a P$-a.s. and $b \leqslant 0 \leqslant a$. Consider*

$$t_1 = 1 - \int_{-\infty}^{b} \frac{\mathrm{d}y}{G(y)}, \qquad t_2 = 1 - \int_a^{\infty} \frac{\mathrm{d}y}{G(y)}, \qquad t^* = t_1 \vee t_2.$$

Then there exists a local solution (Y, Z) of (1) defined on $(t^*, 1] \cap [0, 1]$ which is bounded on $[t_0, 1] \cap [0, 1]$ for each $t_0 > t^*$ and which is also maximal among the bounded local solutions of (1).

Proof. Let $t_0 \in (t^*, 1] \cap [0, 1]$. Let C be large enough such that $-C \leqslant l(t_0) \leqslant l(t) \leqslant u(t) \leqslant u(t_0) \leqslant C$, where $l(t)$ is the minimal solution of $l(t) = b - \int_t^1 G(l(s)) \mathrm{d}s$ and $u(t)$ is the maximal solution of $u(t) = a + \int_t^1 G(u(s)) \mathrm{d}s$. Consider the coefficient

$$f_C^{t_0}(s, \omega, y, z) = \begin{cases} f(s, \omega, \varphi_C(y), z) & t_0 \leqslant s \leqslant 1 \\ 0 & 0 \leqslant s < t_0. \end{cases}$$

Then

$$|f_C^{t_0}(s, \omega, y, z)| \leqslant G(\varphi_C(y)) + AR(z),$$

for some finite constant A. As before, we take $\delta > 0$ and $D(\delta) < \infty$ such that $AR(z) \leqslant \delta + D(\delta)|z|^2$. From the results of Lepeltier and San Martín (1998), there exists a unique maximal solution (X^{t_0}, Λ^{t_0}) of $Eq(f_C^{t_0}, \xi)$ and, from Lemma 3 and similar arguments, we have,

$$\forall t \geqslant t_0, \qquad l(t_0) \leqslant l(t) \leqslant X_t^{t_0} \leqslant u(t) \leqslant u(t_0), \tag{4}$$

which proves that $Y_t = X_t^{t_0}$, $Z_t = \Lambda_t^{t_0}$, $t \geqslant t_0$ defines a bounded local solution of (1) on $[t_0, 1]$.

We shall now prove that if $1 > s_0 > t_0$, then (X^{s_0}, Λ^{s_0}) and (X^{t_0}, Λ^{t_0}) agree on $[s_0, 1]$. In fact this follows from the method developed in Kobylanski (1997) and Lepeltier and San Martín (1998) because after a localization and an exponential change of variable, the maximal solutions are found by approximating the respective coefficients from above by Lipschitz functions. Our a priori estimate (4) implies that the same truncation can be used on $[s_0, 1]$ as on $[t_0, 1]$. Since $f_C^{t_0} = f_C^{s_0}$ on $[s_0, 1]$, we obtain that the Lipschitz functions there constructed also agree on $[s_0, 1]$. By Lemma 5 below we shall have the desired compatibility.

Finally, take (M, Γ) to be any bounded local solution of (1) defined on $[\bar{t}, 1]$. We consider $s_0 = \bar{t} \vee t_0$. If $(\overline{M}, \overline{\Gamma})$ is the extension of the process to $[0, 1]$, then $(\overline{M}, \overline{\Gamma})$ is a solution of the BSDE whose coefficient is $f_C^{s_0}$, implying that $M_t \leqslant X_t^{s_0} = X_t^{t_0} = Y_t$ on $[s_0, 1]$. This completes the proof of the theorem. $\square$

Lemma 5. *Let f satisfy the standard Lipschitz conditions. If (X^{t_i}, Λ^{t_i}) is a solution whose coefficient is f^{t_i}, $i = 1, 2$, then*

$$(X_s^{t_1}, \Lambda_s^{t_1}) = (X_s^{t_2}, \Lambda_s^{t_2}) \qquad \text{on } s \geqslant t_1 \vee t_2 = \bar{t}.$$

Proof. The result follows immediately from Itô's formula applied to the process $e^{\theta s}(X_s^{t_1} - X_s^{t_2})^2$. In fact we obtain

$$\mathrm{E}(e^{\theta \bar{t}}(X_{\bar{t}}^{t_1} - X_{\bar{t}}^{t_2})^2) + \mathrm{E}\left(\int_{\bar{t}}^1 e^{\theta s}(\theta(X_s^{t_1} - X_s^{t_2})^2 + (\Lambda_s^{\bar{t}_1} - \Lambda_s^{\bar{t}_2})^2)\mathrm{d}s\right)$$

$$= 2\mathrm{E}\left(\int_{\bar{t}}^1 e^{\theta s}(X_s^{t_1} - X_s^{t_2})(f(s, \omega, X_s^{t_1}, \Lambda_s^{t_1}) - f(s, \omega, X_s^{t_2}, \Lambda_s^{t_2}))\mathrm{d}s\right)$$

$$\leqslant 2K(1 + C)\mathrm{E}\left(\int_{\bar{t}}^1 e^{\theta s}|X_s^{t_1} - X_s^{t_2}|^2 \,\mathrm{d}s\right) + \frac{2K}{C}\mathrm{E}\left(\int_{\bar{t}}^1 e^{\theta s}|\Lambda_s^{t_1} - \Lambda_s^{t_2}|^2 \,\mathrm{d}s\right).$$

The result follows by taking $C > 2K$ and $\theta > 2K(1 + C)$. $\square$

We now discuss the explosion time for equation (1). For this purpose we will construct recursively a decreasing sequence (s_n) . We take $s_0 = 1$, $b_0 = -\|\xi^-\|_\infty \leqslant \|\xi^+\|_\infty = a_0$. Let s_1 be such that

$$s_0 - s_1 = \int_{a_0}^{\infty} \frac{\mathrm{d}x}{G(x)} \wedge \int_{-\infty}^{b_0} \frac{\mathrm{d}x}{G(x)}.$$

We know that there is a maximal bounded solution (Y^1, Z^1) on $(s_1, 1] \cap [0, 1]$. If $s_1 < 0$, we stop. Otherwise two things may happen. If $\overline{\lim}_{t \searrow s_1} \|Y_t^1\|_\infty = \infty$ then s_1 is an explosion time for (1). If not, take $C > \sup_{t > s_1} \|Y_t^1\|_\infty$ and consider $t_0 > s_1$ such that

$$t_0 - s_1 < \int_C^{\infty} \frac{\mathrm{d}x}{G(x)} \wedge \int_{-\infty}^{-C} \frac{\mathrm{d}x}{G(x)} = t_0 - \bar{t}.$$

We know that the equation

$$X_t = Y_{t_0}^1 + \int_t^{t_0} f(s, \omega, X_s, \Lambda_s)\mathrm{d}s - \int_t^{t_0} \Lambda_s \, \mathrm{d}W_s$$

has a unique bounded maximal solution on $(\bar{t}, t_0] \cap [0, t_0]$. Then $X_t = Y_t^1$ on $(s_1, t_0]$, which in particular shows that $Y_{s_1}^1 = \lim_{t \searrow s_1} Y_t^1$ exists and is also bounded by C. Then we take $b_1 = -\|(Y_{s_1}^1)^-\|_\infty \leqslant a_1 = \|(Y_{s_1}^1)^+\|_\infty$, and we define s_2 by

$$s_1 - s_2 = \int_{a_1}^{\infty} \frac{\mathrm{d}x}{G(x)} \wedge \int_{-\infty}^{b_1} \frac{\mathrm{d}x}{G(x)}.$$

On $(s_2, s_1] \cap [0, 1]$ there exists a unique maximal bounded solution of

$$X_t = Y_{s_1}^1 + \int_t^{s_1} f(s, \omega, X_s, \Lambda_s)\mathrm{d}s - \int_t^{s_1} \Lambda_s \, \mathrm{d}W_s.$$

Let

$$Y_t^2 = \begin{cases} Y_t^1 & t \geqslant s_1, \\ X_t & t \in (s_2, s_1] \cap [0, 1], \end{cases}$$

and define Z^2 in an analogous way.

If $s_2 < 0$, we obtain a solution on $[0, 1]$. Otherwise we start again. In this way we have a sequence (s_n) which may eventually stop after a finite number of steps because

(i) $s_n < 0$, in which case we have a maximal bounded solution on $[0, 1]$, or
(ii) $\sup_{t > s_{n-1}} \|Y_t^{n-1}\|_\infty < \sup_{t > s_n} \|Y_t^n\|_\infty = \infty$ in which case there is a unique maximal bounded solution which explodes at s_n.

We call $\hat{t} = s_n$ the explosion time of (1).

If the procedure continues indefinitely we have $s_n \searrow \hat{t} \geqslant 0$ which is also the explosion time of (1). In fact, since

$$s_n - s_{n-1} = \int_{a_n}^{\infty} \frac{\mathrm{d}x}{G(x)} \wedge \int_{-\infty}^{b_n} \frac{\mathrm{d}x}{G(x)} \underset{n \to \infty}{\to} 0,$$

we have either $\limsup_n a_n = \infty$ or $\liminf_n b_n = -\infty$. The unique maximal solution (Y, Z) on $(\hat{t}, 1]$ agrees with (Y^n, Z^n) on $(s_n, 1]$, and therefore $\limsup_{t \searrow \hat{t}} \|Y_t\|_\infty = \infty$. $\square$

Notice that $\hat{t}$ is the explosion time for the maximal solution of (1). There may be other solutions which do not explode at $\hat{t}$ as the following ODE shows. Let

$$u(t) = \int_t^1 G(u(s))\mathrm{d}s,$$

where

$$G(x) = \begin{cases} 4\sqrt{|x|} & |x| \leqslant 1, \\ 4x^2 & |x| > 1. \end{cases}$$

Then $u \equiv 0$ is a solution on [0, 1], but the maximal solution is

$$u(t) = \begin{cases} 4(1-t)^2 & t \geqslant \frac{1}{2}, \\ (1 - 4(\frac{1}{2} - t))^{-1} & \frac{1}{4} < t < \frac{1}{2}, \end{cases}$$

which explodes at $\hat{t} = \frac{1}{4}$.

We also have a comparison theorem for the maximal solutions for BSDEs.

Theorem 6. *Assume $\xi \geqslant \eta$ are bounded and $\mathcal{F}_1$-measurable. Also assume that $f \geqslant g$, where g is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable and f satisfies Hypothesis H. If (X, Λ) is a bounded local solution of $Eq(\eta, g)$ on $(t_1, 1]$ and (Y, Z) is the maximal bounded local solution of $Eq(\xi, f)$ on $(t_2, 1]$, then $X \leqslant Y$ on $(t_1 \vee t_2, 1]$.*

Proof. The proof follows from the Corollary 2 of Lepeltier and San Martín (1998) after noticing that $f_C^{\bar{t}} \geqslant g_C^{\bar{t}}$ for any $\bar{t}$, C. □

3. The case $f(s, \omega, y, z) = G(y) + A|z|^2$

We now concentrate on the case $f(s, \omega, y, z) = G(y) + A|z|^2$, where $A \geqslant 0$. We shall assume that ξ is bounded and positive, $\|\xi\|_\infty = a$. Recall that a is critical if $\int_a^\infty \mathrm{d}x/G(x) = 1$.

3.1. The case $A = 0$

In this subsection we assume that the drift takes the particular form $f(s, \omega, y, z) = G(y)$, and we study the critical case in more detail.

Theorem 7. *Let (Y, Z) be the maximal bounded solution of*

$$Y_t = \xi + \int_t^1 G(Y_s)\mathrm{d}s - \int_t^1 Z_s \,\mathrm{d}W_s, \tag{5}$$

where $a = \|\xi\|_\infty$ is such that $\int_a^\infty \mathrm{d}x/G(x) = 1$.

(*i*) *If for some $t_0 > 0$, $\|\mathrm{E}(\xi|\mathcal{F}_{t_0})\|_\infty = a' < a$, then (Y, Z) may be extended to a maximal bounded solution on $[0, 1]$.*

(*ii*) *If G is convex, the previous condition is also necessary for the existence of a maximal bounded solution on* [0, 1].

Proof. (i) We take (Y, Z) to be the unique maximal bounded solution on (0, 1]. Since $G \geqslant 0$, we have

$$Y_t = \mathrm{E}\left(\xi + \int_t^1 G(Y_s)\mathrm{d}s \middle| \mathcal{F}_\mathrm{t}\right) \geqslant 0,$$

and since G is increasing on $\mathbb{R}_+$, we obtain, for $t < t_0$,

$$\begin{aligned}\|Y_t\|_\infty &\leqslant \|\mathrm{E}(\xi|\mathcal{F}_t)\|_\infty + \int_t^1 G(\|Y_s\|_\infty)\mathrm{d}s \\ &\leqslant a' + \int_{t_0}^1 G(u(s))\mathrm{d}s + \int_t^{t_0} G(\|Y_s\|_\infty)\mathrm{d}s \\ &= u(t_0) + a' - a + \int_t^{t_0} G(\|Y_s\|_\infty)\mathrm{d}s.\end{aligned}$$

Therefore, the explosion time for the solution of $Eq(Y_{t_0}, G)$ on the interval $[0, t_0]$ is bounded by the explosion time of the ODE

$$v(t) = u(t_0) + a' - a + \int_t^{t_0} G(v(s))\mathrm{d}s,$$

which is given by t^* such that

$$t_0 - t^* = \int_{u(t_0)+a'-a}^\infty \frac{\mathrm{d}x}{G(x)} > \int_{u(t_0)}^\infty \frac{\mathrm{d}x}{G(x)} = \int_a^\infty \frac{\mathrm{d}x}{G(x)} - \int_a^{u(t_0)} \frac{\mathrm{d}x}{G(x)} = 1 - (1 - t_0) = t_0.$$

Hence, $t^* < 0$ and there is an extension of (Y, Z) to the entire interval [0, 1].

(ii) If $a = 0$ then $\xi = 0$ and $(u, 0)$ is the maximal solution of (5), which obviously explodes at $t = 0$. So we can assume that $a > 0$. We reason by contradiction and suppose that $\|\mathrm{E}(\xi|\mathcal{F}_t)\|_\infty = a$ for all $t > 0$. We fix $\epsilon > 0$ and $t_0 > 0$. Consider $A = \{\omega : \mathrm{E}(\xi|\mathcal{F}_{t_0})(\omega) > a - \epsilon\}$; then $P(A) > 0$. For $t \geqslant t_0$, we have $Y_t = \mathrm{E}(\xi + \int_t^1 G(Y_s)\mathrm{d}s|\mathcal{F}_\mathrm{t})$; then

$$\begin{aligned}\mathrm{E}(Y_t 1_A) &= \mathrm{E}(\mathrm{E}(\xi|\mathcal{F}_{t_0})1_A) + \int_t^1 \mathrm{E}(G(Y_s)1_A)\mathrm{d}s \\ &\geqslant (a-\epsilon)P(A) + \int_t^1 G\left(\frac{\mathrm{E}(Y_s 1_A)}{P(A)}\right)P(A)\mathrm{d}s.\end{aligned}$$

Hence $\varphi(t) = \mathrm{E}(Y_t 1_A)/P(A)$ satisfies $\varphi(t) \geqslant (a - \epsilon) + \int_t^1 G(\varphi(s))\mathrm{d}s$, which implies that

$$\varphi(t) \geqslant u^{-\epsilon}(t) = a - \epsilon + \int_t^1 G(u^{-\epsilon}(s))\mathrm{d}s.$$

We notice that since G is convex it is locally Lipschitz, and therefore $u^{-\epsilon}$ is unique.

Moreover, $u^{-\epsilon}(t) \nearrow u(t)$ when $\epsilon \to 0$, which also is the unique solution of (2). Since $u(t) \geqslant \|Y_t\|_\infty \geqslant \varphi(t) \geqslant u^{-\epsilon}(t)$ we conclude that $\|Y_t\|_\infty = u(t)$ for all $t > 0$, and then $\lim_{t \searrow 0} \|Y_t\|_\infty = \infty$, which implies that there is no bounded maximal solution on [0, 1]. $\square$

Remark. We observe that, using the technique developed in the proof of Lemma 7, we actually have that if (Y, Z) is a maximal bounded solution of (5), then $\|Y_t\|_\infty \leqslant w(t)$, where w is the maximal solution of $w(t) = \|\mathrm{E}(\xi|\mathcal{F}_t)\|_\infty + \int_t^1 G(w(s))\mathrm{d}s$, and that, under convexity, equality holds. Also we point out that if $a = |\xi|_\infty$ is such that $\int_a^\infty \mathrm{d}x/G(x) > 1$, we have the existence of a maximal bounded solution on [0, 1] by using the result of Theorem 4 and observing that if $\xi \geqslant 0$, we can take $t^* = t_2 < 0$. A difficult case is the supercritical situation $\int_a^\infty \mathrm{d}x/G(x) < 1$. We can give a sufficient condition for there being no explosion in this case.

Proposition 8. *Assume that* $a = |\xi|_\infty$ *is such that* $\int_a^\infty \mathrm{d}x/G(x) < 1$*, that is, the explosion time for* (2) *is* $t^* > 0$*. If there exists* $t_0 > t^*$ *such that* $a' = \|\mathrm{E}(\xi|\mathcal{F}_{t_0})\|_\infty$ *satisfies*

$$\int_{u(t_0)+a'-a}^{u(t_0)} \frac{\mathrm{d}x}{G(x)} > t^*, \tag{6}$$

where u *is the maximal solution of* (2)*, then there is a unique maximal bounded solution of* (5) *in* [0, 1].

Proof. As in the previous proof, we have that for (Y, Z) the maximal local bounded solution of (5) satisfies, for $t \leqslant t_0$,

$$\|Y_t\|_\infty \leqslant u(t_0) + a' - a + \int_t^{t_0} G(\|Y_s\|_\infty)\mathrm{d}s.$$

Therefore, the explosion time of $Eq(Y_{t_0}, G)$ in the interval $[0, t_0]$ is dominated by $\bar{t}$, where

$$t_0 - \bar{t} = \int_{u(t_0)+a'-a}^{\infty} \frac{\mathrm{d}x}{G(x)} = \int_{u(t_0)+a'-a}^{u(t_0)} \frac{\mathrm{d}x}{G(x)} + \int_{u(t_0)}^{\infty} \frac{\mathrm{d}x}{G(x)} = \int_{u(t_0)+a'-a}^{u(t_0)} \frac{\mathrm{d}x}{G(x)} + t_0 - t^*,$$

which implies that $\bar{t} < 0$ because of condition (6). $\square$

In the case $G(x) = x^2$ and $a > 1$, condition (6) becomes that $t_0 > t^* = 1 - 1/a$ and

$$\int_{u(t_0)+a'-a}^{u(t_0)} \frac{\mathrm{d}x}{G(x)} > 1 - \frac{1}{a}.$$

Under some sufficient conditions we can prove the non-existence of solutions on [0, 1] for the supercritical case.

Theorem 9. *Let* $|\xi|_\infty = a$ *be such that* $\int_a^\infty \mathrm{d}x/G(x) < 1$*, where* G *is non-negative, increasing on* $\mathbb{R}_+$ *and locally Lipschitz.*

(*i*) *Equation* (5) *has no solution on* [0, 1] *when*

$$\exists\, 0 \leqslant \alpha_0 \leqslant a \textit{ such that } \int_{\alpha_0}^{\infty} \frac{\mathrm{d}y}{G(y)} \leqslant 1, \quad G(\alpha_0) > 0, \quad \xi \geqslant \alpha_0.$$

(*ii*) *Under the assumption that G is convex,* (5) *has no bounded solution when*

$$\exists 0 \leqslant \alpha_0 \leqslant a \textit{ such that } \int_{\alpha_0}^{\infty} \frac{\mathrm{d}y}{G(y)} \leqslant 1, \quad G(\alpha_0) > 0, \quad \mathrm{E}(\xi) \geqslant \alpha_0.$$

Proof. (i) Let (Y_t, Z_t) be a solution of (5) on [0, 1]. By the comparison theorem (Pardoux and Peng, 1990) we obtain, for all $n \geqslant 1$, that $Y_t \geqslant Y_t^{(n)}$, where $(Y_t^{(n)}, 0)$ is the unique solution of the BSDE

$$X_t = \alpha_0 + \int_t^1 G(\varphi_n(X_s))\mathrm{d}s - \int_t^1 \Lambda_s \,\mathrm{d}W_s.$$

It is easy to see that $Y_t^{(n)}$ is the unique solution of the ODE

$$X_t = \alpha_0 + \int_t^1 G(\varphi_n(X_s))\mathrm{d}s.$$

Let us assume that $Y_0^{(n)} \leqslant n$. Since $Y^{(n)}$ is decreasing, we have $Y_s^{(n)} \leqslant n$ for all s, and consequently $Y_t^{(n)}$ is a solution of the ODE

$$X_t = \alpha_0 + \int_t^1 G(X_s)\mathrm{d}s,$$

which admits no solution on [0, 1]. Finally, we obtain $Y_0 \geqslant Y_0^{(n)} \geqslant n$ for all n, which is a contradiction.

(ii) In the same way, let (Y_t, Z_t) now be a bounded solution of (5) on [0, 1]. Taking expectations, we obtain, since G is convex,

$$\mathrm{E}(Y_t) = \mathrm{E}(\xi) + \int_t^1 \mathrm{E}(G(Y_s))\mathrm{d}s \geqslant \alpha_0 + \int_t^1 G(\mathrm{E}(Y_s))\mathrm{d}s.$$

Thus, $(\mathrm{E}(Y_s))_{0\leqslant s\leqslant 1}$ is a supersolution of (2) which has no solution on [0, 1], consequently no supersolution, which is a contradiction. □

Remark. In the critical case ($\int_a^\infty \mathrm{d}x/G(x) = 1$) we can also treat the case of ξ not necessarily positive. In fact, let (X, Λ) be the maximal solution of $Eq(\xi^+, G)$. Then the maximal solution (Y, Z) of $Eq(\xi, G)$ satisfies:

$$-\|\xi^-\|_\infty \leqslant Y_t \leqslant X_t.$$

If X_t does not explode then neither does Y_t, and the sufficient condition for this is the existence of $t_0 > 0$ such that $\|\mathrm{E}(\xi^+|\mathcal{F}_{t_0})\|_\infty < a$. Nevertheless, when G is convex, this condition cannot be both sufficient and necessary since we do not have G increasing in $[-\|\xi^-\|_\infty, \infty)$.

In the special case when G is C^1, we can give a uniqueness result.

Theorem 10. *Let G be C^1 and (Y^1, Z^1), (Y^2, Z^2) be two solutions of* (5)*, with Y^1 and Y^2 bounded. Then we have $Y^1 = Y^2$, $Z^1 = Z^2$.*

Proof. Since G is C^1, hence locally Lipschitz, and Y^1, Y^2 bounded, the function G is Lipschitz in the range of Y^1, Y^2, and the uniqueness result of Pardoux and Peng is applicable. $\square$

3.2. The case $A > 0$

We consider the BSDE

$$Y_t = \xi + \int_t^1 (G(Y_s) + A|Z_s|^2)\mathrm{d}s - \int_t^1 Z_s \, \mathrm{d}W_s, \tag{7}$$

where $\|\xi\|_\infty = a$, $\xi \geqslant 0$.

We can reduce the study of this BSDE to case when $A = 0$ by an exponential change of variables. Let $X_t = \mathrm{e}^{2AY_t}$. Then, using Itô's formula, we obtain

$$\begin{aligned}
\mathrm{e}^{2A\xi} &= \mathrm{e}^{2AY_t} + \int_t^1 2A\mathrm{e}^{2AY_s}\,\mathrm{d}Y_s + \frac{1}{2}\int_t^1 4A^2\mathrm{e}^{2AY_s}|Z_s|^2\,\mathrm{d}s \\
&= \mathrm{e}^{2AY_t} + \int_t^1 2A\mathrm{e}^{2AY_s}(-G(Y_s) - A|Z_s|^2)\mathrm{d}s + \int_t^1 2A\mathrm{e}^{2AY_s} Z_s\,\mathrm{d}W_s \\
&\quad + \int_t^1 2A^2\mathrm{e}^{2AY_s}|Z_s|^2\,\mathrm{d}s,
\end{aligned}$$

and therefore

$$X_t = \eta + \int_t^1 2AX_s G\left(\frac{\log(X_s)}{2A}\right)\mathrm{d}s - \int_t^1 \Lambda_s\,\mathrm{d}W_s,$$

where $\eta = \mathrm{e}^{2A\xi}$.

We observe that $Y_t \geqslant 0$, implying that $X_t \geqslant 1$. If G is increasing then so is the function $h(x) = 2AxG(\log(x)/2A)$ for $x \geqslant 1$, and if G is convex then so is h. The explosion time for X is given by

$$1 - t^* = \int_{\mathrm{e}^{2Aa}}^\infty \left(2AxG\left(\frac{\log(x)}{2A}\right)\right)^{-1}\mathrm{d}x = \int_a^\infty \frac{\mathrm{d}x}{G(x)},$$

the same as the one of the initial equation (7), as expected. By Theorem 7 we deduce the following result.

Theorem 11.

(*i*) *Equation* (7) *does not explode if* $\|\mathrm{E}(\mathrm{e}^{2A\xi}|\mathcal{F}_{t_0})\|_\infty < \mathrm{e}^{2Aa}$ *for some* $t_0 > 0$.
(*ii*) *If G is convex, the above condition is also necessary.*

Using Theorems 4, 7 and the comparison theorem, we obtain the following result about the BSDE (1).

Theorem 12. *Assume* $|f(s, \omega, y, z)| \leqslant G(y) + A|z|^2$, *where* G *is a symmetric function, and* $G : \mathbb{R}_+ \to \mathbb{R}_+$ *is increasing. Assume* $\|\xi\|_\infty = a' \leqslant a$, *where* $\int_a^\infty \mathrm{d}y/G(y) = 1$.

(*i*) *If* $a' < a$ *then* (1) *does not explode.*
(*ii*) *In the (critical) case* $a' = a$, *a sufficient condition for non-explosion is that there exists* $t_0 > 0$ *such that* $\|\mathrm{E}(\mathrm{e}^{2A\xi}|\mathcal{F}_{t_0})\|_\infty < \mathrm{e}^{2Aa}$, *when* $A > 0$; *and such that* $\|\mathrm{E}(\xi^+|\mathcal{F}_{t_0})\|_\infty < a$, *when* $A = 0$.

Finally, we investigate the condition

$$\|\xi\|_\infty = a, \qquad \text{and } \exists t_0 > 0 \|\mathrm{E}(\xi|\mathcal{F}_{t_0})\|_\infty < a. \tag{8}$$

For this purpose let us denote by $\mathcal{F}_{\alpha,\beta}$ the completed σ-algebra generated by $W_t - W_u$, $\alpha \leqslant u \leqslant t \leqslant \beta$.

Lemma 13. *Under either of the following conditions, the random variable* ξ *satisfies* (8).

(*i*) $\|\xi\|_\infty = a$, ξ *is not a constant P-a.s. and* $\xi \in \cup_{\epsilon>0} \cup \mathcal{F}_{\epsilon,1}$;
(*ii*) $\xi = F(W_{t_1}, \ldots, W_{t_n})$, *where* $0 < t_1 < t_2 < \ldots < t_n \leqslant 1$, *with* $\|F\|_\infty = a$ *and there exists* $\epsilon > 0$, $K \subseteq R^n$ *compact, such that* $|F(x)| < a - \epsilon$ *for all* $x \notin K$.

Proof. Under (i), $\xi \in \mathcal{F}_{\epsilon,1}$ for some $\epsilon > 0$ and then $\mathrm{E}(\xi|\mathcal{F}_\epsilon) = \mathrm{E}(\xi) < a$, and (8) holds.
Now assume (ii), and take any $0 < t_0 < t_1$. We have

$$\mathrm{E}(F(W_{t_1}, \ldots, W_{t_n})|\mathcal{F}_{t_0}) = \varphi(W_{t_0}),$$

where $\varphi(x) = \mathrm{E}_x(F(W_{t_1-t_0}, \ldots, W_{t_n-t_0}))$. The function φ is continuous and satisfies the inequality

$$|\varphi(x)| \leqslant aP_x((W_{t_1-t_0}, \ldots, W_{t_n-t_0}) \in K) + (a-\epsilon)P_x((W_{t_1-t_0}, \ldots, W_{t_n-t_0}) \notin K) < a.$$

Since $\lim_{x\to\infty} P_x((W_{t_1-t_0}, \ldots, W_{t_n-t_0}) \in K) = 0$, it is deduced that $\|\varphi\|_\infty < a$. Also observe that $\|F(W_{t_1}, \ldots, W_{t_n})\|_\infty = \|F\|_\infty = a$. □

We remark that under (i) or (ii) of Lemma 13 we also have, for all $A > 0$,

$$\|\mathrm{E}(\mathrm{e}^{2A\xi^+}|\mathcal{F}_{t_0})\|_\infty < \mathrm{e}^{2Aa}$$

for the same t_0.

We also give an example where (8) is not satisfied. Take $A_n \in \mathcal{F}_{1/(n+1),1/n}$ for all $n \geqslant 1$, $0 < P(A_n) < 1$, with $\sum_{n\geqslant1} P(A_n) < \infty$, and define $\xi = a - \exp(-\sum_{n\geqslant1} 1_{A_n})$, which is a non-negative bounded random variable. It is easy to prove that $\|\xi\|_\infty = a$, and if $t \in (1/(n+1), 1/n]$, then

$$\mathrm{E}(\xi|\mathcal{F}_t) = a - \exp\left(-\sum_{k\geqslant n+1} 1_{A_k}\right)\mathrm{E}\left(\exp\left(-\sum_{k=1}^{n} 1_{A_k}\right)\Big|\mathcal{F}_t\right)$$

$$\geqslant a - \exp\left(-\sum_{k\geqslant n+1} 1_{A_k}\right).$$

Let $B_{n,m} = A_{n+1} \cap \ldots \cap A_{n+m}$; then

$$P(B_{n,m}) = \prod_{k=n+1}^{n+m} P(A_k) > 0.$$

We have $\mathrm{E}(\xi|\mathcal{F}_t)1_{B_{n,m}} \geqslant (a - \mathrm{e}^{-m})1_{B_{n,m}}$, which proves that $\|\mathrm{E}(\xi|\mathcal{F}_t)\|_\infty = a$ for all $t > 0$. Observe that $\xi = F(W_1)$ with F bounded by 1 but $\lim_{x\to\infty} F(x) = 1$ also gives an example of a random variable which does not satisfy (8).

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References

Kobylanski, M. (1997) Résultats d'existence et d'unicité pour des équations différentielles stochastiques rétrogrades avec des générateurs à croissance quadratique. *C. R. Acad. Sci. Paris*, **324**(1), 81–86.

Lepeltier, J.-P. and San Martín, J. (1997) Backward stochastic differential equations with continuous coefficient. *Statist. Probab. Lett.*, **32**, 425–430.

Lepeltier, J.-P. and San Martín, J. (1998) Existence for BSDE with superlinear-quadratic coefficient. *Stochastics Stochastics Rep.*, **63**, 227–240.

Pardoux, E. and Peng, S. (1990) Adapted solution of a backward stochastic differential equation. *Systems Control Lett.*, **14**, 55–61.

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