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Identification and properties of real harmonizable fractional Lévy motions

ALBERT BENASSI[1], SERGE COHEN[2] and JACQUES ISTAS[3]

[1]*Université Blaise Pascal (Clermont-Ferrand II), LaMP, CNRS UPRESA 6016, 63177 Aubière Cedex, France. E-mail: benassi@opgc.univ-bpclermont.fr*
[2]*Université Paul Sabatier, UFR MIG, Laboratoire de Statistique et de Probabilités, 118 route de Narbonne, 31062 Toulouse, France. E-mail: scohen@cict.fr*
[3]*Département IMSS BSHM, Université Pierre Mendes-France 38000 Grenoble, France. E-mail: jacques.istas@upmf-grenoble.fr*

The class of real harmonizable fractional Lévy motions (RHFLMs) is introduced. It is shown that these share many properties with fractional Brownian motion. These fields are locally asymptotically self-similar with a constant index H, and have Hölderian paths. Moreover, the identification of H for the RHFLMs can be performed with the so-called generalized variation method. Besides fractional Brownian motion, this class contains non-Gaussian fields that are asymptotically self-similar at infinity with a real harmonizable fractional stable motion of index $\tilde{H}$ as tangent field. This last property should be useful in modelling phenomena with multiscale behaviour.

Keywords: identification; local asymptotic self-similarity; second-order fields; stable fields

1. Introduction

The fractional Brownian motion (FBM) $B_H(t)$ of fractional index H, introduced in Mandelbrot and Ness (1968), provides a very powerful model in applied mathematics. It is the only centred Gaussian self-similar process with stationary increments and with index H.

Let us recall the definition of self-similarity. A field $X(x)$, $x \in \mathbb{R}^d$, is self-similar with index H if, for all $\lambda > 0$,

$$(X(\lambda x))_{x\in\mathbb{R}^d} \stackrel{(d)}{=} \lambda^H (X(x))_{x\in\mathbb{R}^d},$$

where $\stackrel{(d)}{=}$ stands for equality in distribution.

Self-similarity is a global property, and for some applications one needs a local version of it. A field $X(x)$ is locally asymptotically self-similar (lass) at point x with index H if

$$\lim_{\lambda\to 0^+} \left(\frac{X(x+\lambda u) - X(x)}{\lambda^H} \right)_{u\in\mathbb{R}^d} \stackrel{(d)}{=} (T(u))_{u\in\mathbb{R}^d},$$

where the non-degenerate field $(T(u))_{u\in\mathbb{R}^d}$ is called the tangent field at x. This definition was introduced in Benassi *et al.* (1997) and Peltier and Lévy Véhel (1995), and various examples of lass Gaussian fields have been studied: filtered white noise (Benassi *et al.* 1998b); multifractional Brownian motion, where the fractional index H is replaced by a fractional function $H(x)$ (Benassi *et al.* 1997; Peltier and Lévy Véhel 1996); and, for irregular fractional functions, the generalized multifractional Brownian motion (Ayache and Lévy Véhel 1999) and the step fractional Brownian motion (Benassi *et al.* 2000). These fields share many properties with FBM. For instance, the sample paths are locally almost surely $C^{H'}$ for every $H' < H(x)$. In these models the identification of the fractional function is a central problem. It has been done with semi-parametric estimators based on generalized quadratic variations (Istas and Lang 1997; Benassi *et al.* 1998a; 2000).

In this paper we propose a class of second-order fields which we refer to as real harmonizable fractional Lévy motion (RHFLM), including non-Gaussian fields and FBM, which have stationary increments, which are lass with index H and tangent FBM, and which have sample paths locally almost surely $C^{H'}$ for every $H' < H$. These fields are obtained by integrating fractionally a Lévy measure $L(\mathrm{d}\xi)$ that has moments of every order:

$$X_H(x) = \int_{\mathbb{R}^d} \frac{\mathrm{e}^{-\mathrm{i}x\cdot\xi} - 1}{\|\xi\|^{d/2+H}} L(\mathrm{d}\xi). \tag{1}$$

When the Lévy measure $L(\mathrm{d}\xi)$ is a Brownian measure $W(\mathrm{d}\xi)$, (1) yields the harmonizable representation of the FBM (Samorodnitsky and Taqqu 1994). Let us also recall that the real harmonizable fractional stable motion (RHFSM) has a representation similar to (1) where $L(\mathrm{d}\xi)$ is a complex isotropic stable α-symmetric measure $M_\alpha(\mathrm{d}\xi)$. Nevertheless, RHFSMs are not RHFLMs since the variance of their increments is infinite. Actually the complex isotropic stable α-symmetric measure can be truncated to yield an RHFLM that has an asymptotic self-similarity at infinity with an index $\tilde{H}$ different from H:

$$\lim_{\lambda\to+\infty} \left(\frac{X(\lambda u)}{\lambda^{\tilde{H}}}\right)_{u\in\mathbb{R}^d} \overset{(d)}{=} (T(u))_{u\in\mathbb{R}^d},$$

where the non-degenerate field $(T(u)_{u\in\mathbb{R}^{(d)}}$ is called the asymptotic field. Roughly speaking, this 'truncated' real harmonizable fractional stable motion can be seen as a bridge between FBM and true RHFSMs. We think that this class of identifable models which exhibit a very different behaviour at low scale ($\lambda \to 0^+$, lass) and at large scale ($\lambda \to +\infty$), where discrete structures (related to the 'big jumps' of $M_\alpha(\mathrm{d}\xi)$) appear, should be useful in certain applications (see Hermann and Roux 1990, for instance). This phenomenon of different asymptotic self-similarities at low and large scales has already been encountered in Benassi and Deguy (1999), but there the asymptotic fields are both FBMs, which is less remarkable. FBM is also often used in finance, although a scaling index that varies with scale has been reported in Bardet (2001). RHFLM may provide a more realistic model in this domain.

From the statistical point of view, the fractional index of an RHFLM can be estimated with the method of generalized quadratic variations which cannot be applied to RHFSM – see Abry *et al.* (2000a; 2000b) for other methods of estimation in the case of RHFSM.

In Section 2 the construction of RHFLMs is discussed. The asymptotic self-similarity properties and the regularity of the sample paths of such fields are studied in Section 3, and Section 4 is devoted to the identification of the fractional index.

2. Construction of non-Gaussian fractional fields

In this section the RHFLMs are obtained and the construction of the Lévy measure $L(\mathrm{d}\xi)$ described in detail. For the sake of simplicity, we work first with a Lévy measure (written $M(\mathrm{d}\xi)$) without Brownian component. Since it is desirable to be able to identify the parameter H with generalized quadratic variations, we require the field X_H to have moments of order 2: $\mathbb{E}(|X_H(x)|^2) < +\infty$, for all $x \in \mathbb{R}^d$. Actually if $M(\mathrm{d}\xi)$ is a symmetric α-stable measure the RHFSM X_H (cf. Samorodnitsky and Taqqu 1994) is self-similar but $X_H(x)$ fails to have moments of order 2. Hence we consider a non-vanishing Lévy measure $M(\mathrm{d}\xi)$ represented by a Poisson random measure $N(\mathrm{d}\xi, \mathrm{d}z)$ in the sense of Section 3.12 of Samorodnitsky and Taqqu (1994) but with a control measure that has moments of order $p \geqslant 2$. See Neveu (1977) for a general discussion of Poisson measures. Specifically, let $N(\mathrm{d}\xi, \mathrm{d}z)$ be a Poisson measure on $\mathbb{R}^{(d)} \times \mathbb{C}$ for which the mean measure $n(\mathrm{d}\xi, \mathrm{d}z) = \mathbb{E}N(\mathrm{d}\xi, \mathrm{d}z) = \mathrm{d}\xi\nu(\mathrm{d}z)$ satisfies

$$\forall p \geqslant 2, \qquad \int_{\mathbb{C}} |z|^p \nu(\mathrm{d}z) < +\infty. \tag{2}$$

Let us recall the basic properties of the compensated Poisson measure

$$\tilde{N} = N - n.$$

For every function $\varphi : \mathbb{R}^d \times \mathbb{C} \to \mathbb{C}$ such that $\varphi \in L^2(\mathbb{R}^d \times \mathbb{C})$, the stochastic integral

$$\int_{\mathbb{R}^d \times \mathbb{C}} \varphi(\xi, z)\tilde{N}(\mathrm{d}\xi, \mathrm{d}z)$$

is defined as the limit in $L^2(\Omega)$ of

$$\int_{\mathbb{R}^d \times \mathbb{C}} \varphi_k(\xi, z)\tilde{N}(\mathrm{d}\xi, \mathrm{d}z),$$

where φ_k is a simple function of the form $\sum_{i \in I} a_i \mathbf{1}_{A_i}$. The set I is finite and

$$\int_{\mathbb{R}^d \times \mathbb{C}} \sum_{i \in I} a_i \mathbf{1}_{A_i} \tilde{N}(\mathrm{d}\xi, \mathrm{d}z) \stackrel{def}{=} \sum_{i \in I} a_i \tilde{N}(A_i),$$

where the Poisson random variables have intensity $n(A_i)$ and are independent if the sets A_i are disjoint. The classical isometry property is:

$$\mathbb{E}\left| \int_{\mathbb{R}^d \times \mathbb{C}} \varphi(\xi, z)\tilde{N}(\mathrm{d}\xi, \mathrm{d}z) \right|^2 = \int_{\mathbb{R}^d \times \mathbb{C}} |\varphi(\xi, z)|^2 n(\mathrm{d}\xi, \mathrm{d}z). \tag{3}$$

Furthermore if φ is real-valued then so is $\int \varphi \, \mathrm{d}\tilde{N}$, and if $\mathcal{R}(z)$ denotes the real part of a complex quantity z then $\mathcal{R}(\int \varphi \, \mathrm{d}\tilde{N}) = \int \mathcal{R}(\varphi)\mathrm{d}\tilde{N}$; the same property is true for the imaginary part $\mathcal{J}$ of stochastic integrals.

It follows that, for all $u, v \in \mathbb{R}$, the characteristic function of the stochastic integral is

$$\mathbb{E} \exp\left(\mathrm{i}\left(u \int \mathcal{R}(\varphi)\mathrm{d}\tilde{N} + v \int \mathcal{J}(\varphi)\mathrm{d}\tilde{N} \right) \right)$$

$$= \exp\left[\int_{\mathbb{R}^d\times\mathbb{C}} [\exp(\mathrm{i}(u\mathcal{R}(\varphi) + v\mathcal{J}(\varphi))) - 1 - \mathrm{i}(u\mathcal{R}(\varphi) + v\mathcal{J}(\varphi))]\mathrm{d}\xi\nu(\mathrm{d}z)\right], \quad (4)$$

where the integral on the right-hand side is convergent since

$$|\exp(\mathrm{i}x) - 1 - \mathrm{i}x| \leqslant C|x|^2 \qquad \forall x \in \mathbb{R}.$$

We observe that the Poisson measure N has to be defined on $\mathbb{R}^d \times \mathbb{C}$ for the field X_H to be real-valued, as in the case of RHFSM. To be more specific, we consider the compensated Poisson measure $\tilde{N} = N - n$, and we can now define the Lévy measure.

Definition 2.1. *We define a Lévy measure by the following integration property:*

$$\int_{\mathbb{R}^d} f(\xi)M(\mathrm{d}\xi) = \int_{\mathbb{R}^d\times\mathbb{C}} [f(\xi)z + f(-\xi)\bar{z}]\tilde{N}(\mathrm{d}\xi, \mathrm{d}z) \quad (5)$$

for every function $f : \mathbb{R}^d \to \mathbb{C}$ *where* $f \in L^2(\mathbb{R}^d)$.

The name 'Lévy measure' is related to Lévy processes, which are processes with stationary and independent increments. The Lévy decomposition of such processes (cf. Theorem 42 in Protter 1990, p. 32) can in some cases be given by

$$X_t = \int \mathbf{1}_{[0,t]}(\xi)z\tilde{N}(\mathrm{d}\xi, \mathrm{d}z),$$

where $\tilde{N}$ is a real-valued compensated Poisson measure. Formally, we can define

$$M(\mathrm{d}\xi) = z\tilde{N}(\mathrm{d}\xi, \mathrm{d}z)$$

and

$$X_t = \int \mathbf{1}_{[0,t]}(\xi)M(\mathrm{d}\xi).$$

This suggests (5) if we neglect the problem of getting real-valued processes. Then if,

$$\forall \xi \in \mathbb{R}^d, \qquad f(-\xi) = \overline{f(\xi)}, \quad (6)$$

then

$$\int_{\mathbb{R}^d} f(\xi)M(\mathrm{d}\xi) = \int_{\mathbb{R}^d\times\mathbb{C}} 2\mathcal{R}(f(\xi)z)\tilde{N}(\mathrm{d}\xi, \mathrm{d}z) = 2\mathcal{R}\left(\int_{\mathbb{R}^d\times\mathbb{C}} f(\xi)z\tilde{N}(\mathrm{d}\xi, \mathrm{d}z)\right) \in \mathbb{R}.$$

Hence X_H is a real-valued field since the integrand

$$\frac{\mathrm{e}^{-\mathrm{i}x\cdot\xi} - 1}{\|\xi\|^{d/2+H}} \quad (7)$$

satisfies (6).

Classically, the field X_H has stationary increments if the control measure $\nu(\mathrm{d}z)$ is assumed to be rotationally invariant. Let P be the map $P(\rho\exp(\mathrm{i}\theta)) = (\theta, \rho) \in [0, 2\pi) \times \mathbb{R}_*^+$. Henceforth the measure satisfies the property

$$P(\nu(\mathrm{d}z)) = \mathrm{d}\theta \nu_\rho(\mathrm{d}\rho), \tag{8}$$

where $\mathrm{d}\theta$ is the uniform measure on $[0, 2\pi)$. When f satisfies (6), for every measurable function a, the equality

$$\int f(\xi)\exp(\mathrm{i}a(\xi))M(\mathrm{d}\xi) \overset{(d)}{=} \int f(\xi)M(\mathrm{d}\xi),$$

where $\overset{(d)}{=}$ means that the random variables have the same distributions, is a consequence of (4). Moreover, the stochastic integral has a symmetric distributions:

$$-\int f(\xi)M(\mathrm{d}\xi) \overset{(d)}{=} \int f(\xi)M(\mathrm{d}\xi).$$

Hence, under assumption (8), X_H is a field with stationary increments, which considerably simplifies the following developments. Moreover, as a consequence of (3), an isometry property holds for the Lévy measure $M(\mathrm{d}\xi)$ when f satisfies (6):

$$\mathbb{E}\left|\int_{\mathbb{R}^d} f(\xi)M(\mathrm{d}\xi)\right|^2 = 4\pi\|f\|^2_{L^2(\mathbb{R}^d)}\int_0^{+\infty}\rho^2\nu_\rho(\mathrm{d}\rho).$$

To study the field X_H moment of order $2p$, $\mathbb{E}((X_H(s) - X_H(t))^{2p})$ is computed with the help of the corresponding L^{2p} norms of the deterministic integrand (7). Actually the characteristic function (4) allows us to compute every moment of the stochastic integrals $\int f(\xi)M(\mathrm{d}\xi)$.

Proposition 2.2. *If $f \in \bigcap_{q=1}^{p} L^{2q}(\mathbb{R}^d)$ and f satisfies (6) then $\int f(\xi)M(\mathrm{d}\xi)$ is in $L^{2p}(\Omega)$ and*

$$\mathbb{E}\left(\left(\int f(\xi)M(\mathrm{d}\xi)\right)^{2p}\right) = \sum_{n=1}^{p}(2\pi)^n\sum_{P_n}\prod_{q=1}^{n}\frac{(2m_q)!\|f\|^{2m_q}_{2m_q}\int_0^{+\infty}\rho^{2m_q}\nu_\rho(\mathrm{d}\rho)}{(m_q!)}, \tag{9}$$

where $\sum_{P_n}$ stands for the sum over the set of partitions P_n of $\{1, \ldots, 2p\}$ into n subsets K_q such that the cardinality of K_q is $2m_q$ with $m_q \geqslant 1$ and where $\|f\|_{2m_q}$ is the $L^{2m_q}(\mathbb{R}^d)$ norm of f.

Proof. A power series expansion of both sides of (4) yields the result. □

We now introduce the real harmonizable fractional Lévy motion.

Definition 2.3. *A real harmonizable fractional Lévy motion (RHFLM) is a real-valued field which admits a harmonizable representation*

$$X_H(x) = \int_{\mathbb{R}^d}\frac{\mathrm{e}^{-\mathrm{i}x\cdot\xi} - 1}{\|\xi\|^{d/2+H}}L(\mathrm{d}\xi),$$

where $L(\mathrm{d}\xi) = aM(\mathrm{d}\xi) + bW(\mathrm{d}\xi)$ is the sum of a Lévy measure $aM(\mathrm{d}\xi)$ and of an independent $bW(\mathrm{d}\xi)$ Wiener measure. We suppose that $M(\mathrm{d}\xi)$ satisfies the finite-moment assumption (2) and the rotational invariance (8).

With this definition X_H is the sum of the process

$$\int_{\mathbb{R}^d} \frac{e^{-ix\cdot\xi} - 1}{\|\xi\|^{d/2+H}} bW(d\xi),$$

which is an FBM, and of

$$\int_{\mathbb{R}^d} \frac{e^{-ix\cdot\xi} - 1}{\|\xi\|^{d/2+H}} aM(d\xi),$$

both processes being independent. In particular, the FBM is an RHFLM obtained for $L(d\xi) = W(d\xi)$.

In the following proposition all the properties obtained for the RHFLMs in the construction are summarized. For the sake of simplicity, let us focus on the case $L(d\xi) = M(d\xi)$.

Proposition 2.4. *Let us consider a Lévy measure* $M(d\xi)$ *that satisfies the finite moment assumption* (2) *and the rotational invariance* (8).

The real harmonizable fractional Lévy motion

$$X_H(x) = \int_{\mathbb{R}^d} \frac{e^{-ix\cdot\xi} - 1}{\|\xi\|^{d/2+H}} M(d\xi)$$

is such that:

- $\mathbb{P}(X_H(x) \in \mathbb{R},\ \forall x \in \mathbb{R}^d) = 1$;
- X_H *has stationary increments such that*

$$\mathbb{E}(X_H(x) - X_H(y))^2 = 4\pi \int_0^{+\infty} \rho^2 \nu_\rho(d\rho) \int_{\mathbb{R}^d} \frac{2(1 - \cos(\xi_1))}{\|\xi\|^{d+2H}} d\xi \|x - y\|^{2H},$$

  where ξ_1 *is the first component of* ξ;
- *for every* $u = (u_1, \ldots, u_n)$ *and* $v = (v_1, \ldots, v_n)$,

$$\mathbb{E} \exp\left(i \sum_{k=1}^{n} v_k X_H(u_k) \right) = \exp\left(\int_{\mathbb{R}^d \times \mathbb{C}} [\exp(f_{u,v,H}(\xi, z)) - 1 - f_{u,v,H}(\xi, z)] d\xi\, d\nu(z) \right),$$

  where

$$f_{u,v,H}(\xi, z) = i2\mathcal{R}\left(z \sum_{k=1}^{n} v_k \frac{e^{-i\epsilon u_k\cdot\xi} - 1}{\|\xi\|^{d/2+H}} \right).$$

The proof of this proposition is a straightforward consequence of the construction of RHFLM.

3. Some properties of real harmonizable fractional Lévy motions

In this section we investigate some properties of self-similarity and regularity types that the RHFLM shares with FBM. In Section 3.1 we prove two asymptotic self-similarity properties for the RHFLM. Since no trivial invariance property is assumed for the Lévy measure, this is the best result we can hope for in that direction. In Section 3.2 we see that the paths of the RHFLM are almost surely Hölder-continuous with a pointwise Hölder exponent H. We suppose for the sake of simplicity in this subsection that $L(\mathrm{d}\xi) = M(\mathrm{d}\xi)$.

3.1. Asymptotic self-similarity

Since we know the characteristic function of stochastic integrals of the measure $M(\mathrm{d}\xi)$ we can prove the local self-similarity of RHFLMs. Actually this is a consequence of the homogeneity property of $1/(\|\xi\|^{d/2+H})$ and of a central limit theorem for the stochastic measure $M(\mathrm{d}\xi)$.

Proposition 3.1. *Real harmonizable fractional Lévy motion is locally self-similar with parameter H in the sense that, for every fixed $x \in \mathbb{R}^d$,*

$$\lim_{\epsilon\to 0^+}\left(\frac{X_H(x+\epsilon u)-X_H(x)}{\epsilon^H}\right)_{u\in\mathbb{R}^d} \overset{(d)}{=} \left(2\pi\int_0^{+\infty}\rho^2\nu_\rho(\mathrm{d}\rho)\right)^{1/2}(B_H(u))_{u\in\mathbb{R}^d}, \tag{10}$$

where the convergence is in distribution on the space of continuous functions endowed with the topology of the uniform convergence on compact sets. The limit is the distribution of an FBM.

Proof. The convergence of the finite-dimensional margins is proved first. Since the RHFLM has stationary increments we only have to prove convergence for $x = 0$. Let us consider the multivariate function

$$g_{u,v,H}(\epsilon,\xi,z) = \mathrm{i}2\mathcal{R}\left(z\sum_{k=1}^{n} v_k\frac{\mathrm{e}^{-\mathrm{i}\epsilon u_k\cdot\xi}-1}{\epsilon^H\|\xi\|^{d/2+H}}\right),$$

where $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are in $\mathbb{R}^n$. Then

$$\mathbb{E}\exp\left(\mathrm{i}\sum_{k=1}^{n} v_k\frac{X_H(\epsilon u_k)}{\epsilon^H}\right) = \exp\left(\int_{\mathbb{R}^d\times\mathbb{C}}[\exp(g_{u,v,H}(\epsilon,\xi,z)) - 1 - g_{u,v,H}(\epsilon,\xi,z)]\mathrm{d}\xi\,\mathrm{d}\nu(z)\right).$$

The change of variable $\lambda = \epsilon\xi$ is applied to the integral on the right-hand side to give

$$\int_{\mathbb{R}^d\times\mathbb{C}}\left[\exp\left(\epsilon^{d/2}g_{u,v,H}(1,\lambda,z)\right) - 1 - \epsilon^{d/2}g_{u,v,H}(1,\lambda,z)\right]\frac{\mathrm{d}\lambda}{\epsilon^d}\,\mathrm{d}\upsilon(z).$$

Then as $\epsilon \to 0^+$ a dominated convergence argument yields that

$$\lim_{\epsilon\to 0^+} \mathbb{E}\exp\left(\mathrm{i}\sum_{k=1}^{n} v_k \frac{X_H(\epsilon u_k)}{\epsilon^H}\right) = \exp\left(\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{C}} g^2_{u,v,H}(1, \lambda, z)\mathrm{d}\lambda\, \mathrm{d}\nu(z)\right).$$

Moreover, (8) allows us to express the logarithm of this limit as

$$-2\pi\int_0^{+\infty} \rho^2\nu_\rho(\mathrm{d}\rho)\int_{\mathbb{R}^d} \frac{\left|\sum_{k=1}^{n} v_k(\mathrm{e}^{-\mathrm{i}u_k\cdot\lambda} - 1)\right|^2}{\|\lambda\|^{d+2H}}\,\mathrm{d}\lambda,$$

which is the variance of $\sum_{k=1}^{n} v_k B_H(u_k)$, and this concludes the proof of the convergence of finite-dimensional margins.

Let us proceed to the proof that the distributions are tight. We need to estimate

$$\mathbb{E}(X_H(x) - X_H(y))^{2p}$$

for sufficiently large p. Unfortunately, when $H > 1 - d/2$ these moments are not finite because of the asymptotic properties of the integrand:

$$g_0(x, \xi) = \frac{\mathrm{e}^{-\mathrm{i}x\cdot\xi} - 1}{\|\xi\|^{d/2+H}}$$

when $\|\xi\| \to 0$. In the case $H > 1 - d/2$ we thus apply a transformation to the integrand g_0 to analyse in two different ways its behaviour at both ends of the spectrum.

Let us first consider the easy case: $H \leqslant 1 - d/2$. Then $g_0(x, .) \in L^{2q}(\mathbb{R}^d)$ for all $q \in \mathbb{N}^+$, and

$$\|g_0(x, .) - g_0(y, .)\|^{2q}_{L^{2q}(\mathbb{R}^d)} = \|x - y\|^{2Hq+d(q-1)}\|g_0(e_1, .)\|^{2q}_{L^{2q}(\mathbb{R}^d)},$$

where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d$. Because of (9) we know that

$$\mathbb{E}(X_H(x) - X_H(y))^{2p} = \sum_{n=1}^{p} D(n)\|x - y\|^{2Hp+d(p-n)}$$

for some non-negative constants $D(n)$. Hence there exists $C < +\infty$ such that

$$\mathbb{E}(X_H(x) - X_H(y))^{2p} \leqslant C\|x - y\|^{2Hp}. \tag{11}$$

Hence if $H \leqslant 1 - d/2$,

$$\mathbb{E}\left(\frac{(X_H(x + \epsilon u) - X_H(x + \epsilon v))^{2p}}{\epsilon^{2Hp}}\right) \leqslant \|u - v\|^{2Hp}$$

and we can take $p > d/2H$ to show tightness.

When $H > 1 - d/2$, let K be an integer such that $K \geqslant 1 + d/2$, let

$$P_K(t) = \sum_{k=1}^{K} \frac{t^k}{k!},$$

and let φ an even C^1-function such that $\varphi(t) = 1$ when $|t| \leqslant 1/2$ and $\varphi(t) = 0$ when $|t| > 1$. Then

$$g_K(x,\xi)=\frac{\mathrm{e}^{-\mathrm{i}x\cdot\xi}-1-P_K(-\mathrm{i}x\cdot\xi)\varphi(\|x\|\,\|\xi\|)}{\|\xi\|^{d/2+H}}$$

is in $L^{2q}(\mathbb{R}^d)$ for every $x\in\mathbb{R}^d$ and $q\in\mathbb{N}^+$.

X_H is then split into two fields $X_H=X_H^+ + X_H^-$, where

$$X_H^+(x)=\int g_K(x,\xi)M(\mathrm{d}\xi) \tag{12}$$

and

$$X_H^-(x)=\int\frac{P_K(-\mathrm{i}x\cdot\xi)}{\|\xi\|^{d/2+H}}\varphi(\|x\|\,\|\xi\|)M(\mathrm{d}\xi). \tag{13}$$

A method similar to the one used for X_H when $H\leqslant 1-d/2$ is applied to X_H^+, and we check that X_H^- has almost surely C^1 paths.

Let us start by observing that

$$\|g_K(x,.)\|^{2q}_{L^{2q}(\mathbb{R}^d)}=\|x\|^{2Hq+d(q-1)}\|g_K(e_1,.)\|^{2q}_{L^{2q}(\mathbb{R}^d)}.$$

As in the easy case, we have to estimate

$$I_\epsilon=\int_{\mathbb{R}^d}|g_K(x,\xi)-g_K(x+\epsilon u,\xi)|^{2q}\,\mathrm{d}\xi$$

when $\epsilon\to 0^+$. Let us split this integral into

$$I_\epsilon^+=\int_{\epsilon\|\xi\|\geqslant 1}|g_K(x,\xi)-g_K(x+\epsilon u,\xi)|^{2q}\,\mathrm{d}\xi$$

and

$$I_\epsilon^-=\int_{\epsilon\|\xi\|<1}|g_K(x,\xi)-g_K(x+\epsilon u,\xi)|^{2q}\,\mathrm{d}\xi$$

as $I_\epsilon=I_\epsilon^+ + I_\epsilon^-$. Actually

$$|g_K(x,\xi)-g_K(y,\xi)|=|g_0(x-y,\xi)|$$

on $\{\epsilon\|\xi\|\geqslant 1\}$ for ϵ small enough, and we obtain, by the change of variable $\lambda=\epsilon\xi$,

$$I_\epsilon^+=\epsilon^{2Hq+d(q-1)}\int_{\|\lambda\|\geqslant 1}\frac{|\mathrm{e}^{-\mathrm{i}e_1\cdot\lambda}-1|^{2q}}{\|\lambda\|^{2Hq+dq}}\,\mathrm{d}\lambda. \tag{14}$$

Then a Taylor expansion is applied to I_ϵ^-:

$$I_\epsilon^-=\int_{\|\xi\|<\frac{1}{\epsilon}}|\mathrm{d}g_K(\theta(x,\epsilon u,\xi),\xi).\epsilon u|^{2q}\,\mathrm{d}\xi,$$

where $\mathrm{d}g_K(\theta(x,\epsilon u,\xi),\xi)$ is the differential of the map $g_K(.,\xi)$ and $\theta(x,\epsilon,\xi)$ is a point in the segment $(x,x+\epsilon u)$. Note that

$$\int_{\|\xi\|<C} \|\mathrm{d}g_K(\theta(x, \epsilon u, \xi), \xi)\|^{2q} \mathrm{d}\xi < +\infty$$

for every fixed C and that

$$\|\mathrm{d}g_K(\theta(x, \epsilon u, \xi), \xi)\|^{2q} = O\Big(\|\xi\|^{2q(1-\frac{d}{2}-H)}\Big)$$

when $\|\xi\| \to +\infty$, hence

$$\left(\int_{\|\xi\|<\frac{1}{\epsilon}} \|\mathrm{d}g_K(\theta(x, \epsilon u, \xi), \xi)\|^{2q}\, \mathrm{d}\xi\right)\epsilon^{2q} = O\Big(\epsilon^{2Hq+d(q-1)}\Big)$$

when $\epsilon \to 0^+$ and

$$|I_\epsilon^-| \leqslant C\epsilon^{2Hq+d(q-1)} \tag{15}$$

when $\epsilon \to 0^+$. Because of (14) and (15) there exists a positive constant C such that

$$\int_{\mathbb{R}^d} |g_K(x, \xi) - g_K(y, \xi)|^{2q}\, \mathrm{d}\xi \leqslant C\|x - y\|^{2Hq+d(q-1)},$$

and consequently

$$\mathbb{E}\big(X_H^+(x) - X_H^+(y)\big)^{2p} \leqslant C\|x - y\|^{2Hp} \tag{16}$$

when $\|x\| \leqslant 1$, $\|y\| \leqslant 1$, which yields that the distributions of

$$\left(\frac{X_H^+(x + \epsilon.) - X_H^+(x)}{\epsilon^H}\right)_{\epsilon>0}$$

are tight. To conclude, let us write X_H^- for $\|x\| < \epsilon$ as

$$\int_{\epsilon\|\xi\|\leqslant 1/2} \frac{P_K(-\mathrm{i}x \cdot \xi)}{\|\xi\|^{d/2+H}} M(\mathrm{d}\xi) + \int_{1/2\leqslant\epsilon\|\xi\|\leqslant 1} \frac{P_K(-\mathrm{i}x \cdot \xi)}{\|\xi\|^{d/2+H}} \varphi(\mathrm{i}\|x\|\,\|\xi\|)M(\mathrm{d}\xi).$$

The first integral of this expression is actually a polynomial in the variables $(x_1, \ldots, x_d)$ with coefficients that are random variables; hence it has almost surely C^1 paths. Let us remark that the integrand of the second integral is bounded with compact support in $\mathbb{R}^d \times \mathbb{C}$ and is C^1 in the variable x, and so the integral has the same properties which yield that X_H^- is almost surely C^1. Then it is clear that

$$\lim_{\epsilon\to 0^+} \left(\frac{X_H^-(x + \epsilon u) - X_H^-(x)}{\epsilon^H}\right)_{u\in\mathbb{R}^d} \overset{(d)}{=} 0,$$

which concludes the proof. $\square$

We now wish to exhibit an example of RHFLM that has asymptotic self-similarity properties when the increment is taken at large scales. Actually if the control measure $\nu_\rho(\mathrm{d}\rho)$ is

$$\frac{\mathrm{d}\rho}{|\rho|^{1+\alpha}}\mathbf{1}(|\rho|<1) \tag{17}$$

where $0<\alpha<2$, we show that at large scales the RHFLM is asymptotically self-similar with parameter $0<\tilde{H}<1$ such that $\tilde{H}+d/\alpha=H+d/2$. Heuristically this means that at large scales the truncation of the Lévy measure disappears. Moreover, the limit field is an RHFSM with parameter $\tilde{H}$. This shows that at large scales the behaviour of an RHFLM can be very far from the Gaussian model even if the RHFLM is a field that has moments of order 2. The RHFLM with control measure (17) can be viewed roughly speaking as between an RHFSM at large scales and an FBM at low scales. Let us now state the asymptotic self-similarity precisely.

Proposition 3.2. *Let us assume that $\tilde{H}$, defined by $\tilde{H}+d/\alpha=H+d/2$, is such that $0<\tilde{H}<1$. The RHFLM with control measure $\nu_\rho(\mathrm{d}\rho)$ given by* (17) *is asymptotically self-similar with parameter $\tilde{H}$,*

$$\lim_{R\to+\infty}\left(\frac{X_H(Ru)}{R^{\tilde{H}}}\right)_{u\in\mathbb{R}^d}\overset{(d)}{=}\left(Y_{\tilde{H}}(u)\right)_{u\in\mathbb{R}^d},$$

where the limit is in distribution for all finite-dimensional margins of the fields and the limit is an RHFSM that has a representation

$$Y_{\tilde{H}}(u)=2\mathcal{R}\int_{\mathbb{R}^d}\frac{\mathrm{e}^{-\mathrm{i}u\cdot\xi}-1}{\|\xi\|^{d/\alpha+\tilde{H}}}M_\alpha(\mathrm{d}\xi),$$

where $M_\alpha(d\xi)$ is complex isotropic stable α-symmetric random measure.

Proof. As in the previous proposition we consider a multivariate function

$$g_{u,v,H}(R,\xi,z)=\mathrm{i}2\mathcal{R}\left(z\sum_{k=1}^n v_k\frac{\mathrm{e}^{-\mathrm{i}Ru_k\cdot\xi}-1}{R^{\tilde{H}}\|\xi\|^{d/2+H}}\right)$$

where u and v are in $\mathbb{R}^n$. Further,

$$\mathbb{E}\exp\left(\mathrm{i}\sum_{k=1}^n v_k\frac{X_H(Ru_k)}{R^{\tilde{H}}}\right)=\exp\left(\int_{\mathbb{R}^d\times\mathbb{C}}[\exp(g_{u,v,H}(R,\xi,z))-1-g_{u,v,H}(R,\xi,z)]\mathrm{d}\xi\,\mathrm{d}\nu(z)\right).$$

Then the change of variable $\lambda=R\xi$ is applied and $\tilde{H}$ is chosen such that the integral in the previous equation is now

$$\int_{\mathbb{R}^d\times[0,2\pi]\times\mathbb{R}^+_*}\left[\exp\left(g_{u,v,H}\left(1,\lambda,R^{d/\alpha}\rho\,\mathrm{e}^{\mathrm{i}\theta}\right)\right)-1-g_{u,v,H}\left(1,\lambda,R^{d/\alpha}\rho\,\mathrm{e}^{\mathrm{i}\theta}\right)\right]\mathbf{1}(|\rho|<1)R^{-d}\mathrm{d}\lambda\,\mathrm{d}\theta\frac{\mathrm{d}\rho}{|\rho|^{1+\alpha}}.$$

Setting $r=R^{d/\alpha}\rho$, the integral becomes

$$I(R) = \int_{\mathbb{R}^d \times [0,2\pi] \times \mathbb{R}} \left[\exp\left(g_{u,v,H}\left(1,\, \lambda,\, r\,\mathrm{e}^{\mathrm{i}\theta}\right)\right) - 1 - g_{u,v,H}\left(1,\, \lambda,\, r\,\mathrm{e}^{\mathrm{i}\theta}\right)\right] \mathbf{1}(|r| < R^{d/\alpha}) \mathrm{d}\lambda\, \mathrm{d}\theta \frac{\mathrm{d}r}{2|r|^{1+\alpha}}.$$

Recall that

$$-C(\alpha)|x|^\alpha = \int_{\mathbb{R}} \left[\mathrm{e}^{\mathrm{i}xr} - 1 - \mathrm{i}xr\mathbf{1}\left(|r| \leqslant R^{d/\alpha}\right)\right] \frac{\mathrm{d}r}{2|r|^{1+\alpha}}$$

for every $R > 0$, where

$$C(\alpha) = \int_0^{+\infty} (1 - \cos(r)) \frac{\mathrm{d}r}{2r^{1+\alpha}}.$$

Write

$$J_R = \int_{\mathbb{R}} [\mathrm{e}^{\mathrm{i}xr} - 1 - \mathrm{i}xr]\mathbf{1}\left(|r| \leqslant R^{d/\alpha}\right) \frac{\mathrm{d}r}{2|r|^{1+\alpha}}.$$

Then

$$\lim_{R \to +\infty} \left(J_R + C(\alpha)|x|^\alpha \right) = \lim_{R \to +\infty} \int_{\mathbb{R}} [\mathrm{e}^{\mathrm{i}xr} - 1]\mathbf{1}\left(|r| > R^{d/\alpha}\right) \frac{\mathrm{d}r}{2|r|^{1+\alpha}}$$
$$= 0.$$

Hence

$$\lim_{R \to +\infty} I(R) = -C(\alpha) \int_{\mathbb{R}^d \times [0,2\pi]} \left| 2\mathcal{R}\left(\mathrm{e}^{\mathrm{i}\theta} \sum_{k=1}^n v_k \frac{\mathrm{e}^{-\mathrm{i}u_k\lambda} - 1}{\|\lambda\|^{d/\alpha + \tilde{H}}} \right) \right|^\alpha \mathrm{d}\lambda\, \mathrm{d}\theta,$$
$$= -C(\alpha) \int_0^{2\pi} |2\cos(\theta)|^\alpha \,\mathrm{d}\theta \int_{\mathbb{R}^d} \left| \sum_{k=1}^n v_k \frac{\mathrm{e}^{-\mathrm{i}u_k\lambda} - 1}{\|\lambda\|^{d/\alpha + \tilde{H}}} \right|^\alpha \mathrm{d}\lambda.$$

Since this last expression is the logarithm of

$$\mathbb{E} \exp\left(\mathrm{i} \sum_{k=1}^n v_k Y_{\tilde{H}}(u_k) \right),$$

the proof is complete. □

3.2. Regularity of the sample paths of the RHFLM

The Kolmogorov theorem (see, for instance, Karatzas and Shreve 1988) and Proposition 2.2 show that H can be considered roughly speaking as the Hölder exponent of the sample paths of the RHFLMs. Recall the definition of the pointwise exponent $H_f(x)$ of a deterministic function f at point x:

$$H_f(x) = \sup\left\{H',\ \lim_{\epsilon\to 0}\frac{f(x+\epsilon)-f(x)}{\|\epsilon\|^{H'}} = 0\right\}.$$

Then the regularity of the sample paths is described by the following proposition.

Proposition 3.3. *For every $H' < H$ there exists a continuous modification of the RHFLM such that*

$$\mathbb{P}\left[\omega;\ \sup_{0<\|x-y\|<\epsilon(\omega),\|x\|\leqslant 1,\|y\|\leqslant 1}\left(\frac{X_H(x)-X_H(y)}{\|x-y\|^{H'}}\right) \leqslant \delta\right] = 1, \tag{18}$$

where $\epsilon(\omega)$ is an almost surely positive random variable and $\delta > 0$. Moreover, at every point x the pointwise exponent $H_{X_H}(x)$ of the RHFLM X_H is almost surely equal to H.

Proof. In the first part of the proof we will use our estimate (11) and the Kolmogorov theorem. When $H \leqslant 1 - d/2$ we already know by (11) that

$$\mathbb{E}(X_H(x) - X_H(y))^{2p} \leqslant C\|x-y\|^{2Hp}$$

when $\|x\| \leqslant 1$, $\|y\| \leqslant 1$ and the Kolmogorov theorem yields (18) for every $H' < H$. When $H > 1 - d/2$ we recall that X_H has been split into

$$X_H = X_H^+ + X_H^-,$$

where X_H^+ and X_H^- are defined in (12) and (13). Furthermore, we know that X_H^+ is H'-Hölder continuous for every $H' < H$ by the Kolmogorov theorem and inequality (16), and that X_H^- has almost surely C^1 sample paths, which concludes the proof of (18).

Because of (18), at every point x the Hölder exponent satisfies $H(x) \geqslant H$. To show $H(x) \leqslant H$ let us use local self-similarity (10). Actually if $H' > H$ we can deduce from (10) that

$$\lim_{\epsilon\to 0^+}\frac{\epsilon^{H'}}{|X_H(x+\epsilon)-X_H(x)|} \overset{(d)}{=} 0,$$

which is also a convergence in probability. Hence we can find a sequence $(\epsilon_n)_{n\in\mathbb{N}} \to 0^+$ such that

$$\lim_{n\to+\infty}\frac{|X_H(x+\epsilon_n)-X_H(x)|}{\epsilon_n^{H'}} = +\infty \text{ almost surely.}$$

This argument concludes the proof of Proposition 3.3. □

4. Identification of the fractional index

Let W be a Brownian measure on $L^2(\mathbb{R}^d)$ and M a Lévy measure satisfying the assumptions of Definition 2.3. Assume that M and W are independent. Define the Lévy measure

$$L = \sigma W + M.$$

Recall that

$$X_H(x) = \int_{\mathbb{R}^d} \frac{\mathrm{e}^{-\mathrm{i}x\cdot\xi} - 1}{\|\xi\|^{d/2+H}} L(\mathrm{d}\xi)$$

is the sum of an FBM and an RHFLM, both motions being independent.

For $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$ and $n \in \mathbb{N}^+$, define

$$\frac{\mathbf{k}}{n} = \left(\frac{k_1}{n}, \ldots, \frac{k_d}{n}\right),$$

$$X_H\left(\frac{\mathbf{k}}{n}\right) = X_H\left(\frac{k_1}{n}, \ldots, \frac{k_d}{n}\right).$$

The aim of this section is to identify the fractional index H in a semi-parametric set-up from discrete observations of the field X_H on $[0, 1]^d$: the variance σ and the control measure $\nu(\mathrm{d}z)$ of M are therefore unknown. X_H is observed at times $(k_1/n, \ldots, k_d/n)$, $0 \leqslant k_i \leqslant n$, $i = 1, \ldots, d$.

Let (a_l), $l = 0, \ldots, K$, be a real-valued sequence such that

$$\sum_{l=0}^{K} a_l = 0, \qquad \sum_{l=0}^{K} l a_l = 0.$$

For $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$, define

$$a_{\mathbf{k}} = a_{k_1} \ldots a_{k_d}.$$

Define the increments of X_H associated with the sequence a:

$$\begin{aligned}\Delta X_{\mathbf{p}} &= \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} X_H\left(\frac{\mathbf{k}+\mathbf{p}}{n}\right) \\ &= \sum_{k_1,\ldots,k_d=0}^{K} a_{k_1} \ldots a_{k_d} X_H\left(\frac{k_1+p_1}{n}, \ldots, \frac{k_d+p_d}{n}\right);\end{aligned}$$

one can take, for instance, $K = 2$, $a_0 = 1$, $a_1 = -2$, $a_2 = 1$. Define the quadratic variations associated with sequence a:

$$Q_n = \frac{1}{(n-K+1)^d} \sum_{\mathbf{p}=\mathbf{0}}^{\mathbf{n}-\mathbf{K}} (\Delta X_{\mathbf{p}})^2.$$

One can check that

$$\log(\mathbb{E}(Q_n)) = -2H \log n + C,$$

where C is a constant, and it is usual to identify H as the slope of a linear regression of $\log(Q_n)$ with respect to $\log n$ for the FBM. Coeurjolly (2001) shows that the quadratic variations are optimal in a Gaussian framework. For the sake of simplicity, we consider that the estimator of fractional index H is

$$\hat{H}_n = \frac{1}{2} \log_2 \frac{Q_{n/2}}{Q_n},$$

but linear regression with $(\log(Q_{n/l}))_{l=1,\dots,L}$ could have been chosen. As pointed out in Section 1, this estimator is the estimator used by Istas and Lang (1997) to estimate the fractional index of an FBM. Other estimators using wavelet coefficients instead of discrete variations are also available in the literature (see Bardet 2000; Flandrin and Abry 1999) but only in the Gaussian and stable framework.

Theorem 4.1. *As* $n \to +\infty$,

$$\hat{H}_n \overset{(\mathbb{P})}{\to} H,$$

where $\overset{(\mathbb{P})}{\to}$ *means a convergence in probability. Moreover, there exists a constant* $C > 0$ *such that, as* $n \to +\infty$,

$$n^{d/2}(\hat{H}_n - H) \overset{(\mathbb{P})}{\to} C.$$

Proof. First define the following constants:

$$A = \sigma^2 + 4\pi \int_{\mathbb{R}^+} \rho^2 \nu_\rho(\mathrm{d}\rho)$$

$$B = 4\pi \int_{\mathbb{R}^+} \rho^4 \nu_\rho(\mathrm{d}\rho).$$

Define the following functional spaces:

$$\mathcal{F}_2 = \{f \in L^2(\mathbb{R}^d),\ f(-\xi) = \overline{f(\xi)},\ \forall \xi \in \mathbb{R}^d\},$$

$$\mathcal{F}_4 = \{f \in L^2(\mathbb{R}^d) \cap L^4(\mathbb{R}^d),\ f(-\xi) = \overline{f(\xi)},\ \forall \xi \in \mathbb{R}^d\}.$$

According to Proposition 2.2, we then have:

- for all $f_1, f_2 \in \mathcal{F}_2$,

$$\mathbb{E} \int f_1(\xi) L(\mathrm{d}\xi) \int f_2(\xi) L(\mathrm{d}\xi) = A \int f_1(\xi) f_2(-\xi) \mathrm{d}\xi; \tag{19}$$

- for all $f_1, f_2, f_3, f_4 \in \mathcal{F}_4$,

$$
\begin{aligned}
&\mathbb{E}\prod_{i=1}^{4}\int f_i(\xi)L(d\xi)\\
&\quad = A^2\Bigg(\int f_1(\xi)f_2(-\xi)\mathrm{d}\xi \times \int f_3(\xi)f_4(-\xi)\mathrm{d}\xi + \int f_1(\xi)f_3(-\xi)\mathrm{d}\xi \times \int f_2(\xi)f_4(-\xi)\mathrm{d}\xi\\
&\qquad + \int f_1(\xi)f_4(-\xi)\mathrm{d}\xi \times \int f_2(\xi)f_3(-\xi)\mathrm{d}\xi\Bigg) + B\Bigg(\int f_1(\xi)f_2(-\xi)f_3(\xi)f_4(-\xi)\mathrm{d}\xi\\
&\qquad + \int f_1(\xi)f_2(\xi)f_3(-\xi)f_4(-\xi)\mathrm{d}\xi + \int f_1(\xi)f_2(-\xi)f_3(-\xi)f_4(\xi)\mathrm{d}\xi\Bigg). \qquad (20)
\end{aligned}
$$

Now define

$$V_n = n^{2H}Q_n.$$

We first calculate the expected value of V_n. We deduce from (19) that

$$\mathbb{E}(\Delta X_{\mathbf{p}})^2 = A\int_{\mathbb{R}^d} \frac{\left|\sum_{\mathbf{k}=0}^{\mathbf{K}} a_{\mathbf{k}} \exp\left(\mathrm{i}\frac{\mathbf{k}}{n}\cdot\xi\right)\right|^2}{\|\xi\|^{d+2H}}\,\mathrm{d}\xi.$$

The change of variables $\lambda = \xi/n$ leads to

$$\mathbb{E}(\Delta X_{\mathbf{p}})^2 = An^{-2H}\int_{\mathbb{R}^d} \frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} \exp(\mathrm{i}\mathbf{k}\cdot\lambda)\right|^2}{\|\lambda\|^{d+2H}}\,\mathrm{d}\lambda,$$

and therefore

$$\mathbb{E}V_n = A\int_{\mathbb{R}^d} \frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} \exp(\mathrm{i}\mathbf{k}\cdot\lambda)\right|^2}{\|\lambda\|^{d+2H}}\,\mathrm{d}\lambda.$$

We can now calculate the variance of V_n. We deduce from (20) that

$$\mathbb{E}\left[\left(\Delta X_{\mathbf{p}}\right)^2\left(\Delta X_{\mathbf{p}'}\right)^2\right] = T_1 + T_2 + T_3 + T_4,$$

with

$$T_1 = A^2 \left(\int_{\mathbb{R}^d} \frac{\left| \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} \exp\left(-\mathrm{i} \frac{\mathbf{k}}{n} \cdot \xi \right) \right|^2}{\|\xi\|^{d+2H}} \, \mathrm{d}\xi \right)^2,$$

$$T_2 = 2A^2 \left(\int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i} \frac{\mathbf{p}-\mathbf{p}'}{n} \cdot \xi} \frac{\left| \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} \exp\left(-\mathrm{i} \frac{\mathbf{k}}{n} \cdot \xi \right) \right|^2}{\|\xi\|^{d+2H}} \, \mathrm{d}\xi \right)^2,$$

$$T_3 = 2B \int_{\mathbb{R}^d} \frac{\left| \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} \exp\left(\mathrm{i} \frac{\mathbf{k}}{n} \cdot \xi \right) \right|^4}{\|\xi\|^{2d+4H}} \, \mathrm{d}\xi,$$

$$T_4 = B \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i} \frac{2\mathbf{p}-2\mathbf{p}'}{n} \cdot \xi} \frac{\left| \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} \exp\left(\mathrm{i} \frac{\mathbf{k}}{n} \cdot \xi \right) \right|^4}{\|\xi\|^{2d+4H}} \, \mathrm{d}\xi.$$

Thus

$$\operatorname{var}(V_n) = G_n + NG_n,$$

with

$$G_n = 2A^2 \frac{n^{4H}}{(n-K+1)^{2d}} \sum_{\mathbf{p},\mathbf{p}'=0}^{\mathbf{n}-\mathbf{K}} \left(\int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i} \frac{\mathbf{p}-\mathbf{p}'}{n} \cdot \xi} \frac{\left| \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} \exp\left(\mathrm{i} \frac{\mathbf{k}}{n} \cdot \xi \right) \right|^2}{\|\xi\|^{d+2H}} \, \mathrm{d}\xi \right)^2,$$

$$\begin{aligned} NG_n = {} & 2Bn^{4H} \int_{\mathbb{R}^d} \frac{\left| \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} \exp\left(\mathrm{i} \frac{\mathbf{k}}{n} \cdot \xi \right) \right|^4}{\|\xi\|^{2d+4H}} \, \mathrm{d}\xi \\ & + B \frac{n^{4H}}{(n-K+1)^{2d}} \sum_{\mathbf{p},\mathbf{p}'=\mathbf{0}}^{\mathbf{n}-\mathbf{K}} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i} \frac{2\mathbf{p}-2\mathbf{p}'}{n} \cdot \xi} \frac{\left| \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} \exp\left(\mathrm{i} \frac{\mathbf{k}}{n} \cdot \xi \right) \right|^4}{\|\xi\|^{2d+4H}} \, \mathrm{d}\xi. \end{aligned}$$

We first study the G_n part of the variance. The change of variables $\lambda = \xi/n$ leads to:

$$\int_{\mathbb{R}^d} e^{-i\frac{\mathbf{p},\mathbf{p}'}{n}\cdot\xi} \frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} \exp\left(-i\frac{\mathbf{k}}{n}\cdot\xi\right)\right|^2}{\|\xi\|^{d+2H}} \,d\xi = n^{-2H} \int_{\mathbb{R}^d} e^{-i(\mathbf{p}-\mathbf{p}')\cdot\lambda} \frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} \exp(i\mathbf{k}\cdot\lambda)\right|^2}{\|\lambda\|^{d+2H}} \,d\lambda.$$

Define the operator

$$\mathbf{D} = \prod_{j=1}^{d} \frac{\partial}{\partial x_i}.$$

Let us suppose that for all j, $p_j \neq p'_j$. Integrating by parts leads to

$$\int_{\mathbb{R}^d} e^{-i(\mathbf{p},\mathbf{p}')\cdot\lambda} \frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} \exp(i\mathbf{k}\cdot\lambda)\right|^2}{\|\lambda\|^{d+2H}} \,d\lambda = i^d \prod_{j=1}^{d} \frac{1}{(p_j - p'_j)} \int_{\mathbb{R}^d} e^{-i(\mathbf{p}-\mathbf{p}')\cdot\lambda} \mathbf{D}\left[\frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} \exp(i\mathbf{k}\cdot\lambda)\right|^2}{\|\lambda\|^{d+2H}}\right] d\lambda.$$

The conditions $\sum_{l=0}^{K} a_l = 0$, $\sum_{l=0}^{K} l a_l = 0$ ensure the convergence of the integral.

Since there exists a constant C_1 such that, as $n \to +\infty$,

$$\left(\frac{1}{n} \sum_{m,m'=0,m\neq m'}^{n-K} \frac{1}{(m-m')^2}\right) \to C_1,$$

as $n \to +\infty$,

$$n^d G_n \to C_2.$$

We now study the NG_n part of the variance. Using the change of variables $\lambda = \xi/n$, we obtain, for T_3,

$$\frac{2B}{n^d} \int_{\mathbb{R}^d} \frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} \exp(i\mathbf{k}\cdot\lambda)\right|^4}{\|\lambda\|^{2d+4H}} \,d\lambda.$$

It remains to study the part of NG_n depending on T_4. This part can be written as:

$$B \sum_{\mathbf{p},\mathbf{p}'=\mathbf{0}}^{\mathbf{n}-\mathbf{K}} \int_{\mathbb{R}^d} e^{2i(\mathbf{p}-\mathbf{p}')\cdot\lambda} \frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} \exp(i\mathbf{k}\cdot\lambda)\right|^4}{\|\lambda\|^{2d+4H}} \,d\lambda.$$

Using integration by parts as previously, we prove that this part is negligible with respect to the previous parts.

To summarize, we have proved that there exists $C > 0$ such that

$$n^d \operatorname{var} V_n \to C,$$

and Theorem 4.1 is proved. □

References

Abry, P., Delbeke, L. and Flandrin, P. (2000a) Wavelet based estimator for the self-similarity parameter of α-stable processes. *Preprint*.

Abry, P., Pesquet-Popescu, B. and Taqqu, M. (2000b) Estimation ondelette des paramètres de stabilité et d'auto similarité des processus α-stables autosimilaires. *Preprint*.

Ayache, A. and Lévy Véhel, J. (1999) Generalized multifractional Brownian motion: Definition and preliminary results. In M. Dekking, J. Lévy Véhel, E. Lutton and C. Tricot (eds), *Fractals: Theory and Applications in Engineering*, pp. 17–32. London: Springer-Verlag.

Bardet, J.M. (2000) Testing for the presence of self-similarity of Gaussian time series having stationary increments. *J. Time Ser. Anal.*, **21**, 497–515.

Bardet, J.M. (2001) Les cours d'actifs sont-ils auto-similaires? J. Société Française de Statistique. To appear.

Benassi, A. and Deguy, S. (1999) Multi-scale fractional Brownian motion: definition and identification. Technical report 83, Laboratoire de Logique, Algorithmique et Informatique de Clermont 1. Available at http://llaic3.u-clermont1.fr/prepubli/prellaic83.ps.gz.

Benassi, A., Bertrand, P., Cohen, S. and Istas, J. (2000) Identification of the Hurst exponent of a step multifractional Brownian motion. *Statist. Inference Stochastic Process.*, **3**, 101–110.

Benassi, A., Cohen, S. and Istas, J. (1998a) Identifying the multifractional function of a Gaussian process. *Statist. Probab. Lett.*, **39**, 337–345.

Benassi, A., Cohen, S., Istas, J. and Jaffard, S. (1998b) Identification of filtered white noises. *Stochastic Process. Appl.*, **75**, 31–49.

Benassi, A., Jaffard, S. and Roux, D. (1997) Gaussian processes and pseudodifferential elliptic operators. *Rev. Mat. Iberoamericana*, **13**(1), 19–89.

Coeurjolly, J.F. (2001) Estimating the parameters of a fractional Brownian motion by discrete variation of its sample paths. *Statist. Inference Stochastic Process*, **4**, 199–227.

Flandrin, P. and Abry, P. (1999) Wavelets for scaling processes. In M. Dekking, J. Lévy Véhel, E. Lutton and C. Tricot (eds), *Fractals: theory and applications in engineering*, pp. 47–64. London: Springer-Verlag.

Hermann, H.J. and Roux, S. (eds) (1990) *Statistical Models for the Fracture of Disordered Media.* Amsterdam: North-Holland.

Istas, J. and Lang, G. (1997) Quadratic variations and estimation of the local Hölder index of a Gaussian process. *Ann. Inst. H. Poincaré Probab. Statist.*, **33**, 407–437.

Karatzas, I. and Shreve, S. (1988) *Brownian Motion and Stochastic Calculus.* Berlin: Springer-Verlag.

Mandelbrot, B. and Ness, J. (1968) Fractional Brownian motion, fractional noises and applications. *SIAM Rev.*, **10**, 422–437.

Neveu, J. (1977) Processus ponctuels. In P.L. Hennequin (ed.), *École d'Été de Probabilités de Saint-Flour VI*, Lecture Notes in Math. 598, pp. 249–445. Berlin: Springer-Verlag.

Peltier, R. and Lévy Véhel, J. (1995) Multifractional Brownian motion: definition and preliminary results. INRIA research report 2645. Available at http://www-syntim.inria.fr/fractales/.

Protter, P. (1990) *Stochastic Integration and Differential Equations.* Berlin: Springer-Verlag.

Samorodnitsky, G. and Taqqu, M.S. (1994) *Stable Non-Gaussian Random Processes*. New York: Chapman & Hall.

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