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On positive spectral density functions

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A necessary and sufficient condition is given for a weakly stationary random field (indexed by the integer lattice of an arbitrary finite dimension) to have a spectral density which is bounded between two positive constants. As a corollary, a necessary and sufficient condition is derived for a positive continuous spectral density. The conditions involve 'linear' dependence coefficients.

Keywords: linear dependence coefficients; spectral density; weakly stationary random fields

Introduction

In the estimation of the spectral density (if it exists) of a stationary random field, an important role is played by basic properties of the spectral density itself, such as continuity, differentiability and positivity; see, for example, Rosenblatt (1985), Zhurbenko (1986), Ivanov and Leonenko (1989) and Miller (1995). In these and other references, such basic properties of spectral density are connected with certain dependence coefficients, and in particular with certain 'linear' dependence coefficients. The purpose of this paper is to re-examine the question of the existence of a positive continuous spectral density in light of certain 'linear' dependence coefficients.

For the sake of simplicity, the discussion here will be restricted to the case of centred (mean zero) random variables. The definitions and results can trivially be transcribed to random variables with non-zero mean. All random variables are defined on a given probability space $(\Omega, \mathcal{F}, P)$. In Definitions 1.1, 1.2 and 1.3, d is an arbitrary fixed positive integer.

Definition 1.1. *A random field $X := (X_k,\ k \in \mathbb{Z}^d)$ is said to be 'centred, complex and weakly stationary' (CCWS) if the following conditions hold: (i) For each $k \in \mathbb{Z}^d$, X_k is a complex-valued random variable such that $\mathrm{E}|X_k|^2 < \infty$ and $\mathrm{E}X_k = 0$. (ii) There exists a function $\gamma : \mathbb{Z}^d \to \mathbb{C}$ such that for any elements $j, k \in \mathbb{Z}^d$, $\mathrm{E}X_j\overline{X_k} = \gamma(j-k)$.*

Here $\bar{z}$ is the complex conjugate of a complex number z. Of course $\gamma(0) = \mathrm{E}|X_0|^2$ and $\gamma(-u) = \overline{\gamma(u)}$ for all $u \in \mathbb{Z}^d$.

Definition 1.2. *Let T denote the unit circle in the complex plane. Let μ_T denote (one-dimensional) normalized Lebesgue measure on T (i.e. normalized so that $\mu_T(T) = 1$). Let $\mu_T^d = \mu_T \times \ldots \times \mu_T$ denote the d-dimensional product measure on T^d. A Borel non-*

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negative integrable function f on T^d is said to be a 'spectral density' for a given CCWS random field $X := (X_k,\ k \in \mathbb{Z}^d)$ if

$$\forall k \in \mathbb{Z}^d, \quad \mathrm{E} X_k \overline{X_0} = \int_{t \in T^d} \mathrm{e}^{\mathrm{i}k\cdot\lambda} f(t) \mathrm{d}\mu_T^d(t), \tag{1.1}$$

where, for a given $t := (t_1, \ldots, t_d) \in T^d$, $\lambda = \lambda(t) := (\lambda_1, \ldots, \lambda_d)$ is the element of $(-\pi, \pi]^d$ such that $t_u = \exp(\mathrm{i}\lambda_u)$ for all $u \in \{1, \ldots, d\}$. In (1.1) the notation $k \cdot \lambda$ denotes the dot product.

Since an integrable function on T^d is uniquely determined (modulo null sets) by its Fourier coefficients, the spectral density (if it exists) is unique (modulo null sets).

The notation for the 'linear dependence coefficients' in (1.3)–(1.11) below may seem somewhat arbitrary. It is chosen to coincide with, or at least avoid conflicting with, the notation in certain other papers that will be cited below.

Suppose $X := (X_k,\ k \in \mathbb{Z}^d)$ is a CCWS random field.

Definition 1.3. *For any two non-empty finite disjoint subsets Q, $S \subset \mathbb{Z}^d$, define the non-negative number*

$$\mathcal{R}(Q, S) := \sup \frac{|\mathrm{E} V \overline{W}|}{\|V\|_2 \|W\|_2}, \tag{1.2}$$

where the supremum is taken over all pairs of (complex-valued) random variables V and W of the form

$$V = \sum_{k \in Q} a_k X_k \quad \textit{and} \quad W = \sum_{k \in S} a_k X_k,$$

where the a_k, $k \in Q \cup S$, are complex numbers. In (1.2) *and in the equations below,* $0/0$ *is interpreted as* 0.

For each positive integer n, define

$$\tilde{q}(n) = \tilde{q}(X, n) := \sup \frac{\left| \mathrm{E} \left(\sum_{k \in Q} X_k \right) \left(\overline{\sum_{k \in S} X_k} \right) \right|}{\left\| \sum_{k \in Q} X_k \right\|_2 \left\| \sum_{k \in S} X_k \right\|_2}, \tag{1.3}$$

$$r(n) = r(X, n) := \sup \mathcal{R}(Q, S). \tag{1.4}$$

Here the supremum is taken over all pairs of non-empty finite sets Q and $S \subset \mathbb{Z}^{\mathrm{d}}$ with the property that there exists $u \in \{1, \ldots, d\}$ such that

$$\begin{aligned} Q &\subset \{k := (k_1, \ldots, k_d) \in \mathbb{Z}^d : k_u \leqslant 0\} \\ S &\subset \{k := (k_1, \ldots, k_d) \in \mathbb{Z}^d : k_u \geqslant n\}. \end{aligned} \tag{1.5}$$

(*Because of weak stationarity, the numbers $\tilde{q}(n)$ and $r(n)$ in* (1.3) *and* (1.4) *will not change if* *in* (1.5) *the inequalities* $k_u \leqslant 0$ *and* $k_u \geqslant n$ *are replaced by* $k_u \leqslant j$ *and* $k_u \geqslant j + n$, *where* j *is an arbitrary integer.)*

For each positive integer n, *define*

$$\zeta(n) = \zeta(X, n) := \sup \frac{\left| \mathrm{E}\left(\sum_{k \in Q} X_k\right)\left(\overline{\sum_{k \in S} X_k}\right)\right|}{\mathrm{card}(Q \cup S)}, \tag{1.6}$$

$$q'(n) = q'(X, n) := \sup \frac{\left| \mathrm{E}\left(\sum_{k \in Q} X_k\right)\left(\overline{\sum_{k \in S} X_k}\right)\right|}{\left\| \sum_{k \in Q} X_k \right\|_2 \left\| \sum_{k \in S} X_k \right\|_2}, \tag{1.7}$$

$$r'(n) = r'(X, n) := \sup \mathcal{R}(Q, S). \tag{1.8}$$

Here each supremum is taken over all pairs of non-empty finite sets Q *and* $S \subset \mathbb{Z}^d$ *with the property that there exist* $u \in \{1, \ldots, d\}$ *and non-empty disjoint sets* $A, B \subset \mathbb{Z}$, *with*

$$\mathrm{dist}(A, B) := \min_{a \in A, b \in B} |a - b| \geqslant n, \tag{1.9}$$

such that

$$\begin{aligned} Q &\subset \{k := (k_1, \ldots, k_d) \in \mathbb{Z}^d : \quad k_u \in A\} \\ S &\subset \{k := (k_1, \ldots, k_d) \in \mathbb{Z}^d : \quad k_u \in B\}. \end{aligned} \tag{1.10}$$

(*It is understood that the sets A and B can be 'interlaced', with each set containing elements that are 'between' elements of the other set.*)

For each positive integer n, *define*

$$r^*(n) = r^*(X, n) = \sup \mathcal{R}(Q, S), \tag{1.11}$$

where the supremum is taken over all pairs of non-empty finite sets Q *and* $S \subset \mathbb{Z}^d$ *such that*

$$\mathrm{dist}(Q, S) := \min_{q \in Q, s \in S} \|q - s\| \geqslant n. \tag{1.12}$$

Here, for $k := (k_1, \ldots, k_d) \in \mathbb{Z}^d$, *one defines* $\|k\| := (k_1^2 + \ldots + k_d^2)^{1/2}$, *the Euclidean norm.*

Remark 1.4. Obviously $r^*(n)$ in (1.11) is non-increasing as n increases; and the same comment applies to each of the other dependence coefficients here. Also, for each $n \geqslant 1$, one has that $r(n) \leqslant r'(n) \leqslant r^*(n)$, that $\tilde{q}(n) \leqslant q'(n) \leqslant r'(n)$, and that $\tilde{q}(n) \leqslant r(n)$. If $q'(n) \to 0$ as $n \to \infty$, then $\zeta(n) \to 0$ as $n \to \infty$, by Bradley (2001, Remark 1.6). If $\sum_{n=0}^{\infty} r(2^n) < \infty$, then $r'(n) \to 0$ as $n \to \infty$, by a result communicated to me by Utev in 1993 (see Bradley and Utev 1994, Theorem 3). If $d = 1$, then (trivially) $r'(n) = r^*(n)$ for each $n \geqslant 1$.

Theorem 1.5. *Suppose d is a positive integer, and $X := (X_k,\ k \in \mathbb{Z}^d)$ is a CCWS random field. Then X has a continuous spectral density function on T^d if and only if*

$$\zeta(X, n) \to 0 \quad \text{as } n \to \infty. \tag{1.13}$$

This is background information from Bradley (2001, Theorem 1.4). Ibragimov (1962, Lemma 2; 1970, Lemma 5.1) had proved that if a CCWS random sequence ($d = 1$) satisfies $\sum_{n=0}^{\infty} r(2^n) < \infty$, then it has a continuous spectral density. See also Ibragimov and Rozanov (1978, p. 182, Lemma 17). In papers such as Bradley (1992), Bradley and Utev (1994) and Miller (1997), that result and its proof were adapted to CCWS random fields under various linear dependence conditions ($\sum r(2^n) < \infty$, $r^*(n) \to 0$, $r'(n) \to 0$). (Miller (1997) dealt with the index set $\mathbb{R}^d$ instead of $\mathbb{Z}^d$.) Theorem 1.5 and its proof, in Bradley (2001), involved an adaptation of Ibragimov's argument and an idea of Peligrad (1997).

The continuous spectral density function in Theorem 1.5 (if (1.13) holds) may have zeros. In this paper, we shall re-examine the question of when that continuous spectral density will be (strictly) positive. This question will be addressed via Theorems 1.6 and 1.7 below, the main results of this paper.

In a related vein, Helson and Sarason (1967, Theorem 5) (see also Sarason (1972, p. 62, the Theorem)) characterized the spectral densities of CCWS random sequences ($d = 1$) satisfying $r(n) \to 0$. Those spectral densities do not need to be either continuous or positive. Ibragimov (1965, p. 104, Corollary 1) had earlier shown, among other properties, that those spectral densities cannot have 'jump discontinuities'. Ibragimov and Rozanov (1978, pp. 179–180, Example 1) discussed specific classes of examples of stationary Gaussian sequences satisfying $r(n) \to 0$, such that the spectral density could (in a critical way) have a zero or be unbounded.

Theorem 1.6. *Suppose d is a positive integer, and $X := (X_k,\ k \in \mathbb{Z}^d)$ is a non-degenerate CCWS random field. Then the following three conditions are equivalent:*

(A) *X has a (not necessarily continuous) spectral density f on T^d such that f is bounded between two positive constants.*

(B) $r^*(X, 1) < 1$.

(C) *One has that*

$$r(X, 1) < 1, \quad \text{and} \tag{1.14}$$

$$\exists n \geqslant 1 \text{ such that } r'(X, n) < 1. \tag{1.15}$$

Theorem 1.6 is largely based on ideas that were developed in earlier papers by various researchers (especially the papers cited after Theorems 1.5 and 1.7). Theorem 1.6 will be proved in Section 2.

Helson and Szegö (1960) (see also Helson and Sarason 1967, Theorem 6) characterized the spectral densities of CCWS random sequences satisfying $r(1) < 1$ (as in (1.14)). From that characterization, one sees that when $r(1) < 1$, the spectral density can have zeros and/or be unbounded.

Theorem 1.7. *Suppose d is a positive integer, and $X := (X_k,\ k \in \mathbb{Z}^d)$ is a non-degenerate CCWS random field. Then the following four conditions are equivalent:*

(*a*) *X has a positive continuous spectral density function on T^d.*
(*b*) *$r^*(X, 1) < 1$, and $r^*(X, n) \to 0$ as $n \to \infty$.*
(*c*) *$r(X, 1) < 1$, and $r'(X, n) \to 0$ as $n \to \infty$.*
(*d*) *Expressions* (1.13), (1.14) *and* (1.15) *hold.*

The equivalence of (a), (b), and (c) is already known. Condition (d) is new here. Kolmogorov and Rozanov (1960) showed that for a CCWS random sequence ($d = 1$), condition (a) in Theorem 1.7 implies $r(n) \to 0$. Adapting their argument, Rosenblatt (1972; 1985, pp. 73–74, Theorem 7 and Lemma 2) showed that for a stationary Gaussian random field (indexed by $\mathbb{Z}^d$), condition (a) in Theorem 1.7 implies $r^*(n) \to 0$ as $n \to \infty$. Building on that work, Bradley (1992, Theorem 2) showed that for CCWS random fields (indexed by $\mathbb{Z}^d$), conditions (a) and (b) in Theorem 1.7 are equivalent, and then Bradley and Utev (1994, Theorem 2) showed that (a) and (c) are equivalent. Of course (b) $\Rightarrow$ (c) $\Rightarrow$ (d) by Remark 1.4. Also, (d) $\Rightarrow$ (a) by Theorems 1.5 and 1.6 and the uniqueness (modulo null sets) of a spectral density function. Hence, once Theorem 1.6 is proved (in Section 2), the proof of Theorem 1.7 will be complete.

Remark 1.8. In Theorem 1.7, condition (d) appears to be in some sense as 'weak' a condition (equivalent to (a)) as one can formulate simply in terms of the dependence coefficients in Definition 1.3. In Section 3, this will be illustrated with three examples.

From the arguments in Section 2 and in Bradley (2001), one can discern other conditions (equivalent to (a)) that are 'weaker' than (but more complicated than) condition (d). One such condition is discussed in Remark 3.5 in Section 3.

Remark 1.9. The definitions, theorems and proofs here carry over from CCWS random fields to arrays $(h_k,\ k \in \mathbb{Z}^d)$ of elements of a complex (or real) Hilbert space such that $\langle h_j, h_k \rangle$ (the inner product) depends only on $j - k$. Alternatively, such 'Hilbert space' versions of Theorems 1.5, 1.6 and 1.7 can be derived from Theorems 1.5, 1.6 and 1.7 themselves via a Hilbert-space isometry. Such techniques are well known from works such as Ibragimov and Rozanov (1978).

2. Proof of Theorem 1.6

The argument will primarily be adapted, with one major change and some minor ones, from the proof in Bradley and Utev (1994, Section 2) that (c) $\Rightarrow$ (a) in Theorem 1.7. The argument will involve a series of lemmas. The first six will involve random sequences (i.e. dimension $d = 1$).

For a given CCWS random sequence $X := (X_k,\ k \in \mathbb{Z})$, the partial sums will be denoted (for $n \geqslant 1$) by

$$S_n := X_1 + \ldots + X_n. \tag{2.1}$$

Lemma 2.1. *Suppose* $X := (X_k,\ k \in \mathbb{Z})$ *is a CCWS random sequence such that* $\mathrm{E}X_k\overline{X_0} \to 0$ *as* $k \to \infty$. *Then either* $\mathrm{E}|S_n|^2 \to \infty$ *as* $n \to \infty$, *or* $\sup_{n\geq 1} \mathrm{E}|S_n|^2 < \infty$. *If* $\sup_{n\geq 1} \mathrm{E}|S_n|^2 < \infty$, *then there exists a CCWS random sequence* $Y := (Y_k,\ k \in \mathbb{Z})$, *with* $\mathrm{E}Y_k\overline{Y_0} \to 0$ *as* $k \to \infty$, *such that for all* $k \in \mathbb{Z}$, $X_k = Y_k - Y_{k-1}$ *almost surely.*

This theorem is due to Leonov. Its proof can be found in Ibragimov and Linnik (1971, p. 323, Theorem 18.2.2). The formulation and proof there involve real-valued random variables, but they extend quite trivially to complex-valued random variables, as in Lemma 2.1. (The property $\mathrm{E}Y_k\overline{Y_0} \to 0$ follows from the assumption $\mathrm{E}X_k\overline{X_0} \to 0$, the definition of Y, and the observation in Ibragimov and Linnik (1971, p. 324, lines 4 and 5).)

Lemma 2.2. *Suppose* $X := (X_k,\ k \in \mathbb{Z})$ *is a CCWS random sequence such that* $r(X, n) < 1$ *for some* $n \geq 1$. *Then* $\mathrm{E}X_k\overline{X_0} \to 0$ *as* $k \to \infty$.

This is well known. It has a fairly elementary proof based on the weak compactness of the unit ball in a (complex) Hilbert space. However, it is also an immediate corollary of Helson and Sarason (1967, Theorem 6), which asserts that the condition $r(X, n) < 1$ is equivalent to the existence of a spectral density with certain properties. By (1.1) and the Riemann–Lebesgue lemma, it follows that $\mathrm{E}X_k\overline{X_0} \to 0$.

Lemma 2.3. *Suppose* $X := (X_k,\ k \in \mathbb{Z})$ *is a non-degenerate CCWS random sequence such that* $r(X, 1) < 1$. *Then* $\mathrm{E}|S_n|^2 \to \infty$ *as* $n \to \infty$.

This has long been part of the folklore. However, it seems hard to find a reference for it. Here is a review of its (well-known) proof.

Proof. By Lemmas 2.1 and 2.2, either $\mathrm{E}|S_n|^2 \to \infty$ as $n \to \infty$, or $\sup_{n\geq 1} \mathrm{E}|S_n|^2 < \infty$. Suppose $\sup_{n\geq 1} E|S_n|^2 < \infty$. We shall obtain a contradiction.

By Lemmas 2.1 and 2.2, there exists a CCWS random sequence $Y := (Y_k,\ k \in \mathbb{Z})$ with $\mathrm{E}Y_k\overline{Y_0} \to 0$ as $k \to \infty$, such that for all $k \in \mathbb{Z}$, $X_k = Y_k - Y_{k-1}$ a.s.

For each $n \geq 1$,

$$Y_0 - n^{-1}\sum_{k=1}^{n} Y_k = n^{-1}\sum_{k=1}^{n}(Y_0 - Y_k) = -n^{-1}\sum_{k=1}^{n}(X_1 + \ldots + X_k). \tag{2.2}$$

Since $\mathrm{E}Y_k\overline{Y_0} \to 0$ as $k \to \infty$, one has by a simple calculation that $\mathrm{E}|Y_1 + \ldots + Y_n|^2 = o(n^2)$ as $n \to \infty$. Hence by (2.2),

$$-n^{-1}\sum_{k=1}^{n}(X_1 + \ldots + X_k) \to Y_0 \text{ in } \mathcal{L}^2 \qquad \text{as } n \to \infty. \tag{2.3}$$

By a similar argument (using $Y_0 - Y_k = X_{k+1} + \ldots + X_0$ for $k \leq -1$), one has that

$$n^{-1} \sum_{k=-n}^{-1} (X_{k+1} + \ldots + X_0) \to Y_0 \text{ in } \mathcal{L}^2 \qquad \text{as } n \to \infty. \tag{2.4}$$

Now $\mathrm{E}|Y_0|^2 > 0$. (Otherwise, since $X_k = Y_k - Y_{k-1}$ a.s., one would have $X_k = 0$ a.s., contradicting the hypothesis of Lemma 2.3.) Hence by (2.3), (2.4), and a simple argument, $r(X, 1) = 1$. But that contradicts the hypothesis of Lemma 2.3. Hence $\mathrm{E}|S_n|^2 \to \infty$ as $n \to \infty$ after all. □

Lemma 2.4. *Suppose $0 \leqslant R < 1$, and A is a positive constant. Then there exists a positive integer $M = M(R, A)$ such that if $X := (X_k, k \in \mathbb{Z})$ is a CCWS random sequence such that $r(X, 1) \leqslant R$, then there exists $n \in \{1, 2, \ldots, M\}$ such that $\mathrm{E}|S_n|^2 \geqslant A \cdot \mathrm{E}|X_0|^2$.*

Under the extra restriction $r(X, n) \to 0$ (and with the integer M involving that 'mixing rate'), versions of Lemma 2.4 were proved for real-valued random variables by Bradley (1988, Lemma 2.2) and for complex-valued random variables by Bradley and Utev (1994, Lemma 2.2). The proof of Lemma 2.4 here will have to be quite different from the arguments in those references, because of the absence of the extra restriction $r(X, n) \to 0$.

Proof. Suppose there does not exist a positive integer M such that the second paragraph of Lemma 2.4 holds. We shall obtain a contradiction.

For each positive integer L, let $X^{(L)} := (X_k^{(L)}, k \in \mathbb{Z})$ be a CCWS random sequence such that

$$r(X^{(L)}, 1) \leqslant R, \quad \text{and} \tag{2.5}$$

$$\forall n \in \{1, 2, \ldots, L\}, \qquad \mathrm{E}|X_1^{(L)} + \ldots + X_n^{(L)}|^2 < A \cdot E|X_0^{(L)}|^2. \tag{2.6}$$

By (2.6), $\mathrm{E}|X_0^{(L)}|^2 = 0$ cannot hold. For each $L \geqslant 1$, we normalize the CCWS sequence $X^{(L)}$ so that

$$E|X_0^{(L)}|^2 = 1. \tag{2.7}$$

Denote $\mathbb{N} := \{1, 2, 3, \ldots\}$. Define the infinite sets $Q_0, Q_1, Q_2, Q_3, \ldots \subset \mathbb{N}$ recursively, as follows. First, let $Q_0 := \mathbb{N}$. Now suppose k is a positive integer and the sets Q_0, $Q_1, \ldots, Q_{k-1}$ are already defined, each being an infinite subset of $\mathbb{N}$. By (2.7), the sequence of complex numbers $\mathrm{E}X_k^{(L)}X_0^{(L)}$, $L \in Q_{k-1}$, is bounded within the closed unit disc, and therefore has a convergent subsequence. Let Q_k be an infinite subset of Q_{k-1} such that $\mathrm{E}X_k^{(L)}X_0^{(L)}$ converges as $L \to \infty$, $L \in Q_k$. That completes the recursive definition of the sets $Q_0, Q_1, Q_2, \ldots$.

Note that $\mathbb{N} \supset Q_1 \supset Q_2 \supset Q_3 \supset \ldots$. Let $L_1, L_2, L_3, \ldots$ be a strictly increasing sequence of positive integers such that for each $k \geqslant 1$, $L_k \in Q_k$ (Cantor diagonalization). Define the set $\Lambda := \{L_1, L_2, L_3, \ldots\}$. Now for each $k \geqslant 1$, one has that $\{L_k, L_{k+1}, L_{k+2}, \ldots\} \subset Q_k$ (since $L_j \in Q_j \subset Q_k$ for $j \geqslant k$), and hence

$$c_k := \lim_{L \to \infty, L \in \Lambda} \mathrm{E}X_k^{(L)}\overline{X_0^{(L)}} \tag{2.8}$$

exists in $\mathbb{C}$. Of course by (2.7), weak stationarity, and trivial arithmetic, the c_k in (2.8) exists for $k \leqslant 0$ as well, with $c_0 = 1$ and $c_k = \overline{c_{-k}}$ for $k \leqslant -1$.

Refer to Doob (1953, p. 473, Theorem 3.1). In the terminology of that theorem, for each $L \geqslant 1$, the sequence of complex numbers $(\mathrm{E}X_k^{(L)}\overline{X_0^{(L)}},\ k \in \mathbb{Z})$ is positive definite. Hence by (2.8) and a simple calculation, the sequence of complex numbers $(c_k,\ k \in \mathbb{Z})$ is positive definite. Hence by Doob (1953, p. 473, Theorem 3.1) itself, there exists a CCWS random sequence $Y := (Y_k,\ k \in \mathbb{Z})$ such that $\mathrm{E}Y_k\overline{Y_0} = c_k$ for all $k \in \mathbb{Z}$. By (2.8) (and weak stationarity), for any (not necessarily disjoint) finite sets $Q, S \subset \mathbb{Z}$ and any choice of complex numbers a_k, $k \in Q$, and b_k, $k \in S$, one has that

$$\mathrm{E}\left(\sum_{j\in Q} a_j Y_j\right)\left(\sum_{k\in S} \overline{b_k}\,\overline{Y_k}\right) = \lim_{L\to\infty, L\in\Lambda} \mathrm{E}\left(\sum_{j\in Q} a_j X_j^{(L)}\right)\left(\sum_{k\in S} \overline{b_k}\,\overline{X_k^{(L)}}\right), \tag{2.9}$$

in particular,

$$\mathrm{E}\left|\sum_{k\in Q} a_k Y_k\right|^2 = \lim_{L\to\infty, L\in\Lambda} \mathrm{E}\left|\sum_{k\in Q} a_k X_k^{(L)}\right|^2, \tag{2.10}$$

and an analogous equation holds for $\sum_{k\in S} b_k Y_k$.

Let us use the notation $\mathrm{Corr}(V, W) := (\mathrm{E}V\overline{W})/(\|V\|_2\|W\|_2)$ for complex, square-integrable, mean-zero random variables V and W. (Interpret $0/0 := 0$.) For any pair of finite sets $Q \subset \{\ldots, -2, -1, 0\}$ and $S \subset \mathbb{N}$, and any choice of complex numbers a_k, $k \in Q$, and b_k, $k \in S$, such that $\mathrm{E}|\sum_{k\in Q} a_k Y_k|^2 > 0$ and $\mathrm{E}|\sum_{k\in S} b_k Y_k|^2 > 0$, one has that

$$\left|\mathrm{Corr}\left(\sum_{k\in Q} a_k Y_k, \sum_{k\in S} b_k Y_k\right)\right| = \lim_{L\to\infty, L\in\Lambda}\left|\mathrm{Corr}\left(\sum_{k\in Q} a_k X_k^{(L)}, \sum_{k\in S} b_k X_k^{(L)}\right)\right| \leqslant R$$

by (2.9), (2.10) and (2.5). Hence

$$r(Y, 1) \leqslant R < 1. \tag{2.11}$$

Also, for any positive integer n,

$$\mathrm{E}|Y_1 + \ldots + Y_n|^2 = \lim_{L\to\infty, L\in\Lambda} \mathrm{E}|X_1^{(L)} + \ldots + X_n^{(L)}|^2 \leqslant A \tag{2.12}$$

by (2.6), (2.7) and (2.10).

Since $\mathrm{E}|Y_0|^2 = c_0 = 1$ (see (2.7) and (2.8)), the CCWS sequence Y is non-degenerate. However, (2.11) and (2.12) together now contradict Lemma 2.3. Hence, after all, there must exist a positive integer M such that the conclusion of Lemma 2.4 holds. This completes the proof of Lemma 2.4. □

Lemma 2.5. *Suppose $0 \leqslant R < 1$, and B is a positive constant. Then there exists a positive integer $N = N(R, B)$ such that if $X := (X_k,\ k \in \mathbb{Z})$ is a CCWS random sequence such that $r(X, 1) \leqslant R$, then, for all $n \geqslant N$,*

$$\mathrm{E}|S_n|^2 \geqslant B \cdot \mathrm{E}|X_0|^2. \tag{2.13}$$

It was actually a version of this statement, but with the extra restriction $r(X, n) \to 0$ (and with N involving that 'mixing rate'), that was proved in the two references cited after Lemma 2.4. Technically, Lemma 2.5 can be omitted from the chain of arguments in this section, but it is a natural and elementary strengthening of Lemma 2.4.

Proof. Define the positive number

$$A := \frac{B}{(1-R)}. \tag{2.14}$$

Define the positive integer $N := M(R, A)$, where $M(R, A)$ is as in Lemma 2.4. Then N is a function of just R and B.

Now suppose $X := (X_k, k \in \mathbb{Z})$ is a CCWS random sequence such that $r(X, 1) \leqslant R$. Suppose $n \geqslant N$ is an integer. To complete the proof of Lemma 2.5, it suffices to prove (2.13) for this n.

From Lemma 2.4 and the above definition of N, there exists a (henceforth fixed) positive integer $m \leqslant N$ such that

$$\mathrm{E}|S_m|^2 \geqslant A \cdot \mathrm{E}|X_0|^2. \tag{2.15}$$

Now $m \leqslant N \leqslant n$. Also, $A \geqslant B$ by (2.14). If $n = m$, then (2.13) holds by (2.15) and we are done. Therefore, assume that $n \geqslant m+1$.

Now

$$\begin{aligned} |\mathrm{E}S_m(\overline{S_n - S_m})| &\leqslant r(X, 1) \cdot \|S_m\|_2 \|S_n - S_m\|_2 \\ &\leqslant R \cdot \tfrac{1}{2}[\mathrm{E}|S_m|^2 + \mathrm{E}|S_n - S_m|^2]. \end{aligned}$$

Hence by (2.15) and (2.14),

$$\begin{aligned} \mathrm{E}|S_n|^2 &= \mathrm{E}|S_m|^2 + \mathrm{E}|S_n - S_m|^2 + 2\,\mathrm{Re}\,\mathrm{E}S_m(\overline{S_n - S_m}) \\ &\geqslant (1-R) \cdot [\mathrm{E}|S_m|^2 + \mathrm{E}|S_n - S_m|^2] \\ &\geqslant (1-R) \cdot A \cdot \mathrm{E}|X_0|^2 \\ &= B \cdot \mathrm{E}|X_0|^2. \end{aligned}$$

Thus (2.13) holds. Lemma 2.5 is proved. □

Lemma 2.6. *Suppose $0 \leqslant R < 1$, and L is a positive integer. Then there exists a positive constant $C = C(R, L)$ such that if $X := (X_k, k \in \mathbb{Z})$ is a CCWS random sequence such that $r(X, 1) \leqslant R$ and $q'(X, L) \leqslant R$, then for every positive integer n,*

$$\mathrm{E}|S_n|^2 \geqslant C \cdot n \cdot \mathrm{E}|X_0|^2. \tag{2.16}$$

Bradley and Utev (1994, Lemma 2.3) proved a version of this lemma, but with the condition $q'(X, L) \leqslant R$ replaced by $r^*(X, n) \to 0$ (and with the constant C involving that 'mixing rate'). The proof here will be somewhat similar, but will involve an adaptation of an argument from Peligrad (1996) in order to accommodate the assumptions here. The use of the dependence coefficient $q'(X, n)$ in results on rates of growth of $\mathrm{E}|S_n|^2$ for random sequences was suggested by Peligrad (1997).

Proof. Define the positive constant

$$B := \frac{4L^2(1+R)^2}{(1-R)^2}. \tag{2.17}$$

Let $N = N(R, B)$ be as in Lemma 2.5. Define the positive constant C by

$$C := \frac{1}{(N+2L)^2}. \tag{2.18}$$

Then C depends only on R and L.

Now suppose $X := (X_k, k \in \mathbb{Z})$ is a CCWS random sequence such that

$$r(X, 1) \leqslant R \quad \textit{and} \quad q'(X, L) \leqslant R. \tag{2.19}$$

Suppose n is a positive integer. To complete the proof of Lemma 2.6, our task is to prove (2.16).

The set $\{1, 2, 3, \ldots, n(N+2L)\}$ will be partitioned into $2n$ blocks of consecutive integers, with no gaps between the blocks. In order, the blocks will be denoted $I(1)$, $J(1)$, $I(2)$, $J(2), \ldots, I(n)$, $J(n)$. The cardinalities will be given by $\mathit{card}\, I(u) = N + L$ and $\mathit{card}\, J(u) = L$ for $u = 1, \ldots, n$. For each $u \in \{1, \ldots, n\}$, define the (complex) random variables

$$V_u := \sum_{k \in I(u)} X_k \quad \textit{and} \quad W_u := \sum_{k \in J(u)} X_k.$$

Then

$$S_{n(N+2L)} = \sum_{u=1}^{n} (V_u + W_u). \tag{2.20}$$

From (2.17), Lemma 2.5, and the above definition of N, one has that

$$\mathrm{E}|S_{N+L}|^2 \geqslant B \cdot \mathrm{E}|X_0|^2 = \frac{4L^2(1+R)^2}{(1-R)^2} \cdot E|X_0|^2.$$

Hence by (weak) stationarity and Bradley (1992, Lemma 1),

$$\begin{aligned} \mathrm{E}\left|\sum_{u=1}^{n} V_u\right|^2 &\geqslant [1 - q'(X, L)] \cdot [1 + q'(X, L)]^{-1} \cdot \sum_{u=1}^{n} \mathrm{E}|V_u|^2 \\ &\geqslant (1-R)(1+R)^{-1} \cdot n \cdot \mathrm{E}|S_{N+L}|^2 \\ &\geqslant \frac{4L^2 n(1+R)}{(1-R)} \cdot \mathrm{E}|X_0|^2. \end{aligned} \tag{2.21}$$

Also, by (weak) stationarity and Bradley (1992, Lemma 1),

$$\begin{aligned} \mathrm{E}\left|\sum_{u=1}^{n} W_u\right|^2 &\leqslant [1 + q'(X, N+L)] \cdot [1 - q'(X, N+L)]^{-1} \cdot \sum_{u=1}^{n} \mathrm{E}|W_u|^2 \\ &\leqslant (1+R)(1-R)^{-1} \cdot n \cdot \mathrm{E}|S_L|^2 \\ &\leqslant \frac{n(1+R)}{(1-R)} \cdot L^2 \mathrm{E}|X_0|^2. \end{aligned} \tag{2.22}$$

By (2.20), (2.21), (2.22) and a simple calculation,

$$\begin{aligned} \|S_{n(N+2L)}\|_2 &\geqslant \left\|\sum_{u=1}^{n} V_u\right\|_2 - \left\|\sum_{u=1}^{n} W_u\right\|_2 \\ &\geqslant \frac{Ln^{1/2}(1+R)^{1/2}}{(1-R)^{1/2}} \cdot \|X_0\|_2 \geqslant n^{1/2}\|X_0\|_2 \end{aligned} \tag{2.23}$$

Also, $S_{n(N+2L)} = \sum_{v=1}^{N+2L}(S_{vn} - S_{(v-1)n})$, and hence by (weak) stationarity, $\|S_{n(N+2L)}\|_2 \leqslant (N+2L)\|S_n\|_2$. Hence by (2.23), $\|S_n\|_2 \geqslant [n^{1/2}/(N+2L)] \cdot \|X_0\|_2$. Hence (2.16) holds by (2.18). Lemma 2.6 is proved. □

We now turn our attention to random fields. For any positive integer d, any CCWS random field $X := (X_k, k \in \mathbb{Z}^d)$, and any positive integer n, define the sum

$$S_n = S_n(X) := \sum_k X_k, \tag{2.24}$$

where the sum is taken over all $k := (k_1, \ldots, k_d) \in \mathbb{Z}^d$ such that, for all $u \in \{1, \ldots, d\}$, $1 \leqslant k_u \leqslant n$. It is the sum of n^d random variables X_k. In the case $d = 1$ it coincides with (2.1).

Lemma 2.7. *Suppose* $0 \leqslant R < 1$*, and* L *is a positive integer. Let the positive constant* $C = C(R, L)$ *be as in Lemma 2.6. If* d *is a positive integer and* $X := (X_k, k \in \mathbb{Z}^d)$ *is a CCWS random field such that* $r(X, 1) \leqslant R$ *and* $q'(X, L) \leqslant R$*, then for every positive integer* n*,*

$$\mathrm{E}|S_n(X)|^2 \geqslant C^d \cdot n^d \cdot \mathrm{E}|X_{(0,\ldots,0)}|^2. \tag{2.25}$$

Bradley and Utev (1994, Lemma 2.4) gave a version of this lemma, but with the condition $q'(X, L) \leqslant R$ replaced by $r'(X, n) \to 0$ (and with the constant C involving that 'mixing rate'). The proof here will be similar – a well-known type of induction argument described (in a similar context) by Gaposhkin (1991) as 'layering'.

Proof. The proof will be done by induction on d.

In the case $d = 1$, Lemma 2.7 is simply a restatement of Lemma 2.6.

Now suppose that $D \geqslant 2$ and Lemma 2.7 holds for $1 \leqslant d \leqslant D-1$. Our task is to verify Lemma 2.7 for $d = D$. Suppose $X := (X_k,\, k \in \mathbb{Z}^D)$ is a CCWS random field such that $r(X, 1) \leqslant R$ and $q'(X, L) \leqslant R$. Suppose n is a positive integer. Our task is to prove (2.25) with $d = D$.

For each $j \in \mathbb{Z}$, define the (complex) random variable $Y_j := \sum_k X_k$, where the sum is taken over all $k := (k_1, \ldots, k_d) \in \mathbb{Z}^D$ such that $k_1 = j$ and $k_u \in \{1, \ldots, n\}$ for all $u \in \{2, \ldots, D\}$. Then by a simple calculation, $Y := (Y_j,\, j \in \mathbb{Z})$ is a CCWS random sequence. Also, $S_n(Y) = S_n(X)$. Also, $r(Y, n) \leqslant r(X, n)$ and $q'(Y, n) \leqslant q'(X, n)$ for all $n \geqslant 1$.

Also, the random field $W := (X_k,\, k \in \{0\} \times \mathbb{Z}^{D-1})$ is CCWS and satisfies $r(W, n) \leqslant r(X, n)$ and $q'(W, n) \leqslant q'(X, n)$ for all $n \geqslant 1$.

Hence by Lemma 2.6 and our induction assumption,

$$\mathrm{E}|S_n(X)|^2 = \mathrm{E}|S_n(Y)|^2 \geqslant C \cdot n \cdot \mathrm{E}|Y_0|^2 \geqslant C \cdot n \cdot (C^{D-1} n^{D-1} \mathrm{E}|X_{(0,\ldots,0)}|^2).$$

Thus (2.25) holds with $d = D$. Lemma 2.7 is proved. □

Lemma 2.8. *Suppose* $0 \leqslant R < 1$*, and* L *is a positive integer. If* d *is a positive integer and* $X := (X_k,\, k \in \mathbb{Z}^d)$ *is a CCWS random field such that* $q'(X, L) \leqslant R$*, then for every positive integer* n*,*

$$\mathrm{E}|S_n(X)|^2 \leqslant [L^d (1+R)^d (1-R)^{-d}] \cdot n^d \cdot \mathrm{E}|X_{(0,\ldots,0)}|^2.$$

This is formulated as a counterpart to Lemma 2.7 (though it does not require $r(X, 1) < 1$). It is a trivial consequence of Bradley (2001, Lemma 1.5).

We are now in a position to prove Theorem 1.6. The proof that (A) ⇒ (B) is a standard argument (part of the folklore), given, for example, in Bradley (1992, p. 365, lines −15 to −3). (That particular calculation only required the spectral density function f to be bounded between two positive constants; it did not require f to be continuous.) Also, (B) ⇒ (C) by Remark 1.4. To complete the proof of Theorem 1.6, all that remains is to prove that (C) ⇒ (A).

Suppose condition (C) in Theorem 1.6 holds. For each $t \in T^d$, define the random field $X^{(t)} := (X_k^{(t)},\, k \in \mathbb{Z}^d)$ by $X_k^{(t)} := \mathrm{e}^{-\mathrm{i}k\cdot\lambda} X_k$, where $\lambda \in (-\pi, \pi]^d$ is related to t as in Definition 1.2 and $k \cdot \lambda$ denotes the dot product. For each $t \in T^d$, by a simple argument, the random field $X^{(t)}$ is CCWS, and $\mathrm{E}|X_{(0,\ldots,0)}^{(t)}|^2 = \mathrm{E}|X_{(0,\ldots,0)}|^2$.

For each $n \geqslant 1$, define the function $f_n : T^d \to [0, \infty)$ as follows:

$$\forall t \in T^d, \qquad f_n(t) := n^{-d} \mathrm{E}|S_n(X^{(t)})|^2. \tag{2.26}$$

Referring to (1.15), let L be a positive integer such that $r'(X, L) < 1$. Referring to (1.14), define the number $R \in [0, 1)$ by $R := \max\{r(X, 1), r'(X, L)\}$. Let $C = C(R, L)$ be the positive constant from Lemma 2.6. For each $t \in T^d$, by a simple argument, $r(X^{(t)}, 1) = r(X, 1) \leqslant R$ and $r'(X^{(t)}, L) = r'(X, L) \leqslant R$. Hence by (2.26) and Lemma 2.7, $f_n(t) \geqslant C^d \mathrm{E}|X_{(0,\ldots,0)}|^2$ for all $t \in T^d$ and all $n \geqslant 1$. By the hypothesis of Theorem 1.6, $\mathrm{E}|X_{(0,\ldots,0)}|^2 > 0$. Hence

$$\theta_1 := \inf\{f_n(t) : t \in T^d,\ n \geqslant 1\} > 0. \tag{2.27}$$

Also, by (2.26) and Lemma 2.8,

$$f_n(t) \leqslant \left[\frac{L(1+R)}{(1-R)}\right]^d E|X_{(0,\ldots,0)}|^2$$

for each $t \in T^d$ and each $n \geqslant 1$. Hence

$$\theta_2 := \sup\{f_n(t) :\ t \in T^d,\ n \geqslant 1\} < \infty. \tag{2.28}$$

Let H denote the real Hilbert space of (equivalence classes of) real Borel square-integrable functions on T^d, with the usual inner product $\langle g, h\rangle := \int gh\mathrm{d}\mu_T^d$ and norm $\|g\|_2 = [\int g^2]^{1/2}$. Here and below, the integrals are taken over $t \in T^d$, with respect to the (probability) measure μ_T^d. By (2.27) and (2.28), $\sup_{n\geqslant 1}\|f_n\|_2 < \infty$. Recall that the closed unit ball of a Hilbert space is weakly compact (see, for example, Halmos 1974, Problem 17). As a consequence, there exists an element $f \in H$ and an infinite set Γ of positive integers such that

$$\lim_{n\to\infty, n\in\Gamma} \langle f_n, h\rangle = \langle f, h\rangle \qquad \forall h \in H. \tag{2.29}$$

Refer to (2.27) and (2.28). To prove condition (A) (under our assumption of (C)) in Theorem 1.6, we shall show that

$$\theta_1 \leqslant f(t) \leqslant \theta_2, \qquad \text{for almost every } t \in T^d, \tag{2.30}$$

and

$$\forall k \in \mathbb{Z}^d, \qquad \mathrm{E}X_k\overline{X_0} = \int \mathrm{e}^{\mathrm{i}k\cdot\lambda} \cdot f(t). \tag{2.31}$$

The type of argument used to prove (2.30) is well known. Suppose $\varepsilon > 0$. Define the set $A := \{t \in T^d : f(t) \geqslant \theta_2 + \varepsilon\}$. Suppose $\mu_T^d(A) > 0$. Let I_A denote the indicator function of A on T^d. Then by (2.28), $\int f_n \cdot I_A \leqslant \theta_2 \cdot \mu_T^d(A)$ for each $n \geqslant 1$, and $\int f \cdot I_A \geqslant (\theta_2 + \varepsilon)\cdot \mu_T^d(A)$, but this contradicts (2.29). Hence $\mu_T^d(A) = 0$ instead. Since $\varepsilon > 0$ was arbitrary, $f(t) \leqslant \theta_2$ for a.e. $t \in T^d$. By a similar argument, $f(t) \geqslant \theta_1$ for a.e. $t \in T^d$. Thus (2.30) holds.

Finally, by a well-known standard argument, for each $k \in \mathbb{Z}^d$,

$$\lim_{n\to\infty} \int \mathrm{e}^{\mathrm{i}k\cdot\lambda} \cdot f_n(t) = \mathrm{E}X_k\overline{X_0}. \tag{2.32}$$

This argument can be found, for example, in Bradley (1992, p. 365, proof of (2.9)). Also, by (2.29) (applied with $h = \cos(k\cdot\lambda)$ and $h = \sin(k\cdot\lambda)$),

$$\lim_{n\to\infty, n\in\Lambda} \int \mathrm{e}^{\mathrm{i}k\cdot\lambda} \cdot f_n(t) = \int \mathrm{e}^{\mathrm{i}k\cdot\lambda} \cdot f(t). \tag{2.33}$$

Now (2.31) holds by (2.32) and (2.33). This completes the proof that (C) $\Rightarrow$ (A), and Theorem 1.6 is proved.

3. A further look at condition (d) in Theorem 1.7

In this section, some extra perspective on condition (d) in Theorem 1.7 will be provided with Examples 3.1, 3.2 and 3.4, and Remark 3.5. Each of the three examples will be a stationary real centred (i.e. mean-zero) Gaussian random sequence (the case $d = 1$).

Example 3.1. For a given $\varepsilon \in (0, 1)$, there exists a stationary real centred Gaussian random sequence $X := (X_k, k \in \mathbb{Z})$ with the following five properties: (i) $r(X, 1) \leqslant r'(X, 1) = r^*(X, 1) \leqslant \varepsilon$; (ii) $r(X, n) \to 0$ as $n \to \infty$; (iii) $\lim_{n\to\infty} n^{-1}\mathrm{E}S_n^2$ fails to exist; (iv) X has a spectral density f_X (defined on T) which is bounded between two positive constants, and (v) the spectral density $f_X(\mathrm{e}^{\mathrm{i}\lambda})$ has (in a critical way) a discontinuity at $\lambda = 0$. Of course (iii) $\Rightarrow$ (v), by Ibragimov and Linnik (1971, Theorem 18.2.1, eq. (18.2.3)).

Such an example was given in Bradley (1999, Theorem 1). It was an embellished version of an example – a CCWS ('complex Gaussian') random sequence with similar properties – studied by Ibragimov (1970, p. 29) and Ibragimov and Rozanov (1978, p. 180, Example 2). The sequence X described above satisfies (1.14) and (1.15), but by Theorem 1.5 it fails to satisfy (1.13). Thus in condition (d) in Theorem 1.7, (1.13) cannot be omitted altogether, or even replaced by $r(X, n) \to 0$.

(A stationary Gaussian sequence very similar to X had been used earlier by Bryc and Dembo (1995) to provide counterexamples in connection with large deviations.)

Example 3.2. Let $W := (W_k, k \in \mathbb{Z})$ be a sequence of independent, identically distributed real normal random variables with mean 0 and variance $\frac{1}{2}$. Define the stationary real centred Gaussian sequence $X := (X_k, k \in \mathbb{Z})$ by $X_k = W_k - W_{k-1}$.

One has that $\mathrm{E}X_0^2 = 1$, $\mathrm{E}X_1X_0 = -\frac{1}{2}$, and $\mathrm{E}X_kX_0 = 0$ for $k \geqslant 2$. This sequence X is 1-dependent, and therefore satisfies (1.15), with $r'(X, 2) = r^*(X, 2) = 0$. By a simple calculation, the sequence X has spectral density f_X on T given by $f_X(\mathrm{e}^{\mathrm{i}\lambda}) = 1 - \cos\lambda$, $\lambda \in (-\pi, \pi]$. This spectral density is continuous on T, and hence (1.13) holds by Theorem 1.5. However, the spectral density $f_X(\mathrm{e}^{\mathrm{i}\lambda})$ has a zero at $\lambda = 0$. Hence by Theorem 1.6 (and the fact that (1.15) holds), inequality (1.14) fails to hold.

In fact, in condition (d) in Theorem 1.7, (1.14) cannot be replaced by the weaker condition $\tilde{q}(X, 1) < 1$. To show this, suppose Q and S are non-empty finite sets such that $Q \subset \{\ldots, -2, -1, 0\}$ and $S \subset \{1, 2, 3, \ldots\}$, and one defines the random variables $U := \sum_{k\in Q} X_k$ and $V := \sum_{k\in S} X_k$. Letting j and ℓ denote the least and greatest elements of S respectively, one has that $V = W_\ell - W_{j-1} + Y$ where Y is a linear combination (with coefficients -1, 0 and/or 1) of the random variables W_k, $j \leqslant k \leqslant \ell - 1$. (If $j = \ell$, then $Y = 0$.) Hence, $\mathrm{E}V^2 \geqslant \mathrm{E}W_\ell^2 + \mathrm{E}W_{j-1}^2 = 1$. Similarly, $\mathrm{E}U^2 \geqslant 1$. Also, by a simple argument, $\mathrm{E}UV = -\mathrm{E}W_0^2 = -\frac{1}{2}$ if $0 \in Q$ and $1 \in S$, and $\mathrm{E}UV = 0$ otherwise. Hence, $\tilde{q}(X, 1) \leqslant \frac{1}{2}$. (In fact $\tilde{q}(X, 1) = \frac{1}{2}$; consider the case $Q = \{0\}$ and $S = \{1\}$.) Since (1.13) and (1.15) both hold (as was noted above), this shows that in condition (d) in Theorem 1.7, (1.14) cannot be replaced by $\tilde{q}(X, 1) < 1$.

Lemma 3.3. (i) *If* $g : [0, \pi] \to \mathbb{R}$ *s a strictly increasing function, then* $\int_0^\pi g(x) \cdot (\cos x)\mathrm{d}x < 0$. (ii) *If* $h : [0, 2\pi] \to \mathbb{R}$ *is a strictly concave function, then* $\int_0^{2\pi} h(x) \cdot (\cos x)\mathrm{d}x < 0$. (iii) *If* $f : [0, \pi] \to \mathbb{R}$ *is strictly increasing and strictly concave, then for each* $n = 1, 2, 3, \ldots,$ $\int_0^\pi f(x) \cdot (\cos nx)\mathrm{d}x < 0$.

This is just some elementary information that will be needed in Example 3.4 below. A function $h : [0, 2\pi] \to \mathbb{R}$ is 'strictly concave' if $-h$ is convex on $[0, 2\pi]$ and h is not linear on any interval $\subset [0, 2\pi]$.

Proof. To see (i), note that $\int_0^\pi g(x) \cdot (\cos x)\mathrm{d}x = \int_0^{\pi/2}[g(x) - g(\pi - x)] \cdot (\cos x)\mathrm{d}x$, and the latter integrand is negative on $[0, \pi/2)$. To see (ii), note that $\int_0^{2\pi} h(x) \cdot (\cos x)\mathrm{d}x = \int_0^\pi [h(x) + h(2\pi - x)] \cdot (\cos x)\mathrm{d}x$, note that $h(x) + h(2\pi - x)$ is strictly increasing for $x \in [0, \pi]$, and then apply (i). To prove (iii), we shall just give the argument for odd integers $n \geqslant 3$. The argument for even integers $n \geqslant 2$ is similar. (For $n = 1$, (iii) holds by (i).)

Suppose $n \geqslant 3$ is an odd integer, and the function f is as in (iii). Represent $n = 2J + 1$, where J is a positive integer. Then

$$\begin{aligned}\int_0^\pi f(x) \cdot (\cos nx)\mathrm{d}x &= \int_{2\pi J/n}^\pi f(x) \cdot (\cos nx)\mathrm{d}x + \sum_{j=1}^J \int_{2\pi(j-1)/n}^{2\pi j/n} f(x) \cdot (\cos\ nx)\mathrm{d}t \\ &= \int_0^\pi f\left(\frac{u}{n} + \frac{2\pi J}{n}\right) \cdot \cos(u + 2\pi J) \cdot (1/n)\mathrm{d}u \\ &\quad + \sum_{j=1}^J \int_0^{2\pi} f\left(\frac{u}{n} + \frac{2\pi(j-1)}{n}\right) \cdot \cos(u + 2\pi(j-1)) \cdot (1/n)\mathrm{d}u \\ &< 0,\end{aligned}$$

with the inequality holding by (i) and (ii), since the functions $u \mapsto f(u/n + \ldots)$ in the last $J + 1$ integrands are strictly increasing and strictly concave. □

Example 3.4. Define the function $g : T \to [0, \infty)$ as follows:

$$g(\mathrm{e}^{\mathrm{i}\lambda}) := \begin{cases} 1/\log(30/|\lambda|), & \text{if } \lambda \in (-\pi, \pi] - \{0\}, \\ 0 & \text{if } \lambda = 0. \end{cases} \tag{3.1}$$

Then g is continuous on T. Also, g is 'symmetric': $g(\mathrm{e}^{\mathrm{i}(-\lambda)}) = g(\mathrm{e}^{\mathrm{i}\lambda})$. Also, for $0 < \lambda < \pi$, $(\mathrm{d}/\mathrm{d}\lambda)g(\mathrm{e}^{\mathrm{i}\lambda}) > 0$ and $(\mathrm{d}^2/\mathrm{d}\lambda^2)g(\mathrm{e}^{\mathrm{i}\lambda}) < 0$, by simple calculus. Hence the function $\lambda \mapsto g(\mathrm{e}^{\mathrm{i}\lambda})$ is strictly increasing and strictly concave on $[0, \pi]$. Applying Lemma 3.3(iii) and using the symmetry of g, one has that

$$\forall n = 1, 2, 3, \ldots, \qquad \int_{-\pi}^\pi (\cos n\lambda) \cdot g(\mathrm{e}^{\mathrm{i}\lambda})\mathrm{d}\lambda < 0. \tag{3.2}$$

Let $W := (W_k, k \in \mathbb{Z})$ be a stationary real centred Gaussian random sequence with spectral density function g. As a trivial variant of calculations in Bradley (1980, first paragraph of p. 97), the function g has the form $g = \exp(u + \tilde{v})$, where u and v are real continuous functions on T, $\|v\|_\infty < \pi/2$, and $\tilde{v}$ is the conjugate function of v. Hence, by theorems of Helson, Sarason and Szegö (see Helson and Szegö 1960; or Helson and Sarason 1967, Theorem 6; and also Sarason 1972, p. 62, the Theorem),

$$r(W, 1) < 1 \quad \text{and} \quad r(W, n) \to 0 \qquad \text{as } n \to \infty. \tag{3.3}$$

Also, by (1.1) and (3.2) (and the symmetry of g), one has that

$$\forall n \geq 1, \qquad \mathrm{E}W_n W_0 = (2\pi)^{-1} \int_{-\pi}^{\pi} (\cos n\lambda) \cdot g(\mathrm{e}^{\mathrm{i}\lambda}) \mathrm{d}\lambda < 0. \tag{3.4}$$

Now for each $n \geq 1$,

$$0 \leq \mathrm{E}(W_1 + \ldots + W_n)^2 = n\mathrm{E}W_0^2 + 2\sum_{k=1}^{n-1}(n-k)\mathrm{E}W_k W_0,$$

and hence $2\sum_{k=1}^{n-1}(1 - k/n)(-\mathrm{E}W_k W_0) \leq \mathrm{E}W_0^2$. For each $k \geq 1$, by (3.4), $I_{\{k<n\}} \cdot (1 - k/n)(-\mathrm{E}W_k W_0) \uparrow (-\mathrm{E}W_k W_0)$ as $n \to \infty$. Hence, by monotone convergence, $2\sum_{k=1}^{\infty}(-\mathrm{E}W_k W_0) \leq \mathrm{E}W_0^2$. (Actually, equality holds there by a further argument using the fact that $g(\mathrm{e}^{\mathrm{i}\lambda}) = 0$ for $\lambda = 0$; but that will not be needed here.) By (3.4), $-\mathrm{E}W_1 W_0 > 0$, and hence, by deleting that term, one obtains the strict inequality

$$2 \cdot \sum_{k=2}^{\infty}(-\mathrm{E}W_k W_0) < \mathrm{E}W_0^2. \tag{3.5}$$

Next, let $X := (X_k, k \in \mathbb{Z})$ be the random sequence defined by $X_k := (-1)^k W_k$. Then X is a stationary real centred Gaussian sequence. By a trivial argument, $r(X, n) = r(W, n)$ for each $n \geq 1$. Hence by (3.3), the sequence X satisfies (1.14) (as well as $r(X, n) \to 0$). Also, by a standard simple argument, the sequence X has spectral density function f_X on T given by $f_X(\mathrm{e}^{\mathrm{i}\lambda}) = g(\mathrm{e}^{\mathrm{i}(\lambda+\pi)})$. Hence f_X is continuous on T, and hence (1.13) holds for the sequence X by Theorem 1.5. However, the spectral density function $f_X(\mathrm{e}^{\mathrm{i}\lambda})$ has a zero at $\lambda = \pi$ (see (3.1) again), and hence by Theorem 1.6 (and the fact that (1.14) holds), inequality (1.15) does not hold.

To complete Example 3.4, we shall prove that

$$q'(X, n) \to 0 \qquad \text{as } n \to \infty. \tag{3.6}$$

That will show that in condition (d) in Theorem 1.7, (1.15) cannot be replaced by the condition that $q'(X, n) < 1$ for some $n \geq 1$, or even by $q'(X, n) \to 0$.

Referring to (3.4) and (3.5), define for each positive integer n the positive number

$$C_n := -\sum_{k=n}^{\infty} \mathrm{E}W_k W_0. \tag{3.7}$$

By (3.5), $\mathrm{E}W_0^2 - 2C_2 > 0$. Also, $C_n \to 0$ as $n \to \infty$. To prove (3.6), it suffices to prove that

$$\forall n \geqslant 1, \qquad q'(X, n) \leqslant \frac{2C_n}{\mathrm{E}W_0^2 - 2C_2}. \tag{3.8}$$

Let $n \geqslant 1$ be arbitrary but fixed. Suppose Q and S are (disjoint) finite subsets of $\mathbb{Z}$ such that $\mathrm{dist}(Q, S) \geqslant n$. To prove (3.8), it suffices to show the following three inequalities:

$$\mathrm{E}\left(\sum_{k\in Q} X_k\right)^2 \geqslant (\mathrm{card}\ Q) \cdot [\mathrm{E}W_0^2 - 2C_2]; \tag{3.9}$$

$$\mathrm{E}\left(\sum_{k\in S} X_k\right)^2 \geqslant (\mathrm{card}\ S) \cdot [\mathrm{E}W_0^2 - 2C_2]; \text{ and} \tag{3.10}$$

$$\left|\mathrm{E}\left(\sum_{k\in Q} X_k\right)\left(\sum_{k\in S} X_k\right)\right| \leqslant 2(\mathrm{card}\ Q)^{1/2}(\mathrm{card}\ S)^{1/2} C_n. \tag{3.11}$$

Now $\mathrm{E}X_0^2 = \mathrm{E}W_0^2$, and $\mathrm{E}X_n X_0 = (-1)^n \mathrm{E}W_n W_0$ for all $n \geqslant 1$. In particular, by (3.4), $\mathrm{E}X_1 X_0 > 0$, $\mathrm{E}X_n X_0 \geqslant \mathrm{E}W_n W_0$ for all $n \geqslant 1$, and $|\mathrm{E}X_n X_0| = -\mathrm{E}W_n W_0$ for all $n \geqslant 1$. These observations and (3.4) and (3.7) will be used in the calculations below. One has that

$$\begin{aligned}
\mathrm{E}\left(\sum_{k\in Q} X_k\right)^2 &= \sum_{k\in Q} \mathrm{E}X_k^2 + \sum_{j\in Q}\ \sum_{k\in Q-\{j\}} \mathrm{E}X_j X_k \\
&\geqslant (\mathrm{card}\ Q)\cdot \mathrm{E}X_0^2 + \sum_{j\in Q}\ \sum_{k\in Q-\{j-1,j,j+1\}} \mathrm{E}X_j X_k \\
&\geqslant (\mathrm{card}\ Q)\cdot \mathrm{E}W_0^2 + \sum_{j\in Q}\ \sum_{k\in Q-\{j-1,j,j+1\}} \mathrm{E}W_j W_k \\
&\geqslant (\mathrm{card}\ Q)\cdot \mathrm{E}W_0^2 + 2\sum_{j\in Q}\ \sum_{\ell=2}^{\infty} \mathrm{E}W_\ell W_0 \\
&= (\mathrm{card}\ Q)\cdot [\mathrm{E}W_0^2 - 2C_2].
\end{aligned}$$

Thus (3.9) holds. By an analogous argument, (3.10) holds. To prove (3.11), we may assume without loss of generality that $\mathrm{card}\, Q \leqslant \mathrm{card}\, S$. (Otherwise, interchange Q and S.) Since $\mathrm{dist}(Q, S) \geqslant n$, one has by (3.4) that

$$\left| \mathrm{E}\left(\sum_{k\in Q} X_k\right)\left(\sum_{k\in S} X_k\right)\right| \leqslant \sum_{j\in Q}\sum_{k\in S} |\mathrm{E}X_j X_k| = \sum_{j\in Q}\sum_{k\in S}(-\mathrm{E}W_j W_k)$$

$$\leqslant \sum_{j\in Q}\sum_{k\in\mathbb{Z}:|k-j|\geqslant n}(-\mathrm{E}W_j W_k) = \sum_{j\in Q}\sum_{\ell=n}^{\infty}(-2\mathrm{E}W_\ell W_0)$$

$$= 2(\text{card } Q)\cdot C_n \leqslant 2(\text{card } Q)^{1/2}(\text{card } S)^{1/2} C_n.$$

Thus (3.11) holds. This completes the proof of (3.8), and of (3.6).

Remark 3.5. Suppose d is a positive integer, and $X := (X_k,\ k \in \mathbb{Z}^d)$ is a non-degenerate CCWS random field. As in the last part of Section 2, for each $t \in T^d$, define the (CCWS) random field $X^{(t)} := (X_k^{(t)},\ k \in \mathbb{Z}^d)$ by $X_k^{(t)} := \mathrm{e}^{-\mathrm{i}k\cdot\lambda} X_k$, where $\lambda \in (-\pi, \pi]^d$ is related to t as in Definition 1.2 and $k \cdot \lambda$ denotes the dot product. Theorem 1.7 still holds if in its condition (d) inequality (1.15), borrowed from Theorem 1.6, is replaced by the following condition:

$$\forall t \in T^d, \qquad \exists n = n(t) \geqslant 1 \text{ such that } q'(X^{(t)}, n) < 1. \tag{3.12}$$

Obviously (1.15) implies (3.12). We only need to prove that if (1.13), (1.14) and (3.12) hold, then condition (a) in Theorem 1.7 holds: X has a positive continuous spectral density on T^d. Here is a sketch of the argument.

Suppose (1.13), (1.14) and (3.12) hold. By (1.13) and Theorem 1.5, X has a continuous spectral density f on T^d. By a well-known application of Fejér's theorem in dimension d, for each $t \in T^d$,

$$f(t) = \lim_{n\to\infty} n^{-d}\mathrm{E}|S_n(X^{(t)})|^2, \tag{3.13}$$

in the terminology of (2.24). (In the proof of Theorem 1.5 that was given in Bradley (2001, Section 2), equation (3.13) was used, after appropriate preliminary work, to define the function f.) For a given $t \in T^d$, observe that $r(X^{(t)}, 1) = r(X, 1)$, define $R = R(t) := \max\{r(X^{(t)}, 1), q'(X^{(t)}, n(t))\}$ and define $L = L(t) := n(t)$, where $n(t)$ is as in (3.12). For a given $t \in T^d$, by (1.14), (3.12) and Lemma 2.7, $\inf_{n\geqslant 1} n^{-d}\mathrm{E}|S_n(X^{(t)})|^2 > 0$, and hence $f(t) > 0$ by (3.13). Thus the spectral density f is both continuous and positive.

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