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Semigroup stationary processes and spectral representation

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We present an extended definition of the second-order stationarity concept. This is based on the theory of harmonic analysis for semigroups with involution. It provides a spectral representation for a wide class of processes which are non-stationary in the usual weak sense, and allows miscellaneous spectral representation results to be unified. Many applications are given to illustrate the concept. Most of these are already known, but some are new, such as the multiplicative-symmetric processes. We are less concerned with proving fundamental results than with opening up a new field of investigation for spectral representation of non-stationary processes.

Keywords: non-stationary processes; positive definite functions; semigroups with involution; spectral representation; stationary processes

1. Introduction

The classical notion of weak stationarity for stochastic processes refers to processes defined on additive subgroups $\mathbb{T}$ of $\mathbb{R}^d$, with $d \geqslant 1$. A zero-mean second-order process $\mathbf{X} = (X_t)_{t\in\mathbb{T}}$ is weakly stationary if some positive definite (p.d.) function R of one variable exists such that

$$r(s, t) = \mathrm{E}[X_t \overline{X}_s] = R(t - s), \qquad s, t \in \mathbb{T}.$$

The covariance kernel then has a spectral (or integral) representation (see Rudin 1962). By the Karhunen–Loève representation theorem (Karhunen 1947), a stationary stochastic process can be written as a stochastic integral. To be specific, we have

$$r(s, t) = \int_\Lambda \mathrm{e}^{\mathrm{i}(t-s)\lambda}\,\mathrm{d}\mu(\lambda) \quad \text{and} \quad X_t = \int_\Lambda \mathrm{e}^{\mathrm{i}t\lambda}\,\mathrm{d}Z(\lambda), \tag{1}$$

for a positive Radon measure μ on some $\Lambda \subset \mathbb{R}^d$ and a second-order process $\mathbf{Z}$ with orthogonal increments and power spectrum $\mathrm{d}\mu(\lambda)$.

Weak stationarity says that the covariance kernel, which is a function of two variables, is invariant under the diagonal shifts. Hence, it may be reduced to a p.d. function of only one variable. This reduction in dimension offers many advantages and is often necessary for statistical applications. A whole statistical tool-box has been developed in this regard and

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applied to various domains, such as the estimation of the covariance kernel via the power spectrum. It allows, for example, conditions for ergodicity of these processes to be stated. The spectral representation also is of great interest for both theory and applications (see, for example, Wentzell 1981). It simplifies the study of many problems, for example, linear prediction problems. The study of the process itself or of its linear transforms is often simplified using the structure of Z.

Extension to general Abelian groups $(\mathbb{T}, \circ)$ has been carried out by Hannan (1965). If $r(s, t) = R(t \circ s^{-1})$ with R bounded, Bochner's theorem or Herglotz's theorem for groups (see Rudin 1962; Hannan 1965) yields the spectral representations

$$r(s, t) = \int_{\tilde{\mathbb{T}}} \rho(t \circ s^{-1}) \mathrm{d}\mu(\rho) \quad \text{and} \quad X_t = \int_{\tilde{\mathbb{T}}} \rho(t) \mathrm{d}Z(\rho), \qquad s, t \in \mathbb{T},$$

using the dual group $\tilde{\mathbb{T}}$ of characters,

$$\tilde{\mathbb{T}} = \{\rho : \mathbb{T} \to \mathbb{C} | \rho(e) = 1, \rho(s \circ t) = \rho(s)\rho(t), \rho(t^{-1}) = \bar{\rho}(t), |\rho(t)| = 1\}.$$

The Fourier function basis $\{t \to \mathrm{e}^{\mathrm{i}t\lambda} : \lambda \in \Lambda\}$ used in the case of subgroups of $(\mathbb{R}^d, +)$ is simply replaced here by the Karhunen–Loève basis $\{t \to \rho(t) | \rho \in \tilde{\mathbb{T}}\}$.

Many attempts have been made to extend these methods to other stochastic processes. One of the approaches leading to exact representations is the first notion of local stationarity, as defined by Silverman (1957) and studied, for example, by Cramér (1961) and Michàlek (1986; 1988). Another approach, initiated by Sampson and Guttorp (1992), consists in identifying non-stationary processes as resulting from stationary processes through a transformation of the index space, such as the M-stationary processes defined by Gray and Zhang (1988). Other attempts at generalization are exponentially convex processes (see Widder 1946; Loève 1948) and symmetric processes (Loève 1948).

Another approach involving approximation is the second notion of local stationarity. Typically, a locally stationary process is generated by a phenomenon changing slowly in time or space, and hence can be approximated by a stationary one on small time intervals; to be specific, for any $x \in \mathbb{T}$,

$$r(s, t) = C(x, t - s) \text{ if } |t - s| \leqslant \frac{l(x)}{2}, \qquad s, t \in \mathbb{T}.$$

The Wigner–Ville spectrum (or time-varying spectrum), defined by Martin and Flandrin (1985) as $\int_{\mathbb{T}} C(\lambda, t) \mathrm{e}^{\mathrm{i}t\lambda} \, \mathrm{d}\lambda$, is associated with these processes. Note that it may take negative values, whereas the covariance operator has a positive spectrum. Mallat *et al*. (1998) determined an approximated Karhunen–Loève basis for representing locally stationary processes, allowing estimation of covariance. Dahlhaus (1997), among others, studied a special class of these processes depending on a parameter, the uniform length $l(x) \equiv \epsilon$ of approximated stationarity. Priestley (1988) defined and developed oscillatory processes such that

$$X_t = \int_{\mathbb{R}} A(t, \lambda) \mathrm{e}^{\mathrm{i}t\lambda} \, \mathrm{d}Z(\lambda), \qquad t \in \mathbb{R},$$

with different assumptions on the function A. These processes are locally stationary (see

Mallat *et al*. 1998; Dahlhaus 1997). Moreover, they have an exact spectral representation, but the non-uniqueness of the so-called evolutionary spectrum $|A(t, \lambda)|^2$ has remained an issue.

All these approaches exhibit similarities and redundancies, as do the proofs of the associated representation theorems.

We are interested in a unique exact representation of the covariance kernel and of the process, so we are not concerned with the notion of locally stationary processes. The field of harmonic analysis on semigroups with involution provides a powerful framework to unify many of the approaches cited above. To our knowledge, there has hitherto been no reference to this in the study of stochastic processes and covariance kernels. Potential use of the theory goes beyond the integral representation of processes; for example, results on moment problems (see Bisgaard 2001) may be used to solve the well-known problem of extension of covariance matrices or functions.

The main concern of this paper is the integral representation of a process, defined on an Abelian semigroup with involution $(\mathbb{T}, \circ, ^*)$, whose covariance kernel is written $r(s, t) = R(t \circ s^*)$. If R is exponentially bounded, then results due to Lindahl and Maserik (1971) and the Karhunen–Loève theorem together yield

$$r(s, t) = \int_{\mathbb{T}^*} \rho(t \circ s^*) \mathrm{d}\mu(\rho) \quad \text{and} \quad X_t = \int_{\mathbb{T}^*} \rho(t) \mathrm{d}Z(\rho)$$

on the set of semicharacters of $\mathbb{T}$,

$$\mathbb{T}^* = \{\rho\colon \mathbb{T} \to \mathbb{C} | \rho(e) = 1, \rho(s \circ t) = \rho(s)\rho(t), \rho(t^*) = \bar{\rho}(t)\}.$$

We give numerous applications of this notion, considering different involutions in addition to the classical inversion for groups where $\mathbb{T}^* = \tilde{\mathbb{T}}$. For example, the random fields on $\overline{\mathbb{R}}^d$ with orthogonal increments are $(\overline{\mathbb{R}}^d, \wedge, Id)$-stationary; the symmetric processes defined on $(\mathbb{T}, +, Id)$ and characterized by $r(s, t) = r(s + t)$ have spectral representations which are Laplace transforms; locally stationary processes are the restriction to the diagonal of $(\mathbb{C}, +, \bar{\cdot})$-stationary processes; the M-stationary processes such that $r(s, t) = R(t/s)$ are $(\mathbb{R}_+^*, \times, \cdot^{-1})$-stationary and their spectral representations are Mellin transforms.

Apart from specific examples, we generally consider one-dimensional and multi-dimensional semigroups built on the complex or real fields and on their subsets. Even in these cases, we consider only such composition laws and involutions as we use later in this paper. We develop especially results generated by considering the identity involution. This is of special interest since the identity is an involution for any Abelian semigroup. A great number of detailed examples illustrate the significant differences from the classical stationarity theory.

The paper is organized as follows. In Section 2 we present the framework of semigroups with involution leading to the fundamental result of spectral representation both for positive kernels and for stochastic processes. We give conditions for a p.d. kernel to be reduced to a p.d. function via semigroup invariance properties. In Section 3 we investigate the influence of index set deformation (i.e., of variable change) on semigroup stationarity. We deal with additive semigroups in Section 4 and multiplicative semigroups in Section 5 with two types of involution. In Section 6 we derive results for semigroups with a product structure.

2. Spectral representation

2.1. Bochner-type theorem on semigroups

Let $(\mathbb{T}, \circ, *)$ be an Abelian semigroup with involution (or $*$-semigroup), that is a non-empty set equipped with a commutative and associative composition $\circ$ admitting a neutral element e, and with an involution $s \to s^*$ satisfying $(s \circ t)^* = t^* \circ s^*$ and $(s^*)^* = s$ for all elements of $\mathbb{T}$.

Note that an involution is a bijective mapping such that $e^* = e$. Any Abelian group is a $*$-semigroup, with the involution $t^* = t^{-1}$. Any Abelian semigroup may also be seen as a $*$-semigroup, with the identity as involution.

Definition 1. *A function $\rho : \mathbb{T} \to \mathbb{C}$ is called a semicharacter if*

$$\rho(e) = 1, \quad \rho(s \circ t) = \rho(s)\rho(t) \quad \text{and} \quad \rho(t^*) = \bar{\rho}(t), \qquad s, t \in \mathbb{T}.$$

Semicharacters are semigroup homomorphisms of $(\mathbb{T}, \circ, *)$ into $(\mathbb{C}, \times, \bar{\cdot})$. If the involution is the identity, they are real-valued.

Let $\mathbb{T}^*$ denote the set of all semicharacters, $\hat{\mathbb{T}}$ the subset of semicharacters ρ such that $\sup_{s\in\mathbb{T}}|\rho(s)| \leqslant 1$, and $\tilde{\mathbb{T}}$ the set of characters, that is, of semicharacters with modulus one. We have $\tilde{\mathbb{T}} \subset \hat{\mathbb{T}} \subset \mathbb{T}^*$.

Note that the characteristic function $\mathbf{1}_{\mathbb{T}}$ is a character for any $*$-semigroup and that the semicharacters of a group are simply characters, since then

$$|\rho(t)|^2 = \rho(t)\bar{\rho}(t) = \rho(t)\rho(t^{-1}) = \rho(t \circ t^{-1}) = \rho(\mathrm{e}) = 1, \qquad t \in \mathbb{T}.$$

Example 1. The idempotent semigroups $(\mathbb{T}, \circ, *)$, such that $t \circ t = t$ for $t \in \mathbb{T}$, have very simple semicharacters. These are precisely the characteristic functions of the sub-semigroups with involution. Indeed, since $\rho(t) = \rho(t \circ t) = \rho^2(t)$, the semicharacters are $\{0, 1\}$-valued. The set $\{t \in \mathbb{T} | \rho(t) = 1\}$ contains the neutral element e and is stable under $\circ$ and $*$. The converse is clear.

Endowing $\tilde{\mathbb{T}}$, $\hat{\mathbb{T}}$ and $\mathbb{T}^*$ with pointwise multiplication as binary operation and conjugation as involution gives them a $*$-semigroup structure too. Moreover, if a $\mathbb{T}^*$ is endowed with the topology inherited from the topology of pointwise convergence on $\mathbb{C}^{\mathbb{T}}$, it has a completely regular Hausdorff space structure and its sub-semigroup $\hat{\mathbb{T}}$ has a compactness property.

We define p.d. functions following Lindahl and Maserick (1971).

Definition 2. *A function $R : \mathbb{T} \to \mathbb{C}$ is said to be positive definite if*

$$\sum_{j,k=1}^{n} c_k \bar{c}_j R(t_k \circ t_j^*) \geqslant 0, \qquad n \in \mathbb{N}, c_j \in \mathbb{C}, t_j \in \mathbb{T}, j = 1, \ldots, n.$$

Semicharacters are p.d. functions, since

$$\sum_{j,k}^{n} c_k \bar{c}_j \rho(t_k \circ t_j^*) = \sum_{j,k}^{n} c_k \bar{c}_j \rho(t_k)\bar{\rho}(t_j) = \left| \sum_{j}^{n} c_j \bar{\rho}(t_j) \right|^2 \geqslant 0.$$

In order to state the spectral representation theorems, we need the following definitions.

Definition 3. *A function $\upsilon : \mathbb{T} \to \mathbb{R}_+$ is called an absolute value if $\upsilon(e) = 1$, $\upsilon(t^*) = \upsilon(t)$ and $\upsilon(s \circ t) \leqslant \upsilon(s)\upsilon(t)$. A function $f : \mathbb{T} \to \mathbb{C}$ is said to be υ-bounded if there exists a constant c such that $|f(t)| \leqslant c\upsilon(t)$ for $t \in \mathbb{T}$. f is said to be exponentially bounded if it is bounded with respect to at least one absolute value. It is said to be bounded if $|f(t)| < c$ for $t \in \mathbb{T}$.*

Using these assumptions, we obtain the following Bochner-type representations (see Berg *et al*. 1984 for (a); Lindahl and Maserick 1971 for (b)).

Theorem 1. (a) *For an exponentially bounded p.d. function R on $\mathbb{T}$, there exists a unique positive Radon measure μ with compact support on $\mathbb{T}^*$ such that*

$$R(t) = \int_{\mathbb{T}^*} \rho(t)\mathrm{d}\mu(\rho), \qquad t \in \mathbb{T}. \tag{2}$$

(b) *If, moreover, R is bounded, one can replace $\mathbb{T}^*$ with $\hat{\mathbb{T}}$ in the integral representation* (2).

This theorem says that most p.d. functions are made up of the elementary bricks which are the semicharacters. For groups, it amounts to the classical Bochner's or Herglotz's theorems (see Rudin 1962; Hannan 1965), since then $\mathbb{T}^* = \hat{\mathbb{T}} = \tilde{\mathbb{T}}$.

Example 2. A function R is p.d. on the idempotent semigroup ($[0, +\infty]$, $\wedge$, *Id*) if and only if it is non-negative and non-decreasing. Considering the semicharacters $\rho_\lambda(t) = \mathbf{1}_{[\lambda,+\infty]}(t)$, for $\lambda > 0$, and the neutral element $e = +\infty$, (2) says that R is its own spectral measure; in other words,

$$R(t) = \int_0^t \mathrm{d}R(\lambda) = \int_0^{+\infty} \rho_\lambda(t)\mathrm{d}R(\lambda), \qquad t \in [0, +\infty].$$

2.2. Semigroup stationary processes

Let $\mathbf{X} = (X_t)_{t\in\mathbb{T}}$ be a stochastic process defined on an Abelian $*$-semigroup ($\mathbb{T}$, $\circ$, $*$), with zero mean and covariance kernel $r(s, t) = \mathrm{E}[X_s \overline{X}_t]$.

Definition 4. *The stochastic process* $\mathbf{X}$ *(or its covariance kernel r) is said to be weakly ($\mathbb{T}$, $\circ$, $*$)-stationary (or simply semigroup stationary) if, for some function $R : \mathbb{T} \to \mathbb{C}$, we have*

$$r(s, t) = R(t \circ s^*), \qquad s, t \in \mathbb{T}. \tag{3}$$

If $\mathbf{X}$ is $(\mathbb{T}, \circ, *)$-stationary, the associated function R is p.d. for the $*$-semigroup structure, since

$$\sum_{j,k=1}^{n} c_k \bar{c}_j R(t_k \circ t_j^*) = \mathrm{E}\left[\left(\sum_{j=1}^{n} c_j X_{t_j}\right)^2\right], \qquad t_1, \ldots, t_n \in \mathbb{T}.$$

Hence, if R is exponentially bounded, Theorem 1 yields

$$r(s,\, t) = \int_{\mathbb{T}^*} \rho(t)\bar{\rho}(s)\mathrm{d}\mu(\rho), \qquad s,\, t \in \mathbb{T}, \tag{$*$}$$

and then the Karhunen–Loève theorem gives the spectral representation

$$X_t = \int_{\mathbb{T}^*} \rho(t)\mathrm{d}Z(\rho), \qquad t \in \mathbb{T}, \tag{$**$}$$

for $\mathbf{X}$, where Z is a second-order stochastic process with orthogonal increments and basis μ.

The set of semicharacters T^* is often identified by isomorphism to a known set Λ, that is to say, $T^* = \{\rho_\lambda | \lambda \in \Lambda\}$, and then the spectral representations of both the covariance kernel r and the process $\mathbf{X}$ take the simpler form

$$r(s,\, t) = \int_\Lambda \rho_\lambda(t \circ s^*)\mathrm{d}\mu(\lambda) \quad \text{and} \quad X_t = \int_\Lambda \rho_\lambda(t)\mathrm{d}Z(\lambda). \tag{4}$$

The above representations $(*)$ to (4) extend the well-known representations of weakly stationary processes to different classes of processes which are not weakly stationary in this usual sense. We will characterize many of them in what follows. We begin with some examples of idempotent $*$-semigroup structures.

Example 3. Let $\xi : (\Omega, \mathcal{B}) \to \mathbb{C}$ be a centred, second-order random measure with covariance functional $r(A, B) = \mathrm{E}(\xi(A)\xi(B))$. Any algebra $\mathcal{B}$ equipped with intersection $\cap$ as composition and with identity as involution, is an idempotent $*$-semigroup. Thus, ξ has orthogonal increments (that is, $r(A, B) = R(A \cap B)$) if and only if ξ is $(\mathcal{B}, \cap, Id)$-stationary. It follows that random fields on $\overline{\mathbb{R}}^d$ have orthogonal increments if and only if they are $(\overline{\mathbb{R}}^d, \wedge, Id)$-stationary, where $(s_1, \ldots, s_d) \wedge (t_1, \ldots, t_\mathrm{d}) = (s_1 \wedge t_1, \ldots, s_\mathrm{d} \wedge t_\mathrm{d})$.

Example 4. Stochastic integrals

$$X_t = \int_0^t f(s)\mathrm{d}W(s), \qquad t \in \mathbb{T},$$

where W is a standard Brownian motion and f a locally square-integrable function, are $([0, +\infty], \wedge, Id)$-stationary, with plainly (see Examples 1 and 2)

$$\rho_\lambda(t) = \mathbf{1}_{[\lambda, +\infty]}(t), \quad \lambda \in \Lambda = [0, +\infty], \quad \mathrm{d}\mu(\lambda) = f^2(\lambda)\mathrm{d}\lambda \quad \text{and} \quad \mathrm{d}Z(\lambda) = f(\lambda)\mathrm{d}W(\lambda).$$

Example 5. Let $\mathbf{X} = (X_t)_{t \in \mathbb{G}}$ be a multivariate random process indexed by a binary tree $(\mathbb{G}, \mathcal{E})$, where $\mathbb{G}$ is the set of nodes and $\mathcal{E}$ is the set of directed edges. Let ρ denote the root of the tree and $p(t)$ the parent of node t. Huang and Cressie (2001) considered a general tree-structured Markov process evolving from parents to children according to the model

$$X_\rho = V_\rho \quad \text{and} \quad X_t = X_{p(t)} + V_t, \qquad t \neq \rho, \ t \in \mathbb{G},$$

where V_t are independent zero-mean Gaussian vectors.

Huang and Cressie (2001, p. 87) recommend that we 'notice the blocky nature of the correlation function', and remark that 'while it is not stationary, *en gros* it is nearly so'.

Actually, adding a formal new element ∞ to $\mathbb{G}$ endows $\mathbb{T} = \mathbb{G} \cup \{\infty\}$ with an idempotent $*$-semigroup structure with binary operation $s \circ t$ equal to the first common ancestor of both s and t, the extra element ∞ as neutral element and identity as involution. Such a Markovian process is clearly $(\mathbb{T}, \circ, Id)$-stationary since

$$r(s, t) = \mathrm{E}(X_t X_s) = \mathrm{E}(X^2_{t \circ s}) = R(t \circ s), \qquad s, t \in \mathbb{T},$$

for the variance function R of the process.

2.3. Conditions for semigroup stationarity

Semigroup stationarity has also an interpretation in terms of invariance properties of the covariance kernel under the actions of a semigroup of transformations.

Lemma 1. *A kernel $r : \mathbb{T} \times \mathbb{T} \to \mathbb{C}$ is $(\mathbb{T}, \circ, *)$-stationary if and only if*

$$r(s \circ a, t \circ b) = r(s \circ b^*, t \circ a^*), \qquad s, t, a, b \in \mathbb{T}. \tag{5}$$

Proof. The 'only if' part results from the equality

$$(t \circ b) \circ (s \circ a)^* = (t \circ a^*) \circ (s \circ b^*)^*.$$

The converse follows from the usual argument. Since

$$r(s, t) = r(e \circ s, t \circ e) = r(e \circ e^*, t \circ s^*) = r(e, t \circ s^*),$$

the function $R(u) = r(\mathrm{e}, u)$ clearly satisfies (3). □

The relation $(s, t)\mathcal{R}(a, b)$ if and only if $t \circ s^* = b \circ a^*$ is an equivalence relation. Its quotient set $\mathcal{Q} = \mathbb{T} \times \mathbb{T}/\mathcal{R}$, with cosets denoted by $[s, t]$, inherits a $*$-semigroup structure by setting $[s, t] \circ [a, b] = [s \circ a, t \circ b]$ and $[s, t]^* = [s^*, t^*]$. Moreover, the application $[s, t] \to t \circ s^*$ is a $*$-semigroup isomorphism between $\mathcal{Q}$ and $\mathbb{T}$. Hence, r is semigroup stationary if and only if r is constant on cosets.

For a group, property (5) reduces to the classical diagonal shift invariance property,

$$r(s \circ a, t \circ a) = r(s, t), \qquad s, t, a \in \mathbb{T}.$$

For a $*$-semigroup, it can be expressed in the following way.

Proposition 1. *Let $r : \mathbb{T} \times \mathbb{T} \to \mathbb{C}$. The two following statements are equivalent:*

(a) *The kernel r is semigroup stationary for some* $*$*-semigroup structure on* $\mathbb{T}$.
(b) *There exists an Abelian* $*$*-semigroup* $(\mathcal{F}, \circ, *)$ *where* $\mathcal{F} = \{f_a : \mathbb{T} \to \mathbb{T} | a \in \mathbb{T}\}$, *with the usual composition law* $(f \circ g)(t) = f(g(t))$ *and neutral element* f_e *such that* $f_a(e) = a$, *for which*

$$r(f(s), g(t)) = r(g^*(s), f^*(t)), \qquad s, t \in \mathbb{T}, f, g \in \mathcal{F}.$$

Proof. Let us suppose that r satisfies relation (3) for $(\mathbb{T}, \bullet, *)$. Let τ_a denote the usual translation of $a \in \mathbb{T}$ defined by $\tau_a(t) = t \bullet a$ for $t \in \mathbb{T}$. The set $\mathcal{F} = \{\tau_a | a \in \mathbb{T}\}$ endowed with the composition law $\tau_a \circ \tau_b = \tau_{a \bullet b}$ and the involution $(\tau_a)^* = \tau_{a^*}$ inherits an Abelian $*$-semigroup structure too. Relation (5) can thus be written $r(\tau_a(s), \tau_b(t)) = r(\tau_b^*(s), \tau_a^*(t))$.

Conversely, if $f_a \circ f_b = f_c$ and $(f_a)^* = f_d$ for two well-defined elements c and d, the binary operation $a \bullet b = c$ and the involution $a^* = d$ endow $\mathbb{T}$ with a $*$-semigroup structure. The neutral element of $\mathbb{T}$ is e. Actually, the transformation f_a is the above translation operation τ_a, since $f_a(t) = f_a(f_t(\mathrm{e})) = f_{a \bullet t}(e) = a \bullet t$. □

We prove the following general result for transitive (i.e., for all $s, t \in \mathbb{T}$ there exists $u \in \mathbb{T}$ such that $s = t \circ u$) and 2-divisible (i.e., each $t \in \mathbb{T}$ may be written $t = s \circ s$ for some $s \in \mathbb{T}$) semigroups. Note, for example, that $(\mathbb{Q}, +)$ and $(\mathbb{C}, \times)$ are both transitive and 2-divisible.

Proposition 2. *All bounded positive definite functions on transitive and 2-divisible semigroups with the identity involution are constant.*

Proof. If R is p.d., then $|R(t)|^2 = |R(t \circ e)|^2 \leqslant R(t \circ t)R(e)$, hence $R \equiv 0$ if $R(e) = 0$. We can therefore assume without loss of generality that $R(e) = 1$ if $R(e) \neq 0$. If R is bounded, set $m = \sup_{t \in \mathbb{T}} |R(t)| < +\infty$. Then

$$m^2 = \sup_{t \in \mathbb{T}} |R(t)|^2 \leqslant \sup_{t \in \mathbb{T}} R(t \circ t) \leqslant \sup_{s \in \mathbb{T}} |R(s)| = m,$$

thus $m \leqslant 1$ and $|R(t)| \leqslant R(e) = 1$, $t \in \mathbb{T}$.

If $\mathbb{T}$ is transitive, for all $u \in \mathbb{T}$, there exists some $s \in \mathbb{T}$ such that $e = u \circ s$, and hence

$$1 = R(e) = R(u \circ s) \leqslant R^{1/2}(u \circ u) R^{1/2}(s \circ s) \leqslant 1,$$

which proves that $R(u \circ u) = 1$. Moreover, if $\mathbb{T}$ is 2-divisible then, for all $t \in \mathbb{T}$, we have $R(t) = R(u \circ u) = 1$ for some u. □

3. Semigroup stationarity by space deformation

Semigroup stationarity includes the concept of reducibility to stationarity through a deformation Φ of the index set.

Reducibility to weak stationarity was introduced by Sampson and Guttorp (1992) in an application to environmental data, and formalized by Perrin and Senoussi (1999; 2000). The processes concerned can be written $X_t = Y_{\Phi(t)}$, where the process $\mathbf{Y}$ so defined is weakly stationary. We extend this notion of reducible processes to semigroup stationarity.

Proposition 3. *Let $\mathbb{T}$ be a given set and $(\mathbb{S}, \bullet, *)$ a $*$-semigroup. Let $\Phi : \mathbb{T} \to \mathbb{S}$ be a bijective transformation. Let r be a covariance kernel on $\mathbb{T}$ of the form*

$$r(s, t) = R(\Phi(t) \bullet \Phi(s)^*), \qquad s, t \in \mathbb{T}.$$

If R is an exponentially bounded positive definity function, then r has a spectral representation.

Proof. Every bijection $\Phi : \mathbb{T} \to \mathbb{S}$ from a general set $\mathbb{T}$ to a $*$-semigroup $(\mathbb{S}, \bullet, *)$ generates on $\mathbb{T}$ a $*$-semigroup structure $(\mathbb{T}, \circ, *)$ by setting

$$s \circ t = \Phi^{-1}(\Phi(s) \bullet \Phi(t)) \quad \text{and} \quad t^* = \Phi^{-1}(\Phi(t)^*).$$

The transformation Φ is clearly a $*$-semigroup isomorphism. The semicharacters (respectively p.d. functions) of $\mathbb{T}$ are of the form $\tilde{\rho} = \rho \circ \Phi$ (respectively $\tilde{R} = R \circ \Phi$) for some semicharacter ρ (respectively p.d. function R) of $\mathbb{S}$. Thus the p.d. function $\tilde{R}$ can be written

$$\tilde{R}(t) = R(\Phi(t)) = \int_{\mathbb{S}^*} \rho(\Phi(t)) \mathrm{d}\mu(\rho) = \int_{\mathbb{T}^*} \tilde{\rho}(t) \mathrm{d}\mu_\Phi(\tilde{\rho}), \qquad t \in \mathbb{T},$$

where μ_Φ denotes the image measure of μ by the function $\rho \to \tilde{\rho} = \rho \circ \Phi$ from $\mathbb{S}^*$ to $\mathbb{T}^*$. □

In particular, if $\mathbf{X} = (X_t)_{t \in \mathbb{T}}$ is a process with covariance kernel

$$r(s, t) = R(\Phi(t) \bullet \Phi(s)^*), \qquad s, t \in \mathbb{T},$$

and if $\mathbb{S}^* = \{\rho_\lambda | \lambda \in \Lambda\}$ for a given set Λ, then the spectral representations (4) take the form

$$r(s, t) = \int_\Lambda \rho_\lambda(\Phi(t)) \bar{\rho}_\lambda(\Phi(s)) \mathrm{d}\mu(\lambda) \quad \text{and} \quad X_t = \int_\Lambda \rho_\lambda(\Phi(t)) \mathrm{d}Z(\lambda).$$

The logarithm transform is of particular interest: see the M-stationary processes in Section 5.1, the M-symmetric processes in Section 5.2 and the H-self-similar processes in Example 14.

Example 6. A normalized Brownian sheet $\mathbf{X}$ is a centred Gaussian field on $\mathbb{T} = \mathbb{R}_+^* \times \mathbb{R}_+^*$ with covariance kernel

$$r(s, t) = \frac{\|s\| + \|t\| - \|t - s\|}{2\sqrt{\|s\| \|t\|}}, \qquad s, t \in \mathbb{T}.$$

The transformation $\Phi(s_1, s_2) = (\ln \|s\|, \arctan(s_2/s_1))$ where $s = (s_1, s_2)$, reduces $\mathbf{X}$ to a weakly stationary random field on $\mathbb{S} = \mathbb{R} \times \mathbb{R}$, since $r(s, t) = R(\Phi(t) - \Phi(s))$ with

$$R(u) = \cosh(u_1/2) - \sqrt{[\cosh(u_1/2) - \cos u_2]/2}, \qquad u = (u_1, u_2) \in \mathbb{S}.$$

Note that not all the covariance kernels are reducible to stationarity by transformations of the index set. Perrin and Senoussi (1999; 2000) give a necessary and sufficient condition for a continuous-time process to be reducible to stationarity and examples and counterexamples of stationary reducible processes; see the earlier paper for the one-dimensional case and the later one for the multidimensional case.

4. Additive semigroup stationarity

4.1. Determination of semicharacters

We present here the continuous semicharacters of classical semigroups constructed from $(\mathbb{R}, +)$ and the induced spectral representations for p.d. functions. Non-continuous semicharacters do exist for these semigroups, for example $\mathbf{1}_{\{0\}}$, but they generally play no part in the spectral representation of continuous covariance kernels (see Theorem 2 below).

4.1.1. Groups

For the group $(\mathbb{R}, +, (\cdot)^{-1})$ (respectively $(\mathbb{Z}, +, (\cdot)^{-1})$) with inversion as involution, the semigroup stationarity is just weak stationarity. So, for the sake of comparison, we recall that the semicharacters are characters, have the form $\rho(t) = \mathrm{e}^{\mathrm{i}\lambda t}$, with $\lambda \in \Lambda$ where $\Lambda = \mathbb{R}$ (respectively $\Lambda = \Pi = [-\pi, \pi[$) and that (1) holds for any bounded weakly stationary covariance kernel and weakly stationary process.

4.1.2. Semigroups with the identity involution

If $\mathbb{T}$ is one of the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}_+$, $\mathbb{Q}$, Π, $\mathbb{R}_+$ or $\mathbb{R}$, we can easily determine that continuous characters of the semigroup $(\mathbb{T}, +, Id)$ with the identity involution have the form $\rho_\lambda(t) = \mathrm{e}^{\lambda t}$, with $\lambda \in \mathbb{R}$.

We can also easily determine that $\hat{\mathbb{T}} = \{\mathbf{1}_\mathbb{T}\}$ if $\mathbb{T}$ is $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or Π, while for $\mathbb{N}$, $\mathbb{Q}_+$ or $\mathbb{R}_+$ we obtain $\hat{\mathbb{T}} = \{\rho_\lambda | \rho_\lambda(t) = \mathrm{e}^{-\lambda t}, \lambda \in \mathbb{R}_+\}$ and $\tilde{\mathbb{T}} = \{\mathbf{1}_\mathbb{T}\}$.

In the literature, bounded $(\mathbb{T}, +, Id)$-p.d. functions are generally said to be exponentially convex (see, for example, Loève 1948; Akhieser 1965; Nussbaum 1972).

Note that if the semicharacters of $(\mathbb{R}, +, Id)$ are written in the convenient equivalent form $\rho_\lambda(t) = \lambda^t$ for $\lambda \in \mathbb{R}_+$, then (4) becomes

$$r(s, t) = \int_{\mathbb{R}_+} \lambda^{s+t} \mathrm{d}\mu(\lambda), \qquad s, t \in \mathbb{R}.$$

The case $T = \mathbb{N}$ has direct links to the classical moment problem, since then

$$r(n, m) = R(n+m) = \int_{[-1,1]} \lambda^{n+m} \mathrm{d}\mu(\lambda), \qquad n, m \in \mathbb{N}. \tag{6}$$

4.1.3. Semigroups with index permutation involution

For multidimensional index sets, several involutions may be considered. The extension of the above results through the product structure will be considered in Section 6 below. Many other possibilities exist.

For example, the permutation operation $(t_1, t_2)^* = (t_2, t_1)$ is a well-defined involution on $\mathbb{T} = (\mathbb{R}^2, +)$. For this structure, we obtain $\mathbb{T}^* = \{\rho_\lambda : \mathbb{T} \to \mathbb{C} | \rho_\lambda(t) = \lambda^{t_1}\overline{\lambda}^{t_2}, \lambda \in \mathbb{C}\}$, $\hat{\mathbb{T}} = \{\rho_\lambda : \mathbb{T} \to \mathbb{C} | \rho_\lambda(t) = \lambda^{t_1}\overline{\lambda}^{t_2}, |\lambda| \leqslant 1\}$ and $\tilde{\mathbb{T}} = \{\mathbf{1}_\mathbb{T}\}$. It can be extended to $\mathbb{R}^d$, $d \geqslant 3$, for any permutation σ of coordinates such that $\sigma \circ \sigma(i) = i$, for $i = 1, \ldots, d$.

This involution is also of interest for discrete index-set processes, especially since, as above in (6), the form of the semicharacters links the spectral representation to the complex classical multidimensional moment problem.

4.2. Symmetric processes

If $\mathbf{X}$ is an $(\mathbb{R}^d, +, Id)$-stationary process, then $r(s, t) = R(s + t)$, that is, $\mathrm{cov}(X_s, X_t) = \mathrm{var}(X_{(s+t)/2})$. Following Michàlek (1988), we will say that such semigroup stationary processes are symmetric. Note that Loève (1946; 1948) called exponentially convex any such second-order stochastic process with zero mean and bounded covariance kernel.

Example 7. The sinusoidal signal $X_n = A\cos(\nu n - \pi/4)$, for $n \in \mathbb{Z}$, where $\nu \sim \mathcal{U}(0, \pi)$ and A are independent random variables, is centred if A is centred. Moreover, X is symmetric since

$$r(n, m) = \begin{cases} \mathrm{E}[A^2]/2 & \text{if } n = m, \\ 0 & \text{if } n = -m, \\ [1 - (-1)^{n+m}]\mathrm{E}[A^2]/2\pi(n+m) & \text{if } n \neq \pm m. \end{cases}$$

Using the semigroup properties of $(\mathbb{Q}_+, +, Id)$, Berg *et al*. (1984, Theorem 5.11, p. 212) prove the following result, thus extending and synthesizing earlier results by Widder (1946, Theorem 21, p. 273), Devinatz (1955) and Akhieser (1965, Theorem 5.5.4 and Problem 17).

Theorem 2. *Any continuous $(\mathbb{R}^d, +, Id)$-p.d. function R has the representation*

$$R(u) = \int_{\mathbb{R}^d} \mathrm{e}^{\langle\lambda, u\rangle}\, \mathrm{d}\mu(\lambda), \qquad u \in \mathbb{R}^d, \tag{7}$$

for a uniquely determined positive Radon measure μ. And R can be extended to an entire holomorphic function on $\mathbb{C}^d$.

This result does not require the exponential boundedness assumption of Theorem 1. Therefore, by the Karhunen–Loève theorem, for any symmetric and continuous (in the mean-square sense) process $\mathbf{X}$, a second-order stochastic process Z with orthogonal increments and basis μ exists such that

$$X_t = \int_{\mathbb{R}^d} e^{\langle \lambda, t \rangle} \, dZ(\lambda), \qquad t \in \mathbb{R}^d. \tag{8}$$

Note that Michàlek (1988) obtained a spectral representation for these processes by enlarging the index set to $\mathbb{C}$ (see Section 6 below). He also derived an inversion formula for the spectral measure.

Example 8. A simple illustration of (8) is given by the continuous process

$$X_t = \sum_{i=1}^{n} U_i e^{\langle \lambda_i, t \rangle}, \qquad t \in \mathbb{R}^d,$$

where $\lambda_1, \ldots, \lambda_n$ are deterministic points in $\mathbb{R}^d$ and $U_1, \ldots, U_n$ are centred and orthogonal random variables with variances $\sigma_i^2 = \sigma^2(\lambda_i)$. Both the second-order stochastic measure $dZ = \sum_{i=1}^n U_i \delta_{\lambda_i}$ and its basis measure $d\mu = \sum_{i=1}^n \sigma_i^2 \delta_{\lambda_i}$ have discrete supports. The covariance kernel representation (7) of **X** can be written

$$r(s, t) = \sum_{i=1}^{n} \sigma_i^2 e^{\langle \lambda_i, s+t \rangle}, \qquad s, t \in \mathbb{R}^d.$$

Example 9. The link between a stationary process and a symmetric one can sometimes be given by a standardization procedure. Indeed, taking a Gaussian basis measure μ on $\mathbb{R}$ in (7) yields

$$R(u) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\lambda u} e^{-\lambda^2/2} d\lambda = e^{u^2/2}, \qquad u \in \mathbb{R}.$$

Hence, a zero-mean Gaussian symmetric process **X** exists with covariance $r_{\mathbf{X}}(s, t) = R(s+t)$ and standard deviation $\sigma(t) = \exp(t^2)$. Its normalized process **Y** defined by $Y_t = X_t/\sigma(t)$ is weakly stationary since $r_Y(s, t) = \exp(-(s-t)^2/2)$. Figures 1 and 2 show samples of such processes **X** and **Y**.

5. Multiplicative semigroup stationarity

The multiplicative groups are of increasing interest in the theory of stochastic processes (see, for example, Mandelbrot 200l; Gray and Zhang 1988). We consider here the multiplicative structure of $\mathbb{R}\backslash\{0\}$ and of some of its subsets endowed with the inverse or with the identity involution. The case of $\mathbb{C}$ will be studied through product structures in Section 6 below.

5.1. Groups and M-stationary processes

The triplet $(\mathbb{R}\backslash\{0\}, \times, (\cdot)^{-1})$, with the usual product as composition and inverse as involution, is a group with dual character set

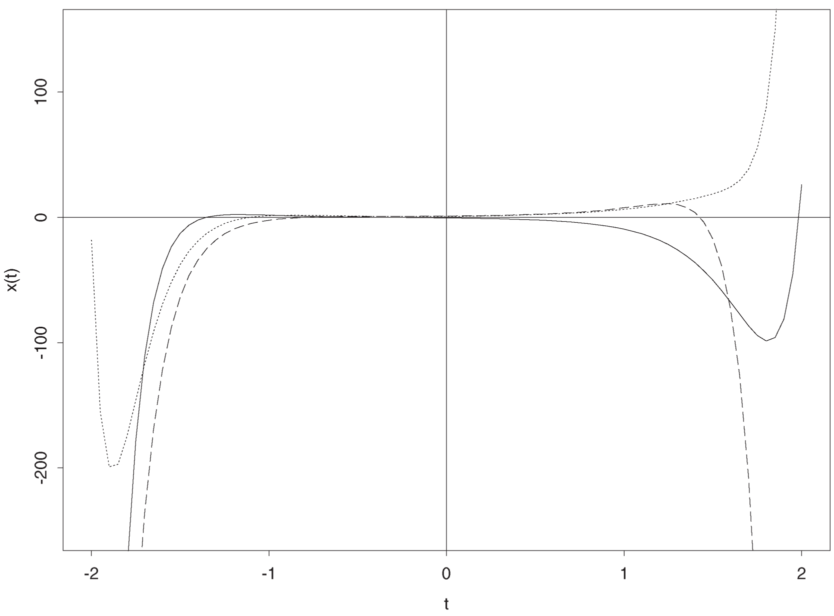

Figure 1. Three samples of a zero-mean Gaussian symmetric process **X** with covariance $r_{\mathbf{X}}(s,\ t) = \exp[(s+t)^2/2]$.

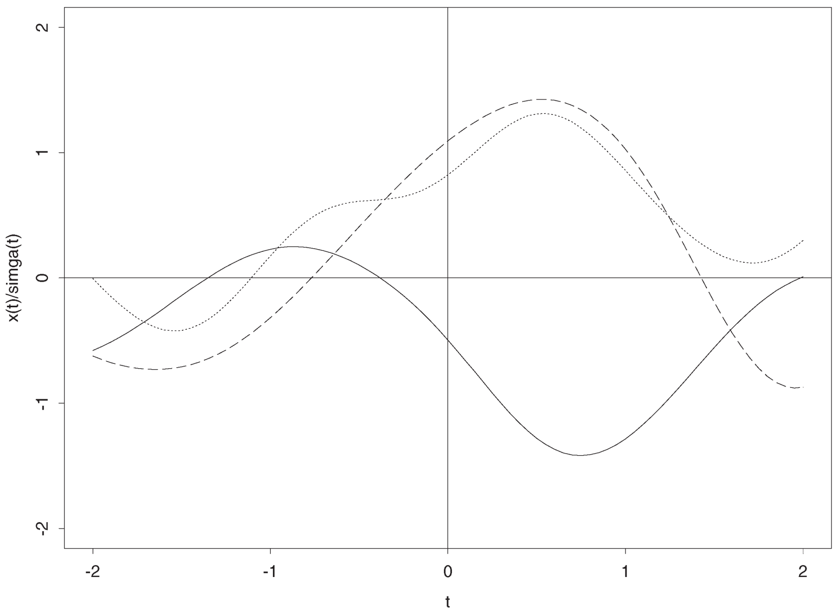

Figure 2. The samples of the stationary transform $\mathbf{Y} = \mathbf{X}/\sigma$ with covariance $r_{\mathbf{Y}}(s,\ t) = \exp(-(s-t)^2/2)$, corresponding to those in Figure 1.

$$(\mathbb{R}\backslash\{0\})^* = \widetilde{\mathbb{R}\backslash\{0\}} = \{\rho_\lambda : \mathbb{R}\backslash\{0\} \to \mathbb{C} | \rho_\lambda(t) = |t|^{\mathrm{i}\lambda}, \lambda \in \mathbb{R}\}.$$

We can also prove easily that

$$(\mathbb{R}_+\backslash\{0\})^* = \widetilde{\mathbb{R}_+\backslash\{0\}} = \{\rho_\lambda : \mathbb{R}\backslash\{0\} \to \mathbb{C} | \rho_\lambda(t) = t^{\mathrm{i}\lambda}, \lambda \in \mathbb{R}\}. \tag{9}$$

The covariance kernels of $(\mathbb{R}_+\backslash\{0\}, \times, (\cdot)^{-1})$-stationary processes satisfy $r(s, t) = R(t/s)$. These processes were called multiplicative stationary (or M-stationary) by Gray and Zhang (1988) who studied them thoroughly through the time transformation $\Phi(t) = \ln t$ which reduces them to stationarity. A statistical point of view is developed in Girardin and Rachdi (2003). Using (9), we obtain the spectral representation of these covariance kernels directly as Mellin transforms (for details on this transformation, see Jerri 1992), namely

$$r(s, t) = \int_{\mathbb{R}} (t/s)^{\mathrm{i}\lambda} \mathrm{d}\mu(\lambda).$$

Example 10. Fractional Brownian motion is a centred Gaussian process indexed by $\mathbb{T} = \mathbb{R}_+$ whose covariance kernel, depending on a parameter $a \in [0, 1]$, is defined by

$$r(s, t) = \frac{s^{2a} + t^{2a} - |t - s|^{2a}}{2(st)^a}.$$

It is clearly M-stationary, with $R(u) = (1 + u^{2a} - |1 - u|^{2a})/(2u^a)$.

In particular, if $a = 1/2$, its dual stationary transformed process is the classical Wiener process, (see Samorodnitsky and Taqqu 1994).

Mandelbrot (2001, Chapter 6) presented some important examples of non-random and random self-affine functions, for which 'moving on from the clock time to the log time' results in new insights. He pointed out that the log time makes the law of the iterated logarithm obvious for the Brownian motion.

5.2. Semigroups and M-symmetric processes

For the semigroup structure $(\mathbb{T}, \times, Id)$, all the sets $\mathbb{T} = \mathbb{N}\backslash\{0\}$, $\mathbb{Z}\backslash\{0\}$, $R_+\backslash\{0\}$ or $\mathbb{R}\backslash\{0\}$ have the same set of semicharacters,

$$\mathbb{T}^* = \{\rho_\lambda : \mathbb{T} \to \mathbb{C} | \rho_\lambda(t) = |t|^\lambda, \lambda \in \mathbb{R}\}.$$

If $\mathbb{T}$ is $\mathbb{N}\backslash\{0\}$ or $\mathbb{Z}\backslash\{0\}$, then

$$\hat{\mathbb{T}} = \{\rho_\lambda : \mathbb{T} \to \mathbb{C} | \rho_\lambda(t) = |t|^{-\lambda}, \lambda \in \mathbb{R}_+\}, \tag{10}$$

while $\hat{\mathbb{T}} = \{\mathbf{1}_{\mathbb{T}}\}$ for $\mathbb{R}\backslash\{0\}$ and $\mathbb{R}_+\backslash\{0\}$, since they are both transitive and 2-divisible.

The covariance kernels of $(\mathbb{R}_+\backslash\{0\}, \times, Id)$-stationary processes satisfy $r(s, t) = R(st)$. We can call these processes M-symmetric, since the time transformation $\Phi(t) = \ln t$ makes the M-symmetric processes reducible to symmetric processes in the same way as the M-stationary processes are reducible to stationary processes. The characterization of semicharacters given in (10) yields the spectral representation

$$r(s, t) = \int_{\mathbb{R}_+} (st)^{-\lambda} \mathrm{d}\mu(\lambda), \qquad s, t \in \mathbb{R}_+.$$

More elaborated products can be considered, as in the following example.

Example 11. Consider $(\mathbb{R}_+, \circ, Id)$, where $s \circ t = (s+1)(t+1)$. Using the transformation $\Phi(t) = \ln(1+t)$ from $(\mathbb{R}_+, \circ, Id)$ to $(\mathbb{R}_+, +, Id)$, the bounded semicharacters of $(\mathbb{R}_+, \circ, Id)$ are proven to be $\rho_\lambda(t) = (1+t)^{-\lambda}$, for $\lambda \in \mathbb{R}_+$ (see the proof of Proposition 3). A stochastic process $\mathbf{X}$ is $(\mathbb{R}_+, \circ, Id)$-stationary if and only if $\mathbf{Y}$ defined by $Y_u = X_{\mathrm{e}^u - 1}$ is symmetric, and then

$$r_{\mathbf{X}}(s, t) = \int_{\mathbb{R}_+} [(s+1)(t+1)]^{-\lambda} \mathrm{d}\mu(\lambda), \qquad s, t \in \mathbb{R}_+.$$

6. Product semigroups

6.1. Semicharacters

Most semigroup structures constructed on $\mathbb{C}$ (or $\mathbb{R}^2$ or $\mathbb{C}^n$) can be deduced from isomorphisms with semigroup products of $\mathbb{R}$. Many other semigroup structures constructed on a set product can be dealt with by the following general result (see Berg *et al*. 1984, Exercise 4.2.13).

Proposition 4. *Let $(\mathbb{T}_1, \circ, *)$ and $(\mathbb{T}_2, \bullet, \star)$ be two semigroups. Let $\mathbb{P} = \mathbb{T}_1 \times \mathbb{T}_2$ be endowed with the composition $(s_1, s_2) \otimes (t_1, t_2) = (s_1 \circ t_1, s_2 \bullet t_2)$ and the involution $(t_1, t_2)^\dagger = (t_1^*, t_2^\star)$. Then $(\mathbb{P}, \otimes, ^\dagger)$ is also a $*$-semigroup. Furthermore, there exists a topological semigroup isomorphism of $\mathbb{T}_1^* \times \mathbb{T}_2^*$ onto $\mathbb{P}^*$, which maps $\hat{\mathbb{T}}_1 \times \hat{\mathbb{T}}_2$ onto $\widehat{\mathbb{P}}$ and yields*

$$\mathbb{P}^* = \{\rho = \rho_1\rho_2 | \rho_1 \in \mathbb{T}_1^*, \rho_2 \in \mathbb{T}_2^*\}. \tag{11}$$

It is worth considering different involutions on some additive or multiplicative product structures of the complex field $\mathbb{C}$. We first list the corresponding continuous semicharacters obtained through Proposition 4 and then show some of their uses.

6.1.1. The additive complex field

Considering $(\mathbb{C}, +) = (\mathbb{R}, +) \times (\mathbb{R}, +)$ is equivalent to representing the complex numbers as $\mathbf{t} = t_1 + \mathrm{i}t_2$. Hence if the involution is

- inversion, $\mathbb{C}$ has its usual group structure, and

$$(\mathbb{C}, +, (\cdot)^{-1})^* = (\mathbb{C}, \widetilde{+, (\cdot)^{-1}}) = \{\rho_\lambda : \mathbb{C} \to \mathbb{C} | \rho_\lambda(\mathbf{t}) = \mathrm{e}^{\mathrm{i}(\lambda_1 t_1 + \lambda_2 t_2)}, \lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2\};$$

- conjugation, then $(\widetilde{\mathbb{C}, +, \bar{\cdot}}) = \{\rho_\lambda : \mathbb{C} \to \mathbb{C} | \rho_\lambda(\mathbf{t}) = e^{i\lambda t_2}, \lambda \in \mathbb{R}\}$, and

$$(\mathbb{C}, +, \bar{\cdot})^* = \{\rho_\lambda : \mathbb{C} \to \mathbb{C} | \rho_\lambda(\mathbf{t}) = e^{\lambda_1 t_1} e^{i\lambda_2 t_2}, \lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2\}; \qquad (12)$$

- identity, then $(\widetilde{\mathbb{C}, +, Id}) = \{\mathbf{1}_\mathbb{C}\}$ and

$$(\mathbb{C}, +, Id)^* = \{\rho_\lambda : \mathbb{C} \to \mathbb{C} | \rho_\lambda(\mathbf{t}) = e^{\lambda_1 t_1 + \lambda_2 t_2}, \lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2\}.$$

6.1.2. The multiplicative complex field

In the same way, considering $(\mathbb{C}, \times) = (\mathbb{R}_+, \times) \times (\Pi, +)$ is equivalent to representing the complex numbers as $\mathbf{t} = t_1 e^{it_2}$. Hence if the involution is

- inversion, $\mathbb{C}\backslash\{0\}$ has a group structure and

$$(\mathbb{C}\backslash\{0\}, \times, (\cdot)^{-1})^* = (\widetilde{\mathbb{C}\backslash\{0\}, \times, (\cdot)^{-1}}) = \{\rho_\lambda : \mathbb{C} \to \mathbb{C} | \rho_\lambda(\mathbf{t}) = t_1^{i\lambda_1} e^{i\lambda_2 t_2}, \lambda_1 \in \mathbb{R}, \lambda_2 \in \mathbb{Z}\};$$

- conjugation, then $(\widetilde{\mathbb{C}\backslash\{0\}, \times, \bar{\cdot}}) = \{\rho_\lambda : \mathbb{C} \to \mathbb{C} | \rho_\lambda(\mathbf{t}) = e^{i\lambda t_2}, \lambda \in \mathbb{Z}\}$, and

$$(\mathbb{C}\backslash\{0\}, \times, \bar{\cdot})^* = \{\rho_\lambda : \mathbb{C}\backslash\{0\} \to \mathbb{C} | \rho(\mathbf{t}) = t_1^{\lambda_1} e^{i\lambda_2 t_2}, \lambda_1 \in \mathbb{R}, \lambda_2 \in \mathbb{Z}\};$$

- identity, we can consider the whole field $\mathbb{C}$. But some of its subsets, for example the unit disk $\mathbb{U}$ for which $0 \leqslant t_1 \leqslant 1$, or the superior half-plane $\mathbb{H}$ for which $0 \leqslant t_2 < \pi$, are interesting too. We compute $(\mathbb{T}, \times, Id)^* = \{\rho_\lambda : \mathbb{T} \to \mathbb{C} | \rho_\lambda(t) = t_1^{\lambda_1} e^{\lambda_2 t_2}, \lambda_1, \lambda_2 \in \mathbb{R}\}$ and $(\widetilde{\mathbb{T}, \times, Id}) = \{\mathbf{1}_\mathbb{T}\}$ if $\mathbb{T}$ is $\mathbb{C}$, $\mathbb{U}$ or $\mathbb{H}$. But $\widehat{\mathbb{C}} = \{\mathbf{1}_\mathbb{C}\}$, $\widehat{\mathbb{U}} = \{\rho_\lambda : \mathbb{U} \to \mathbb{C} | \rho_\lambda(t) = t_1^\lambda,\ \lambda \in \mathbb{R}\}$ and $\widehat{\mathbb{H}} = \{\rho_\lambda : \mathbb{H} \to \mathbb{C} | \rho_\lambda(t) = t_1^{\lambda_1} e^{\lambda_2 t_2},\ \lambda_1 \in \mathbb{R}, \lambda_2 \in \mathbb{Z}\}$.

6.2. Spectral representations

Proposition 4 also provides a sufficient condition for the existence of a spectral representation of random fields indexed by product spaces. Hence a p.d. kernel r on $(\mathbb{P}, \otimes, \dagger)$ that satisfies $r(\mathbf{s}, \mathbf{t}) = R(\mathbf{t} \otimes \mathbf{s}^\dagger)$, for some exponentially bounded R, has the spectral representation

$$r(\mathbf{s}, \mathbf{t}) = \int_{\mathbb{P}^*} \rho(\mathbf{t}) \bar{\rho}(\mathbf{s}) d\mu(\rho), \qquad \mathbf{s}, \mathbf{t} \in \mathbb{P}, \qquad (13)$$

for ρ as in (11).

On the other hand, obviously not all the semicharacters are always necessary for such a representation. The characterization of classes of p.d. functions in the semigroup sense via the support of their spectral measures is not yet as clear as in classical group theory. This problem would deserve more investigation, even if in the literature most of the existing spectral representations seem merely to have been obtained either by restricting a product semigroup index set to one of its subsets or by extending the index set to some product structure.

6.2.1. Induced spectral representations

Let $(X_t)_{t\in\mathbb{T}}$ be a stochastic process whose covariance kernel r satisfies $r(s, t) = R(t \circ s^*, t \bullet s^\star)$, where R is p.d. for the product semigroup structure $(\mathbb{P}, \otimes, \dagger) = (\mathbb{T}, \circ, *) \times (\mathbb{T}, \bullet, \star)$. Since $r(s, t) = R((t, t) \otimes (s, s)^\dagger)$, the process (X_s) can be seen as the restriction on the diagonal set $\mathbb{D} = \{(t, t), t \in \mathbb{T}\}$ of a process $(\mathbf{X_t})$ defined on $\mathbb{P}$.

If R is exponentially bounded, r clearly inherits from (13) the spectral representation

$$r(s, t) = \int_{(\mathbb{T}\times\mathbb{T})^*} \rho(t, t)\bar{\rho}(s, s)\mathrm{d}\mu(\rho), \qquad s, t \in \mathbb{T},$$

and a spectral representation for the process follows.

Example 12. A Brownian bridge is the restriction to the diagonal set of a zero-mean $([0, 1], \wedge, Id) \times ([0, 1], \vee, Id)$-stationary Gaussian process $(\mathbf{X_t})$ whose covariance is

$$\mathrm{E}[\mathbf{X_s X_t}] = R((t_1, t_2) \otimes (s_1, s_2)^\dagger) = (t_1 \wedge s_1) \times ((1 - t_2 \vee s_2)).$$

Note that R is known to be p.d. by the Schur product theorem.

Such inherited structures concern other subsets than the diagonal set $\mathbb{D}$. Let us consider some examples.

Many kinds of processes indexed by $\mathbb{R}$ can be seen as restrictions of a second-order $(\mathbb{C}, +, \bar{\cdot})$-stationary stochastic process $\mathbf{X} = (X_\mathbf{t})_{\mathbf{t}\in\mathbb{C}}$. For instance, if $r_\mathbf{X}(\mathbf{s}, \mathbf{t}) = R(\mathbf{t} + \bar{\mathbf{s}})$ for an exponentially bounded R, the spectral representation (13) of r is obtained through the characterization of semicharacters given in (12), in the form

$$r(\mathbf{s}, \mathbf{t}) = \int_{\mathbb{R}^2} \mathrm{e}^{\lambda_1(s_1+t_1)}\mathrm{e}^{\mathrm{i}\lambda_2(t_2-s_2)}\mathrm{d}\mu(\lambda_1, \lambda_2), \qquad \mathbf{s}, \mathbf{t} \in \mathbb{C}.$$

So a symmetric process can be seen as the restriction of such an $\mathbf{X}$ to the positive real line $\mathbb{R}_+$ and a weakly stationary one as its restriction to the imaginary line $\mathrm{i}\mathbb{R}$.

The restriction to the diagonal subset $(1 + i)\mathbb{R}$ of such an $\mathbf{X}$ yields a process X indexed by $\mathbb{R}$ with covariance kernel such that $r(s, t) = R(t + s, t - s)$. This is not a semigroup stationary structure, but r and X inherit from (13) the spectral representations

$$r(s, t) = \int_{\mathbb{C}} \mathrm{e}^{\lambda t}\mathrm{e}^{\bar{\lambda} s}\mathrm{d}\mu(\lambda), \qquad s, t \in \mathbb{R} \tag{14}$$

and

$$X_t = \int_{\mathbb{C}} \mathrm{e}^{\lambda t}\mathrm{d}Z(\lambda), \qquad t \in \mathbb{R}. \tag{15}$$

Processes indexed by $\mathbb{Z}$ and having such a representation were studied by Cramér (1961) through links with shift operators.

When the spectral measure μ in (14) is supported within the diagonal subset $\mathbb{D} = (1 + \mathrm{i})\mathbb{R}$, the process is an oscillatory process (see Priestley 1988).

The covariance kernels of the form $r(s, t) = r_1(s + t)r_2(t - s)$, where r_1 is a non-negative function and r_2 is a weakly stationary p.d. function, are a subclass of the ones above. They were called locally stationary by Silverman (1957), and exponentially convex locally stationary if r_1 is the Laplace transform of some non-negative function. If R is exponentially bounded, (14) holds and precisely characterizes the so-called normal covariances. Michàlek (1988) proved that a normal covariance is exponentially convex locally stationary if and only if its spectral measure is a product measure.

Example 13. The sinusoidal signal $X_t = \mathrm{e}^{At}\cos(\nu t + \phi)$, for $t \in \mathbb{R}$, where $\phi \sim \mathcal{U}(0, 2\pi)$, ν and A are independent random variables, is an exponentially convex locally stationary process. It is the restriction to the diagonal subset $\mathbb{D}$ of the process $\mathbf{X_t} = \mathrm{e}^{At_1}\cos(\nu t_2 + \varphi)$ for $\mathbf{t} = (t_1, t_2) \in \mathbb{C}$. We have $\mathrm{E}[\mathbf{X_t}] = 0$ and

$$\mathrm{E}[\mathbf{X_s X_t}] = \mathrm{E}[\mathrm{e}^{A(t_1+s_1)}]\mathrm{E}[\cos(\nu(t_2 - s_2))] = R(\mathbf{t} + \bar{\mathbf{s}}).$$

Example 14. If $\mathbf{X}$ is an H-self-similar stochastic process indexed by $\mathbb{R}_+$, then $\mathbf{Y}$ defined by $Y_u = X_{\mathrm{e}^u}$, for $u \in \mathbb{R}$, is normal. Indeed, the process $\mathbf{Z}$ defined by $Z_t = \mathrm{e}^{tH} X_{\mathrm{e}^t}$ is weakly stationary (see Samorodnitsky and Taqqu 1994, Proposition 7.1.4), and $r_{\mathbf{Y}}(u, v) = \mathrm{e}^{(u+v)H} r_{\mathbf{Z}}(u, v)$. Representations (14) and (15) both hold for $\mathbf{Y}$, and Proposition 3 thus provides spectral representations to H-self-similar processes via the logarithm transform, $\mathbb{T} = \mathbb{R}_+$ and $\mathbb{S} = \mathbb{R}$; specifically,

$$r_{\mathbf{X}}(s, t) = \int_{\mathbb{C}} t^{\lambda} s^{\bar{\lambda}} \mathrm{d}\mu(\lambda) \quad \text{and} \quad X_t = \int_{\mathbb{C}} t^{\lambda} \mathrm{d}Z(\lambda), \qquad s, t \in \mathbb{R}.$$

6.2.2. Extending spectral representations

The problem of extending a function p.d. in the group sense to a larger group is well known (see, for example, Rudin 1963). To make it possible, additional assumptions are generally needed. This also seems to be the case for functions p.d. in the semigroup sense.

For example, we readily obtain the main results of Michàlek (1988) on the extension to $\mathbb{C}$ and spectral representation of a symmetric stochastic process X defined on $\mathbb{R}$ with covariance $r(s, t) = R(s + t)$ such that R is continuous. By Theorem 4.2, R is holomorphic, hence

$$R(t) = \sum_{n \geqslant 0} \frac{t^n}{n!} R^{(n)}(0), \qquad t \in \mathbb{R},$$

and then by Loève (1978, Section 37.2), we obtain

$$X_t = \sum_{n \geqslant 0} \frac{t^n}{n!} X_0^{(n)}, \qquad t \in \mathbb{R}. \tag{16}$$

This is not (and cannot be) the spectral representation of X since the $X^{(n)}(0)$ are not orthogonal. But its extension to $\mathbb{C}$, namely

$$\mathbf{X_t} = \lim_{L^2} \sum_{n=0}^{N} \frac{\mathbf{t}^n}{n!} X_0^{(n)}, \qquad \mathbf{t} \in \mathbb{C}, \tag{17}$$

exists, is $\mathbb{C}$-valued and is $(\mathbb{C}, +, \bar{\cdot})$-stationary, with

$$r_{\mathbf{X}}(\mathbf{s}, \mathbf{t}) = R_{\mathbf{X}}(\mathbf{t} + \bar{\mathbf{s}}) = \int_{\mathbb{R}} e^{\lambda(\mathbf{t}+\bar{\mathbf{s}})} d\mu(\lambda), \qquad \mathbf{s}, \mathbf{t} \in \mathbb{C}.$$

Actually, the decomposition (16) of a given analytic stochastic process X indexed by $\mathbb{R}$ is orthogonal if and only if its covariance kernel can be written $r(s, t) = R(st)$ (see Loève 1978, Theorem of Section 37.5.A). This merely means that X is M-symmetric. In this case, its extension to $\mathbb{C}$ defined by (17) is $(\mathbb{C}, \times, \bar{\cdot})$-stationary with covariance

$$r(\mathbf{s}, \mathbf{t}) = R(\mathbf{t}\bar{\mathbf{s}}) = \int_{\mathbb{N}} (\mathbf{t}\bar{\mathbf{s}})^n d\mu(n),$$

where the support of the spectral measure μ is $\mathbb{N}$, with

$$\mu(\{n\}) = \frac{R^{(n)}(0)}{n!} = \mathrm{E}\left[\left(\frac{X^{(n)}}{n!}\right)^2\right], \qquad n \in \mathbb{N}.$$

In these two examples, the holomorphy implied by the symmetry and continuity assumptions is crucial for extension purposes. Observation of the process on the real line thus amounts to knowledge of it on the whole complex plane. This clearly relates back to the problem of characterizing classes of p.d. functions via the supports of their spectral measures.

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