

Chapter 13

SQUARE ROOTS, PYTHAGORAS, AND SIMILAR TRIANGLES



Juan Gris *Jar, Bottle, and Glass*, 1911

The diagonal of an oblong produces by itself both the areas which the two sides of the oblong produce separately. — Baudhayana, Sulbasutram, Sutra 48 [AT: Baudhayana]

This chapter is devoted to a subject that is not only of great theoretical interest but also of recreational interest, for dissection theory plays an important role in a rigorous development of area and volume and also furnishes a seemingly endless variety of attractive and challenging puzzles.

In the last chapter, we showed that two polygons on a sphere with the same area are equivalent by dissection because both are equivalent to the same biangle. In this chapter, we will prove an analogous result on the plane by showing that every planar polygon is equivalent by dissection to a square. This result is the basis for many popular dissection puzzles. Before reading this chapter, you should go through the Introduction to Dissection Theory and Problem **12.1** at the start of Chapter 12.

In the process of exploring this dissection theory, we will follow a path through a corner of the forest of mathematics — a path that has delighted and surprised the authors many times. We will bring with us the question, What are square roots? Along the way we will confront relationships between geometry and algebra of real numbers, in addition to similar triangles, the Pythagorean Theorem (the quote above is a statement of this theorem written before Pythagoras), and possibly the oldest written proof in geometry (at least 2600 years old). This path will lead to the solutions of quadratic and cubic equations in Chapter 19. We will let David's personal experience start us on this path.

SQUARE ROOTS

When David was in eighth grade, he asked his teacher, “What is a square root?” David knew that the square root of N was a number whose square was equal to N , but where would one find it? (Hidden in that question is “How does one know it will always exist?”) He knew what the square roots of 4 and 9 were — no problem there. He even knew that the square root of 2 was the length of the diagonal of a unit square, but what of the square root of 2.5 or of π ?

At first, the teacher showed David a square root table (a table of numerical square roots), but he soon discovered that if one took the number listed in the table as the square root of 2 and squared it, you got 1.999396, not 2. (Modern-day calculators give rise to the same problem but hide it reasonably effectively by calculating more digits than they display and then rounding off.) So, David persisted in asking his question, What is the square root? Then the teacher gave him THE ANSWER — the Square Root Algorithm. Do you remember the Square Root Algorithm — that procedure, similar to long division, by which it is possible to calculate the square root? Or perhaps more recently you were taught the “divide and average” method, a more efficient method that was known to Archytas of Tarentum (428–350 B.C., Greek) but today is often called *Newton’s Method* after Isaac Newton (English, 1643–1727).

If A_1 is an approximation of the square root of N , then the average of A_1 and N/A_1 is an even better approximation, which we could call A_2 . And then the next approximation A_3 is the average of A_2 and N/A_2 . In equation form this becomes

$$A_{n+1} = (1/2)(A_n + (N/A_n)).$$

For example, if $A_1 = 1.5$ is an approximation of the square root of 2, then

$$A_2 = 1.417\cdots, \quad A_3 = 1.414216\cdots$$

and so forth are better and better approximations.

But wait! Most of the time these algorithms do not calculate the square root — they only calculate approximations to the square root. The algorithms have an advantage over the tables because we could, at least in theory, calculate approximations as close as we wished. However, they are still only approximations and David’s question still remained — What is the square root these algorithms approximate?

David’s eighth-grade teacher then gave up, but later in college David found out that some modern mathematicians answer his question in the following way: “We make an assumption (the Completeness Axiom) that implies that the sequence of approximations from the Square Root Algorithm must converge to some real number.” And, when he continued to ask his question, he found that in modern mathematics the square root is a certain equivalence class of Cauchy (Augustin-Louis Cauchy, 1789–1857, French) sequences of rational numbers, or a certain Dedekind (Julius Dedekind, 1831–1916,

German) cut. Finally, David let go of the question and forgot it in the turmoil of graduate school, writing his thesis, and beginning his mathematical career.

Later, David started teaching the geometry course that is the basis for this book. One of the problems in the course was the following Problem 13.1. [For this chapter you will need, at times, to assume the Archimedean Postulate, **AP** — when you do, note it.]

PROBLEM 13.1 A RECTANGLE DISSECTS INTO A SQUARE

Show that, on the plane, every rectangle is equivalent by dissection to a square.

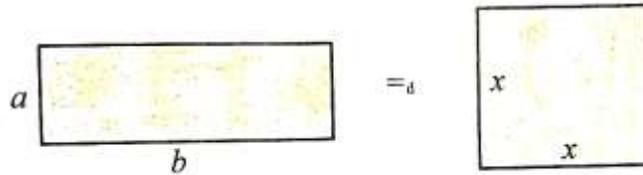


Figure 13.1 Dissecting a rectangle into a square

SUGGESTIONS

Problems **13.1** and **13.2** can be done in any order. So, if you get stuck on one problem, you can still go on to the other. In Problems **13.1** and **13.2**, it is especially important to make accurate models and constructions — rough drawings will not show the necessary length and angle relationships.

Problem **13.1** is one of the oldest problems in geometry, so you may have guessed (correctly!) that it is one of the more complex ones. This problem is interesting for more than just historical reasons. You are asked to prove that you can cut up any rectangle (into a finite number of pieces) and rearrange the pieces to form a square, as in Figure 13.1.

Because you are neither adding anything to the rectangle nor removing anything, the area must remain the same, so $ab = x^2$, or $x = \sqrt{ab}$. What you are really finding is a geometric interpretation of a square root. This was done already by Pythagoreans (6th century B.C.), who called x the *geometric mean* of a and b .

Let us look at a proof that is similar to the proofs in many standard geometry textbooks:

Let $s = \sqrt{ab}$ be the side of the square equivalent by dissection to the rectangle with sides a and b . Place the square, $AEFH$, on the rectangle, $ABCD$, as shown in Figure 13.2. Draw ED to intersect BC in R and HF in K . Let BC intersect HF in G . [In the case that $ABCD$ is so long and skinny that K ends up between G and F , we can, by cutting $ABCD$ in half and stacking the halves, reduce the proof to the case above. Note that **AP** is implicitly being used here.] From the similar triangles $\triangle KDH$ and $\triangle EDA$ we have $HK/AE = HD/AD$, or

$$HK = (AE)(HD)/AD = s(a - s)/a = s - s^2/a = s - b.$$

Therefore, we have $\triangle EFK \cong \triangle RCD$, $\triangle EBR \cong \triangle KHD$.

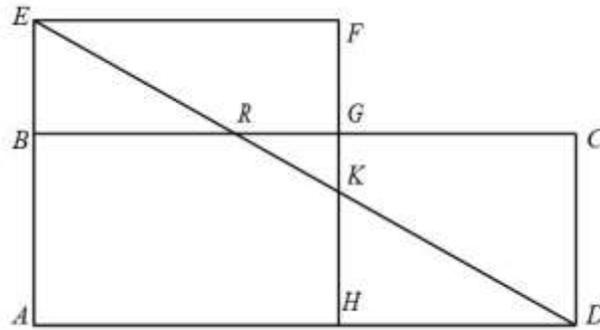


Figure 13.2 Textbook proof

David was satisfied with this proof until, in the second year of teaching the geometry course, he started sensing student uneasiness with it. As he listened to their comments, he noticed questions being asked by the students: “What is \sqrt{ab} ?” How do you find it?” Those used to be his questions!

The students and David also noticed that the facts used about similar triangles in the proof above are usually proved using the theory of areas of triangles. Thus, the proof could not be used as part of a concrete theory of areas of polygons, which was our purpose in studying dissection theory in the first place. Notice that the above proof also assumes the existence of the square, which in analysis is based on what is called the *Completeness Axiom*. The conclusion here seems to be that it would be desirable instead to construct the square root x . That started David on an exploration that continued on and off for many years.

Now let us solve a few problems in dissection theory.

Here are three methods for constructing x . For all three constructions we will use a rectangle like the one shown in Figure 13.1, with the longer side b as the base and the shorter side a as the height.

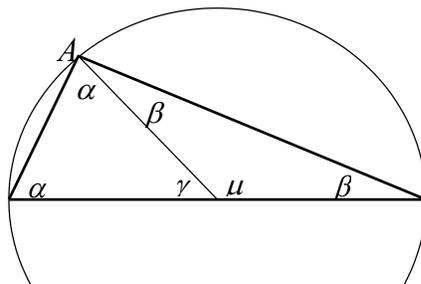


Figure 13.3 The angle at A is a right angle

For both the first and second constructions, you need to know that a triangle with all vertices on the circumference of a circle and one side a diameter of the circle is a right triangle. See Figure 13.3. That the angle not on the diameter is a right angle follows from a more general result, which we will prove in Problem **15.1a**, but the special case we need is easy to see now. Draw the line segment from the vertex to the center of the circle and note that two isosceles triangles are formed. By ITT, Problem **6.2**, there are congruent

angles as marked; and, because the sum of the angles in a planar triangle is 180° (Problems 7.3b or 10.2) we can compute that

$$(2\alpha + \gamma) + (2\beta + \mu) = 360^\circ$$

and, because $\gamma + \mu = 180^\circ$, it follows immediately that $\alpha + \beta = 90^\circ$. See Problem 15.1 for a more general result.

For the first construction, Figure 13.4, take the rectangle and lay a out to the left of b . Use this base line as the diameter of a circle. The length x that you are looking for is the perpendicular line from the left side of the rectangle to where it intersects the circle.

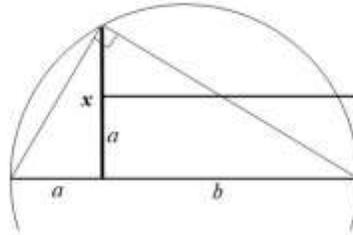


Figure 13.4 First construction of x

The second construction, Figure 13.5, is similar, but this time put a on the inside of the base of the rectangle. Now the side x you are looking for is the segment from the lower left corner of the rectangle to the point of intersection on the circle with the perpendicular rising from the point a in from the corner as indicated in the figure.

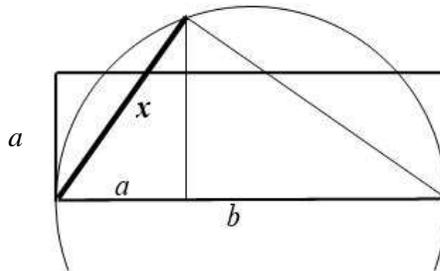


Figure 13.5 Second construction of x

The third construction, Figure 13.6, is a bit more algebraic than the others, and doesn't directly involve a circle.

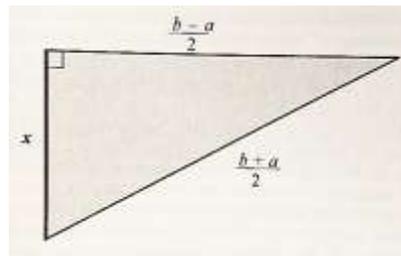


Figure 13.6 Third construction of x

This construction can be used together with the result of Problem 13.2 in order to obtain a proof by subtraction.

For all of these constructions, it is imperative that you use accurate models. Whatever method you choose, make the rectangle and the square overlap as much as possible, and see how to fit the other pieces in. Then you have to prove that all of the sides and angles line up properly. Note that it is much better to solve this problem geometrically rather than by only trying to work out the algebra — actually do the construction and proceed from there. Finally, you don't have to use one of the constructions shown here. If these don't make sense to you, then find one of your own that does.

And remember not to use results about similar triangles because we normally need results about areas in order to prove these results as we will do in Problem 13.4.



Pause, explore, and write out your ideas before reading further.

BAUDHAYANA'S SULBASUTRAM

While reading an article, David ran across an item that said that the problem of changing a rectangle into a square appeared in the *Sulbasutram*. “Sulbasutram” means “rules of the cord,” and the several Sanskrit texts collectively called the *Sulbasutra* were written by the Vedic Hindus starting before 600 B.C. and are thought to be compilations of oral wisdom that may go back to 2000 B.C. (See, for example, A. Seidenberg *The Ritual Origin of Geometry* [HI: Seidenberg].) These texts have prescriptions for building fire altars, or *Agni*. However, contained in the *Sulbasutra* are sections that constitute a geometry textbook detailing the geometry necessary for designing and constructing the altars. As far as we have been able to determine, these are the oldest geometry (or even mathematics) textbooks in existence. There are at least four versions of the *Sulbasutram* by Baudhayana, Apastamba, Katyayana, and Manava. The geometric descriptions are very similar in these four books, and we will only use Baudhayana's version here (see [AT: Baudhayana]).

The first chapter of Baudhayana's *Sulbasutram* contains geometric statements called “Sutra.” Sutra 54 is what we asked you to prove in Problem 13.1. It states the following:

If you wish to turn an oblong [here “oblong” means “rectangle”] into a square, take the shorter side of the oblong for the side of square. Divide the remainder into two parts and inverting join those two parts to two sides of the square. Fill the empty place by adding a piece. It has been taught how to deduct it. [See Figure 13.7.]

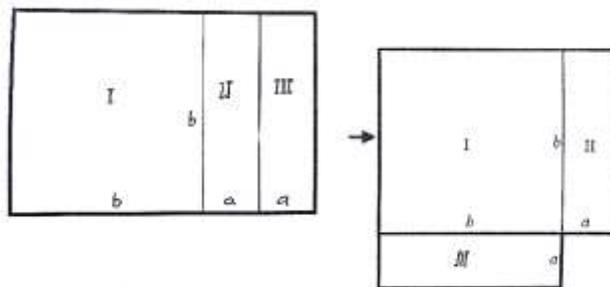


Figure 13.7 Sutra 54

So, our rectangle has been changed into a figure with an “empty place,” which can be filled “by adding a piece” (a small square). The result is a large square from which a small square has to be “deducted.”

Now Sutra 51 (Refer to Figure 13.8.):

If you wish to deduct one square from another square, cut off a piece from the larger square by making a mark on the ground with the side of the smaller square which you wish to deduct; draw one of the sides across the oblong so that it touches the other side; by this line which has been cut off the small square is deducted from the large one.

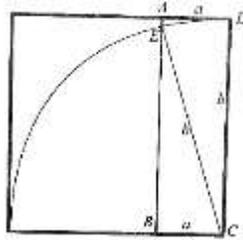


Figure 13.8 Sutra 51: Construction of side of square

We wish to deduct the small square (a^2) from the large square. Sutra 51 tells us to “scratch up” with the side of the smaller square — this produces the line AB and the oblong $ABCD$. Now, if we “draw” the side CD of the large square to produce an arc, then this arc intersects the other side at the point E . The sutra then claims that BE is side of the desired square whose area equals the area of the large square minus the area of the small square. This last assertion follows from Sutra 50, which we ask you to prove in Problem 13.2.

See Figure 13.9 for the drawing that goes with Sutra 50:

If you wish to combine two squares of different size into one, scratch up with the side of the smaller square a piece cut off from the larger one. The diagonal of this cutoff piece is the side of the combined squares.

Be sure you see why Sutra 50 is a statement of what we call the Pythagorean Theorem.

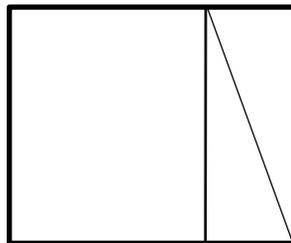


Figure 13.9 Sutra 50

A. Seidenberg, in an article entitled *The Ritual Origin of Geometry* [HI: Seidenberg], gives a detailed discussion of the significance of Baudhayana’s *Sulbasutram*. He argues that it was written before 600 B.C. (Pythagoras lived about 500 B.C. and Euclid about 300 B.C.). He gives evidence to support his claim that it contains codification of

knowledge going “far back of 1700 B.C.” and that knowledge of this kind was the common source of Indian, Egyptian, Babylonian, and Greek mathematics. Together Sutras 50, 51, and 54 describe a construction of a square with the same area as a given rectangle (oblong) and a proof (based on the Pythagorean Theorem) that this construction is correct. You can find stated in many books and articles that the ancient Hindus, in general, and the *Sulbasutram*, in particular, did not have proofs or demonstrations, or they are dismissed as being “rare.” However, there are several sutras in [AT: Baudhayana] similar to the ones discussed above. We suggest you decide for yourself to what extent they constitute proofs or demonstrations.

Baudhayana avoids the Completeness Axiom by giving an explicit construction of the side of the square. The construction can be summarized in Figure 13.10.

This is the same as Euclid’s construction in Proposition II-14 (see [AT: Euclid, *Elements*], page 409). But Euclid’s proof is much more complicated. Note that neither Baudhayana nor Euclid gives a proof of Problem 13.1 because the use of the Pythagorean Theorem obscures the dissection. However, they do give a concrete construction and a proof that the construction works.

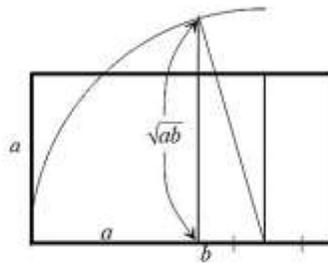


Figure 13.10 Baudhayana’s construction of square root

Both Baudhayana and Euclid prove the following theorem, which uses equivalence by subtraction.

THEOREM. *For every rectangle R there are squares S_1 and S_2 such that $R + S_2$ is equivalent by dissection to $S_1 + S_2$ and thus R and S_1 have the same area.*

Notice that both Baudhayana’s and Euclid’s proofs of this theorem and your proof of Problem 13.1 avoid assuming that the square root exists (and, thus, avoid the Completeness Axiom). They also avoid using any facts about similar triangles. These proofs explicitly construct the square and show in an elementary way that its area is the same as the area of the rectangle. There is no need for the area or the sides of the rectangle to be expressed in numbers. Also given a real number, b , the square root of b can be constructed by using a rectangle with sides b and 1. In Problem 13.3 we will use these techniques to prove basic properties about similar triangles.

So, finally, we have an answer to our question, What is a square root? Also, see David’s chapter, “Square Roots in the Sulba Sutra,” in the book [EG: Gorini], which

describes a geometric method based on Baudhayana that finds arbitrarily accurate numerical approximations to many square roots and does so in a way that is computationally more efficient than the “divide and average” method (*Archytas/Newton’s Method*) described at the beginning of this chapter.

PROBLEM 13.2 EQUIVALENCE OF SQUARES

Prove the following: On the plane, the union of two squares is equivalent by dissection to another square.

SUGGESTIONS

This is closely related to the Pythagorean Theorem. There are two general ways to approach this problem: You can use Problem 13.1 or you can prove it on its own, which will result in a proof of the Pythagorean Theorem — hence, you can’t use the Pythagorean Theorem to solve this problem because you will be proving it!

To see how this problem relates to the Pythagorean Theorem, think about the following statement of the Pythagorean Theorem:

The square on the hypotenuse is equal to the sum of the squares on the other two sides.

This is not just an algebraic equation — **the squares referred to are actual geometric squares.**

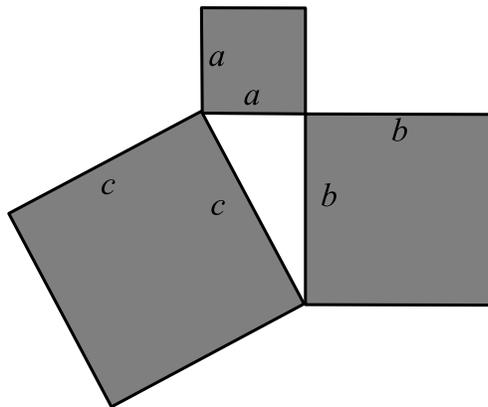


Figure 13.11 Pythagorean Theorem

As in the other dissection problems, make the three squares coincide as much as possible, and see how you might get the remaining pieces to overlap. You might start by reflecting the square with side c over the side c on the triangle. Then prove that the construction works as you say it does.

ANY POLYGON CAN BE DISSECTED INTO A SQUARE

If we put together Problems 12.1, 13.1, and 13.2, we get the surprising

THEOREM 13.2a *On the plane, every simple polygon is equivalent by dissection to a square.*

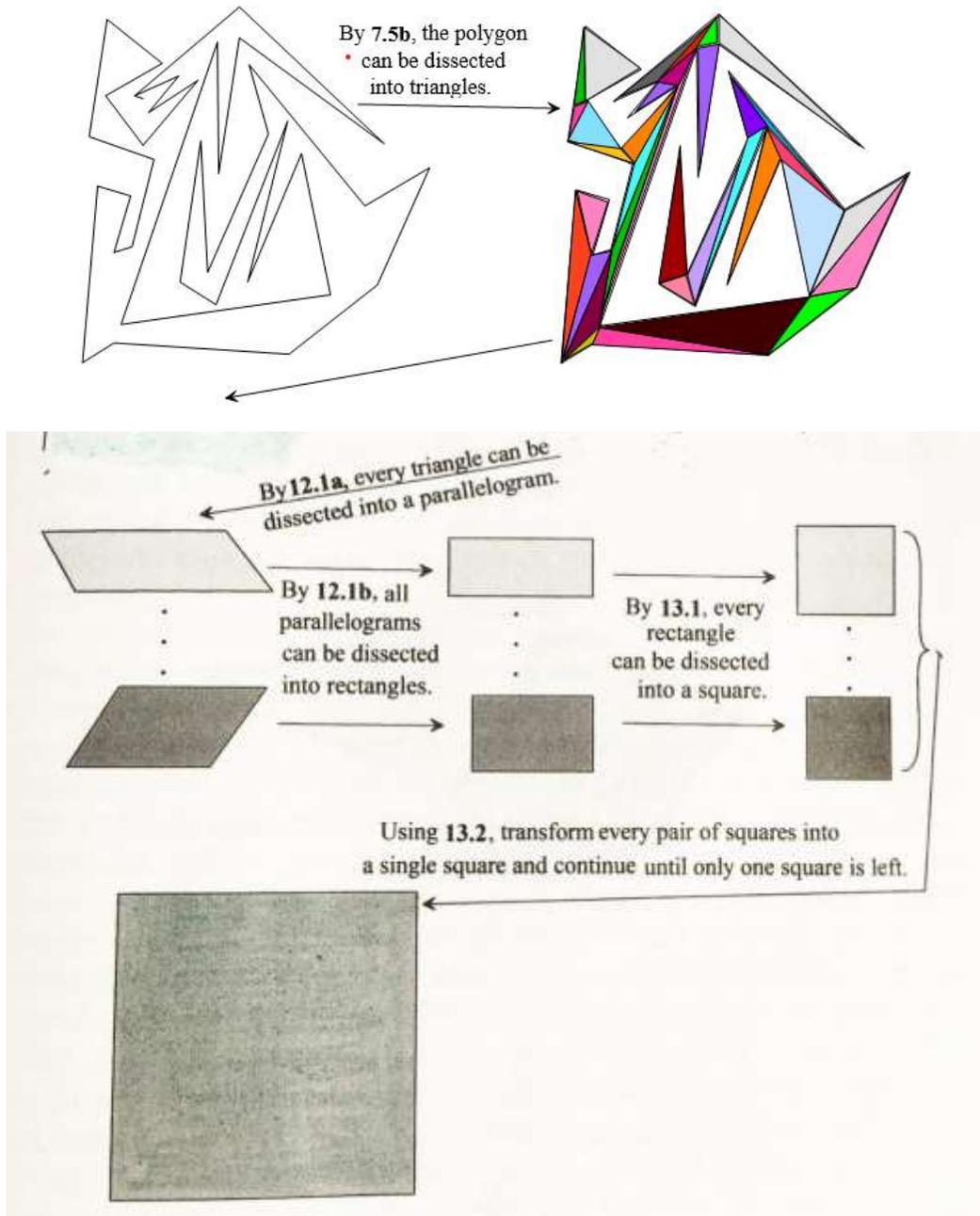


Figure 13.12 Every polygon dissects into a square

HISTORY OF DISSECTIONS

We have now seen dissection proofs in Chapter 12, Problems **13.1** and **13.2**, and we will see more in later chapters (for example, the dissection proof of the Law of Cosines in **20.2a**). In the section History of Dissections in the Theory of Area of Chapter 12 we outlined how dissection have been used to prove results about area and to develop a rigorous theory of area. In this chapter we have seen more examples of this.

Note that the result in Theorem 13.2a implies, on the plane, that

THEOREM 13.2b *Any two polygons with the same area are equivalent by dissection to each other.*

This theorem is true not only on the plane but also on spheres (Problem **12.4**) and hyperbolic planes (Theorem 12.4). On the plane the theorem is often called the *Bolyai-Gerwien-Wallace Theorem* (with sometimes Wallace's name left off). It was proved independently by Wallace in 1831, Bolyai in 1832, and Gerwien in 1833.

Note that Theorem 13.2 does not tell us anything about how many pieces would be needed in any particular dissection. In fact, the number will, in general, be very large. This leads to the question of making dissections with the least number of pieces, or with the most symmetry, or, we can say, with the most elegance. A large number of polygonal dissection puzzles and recreations arises from these questions. There is a surprisingly rich history associated with recreational dissections. See, for example [DI: Frederickson 1997] and [DI: Frederickson 2002]. Refer to these books for further history and about 1000 dissection pictures.

As an example, the dissection in Figure 13.13 is a famous dissection of an equilateral triangle into a square that requires only four pieces. The discovery of this dissection is usually credited to Henry Dudeney (English, 1857–1930) in 1902, but [DI: Frederickson 1997], p. 136, raises some doubts about this attribution.

In this dissection, the segments AD , DB , BE , EC , FG are all equal to half the side of the triangle; EF is equal to the side of the equivalent square; DJ and GK are each perpendicular to EF . If the four pieces are successively hinged to one another at the points, D , E , G , then holding piece I fixed and swinging the connected set of pieces IV-III-II clockwise, the equilateral triangle is neatly carried into the square.

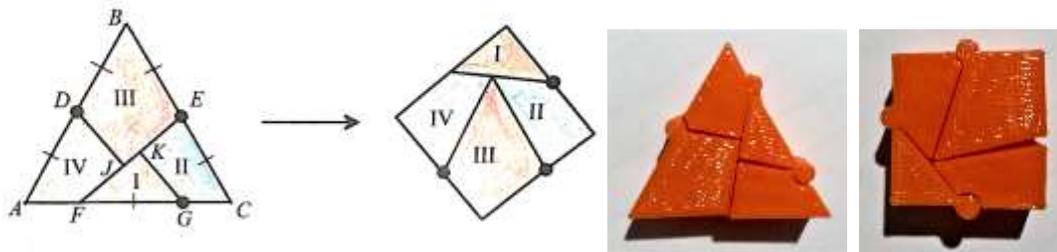


Figure 13.13 Four-piece dissection of triangle to square, 3D printed design by Laura Taalman (mathgrrl.com)

We could build a set of four connected tables based on this fact; swinging the tables in one direction causes the tops to fit together into a single equilateral triangular table and swinging them in the other direction causes the tops to fit together into a single square table. This is an example of a hinged dissection. If you go back to your dissections in Chapter 12 and **13.1** and **13.2**, you will probably find that some of them can also be hinged. The [DI: Frederickson 2002] book is entirely about hinged dissections and what he calls “twisting dissections.”

The 18th and 19th centuries produced a variety of puzzle books that contained dissection puzzles. Articles about dissections also appeared in scientific journals. In the early 19th century the Bolyai-Gerwien-Wallace Theorem was proved. Dissection puzzles had great popularity during 1880s and 1890s, and at the end of 19th century puzzle columns by Sam Loyd and Henry E. Dudeney began to appear in newspapers and magazines. The 20th century brought an increase in sophistication to dissections. Articles and problems appeared in periodicals such as *Mathematical Gazette*, *American Mathematical Monthly*, and *Scientific American*. Geometric dissection has become a frequent topic in books on mathematical recreations, but there are books that focus exclusively on dissections, [DI: Lindgren 1964 & 1972] and [DI: Boltyanski 1980].

PROBLEM 13.3 MORE DISSECTION-RELATED PROBLEMS

Some of these dissection problems are easy and some are hard. We do not tell you which; but, in any case, what is hard for us may be easy for you.

- a. *Show that the four-piece dissection of an equilateral triangle to a square depicted in Figure 13.13 actually works as claimed.*
- b. *Show that any obtuse triangle (one angle is greater than a right angle) can be dissected into acute triangles (all angles are less than a right angle). What is the dissection with the smallest number of pieces you can find?*
- c. *Prove that the figure formed when the midpoints of the sides of a quadrilateral are joined in order is a parallelogram, and its area is half that of the quadrangle. Remember that quadrilaterals need not be convex; see Figure 13.14.*

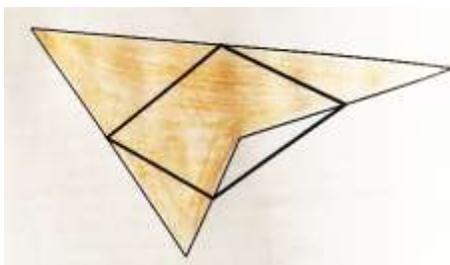


Figure 13.14 The Varignon parallelogram of a quadrilateral

This parallelogram is often referred to as the *Varignon parallelogram* of the quadrilateral, because the result first appeared in a 1731 posthumous publication by Pierre Varignon (1654–1722, French).

- d. *Show that if one diagonal divides a quadrangle into two triangles of equal area, it bisects the other diagonal. Conversely, if one diagonal bisects the other, it bisects the area of the quadrangle.*
- e. *Divide a triangle by two straight lines into three parts, which when properly arranged shall form a parallelogram whose angles are of given magnitudes.*
- f. *From the arbitrary point within a triangle, draw three lines to the sides of the triangle that shall trisect its area.*
- g. *Dissect a regular hexagon by straight cuts into six pieces that can be reassembled to form an equilateral triangle. Does there exist a five-piece solution?*
- h. *Given $6 \geq n$, show that it is possible to fit together n isosceles right triangles, all of different sizes, so as to make a single isosceles right triangle. Can the problem be solved for $n = 5$? Can a right isosceles triangle be dissected into a finite number of right isosceles triangles, no two having a common side?*
- i. *Show that any given triangle can be dissected by straight cuts into four pieces that can be arranged to form two triangles similar to the given triangle.*
- j. *Dissect a regular pentagon by straight cuts into six pieces that can be put together to form an equilateral triangle.*
- k. *Draw a straight line that will bisect both the area and the perimeter of a given quadrilateral.*

THREE-DIMENSIONAL DISSECTIONS AND HILBERT'S THIRD PROBLEM

In 1900, David Hilbert delivered a lecture before the International Congress of Mathematicians in which he listed ten problems “from the discussion of which an advancement of science may be expected.” In a later paper he expanded these to 23 problems, which are now called Hilbert’s Problems.

Hilbert’s Third Problem. *Is it possible to specify two tetrahedra of equal bases and equal altitudes that can in no way be split up into congruent tetrahedra and that cannot be combined with congruent tetrahedra to form two polyhedra that*

themselves could not be split up into congruent tetrahedra? See [DI: Sah], [DI: Boltyanski 1978], and [TX: Hartshorne] for a discussion of this problem and its history.

Shortly after Hilbert's lecture, Max Dehn (1878–1952, Germany, United States) found such tetrahedra and also proved that a regular tetrahedron is not equivalent by dissection to a cube. Thus, there is no possibility of dissecting polyhedra into cubes. To show these results, Dehn proved the following:

THEOREM 13.4 *If P and Q are two polyhedra in 3-space that are equivalent by dissection (or by subtraction), then the dihedral angles (see Chapter 21) of P are, mod π , a rational linear combination of the dihedral angles of Q . That is, if α_i are the dihedral angles of P and β_j are the dihedral angles of Q , then there are integers n_i, m_j, a, b such that*

$$\sum_i n_i \alpha_i + a\pi = \sum_j m_j \beta_j + b\pi.$$

This interesting fact is of far-reaching importance in the theory of volumes. It implies that although a theory of areas of polygons can be constructed without continuity considerations, such is not possible for a complete theory of volumes of polyhedra. Thus, the ordinary treatment of volumes, such as we find in Euclid's *Elements*, involves infinitesimals, which do not occur in the theory of polygonal areas. In addition, we showed in Problem 7.5b that though it is always possible to dissect a polygon into a finite number of triangles having their vertices only at the vertices of the polygon, there exist polyhedra, called *Lenne's polyhedra*, which cannot be dissected into finite numbers of tetrahedra having their vertices only at the vertices of the polyhedra. [See more about polyhedra in *Shaping Space: Exploring Polyhedra in Nature, Art, and Geometrical Imagination*, ed. Marjorie Senechal, Springer, 2013] About disproving triangulation conjecture read here: <https://www.quantamagazine.org/triangulation-conjecture-disproved-20150113/>

PROBLEM 13.4 SIMILAR TRIANGLES

Near the beginning of this chapter we gave a textbook proof of Problem 13.1, which used properties of similar triangles. Later you found a proof that did not need to use similar triangles. Remember that in our discussion of AAA we said that two triangles were *similar* if their corresponding angles are congruent. Now you are ready to give a dissection proof of the following:

- a. **AAA similarity criterion:** *If two triangles are similar, then the corresponding sides of the triangles are in the same proportion to one another.*

SUGGESTIONS

Look at your proof of Problem 13.1. It probably shows implicitly that Problem 13.4 holds for a pair of similar triangles in your construction. For more generality, let θ be one of the

angles of the triangles and place the two θ 's in VAT position in such a way that they form two parallelograms as in Figure 13.15.

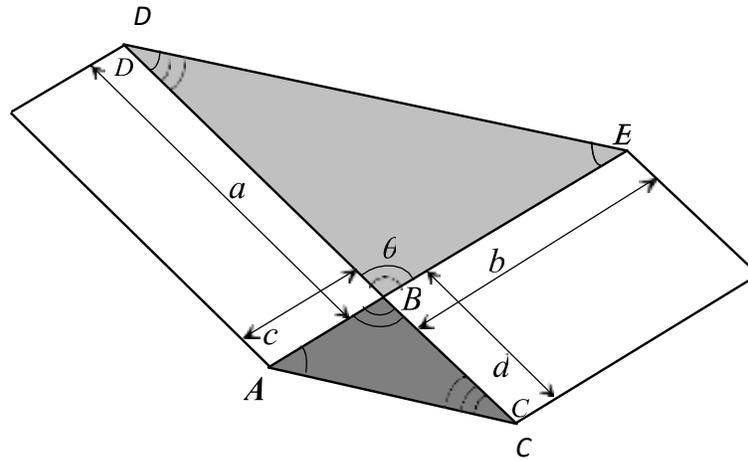


Figure 13.15 Similar triangles

Show that the two parallelograms are equivalent by dissection and use that result to show that $ac = bd$ or, in equivalent form, $a/d = b/c$. You may find it clearer if you start by looking at the special case of $\theta = \pi/2$.

b. SAS similarity criterion: *If two triangles on a plane have an angle in common and if the corresponding sides of the angle are in the same proportion to each other, then the triangles are similar.*

Draw the two triangles with their common angle coinciding. Then show that the opposite sides are parallel. You will probably have to use part **a**.

These results about similar triangles will be used in most of the remaining chapters of the book.



Max Bill *Half-Sphere around Two Axes*, marble, 1966 (The Museum of Modern Art, New York)