Chapter 8

Parallel Transport

Parallel straight lines are straight lines lying in a plane which do not meet if continued indefinitely in both directions. — Euclid, Elements, Definition 23

In this chapter we will further develop the notion of *parallel transport* that was introduced in Chapter 7. This chapter may be studied independently of Chapter 7 if you read the section in Chapter 7 entitled Introducing Parallel Transport. The basic idea of Chapter 8 is to collect all the results related to parallelism that can be examined without assuming any special properties on the plane about parallel lines or about the sum of the angles on the plane. These properties (postulates) will be discussed in detail in Chapter 10.

Problem 8.1 Euclid's Exterior Angle Theorem (EEAT)

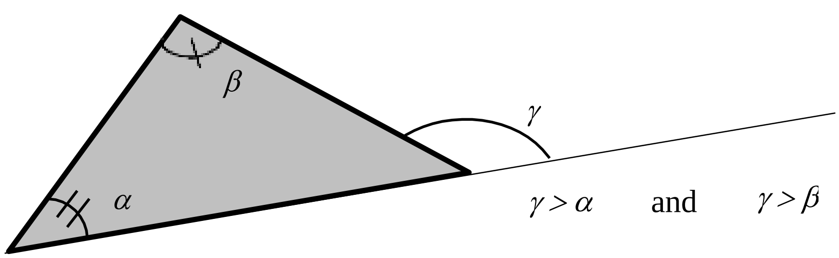

Figure 8.1 EEAT

a. *Any exterior angle of a triangle is greater than each of the opposite interior angles.*
Warning: Euclid's EAT is not the same as the Exterior Angle Theorem usually studied in high school.

b. *When is EEAT true on the plane, on a sphere, and on a hyperbolic plane?*

SUGGESTIONS

You may find the following hint (which is found in Euclid's proof) useful: Draw a line from the vertex of α to the midpoint, M, of the opposite side, BC. Extend that line beyond M to a point A' in such a way that $AM \cong MA'$. Join A' to C. This hint will be referred to as *Euclid's hint* and is pictured in Figure 8.2.

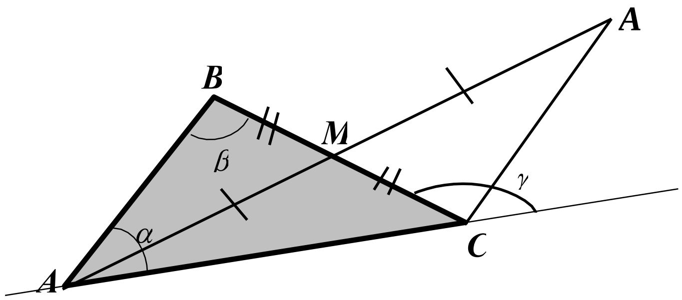

Figure 8.2 Euclid's hint

Be cautious transferring this hint to a sphere. It will probably help to draw Euclid's hint directly on a physical sphere.

It is not necessary to use Euclid's hint to prove EEAT, and in fact many people don't "see" the hint. Another perfectly good way to prove EEAT is to use Problem **8.2**. Problems **8.1** and **8.2** are very closely related, and they can be done in either order. It is also fine to use Problem **8.1** to prove Problem **8.2** or use Problem **8.2** to prove Problem **8.1**, but of course don't do both. As a final note, remember you do not have to look at figures using only one orientation — rotations and reflections of a figure do not change its properties, so if you have trouble "seeing" something, check to see if it's something you're familiar with by orienting it differently on the page.

EEAT is not always true on a sphere, even for small triangles. Look at a counterexample as depicted in Figure 8.3. Then look at your proof of EEAT on the plane. It is very likely that your proof uses properties of angles and triangles that are true for small triangles on the sphere. Thus, it may appear to you that your planar proof is also a valid proof of EEAT for small triangles on the sphere. But there is a counterexample.

This could be, potentially, a very creative situation for you — **whenever you have a proof and counterexample of the same result, you have an opportunity to learn something deep and meaningful**. Try out your planar proof of EEAT on the counterexample in Figure 8.3 and see what happens. Then try it on both large and small spherical triangles. If you can determine exactly which triangles satisfy EEAT and which triangles don't satisfy EEAT, then this information will be useful (but not crucial) to you in later problems.

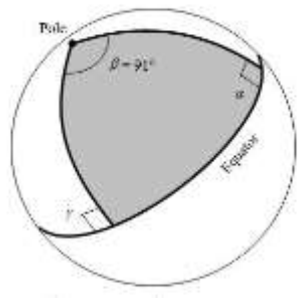

Figure 8.3 Counterexample to EEAT on a sphere

PROBLEM 8.2 SYMMETRIES OF PARALLEL TRANSPORTED LINES

Consider two lines, r and r', that are parallel transports of each other along a third line, l. Consider now the geometric figure that is formed by the three lines and look for the symmetries of that geometric figure. See Figure 8.4.
What can you say about the lines r and r'? Do they intersect? If so, where? Look at the plane, spheres, and hyperbolic planes.

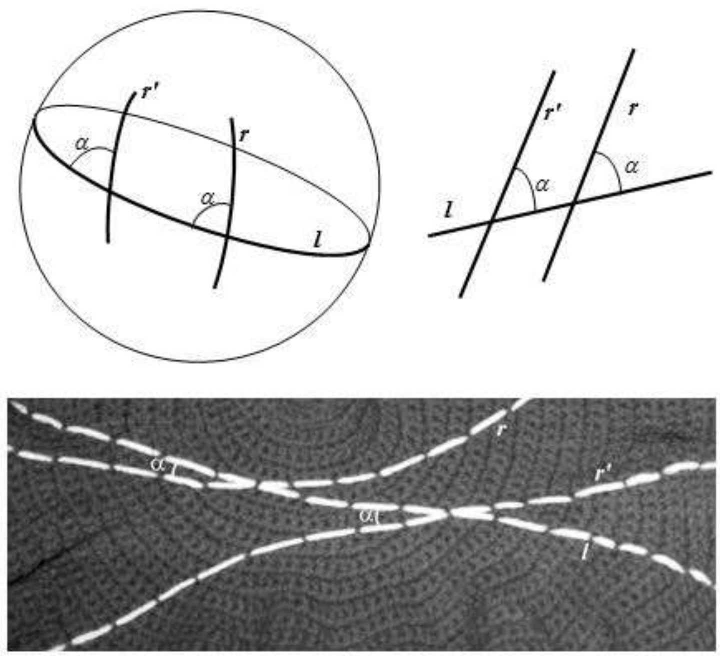

Figure 8.4 What can you say about ***r*** and ***r***'?

SUGGESTIONS

Parallel transport was already *informally* introduced in Chapter 7. In Problem **8.2** you have an opportunity to explore the concept further and prove its implications on the plane and a sphere. You will study the relationship between parallel transport and parallelism, as well.

It is common in high school to use Euclid's definition of parallel lines as "straight lines lying in a plane which do not meet if continued indefinitely in both directions." But this is an inhuman definition — there is no way to check all points on both lines to see if they ever meet. This definition is also irrelevant on a sphere because we know that all geodesics on a sphere *will* cross each other. But we *can* measure the angles of a transversal. This is why it is more useful to talk about lines as parallel transports of one another rather than as parallel. So, the question becomes

> *If a transversal cuts two lines at congruent angles, are the lines in fact parallel in the sense of not intersecting?*

There are many ways to approach this problem. First, be sure to look at the symmetries of the local portion of the figure formed by the three lines. See what you can say about global symmetries from what you find locally. For the question of parallelism, you can use EEAT, but not if you used this problem to prove EEAT previously. Also, don't underestimate the power of symmetry when considering this problem. Many ideas that work on the plane will also be useful on a sphere and a hyperbolic plane, so try your planar proof on a sphere and a hyperbolic plane before attempting something completely different.

In Chapter 1, we said that an isometry is a *symmetry* of a geometric figure if it transforms that figure into itself. That is, the figure looks the same before and after the isometry. Here, we are looking for the symmetries of the figure on the plane and sphere and hyperbolic plane.

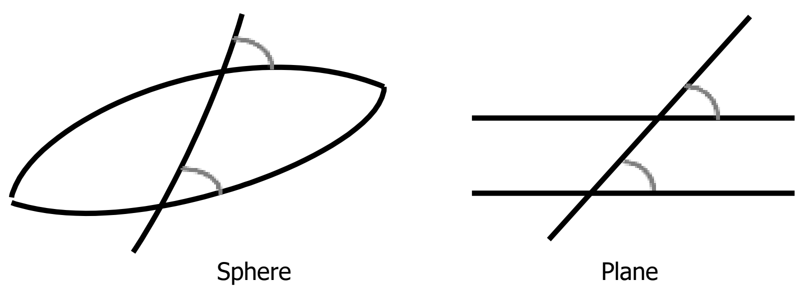

Figure 8.5 What are the symmetries of these figures?

From Figure 8.5, we can see that on a sphere we are looking for the symmetries of a *lune* cut at congruent angles by a geodesic. A ***lune*** is a spherical region bounded by two half great circles (see Problem **7.1**).

You may be inclined to use one or both of the following results that are true on the plane: Any transversal of a pair of parallel lines cuts these lines at congruent angles (Problem **10.1**). And, the angles of any triangle add up to a straight angle (Problems **7.3b** and **10.2**). The use of these results should be avoided for now, as they are both false on both a sphere and a hyperbolic plane. We have been investigating what is common between the plane, spheres, hyperbolic planes — trying to use common proofs whenever possible. You may be tempted to use other properties of parallel lines that seem familiar to you, but in each case ask yourself whether or not the property is true on a sphere and on a hyperbolic plane. If it is not true on these surfaces, then don't use it here because it is not needed.

PROBLEM 8.3 TRANSVERSALS THROUGH A MIDPOINT

a. *Prove: If two geodesics r and r' are parallel transports along another geodesic l, then they are also parallel transports along any transversal passing through the midpoint of the segment of l between r and r'. Does this hold for the plane, spheres, and hyperbolic planes?* See Figure 8.6.

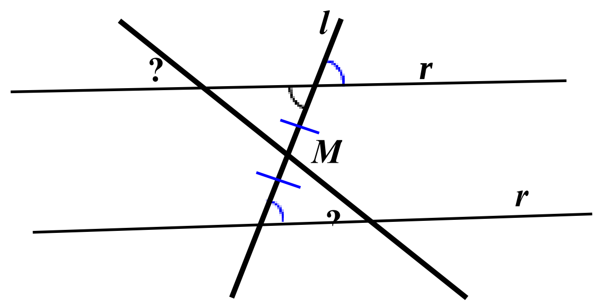

Figure 8.6 Transversals through a midpoint

b. *On a sphere or hyperbolic plane, are these the only lines that will cut r and r' at congruent angles? Why?*

c. *Prove: Two geodesics (on the plane, spheres, or hyperbolic planes) are parallel transports of each other if and only if they have a common perpendicular.*
Is there only one common perpendicular?

All parts of this problem continue the ideas presented in Problem **8.2**. In fact, you may have proven this problem while working on **8.2** without even knowing it. There are many ways to approach this problem. Using symmetry, is always a good way to start. You can also use some of the triangle congruence theorems that you have been working with in Chapter 6. Look at the things you have discovered about transversals from Problems **8.1** and **8.2**; they are very applicable here. For the hyperbolic plane you may want to use results from Chapters 5 and/or 7.

PROBLEM 8.4 WHAT IS "PARALLEL"?

Since Chapter 7, you have been dealing with issues of parallelism. Parallel transport gives you a way to check parallelism. Even though parallel transported lines intersect on the sphere, there is a *feeling of local "parallelness"* about them. In most applications of parallel lines, the issue is not whether the lines ever intersect, but whether a transversal intersects them at congruent angles at certain points; that is, whether the lines are *parallel transports of each other along the transversal*. You may choose to avoid definitions of "parallel" that do not give you a direct method of verification, such as these common definitions for parallel lines in the plane:

1. *Parallel lines are lines that never intersect;*

2. *Parallel lines are lines such that every transversal cuts them at congruent angles; or*

3. *Parallel lines are lines that are everywhere equidistant.*

a. *Check for each of these three definitions whether they apply to parallel transported lines on a sphere or on a hyperbolic plane.*

This is closely related to Problems **8.2** and **8.3**.

b. *Show that there are pairs of geodesics on a hyperbolic plane that do not intersect and yet there are* ***no*** *transversals that cut at congruent angles. That is, the geodesics are parallel (in the sense of not intersecting) but not parallel transports of each other.*

Use the results of **8.2** and **8.3** and look on your hyperbolic plane for the boundary between geodesics that intersect and geodesics that are parallel transports. We call a pair of geodesics that satisfy part **b** by the name **asymptotic geodesics**. Be warned that in many texts these geodesics are called simply parallel.

c. *Show that there are pairs of geodesics on any cone with cone angle greater than 360° that do not intersect and yet there are no transversals that cut at congruent angles. That is, the geodes- ics are parallel (in the sense of not intersecting) but not parallel transports of each other along any straight (in the sense of symmetry) line.*

Experiment with a paper cone with cone angle greater than 360°.

These examples should help you realize that parallelism is not just about non-intersecting lines and that the meaning of parallel is different on different surfaces. You will explore and discuss these various notions of parallelism and parallel transport further in Chapters 9–13. Because we have so many (often unconscious) connotations and assumptions attached to the word "parallel," we find it best to avoid using the term "parallel" as much as possible in this discussion. Instead we will use terms such as "parallel transport," "non-intersecting," and "equidistant," which make explicit the meaning that is intended.

In Chapter 9, we will continue our explorations of triangle congruence theorems, some of which involve parallel transport.

In Chapter 10, we will consider various parallel postulates and explore how they apply on the plane, spheres, and hyperbolic planes. We will assume the parallel postulates on the plane and use them to prove the properties of non-intersecting and parallel transported lines on the plane. In the process, you may learn something about the history and philosophy of parallel lines and the postulates that have been used in attempts to understand parallelism.

In Chapter 11, our understanding of different notions of "parallel" will help us to explore isometries and patterns.

In Chapter 12, we will study parallelograms and rectangles (and their analogues on spheres and hyperbolic planes) and in the process show that, on the plane, non-intersecting lines are equidistant.

In Chapter 13, we will use the results from Chapter 12 to explore results that are only true on the plane, such as the Pythagorean Theorem and results about similar triangles.