Chapter 5

STRAIGHTNESS ON HYPERBOLIC PLANES

[To son János:] For God's sake, please give it [work on hyperbolic geometry] up. Fear it no less than the sensual passion, because it, too, may take up all your time and deprive you of your health, peace of mind and happiness in life. — Wolfgang Bolyai (1775–1856) [**EM**: Davis and Hersh], page 220

We will now study some hyperbolic geometry. As with the cone and cylinder, we must use an intrinsic point of view on hyperbolic planes. This is especially true because, as we will see, there is no standard embedding of a complete hyperbolic plane into 3-space.

A SHORT HISTORY OF HYPERBOLIC GEOMETRY

Hyperbolic geometry initially grew out of the Building Structures Strand through the work of János Bolyai (1802–1860, Hungarian), and N. I. Lobachevsky (1792–1856, Russian). Hyperbolic geometry is special from a formal axiomatic point of view because it satisfies all the postulates (axioms) of Euclidean geometry except for the parallel postulate. In hyperbolic geometry straight lines can converge toward each other without intersecting (violating Euclid's Fifth Postulate), and there is more than one straight line through a point that does not intersect a given line (violating the usual high school parallel postulate, which states that through any point *P* not on a given line *l* there is one and only one line through *P* not intersecting *l*). See Figure 5.1.

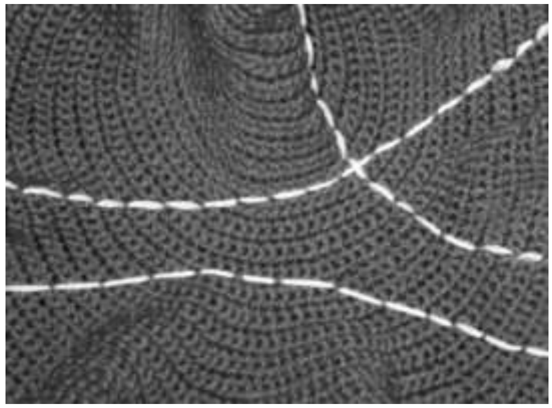


Figure 5.1 Two geodesics through a point not intersecting a given geodesic

The reader can explore more details of the axiomatic nature of hyperbolic geometry in Chapter 10. Note that the 450° cone also violates the two parallel postulates mentioned above. Thus the 450° cone has some of the properties of the hyperbolic plane.

Hyperbolic geometry has turned out to be useful in various branches of higher mathematics. For example, in the classical theory of modular functions, algebraic geometry, differential geometry, complex variables, and dynamic systems. Hyperbolic geometry is used in biology and medicine, cosmology, physics, quantum computing, chemistry, architecture. The geometry of binocular visual space appears experimentally to be best represented by hyperbolic geometry (see **[HY**: Zage]). In addition, hyperbolic geometry was considered as one of the possible geometries for our three-dimensional physical universe — we will explore this connection more in Chapters 18 and 24.

In many books hyperbolic geometry and non-Euclidean geometry are treated as being synonymous, but as we have seen there are other non-Euclidean geometries, especially spherical geometry. It is also not accurate to say (as many books do) that non-Euclidean geometry was discovered about 200 years ago. As we discussed in Chapter 2, spherical geometry (which is clearly not Euclidean) was in existence and studied (within the Navigation/Stargazing Strand) by at least the ancient Babylonians, Indians, and Greeks more than 2000 years ago. For more detailed discussion of the history and applications of hyperbolic geometry see [Taimina, *Crocheting Adventures with the Hyperbolic Planes*, 2nd ed., 2018; ch.5 and 9]

Most texts and popular books introduce hyperbolic geometry either axiomatically or via "models" of the hyperbolic geometry in the Euclidean plane. These models are like our familiar map projections of the surface of the earth. Like these maps of the earth's surface, intrinsic straight lines on the hyperbolic plane are not, in general, straight in the model (map) and the model, in general, distorts distances and angles. We will return to the subject of projection and models in Chapter 17. These "models" grew out of the Art/Pattern Strand.

In this chapter we will introduce the geometry of the hyperbolic plane as the intrinsic geometry of a particular surface in 3-space, in much the same way that we introduced spherical geometry by looking at the intrinsic geometry of the sphere in 3-space. This is more in the flavor of the Navigation/Stargazing Strand. Such a surface is called an ***isometric embedding*** of the hyperbolic plane into 3-space. We will construct such a surface in the next section. Nevertheless, many texts and popular books say that David Hilbert (1862–1943, German) proved in 1901 that it is not possible to have an isometric embedding of the hyperbolic plane onto a closed subset of Euclidean 3-space. These authors miss what Hilbert actually proved. In fact, Hilbert [**HY**: Hilbert] proved that there is no ***real analytic*** isometry (that is, no isometry defined by real-valued functions that have convergent power series). In 1902 Holmgren improved Hilbert's theorem showing that given a smooth embedding of a piece of the hyperbolic plane in three-dimensional space, the embedding cannot be extended isometrically and smoothly beyond the finite distance *d*. Unfortunately, *d* depends on the local embedding, and there is not a uniform bound for the size of the "largest" piece of the hyperbolic plane that can be isometrically embedded in 3-space. Hilbert's theorem was also improved by Amsler in 1955, who showed that every sufficiently smooth immersion of the hyperbolic plane into 3-space has a singular "edge,"i.e., a one-dimensional submanifold beyond which the embedding is no longer smooth. In 1964, N. V. Efimov [**HY**: Efimov] extended Hilbert's result by proving that there is no isometric embedding defined by functions whose first and second derivatives are continuous. Without giving an explicit construction, N. Kuiper [**HY**: Kuiper] showed in 1955 that there is a differentiable isometric embedding onto a closed subset of 3-space.


The first hyperbolic plane model made by E. Beltrami in 1868 and David's model made more than 100 years later

The construction used here was shown to David by William Thurston (b.1946-2012, American) in 1978; and it is not defined by equations at all, because it has no definite embedding in Euclidean space. The idea for this construction is also included in [**DG**: Thurston], pages 49 and 50, and is discussed in [**DG**: Henderson], page 31. In Problem **5.3** we will show that our isometric model is locally isometric to a certain smooth surface of revolution called the ***pseudosphere***, which is well known to locally have hyperbolic geometry. Later, in Chapter 17, we will explore the various (non-isometric) models of the hyperbolic plane (these models are the way that hyperbolic geometry is presented in most texts) and prove that these models and the isometric constructions here produce the same geometry.

DESCRIPTION OF ANNULAR HYPERBOLIC PLANES

In Appendix A we describe the details for five different isometric constructions of hyperbolic planes (or approximations to hyperbolic planes) as surfaces in 3-space. It is very important that you actually perform at least one of these constructions. The act of constructing the surface will give you a feel for hyperbolic planes that is difficult to get any other way. We will focus our discussions in the text on the description of the hyperbolic plane from annuli that was proposed by W. Thurston.

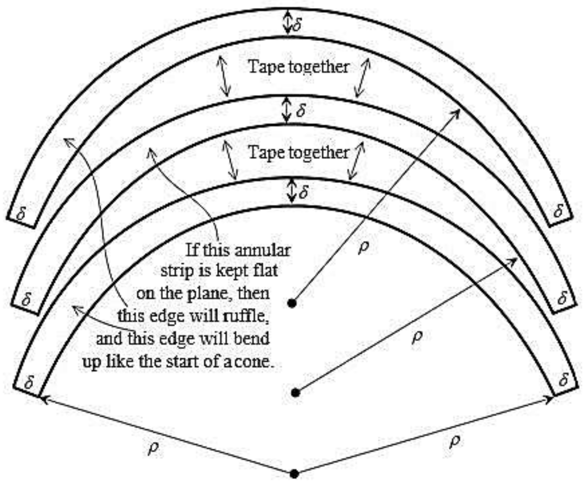


Figure 5.2 Annular strips for making an annular hyperbolic plane

A paper model of the hyperbolic plane may be constructed as follows: Cut out many identical annular ("annulus" is the region between two concentric circles) strips as in Figure 5.2.(See template in Appendix). Attach the strips together by taping the inner circle of one to the outer circle of the other. It is crucial that all the annular strips have the same inner radius and the same outer radius, but the lengths of the annular strips do not matter. You can also cut an annular strip shorter or extend an annular strip by taping two strips together along their straight ends. The resulting surface is of course only an approximation of the desired surface. The actual hyperbolic plane is obtained by letting $\delta \to 0$ while holding the radius ρ fixed. Note that since the surface is constructed (as $\delta \to 0$) the same everywhere it is ***homogeneous*** (that is, intrinsically and geometrically, every point has a neighborhood that is isometric to a neighborhood of any other point). We will call the results of this construction the ***annular hyperbolic plane.*** We strongly suggest that the reader take the time to cut out carefully several such annuli and tape them together as indicated**.**

Daina discovered a process for crocheting the annular hyperbolic plane as described in Appendix A. The result is pictured in Figures 5.1 and 5.3 and other photos in this book.

Figure 5.3 Daina's first crocheted annular hyperbolic plane (1997)

There is also a polyhedral construction of the hyperbolic plane that is not directly related to the annular constructions but is easier for students (and teachers!) to construct. This construction (invented by David's son Keith Henderson) is called the *hyperbolic soccer ball*. See Appendix for the details of the constructions (and templates) and Figure 5.4 for a picture. It also has a nice appearance if you make the heptagons a different color from the hexagons. As with any polyhedral construction we cannot get closer and closer approximations to the hyperbolic plane. There is also no apparent way to see the annuli.

Figure 5.4 Keith Henderson with his hyperbolic soccer ball

HYPERBOLIC PLANES OF DIFFERENT RADII (CURVATURE)

Note that the construction of a hyperbolic plane is dependent on ρ (the radius of the annuli), which we will call the ***radius of the hyperbolic plane***. As in the case of spheres, we get different hyperbolic planes depending on the value of ρ. In Figures 5.5–5.7 there are crocheted hyperbolic planes with radii approximately 4 cm, 8 cm, and 16 cm. The pictures were all taken from approximately the same perspective and in each picture, there is a centimeter rule to indicate the scale.

Figures 5.5-5.7 Hyperbolic planes with $\rho \approx 4$ cm, $\rho \approx 8$ cm and $\rho \approx 16$ cm

Note that as ρ increases, a hyperbolic plane becomes flatter and flatter (has less and less curvature). Both spheres and hyperbolic planes, as ρ goes to infinity, become indistinguishable from the ordinary flat (Euclidean) plane. Thus, the plane can be called a sphere (or hyperbolic plane) with infinite radius. In Chapter 7, we will define the *Gaussian Curvature* and show that it is equal to $1/\rho^2$ for a sphere and $-1/\rho^2$ for a hyperbolic plane.

PROBLEM 5.1 WHAT IS STRAIGHT IN A HYPERBOLIC PLANE?

a. *On a hyperbolic plane, consider the curves that run radially across each annular strip. Argue that these curves are intrinsically straight. Also, show that any two of them are asymptotic, in the sense that they converge toward each other but do not intersect.*

Look for the local intrinsic symmetries of each annular strip and then global symmetries in the whole hyperbolic plane. Make sure you give a convincing argument why the symmetry holds in the limit as $\delta \to 0$.

We shall say that two geodesics that converge in this way are ***asymptotic geodesics***. Note that there are no geodesics (straight lines) on the plane that are asymptotic.

b. *Find other geodesics on your physical hyperbolic surface. Use the properties of straightness (such as symmetries) you talked about in* Problems **1.1**, **2.1**, *and* **4.1**.

Try holding two points between the index fingers and thumbs on your two hands. Now pull gently — a geodesic segment with its reflection symmetry should appear between the two points. If your surface is durable enough, try folding the surface along a geodesic. Also, you may use a ribbon to test for geodesics.

c. *What properties do you notice for geodesics on a hyperbolic plane? How are they the same as geodesics on the plane or spheres, and how are they different from geodesics on the plane and spheres?*

Explore properties of geodesics involving intersecting, uniqueness, and symmetries. Convince yourself as much as possible using your model — full proofs for some of the properties will have to wait until Chapter 17.

PROBLEM 5.2 COORDINATE SYSTEM ON ANNULAR HYPERBOLIC PLANE

First, we will define coordinates on the annular hyperbolic plane that will help us to study it in Chapter 17. Let ρ be the fixed inner radius of the annuli and let H_δ be the approximation of the annular hyperbolic plane constructed from annuli of radius ρ and thickness δ. On H_δ pick the inner curve of any annulus, calling it the *base curve*; and on this curve pick any point as the origin O and pick a positive direction on this curve. We can now construct an (intrinsic) coordinate system $\boldsymbol{x}_\delta: R^2 \to H_\delta$ by defining $\boldsymbol{x}_\delta(0,0) = O$, $\boldsymbol{x}_\delta(w,s)$ to be the point on the base curve at a distance w from O, and $\boldsymbol{x}_\delta(w,s)$ to be the point at a distance s from $\boldsymbol{x}_\delta(w,0)$ along the radial geodesic through $\boldsymbol{x}_\delta(w,0)$, where the positive direction is chosen to be in the direction from outer to inner curve of each annulus. Such coordinates are often called ***geodesic rectangular coordinates***. See Figure 5.8.

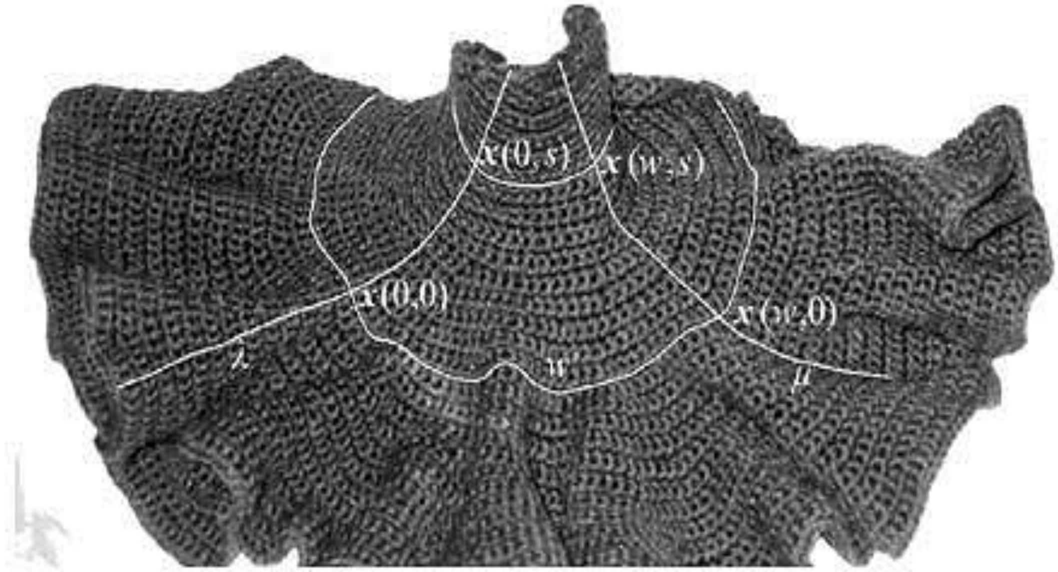


Figure 5.8 Geodesic rectangular coordinates on annular hyperbolic plane

a. *Show that the coordinate map* ***x*** *is one-to-one and onto from the whole of* $\mathbf{R}^2$ *onto the whole of the annular hyperbolic plane. What maps to the annular strips, and what maps to the radial geodesics?*

b. *Let λ and μ be two of the radial geodesics described in part* ***a***. *If the distance between λ and μ along the base curve is w, then show that the distance between them at a distance $s = n\delta$ from the base curve is, on the paper hyperbolic model,*

$$w\left(\frac{\rho}{\rho+\delta}\right)^{n} = w\left(\frac{\rho}{\rho+\delta}\right)^{s/\delta}$$

Now take the limit as $\delta \to 0$ to show that the distance between λ and μ on the annular hyperbolic plane is $w\exp(-s/\rho)$.

Thus, the coordinate chart ***x*** preserves (does not distort) distances along the (vertical) second coordinate curves but at $\boldsymbol{x}(a, b)$ the distances along the first coordinate curve are distorted by the factor of $\exp(-b/\rho)$ when compared to the distances in $\boldsymbol{R}^2$.

PROBLEM 5.3 THE PSEUDOSPHERE IS HYPERBOLIC

Show that locally the annular hyperbolic plane is isometric to portions of a (smooth) surface defined by revolving the graph of a continuously differentiable function of ***z*** *about the* ***z****-axis. This is the surface usually called the* ***pseudosphere****.*

OUTLINE OF PROOF

1. Argue that each point on the annular hyperbolic plane is like any other point. (Think of the annular construction. About a point consider a neighborhood that keeps *its* size as the width of the *annular strips*, δ, shrinks to zero.)
2. Start with one of the annular strips and complete it to a full annulus in a plane. Then construct a surface of revolution by attaching to the inside edge of this annulus other annular strips as described in the construction of the annular hyperbolic plane. (See Figure 5.9.) Note that the second and subsequent annuli form truncated cones. Finally, imagine the width of the annular strips, δ, shrinking to zero.
3. Derive a differential equation representing the coordinates of a point on the surface using the geometry inherent in Figure 5.9. If $f(r)$ is the height (z-coordinate) of the surface at a distance of r from the z-axis, then the differential equation should be (remember that ρ is a constant)

$$\frac{dr}{dz} = \frac{-r}{\sqrt{\rho^2 - r^2}}\,. \quad Why?$$

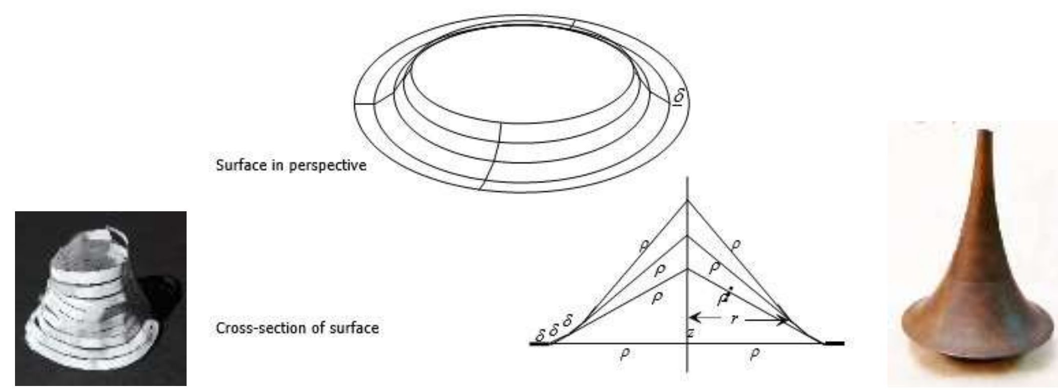


Figure 5.9 Hyperbolic surface of revolution — pseudosphere

4. Solve (using tables or computer algebra systems) the differential equation for $z = f(r)$ as a function of r. Note that you are not getting r as a function of z. This curve is usually called the ***tractrix***.
5. Then argue (using a theorem from first-semester calculus) that r is a continuously differentiable function of z.

Beltrami model wrapped as pseudosphere

Pseudosphere models (made by Edmund Harriss)

You can make a pseudosphere from annuli as you can see in Figure 5.9. Make your own model, it does not have to be perfect. But it will give you a tactile sense of the surface anyway. We can also crochet a pseudosphere by starting with 5 or 6 chain stitches and continuing in a spiral fashion, increasing as when crocheting the hyperbolic plane. See Figure 5.10. Note that, when you crochet beyond the annular strip that lays flat and forms a complete annulus, the surface forms ruffles and is no longer a surface of revolution (nor smooth).

Figure 5.10 Crocheted pseudospheres with ruffles

The term “pseudosphere” seems to have originated with Hermann von Helmholtz (1821–1894, German), who was contrasting spherical space with what he called pseudospherical space. However, Helmholtz did not actually find a surface with this geometry. In 1866 Eugenio Beltrami (1835–1900, Italian) constructed the surface which he called “pseudospherical” and showed that its geometry is locally the same as (locally isometric to) the hyperbolic geometry constructed by Lobachevsky. Beltrami in his construction also used annuli. Unfortunately, all the known (until 1960) surfaces with constant negative curvature could not be extended indefinitely. Beltrami partially overcame this problem by considering a surface wrapped infinitely many times around a pseudospherical surface. Beltrami's model of this wrapping is the first known physical model of the hyperbolic plane. For more historical discussion, see ([**HI**: Katz], pages 781–783.) Mathematicians searched further for a surface (in those days "surface" meant "real analytic surface") that would be the whole of the hyperbolic plane (as opposed to only being locally isometric to it). This search was halted when Hilbert proved that such a surface was impossible (in his theorem that we discussed above at the end of the first section in this chapter, A Short History of Hyperbolic Geometry).

INTRINSIC/EXTRINSIC, LOCAL/GLOBAL

On the plane or on spheres, rotations and reflections are both ***intrinsic*** in the sense that they are experienced by a 2-dimensional bug as rotations and reflections. These intrinsic rotations and reflections are also ***extrinsic*** in the sense that they can also be viewed as isometries of 3-space. (For example, the reflection of a sphere through a great circle can also be viewed as a reflection of 3-space through the plane containing the great circle.) Thus, rotations and reflections are particularly easy to see on planes and spheres. In addition, on the plane and sphere all rotations and reflections are ***global*** in the sense that they take the whole plane to itself or whole sphere to itself. (For example, any intrinsic rotation about a point on a sphere is always a rotation of the whole sphere.) On cylinders and cones, intrinsic rotations and reflections exist locally because cones and cylinders are locally isometric with the plane. However, some intrinsic rotations on cones and cylinders are extrinsic and global: for example, rotations about the cone point on a circular cone with cone angle <360°, or half turns about any point on a cylinder. Rotations about the cone point on (>360°)-cones are global but not extrinsic. Rigid-motion-along- geodesic symmetries are extrinsic and global on cylinders but are neither on any cone. (Do you see why?) Reflections, in general, are neither extrinsic nor global (Can you see the exceptions on cones and cylinders?).

PROBLEM 5.4 ROTATIONS AND REFLECTIONS ON SURFACES

We can see from our physical hyperbolic planes that geodesics exist joining every pair of points and that these geodesics each have reflection- in-themselves symmetry. (If you did not see this in Problem **5.1c**, then go back and explore some more with your physical model. In Chapter 17 we will prove rigorously that this is in fact true by using the upper half-plane model.) In Chapter 17 we will show that these reflections are global reflections of the whole hyperbolic space. Note that there do not exist extrinsic reflections of the hyperbolic plane (embedded in Euclidean 3-space). Given all this, it is not clear that there exist intrinsic rotations, nor is it necessarily clear what exactly intrinsic rotations are.

a. *Let l and m be two geodesics on the hyperbolic plane that inter- sect at the point P. Look at the composition of the reflection Rl through l with the reflection* R*m through m. Show that this composition* $\mathrm{R}_m\mathrm{R}_l$ *deserves to be called a rotation about P. What is the angle of the rotation?*

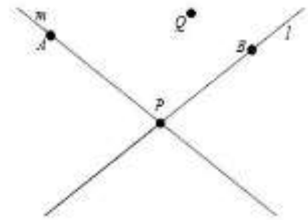


Figure 5.11 Composition of two reflections is a rotation

Let *A* be a point on *m* and *B* be a point on *l*, and let *Q* be an arbitrary point (not on *m* or *l*). Investigate where *A*, *B*, and *Q* are sent by R_l and then by $\mathrm{R}_m\mathrm{R}_l$. See Figure 5.11. Why are all points (except *P*) rotated through the same angle *and in the same direction*?

We will study symmetries and isometries in more detail in Chapter 11. In that chapter we will show that every isometry (on the plane, spheres, and hyperbolic planes) is a composition of one, two, or three reflections.

b. *Show that Problem* **3.2** *(VAT) holds on cylinders, cones (including the cone points), and hyperbolic planes.*

If you check your proof(s) of Problem **3.2** and modify them (if necessary) to involve only symmetries, then you will be able to see that they hold also on the other surfaces.

c. *Define "rotation of a figure about P through an angle θ" without mentioning reflections in your definition. What does a rotation do to a point not at P*?

d. *One of high school textbooks* ***defines*** *a rotation as the composition of two reflections. Is this a good definition? Why or why not?*

Exploring Curvature

[At this point David had made a note that there should be a new chapter on exploring curvature to have more experience with intrinsic and extrinsic properties of surfaces. He was thinking of explorations we described in *Crocheting Adventures with the Hyperbolic Planes*. I am just inserting that part of the book here.]

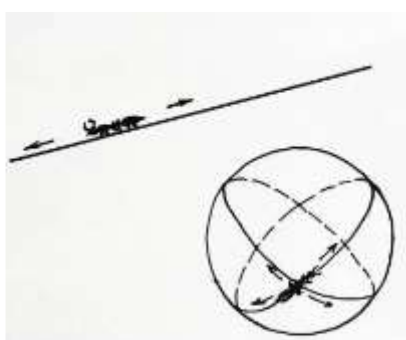
Curvature is a mathematical notion widely used in differential geometry. What does it mean in simple words? If you look, for example, on the surface of your desk or on the floor in your room, you will notice that it is flat—there is no curvature, or we say that the curvature is *zero*. If you look at an orange or an egg, you will see that it is curved "outward"—we say that the curvature is *positive*. The egg is curved more at its tips and curved less in between and thus does not have constant positive curvature, but many oranges have almost constant positive curvature.

Two surfaces with positive curvature: on an egg it is varying, but on an orange it is (almost) constant

Now look at the surface of the pear. Most of its surface has positive curvature like an orange or an egg, but there are some points were the curvature is different. How can we describe this difference?

Not all points on the pear have positive curvature

In the early 19th century Carl Friedrich Gauss (1777–1855), explored the idea that surfaces can be distinguished by their *curvature* at different points, which can be positive, negative, or zero. In the 1820s Gauss was a professional surveyor. This work inspired him to study the intrinsic geometry of surfaces. His concern was how one can determine the curvature of an arbitrary surface without knowing anything about how this surface might be embedded in space—in other words, asking whether and how one could determine the curvature of a surface through measurements made only along this surface, without knowing anything about the shape of this surface.



An ant on a curve can go in only two directions, but
on a surface there are more choices

Consider an ant crawling on a big sphere, where the ant cannot see that it is on a sphere; how could this ant distinguish whether it is on a flat surface or on a curved surface? If the ant crawls on a straight line or on any other one-dimensional curve, it can move only in two directions: forward or backward. If the ant is on a two-dimensional surface, then it can choose to go backward or forward but also right or left. Still, the ant cannot see the surface it is on from the outside; it can explore the surface only intrinsically, not leaving this surface. Another great mathematician, Leonard Euler (1707–1783), had already introduced a notion of surface curvature, which was used in eighteenth-century calculus. But to use Euler's method for surfaces, you had to know how the surface is embedded in space. This means that you had to be able to see the surface from another dimension. Gauss was able to prove that it is possible to find a way to determine the curvature of a surface that depends only on intrinsic properties of the surface. Consider the same ant on a sphere. Suppose that she plants a post at point *P*, ties one end of a length of rope to the pole and ties the other end to herself. She walks away from the pole until the rope is taunt and marks that spot. Keeping the rope taunt, she walks around in a circle until she returns to that mark, measuring how far she walks. If the ant is on a flat surface than the distance of her walk, d, should equal the circumference of a circle with radius equal to the length of her rope, r. If $d < 2\pi r$, then the ant is on a sphere.

If a surface has constant positive curvature, it will become closed, and it is called a sphere. Gauss used the radius of the sphere to determine the magnitude of the curvature. Every point on a sphere is the intersection of two great circles with the radius R. As defined earlier, the curvature of each of those circles is $1/R$, so Gauss measured the curvature of a sphere as the product of the curvatures of these two great circles. Therefore, he defined the curvature of a sphere with radius R as the quantity $1/R^2$; so, as the radius, R, gets larger, the curvature, $1/R^2$, gets smaller.

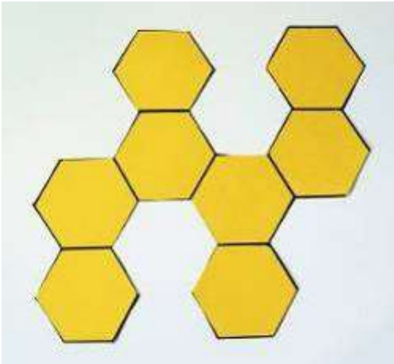
Let us now look at a sketch of a three-dimensional landscape. There are hills, a pass, and the bowl-shaped bottom of the bay. We can draw two intersecting curves to define a point on the top of a hill and a point in the bay. Then, surface curvature at each of these points will be the product of curvatures of the two intersecting curves. Therefore, in both cases—on the top of a hill and in the bottom of the bay—we have positive curvature, since positive times positive equals positive and negative times negative equals positive. But at the pass, where there is one positive curve and one negative curve, the surface curvature will be negative. In general, we will define the curvature of the surface at a point in terms of the one-dimensional curvatures of two curves on the surface that intersect at that point.

Positive curvature is on the top of a hill (as a product of two positive curvatures) and in the bay (as a product of two negative curvatures), but on a pass the curvature is negative (as a product of positive and negative curvatures).

Cut from paper some regular hexagons. Try to put them next to each other. You can see that around each hexagon you can place six other hexagons, and then at each vertex there will be three hexagons and they will lay flat on the table. If you continue adding more hexagons, it looks like the surface of a honeycomb.

Regular hexagons tile the plane without gaps or overlaps.

Design of this sidewalk also uses a property that regular hexagons tile a plane



A. Some of the hexagons are removed, so that at each vertex there are only two hexagons

B. Gluing the hexagons so that there is a pentagonal hole surrounded by five hexagons causes the surface to bend

Glue five hexagons together (like in picture A). Notice that this surface is no longer flat. If you continue doing (see the figures A, B, and C) this then you will get a polyhedron that is sometimes called a truncated icosahedron or an approximation of a soccer ball (in most of the world this is called a football).

Five hexagons around a pentagon make a soccer ball (football) with approximately constant positive curvature

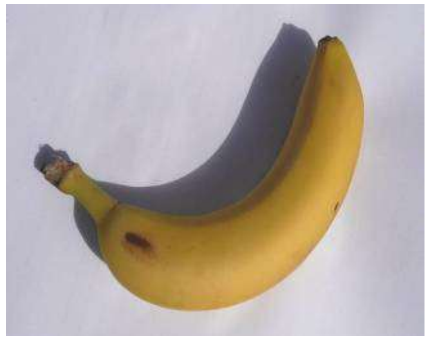
Finally, see what happens if instead you surround with hexagons a heptagon, a regular seven-sided polygon, with the same side lengths as the hexagons. Notice that this surface also is not flat. What are the differences between these two non-flat surfaces?

Seven hexagons around a heptagon approximates a hyperbolic plane (constant negative curvature)

Notice that the surface with constant positive curvature will close in on its self, but the surface with negative constant curvature will extend out indefinitely. Another way to think about this is looking at what happens when you "flatten" a surface so that it lies in the plane. For constant positive curvature, the surface covers less area than the plane. For constant negative curvature, the surface covers more area than the plane.

Flattening positive and negative curvature

Why are we talking about surfaces with *constant* curvature? In order to talk about a surface having a *geometry* (spherical, Euclidean, or hyperbolic), we need it to be “the same” everywhere. There are other surfaces with both positive and negative curvatures that are not constant (for example, the surface of a banana or a pear).

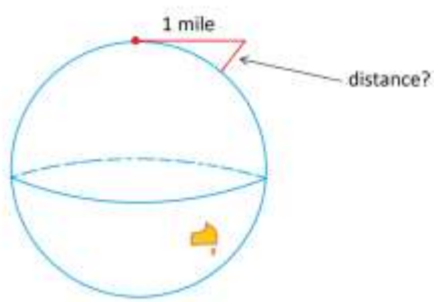


On the surface of a banana, there are both positive and negative curvatures

James Casey in his book *Experiencing Curvature* (Vieweg, 1996, p.190) suggests a following exploration of the intrinsic geometry of a sphere:

> Take a thin-walled plastic ball and cut it into two unequal pieces. Draw some identical figures on both pieces (intersecting lines; triangles). Bend one of the pieces and compare the figures on it to those on the other piece. Measure some intrinsic properties and discuss your findings.

This question was asked on Twitter by James Tanton: Have you an intuitive sense of the curvature of the Earth? Draw a 1-mile segment tangent to surface. What's your guess as to the distance back to surface at endpoint? What is the actual value? (Assume perfect sphere, R=3963 miles.)



Curvature is important not only in geometry. It affects, for example, chemical properties of carbon. It is already known how to mold it into precious diamonds or as the graphite in pencils and in graphene – the strongest material on Earth. In 2018 a new form of carbon has been created “*schwarzites*”. Schwarzites have long been predicted by chemists, who suggested they would have unique properties that make them useful in batteries and as catalysts. Fullerenes are fully composed from carbon molecules and are positively curved, graphene has no curvature, but schwarzites are negatively curved.

Einstein discovered that gravitation is the curvature of space-time, which can curl in on itself, pinching off into black holes. In 2013 a group of physicists from Austria proposed the existence of a new force called "blackbody force". In 2017 study a team from Brazil theoretically demonstrated that the blackbody force depends not only on the geometry of the bodies themselves, but also on both the surrounding spacetime geometry and topology. In some cases, the local curvature significantly increases the strength of the blackbody force. They studied spherical and cylindrical blackbodies.

4x100m shows how areas change in hyperbolic plane. Each of the four colors is crocheted with 100m of the yarn. White hyperbolic plane is crocheted adding a stitch in everyone (2:1 ratio). It shows that hyperbolic plane can be enclosed with a sphere whose radius depends on the radius of the hyperbolic plane (your challenge – find this relationship!)