PREFACE

David Hilbert (1862–1943) in the Preface to *Geometry and the Imagination* by Hilbert and S. Cohn-Vossen emphasized the point we wish to make in this book:

Meaning is important in mathematics and geometry is an important source of that meaning.

We believe that mathematics is a natural and deep part of human experience and that experiences of meaning in mathematics are accessible to everyone. Much of mathematics is not accessible through formal approaches except to those with specialized learning. However, using nonformal experience and geometric imagery, many levels of meaning in mathematics can be opened up in a way that most humans can experience and find intellectually challenging and stimulating.

Formalism contains the power of the meaning but not the meaning. It is necessary to bring the power back to the meaning.

A formal proof as we normally conceive of it is not the goal of mathematics — it is a tool — a means to an end. The goal is understanding. Without understanding we will never be satisfied — with understanding we want to expand that understanding and to communicate it to others. This book is based on a view of proof as a *convincing communication that answers — Why?*

Many formal aspects of mathematics have now been mechanized and this mechanization is widely available on personal computers or even smartphones, but the experience of meaning in mathematics is still a human enterprise that is necessary for creative work.

In this book we invite the reader to explore the basic ideas of geometry from a more mature standpoint. We will suggest some of the deeper meanings, larger contexts, and interrelations of the ideas. We are interested in conveying a different approach to mathematics, stimulating the reader to take a broader and deeper view of mathematics and to experience for herself/himself a sense of mathematizing. Through an active participation with these ideas, including exploring and writing about them, people can gain a broader context and experience. This active participation is vital for anyone who wishes to understand mathematics at a deeper level, or anyone wishing to understand something in their experience through the vehicle of mathematics.

This is particularly true for teachers or prospective teachers who are approaching related topics in the school curriculum. All too often we convey to students that mathematics is a closed system, with a single answer or approach to every problem, and often without a larger context. We believe that even where there are strict curricular constraints, there is room to change the meaning and the experience of mathematics in the classroom.

This book is based on a junior/senior-level course that David started teaching in 1974 at Cornell for mathematics majors, high school teachers, future high school teachers, and others. Most of the chapters start intuitively so that they are accessible to a general reader with no particular mathematics background except imagination and a willingness to struggle with ideas. However, the discussions in the book were written for mathematics majors and mathematics teachers and thus assume of the reader a corresponding level of interest and mathematical sophistication.

The book emphasizes learning geometry using reason, intuitive understanding, and insightful personal experiences of meanings in geometry. To accomplish this the students are given a series of inviting and challenging problems and are encouraged to write and speak their reasonings and understandings.

Most of the problems are placed in an appropriate history perspective and approached both in the context of the plane and in the context of a sphere or hyperbolic plane (and sometimes a geometric manifold). We find that by exploring the geometry of a sphere and a hyperbolic plane, our students gain a deeper understanding of the geometry of the (Euclidean) plane.

We introduce the modern notion of "parallel transport along a geodesic," which is a notion of parallelism that makes sense on the plane but also on a sphere or hyperbolic plane (in fact, on any surface). While exploring parallel transport on a sphere, students can appreciate more fully that the similarities and differences between the Euclidean geometry of the plane and the non-Euclidean geometries of a sphere or hyperbolic plane are not adequately described by the usual Parallel Postulate. We find that the early interplay between the plane and spheres and hyperbolic planes enriches all the later topics whether on the plane or on spheres and hyperbolic planes.

This far the preface is the same as it was for the 3rd edition of the book. David and I met at the conference in Sicily September 1995. David had just received the first printed copies of *Experiencing Geometry*, the book which was his view on geometry and teaching mathematics. We worked together on two following editions, adding more material and our joint teaching experiences to the text. Encouraged by many instructors who still use this book for their geometry classes we were ready to work on the fourth edition of *Experiencing Geometry*. In 2018, at this same desk I am working now, David was reviving the more than 15-year-old files of *Experiencing Geometry*. He finished updating Chapter 1, which is included in this book exactly how he did it. Then came the darkest day of 2018 and our lives: David was hit by a pickup truck on pedestrian crosswalk, and we lost him forever on December 20, 2018.

Keeping *Experiencing Geometry* alive and widely available was David's wish. Through this book his spirit, his thoughts, his deep insights and masterful teaching can still be with us. There cannot be a 4th edition without David. Instead this book is David's memorial edition and therefore his biography (*In Search of Meaning)* and pictures throughout the book are included. In the early 1970's David left highly successful mathematics career in order to make mathematics more human and accessible. He was ahead of his time.

I have kept the structure of the book the same as it was for *Experiencing Geometry: Euclidean and Non-Euclidean with History* (Pearson/Prentice Hall, 2005). Figures with numbers are the same as they were in 3rd edition; figures with just captions are new to this edition. I updated the text throughout the book with notes we had and some new references.

Working on this book was my work through the grief and fulfilling my promise to David. I am grateful for all support from family, especially children Keith, Becky, Lelde, Linda; friends, colleagues, former students – too many to name. You gave me strength to live further. My thanks to Jeff Weeks who helped to update Chapter 24. David was always the one who did the typesetting for all his books and mine. Beverly West offered her help with that and spent many hours with this text, and I learned a great deal about formatting thanks to her. All imperfections left in this book are mine.

Daina Taimina

Ithaca, NY, February 2020

Mathematics is a way of looking at your world. Take charge of it — make it yours. Understand how you see things and see how you understand things. Mathematics can say something about you.

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Think of what we are embarking on together as a stroll in the woods. I will show you paths that I have trod and point to what I have seen. I have been there many times before — but oh! — there is much more to see that I have not yet seen. You with your unique vision will point out to me things that I have passed by. Feel free to explore — look under stones — down in hollow stumps. You can look up a straight tree and see the sky — You can find a riverlet of water that if followed will lead to the sea. But just as valuable are the little surprises that spark something inside — a trillium in a spot of sun — a scarlet eft wriggling under a fallen leaf. Do not hurry "to get some where" — linger a while — listen — see what's around where you are.

David W. Henderson