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Anthony W. Knapp

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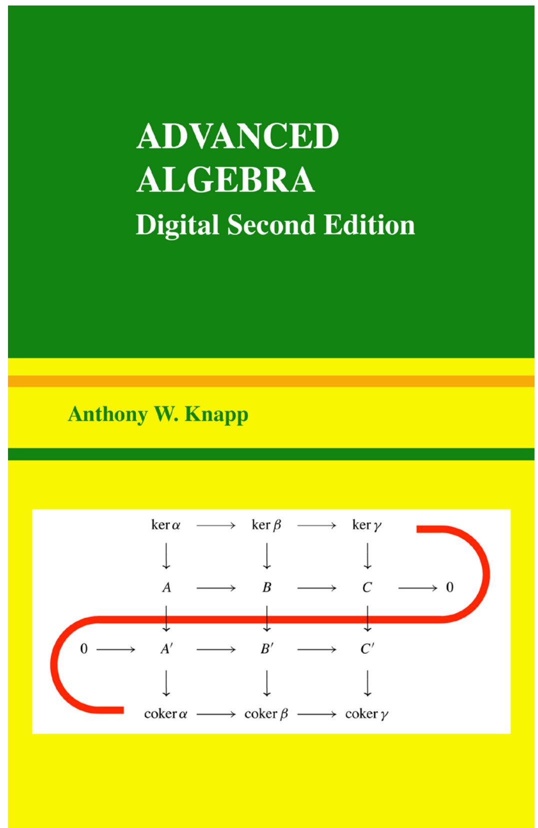

Anthony W. Knapp
81 Upper Sheep Pasture Road
East Setauket, N.Y. 11733–1729, U.S.A.
Email to: `aknapp@math.stonybrook.edu`
Homepage: `www.math.stonybrook.edu/~aknapp`

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CHAPTER IV

Homological Algebra

Abstract. This chapter develops the rudiments of the subject of homological algebra, which is an abstraction of various ideas concerning manipulations with homology and cohomology. Sections 1–7 work in the context of good categories of modules for a ring, and Section 8 extends the discussion to abelian categories.

Section 1 gives a historical overview, defines the good categories and additive functors used in most of the chapter, and gives a more detailed outline than appears in this abstract.

Section 2 introduces some notions that recur throughout the chapter—complexes, chain maps, homotopies, induced maps on homology and cohomology, exact sequences, and additive functors. Additive functors that are exact or left exact or right exact play a special role in the theory.

Section 3 contains the first main theorem, saying that a short exact sequence of chain or cochain complexes leads to a long exact sequence in homology or cohomology. This theorem sees repeated use throughout the chapter. Its proof is based on the Snake Lemma, which associates a connecting homomorphism to a certain kind of diagram of modules and maps and which establishes the exactness of a certain 6-term sequence of modules and maps. The section concludes with proofs of the crucial fact that the Snake Lemma and the first main theorem are functorial.

Section 4 introduces projectives and injectives and proves the second main theorem, which concerns extensions of partial chain and cochain maps and also construction of homotopies for them when the complexes in question satisfy appropriate hypotheses concerning exactness and the presence of projectives or injectives. The notion of a resolution is defined in this section, and the section concludes with a discussion of split exact sequences.

Section 5 introduces derived functors, which are the basic mathematical tool that takes advantage of the theory of homological algebra. Derived functors of all integer orders ≥ 0 are defined for any left exact or right exact additive functor when enough projectives or injectives are present, and they generalize homology and cohomology functors in topology, group theory, and Lie algebra theory.

Section 6 implements the two theorems of Section 3 in the situation in which a left exact or right exact additive functor is applied to an exact sequence. The result is a long exact sequence of derived functor modules. It is proved that the passage from short exact sequences to long exact sequences of derived functor modules is functorial.

Section 7 studies the derived functors of Hom and tensor product in each variable. These are called Ext and Tor, and the theorem is that one obtains the same result by using the derived functor mechanism in the first variable as by using the derived functor mechanism in the second variable.

Section 8 discusses the generalization of the preceding sections to abelian categories, which are abstract categories satisfying some strong axioms about the structure of morphisms and the presence of kernels and cokernels. Some generalization is needed because the theory for good categories is insufficient for the theory for sheaves, which is an essential tool in the theory of several complex variables and in algebraic geometry. Two-thirds of the section concerns the foundations, which involve unfamiliar manipulations that need to be internalized. The remaining one-third introduces an

artificial definition of "member" for each object and shows that familiar manipulations with members can be used to verify equality of morphisms, commutativity of square diagrams, and exactness of sequences of objects and morphisms. The consequence is that general results for categories of modules in homological algebra requiring such verifications can readily be translated into results for general abelian categories. The method with members, however, does not provide for constructions of morphisms member by member. Thus the construction of the connecting homomorphism in the Snake Lemma needs a new proof, and that is given in a concluding example.

1. Overview

This chapter develops the rudiments of the subject of homological algebra. The only prerequisite within the present volume is the self-contained Section III.5 entitled "Digression on Cohomology of Groups," which is helpful primarily as motivation. The definitions of category, functor, object, morphism, natural transformation, product, and coproduct as in Chapters IV and VI of *Basic Algebra* will be taken as known, and it will be helpful as motivation to know also the material from Chapter VII of *Basic Algebra* on group extensions and cohomology of groups. The present chapter will make some allusions to notions from algebraic topology, particularly in this first section, and the reader is encouraged to skip lightly over anything of this kind that might be an impediment to continuing with the remainder of the chapter.

Homology and cohomology have their origins in attempts to assign algebraic invariants to topological obstructions. One example historically was the holes in a domain of the Euclidean plane that can make line integrals that are locally independent of the path fail to be globally independent of the path. Another was the handles on 2-dimensional closed surfaces. These obstructions were originally viewed as numbers (Betti numbers for example) and later viewed as algebraic objects such as abelian groups or vector spaces. A big advance was to regard them not just as objects attached to geometric configurations but as functors that attach objects to geometric configurations and also attach functions between such objects to reflect the behavior of functions between geometric configurations.

Hints of connections with algebra on a deeper level and hints that homology and cohomology could be computed quite flexibly began with work of W. Hurewicz in 1936 and H. Hopf in 1942. Hurewicz considered the following situation: M is a finite connected simplicial complex, U is its universal cover, and G is the fundamental group of M. Suppose that U is contractible. The group G acts freely on the group $C_*(U)$ of simplicial chains of U (with integer coefficients). The boundary operator then gives us an exact sequence

$$0 \longleftarrow \mathbb{Z} \longleftarrow C_0(U) \longleftarrow C_1(U) \longleftarrow C_2(U) \longleftarrow \cdots$$

of abelian groups with an action of G on each $C_j(U)$ by automorphisms in such a way that each $C_j(U)$ in effect is a free $\mathbb{Z}G$ module. Applying $(\,\cdot\,) \otimes_{\mathbb{Z}G} \mathbb{Z}$, we obtain the complex

$$0 \leftarrow C_0(M) \leftarrow C_1(M) \leftarrow C_2(M) \leftarrow \cdots .$$

The homology $H_0(M)$ is just $\mathbb{Z}$ because M is connected, and $H_1(M)$ is just the quotient of G by its commutator subgroup; thus $H_0(M)$ and $H_1(M)$ depend only on G. What Hurewicz showed is that all higher $H_i(M)$ depend only on G; he did not address existence of such spaces M and U for G.

Hopf clarified the situation and drew attention to it by making an explicit calculation: Dropping all assumptions on U other than its simple connectivity, he gave a formula for the quotient of $H_2(M)$ modulo the subgroup of "spherical homology classes" in terms of G. Later he obtained a result for higher-degree homology. In effect, Hopf was giving formulas for $H_n(G, \mathbb{Z})$ by discovering and applying the homology analog of the cohomology result given as Theorem 3.31 in Section III.5.

Meanwhile, S. Eilenberg in 1944 made an adjustment to Lefschetz's singular homology theory and showed for locally finite polyhedra that his adjusted theory gives the same groups as the more traditional simplicial theory. His method was to introduce a third complex, to exhibit chain maps from this to each of the complexes under study, and show that the chain maps possess inverses in a suitable sense.

In addition to the people mentioned above, some others who pursued these matters in the mid 1940s were R. Baer, B. Eckmann, H. Freudenthal, and S. Mac Lane. One thing that mathematicians gradually realized was that homology and cohomology in various situations can be calculated from suitable kinds of abstract resolutions, a fact that lies at the heart of the subject of homological algebra. Another was that the subject of cohomology of groups made sense on an abstract level without any reference to topology and that the theory of factor sets for group extensions, as had been introduced by O. Schreier in the 1920s, was actually one aspect of this theory.

With a great leap of generality, H. Cartan and Eilenberg set down such a theory in their celebrated book *Homological Algebra*, whose publication was delayed until 1956. Homology and cohomology became things attached to complexes, no longer dependent on topology, and the book developed enormous machinery for working with such complexes and homology/cohomology. By the time that Cartan and Eilenberg had published their book, other special cases of homological algebra had already arisen. One was the cohomology theory of Lie algebras, developed by C. Chevalley in the 1940s and by J.-L. Koszul in 1950. Another was the cohomology theory of sheaves, used in the subject of several complex variables starting about 1950 by K. Oka and H. Cartan; sheaves themselves had been introduced in 1946 by J. Leray in connection with partial differential equations.

In the eventual theory the fundamental notion is that of a "derived functor": homology or cohomology is obtained by starting from some kind of resolution, or exact complex, passing to another complex by means of a functor with some special properties, and then extracting the homology or cohomology of the image complex. Two categories are thus involved, one for the resolution and one for the values of the functor. From an expository point of view, it seems wise to start with concrete categories and not to try to identify the most general categories for which the theory makes sense. For much of the chapter, we shall work with a category not much more general than the category $\mathcal{C}_R$ of all unital left R modules, where R is a ring with identity, and our functors will pass from one such category to another. Use of categories $\mathcal{C}_R$ subsumes the following applications:

(i) manipulations with basic homology and cohomology in topology, in which one begins with the ring $R = \mathbb{Z}$ of integers. For more advanced applications in topology, one moves from $\mathbb{Z}$ to more general rings.

(ii) homology and cohomology of groups, in which one initially uses group rings of the form $\mathbb{Z}G$, where G is any group and $\mathbb{Z}$ is the ring of integers.

(iii) homology and cohomology of Lie algebras. If $\mathfrak{g}$ is a Lie algebra over a field such as $\mathbb{C}$, then $\mathfrak{g}$ has a "universal enveloping algebra" $U(\mathfrak{g})$ and a canonical mapping $\iota : \mathfrak{g} \to U(\mathfrak{g})$. Here $U(\mathfrak{g})$ is a complex associative algebra with identity, ι is a Lie algebra homomorphism, and the pair $(U(\mathfrak{g}), \iota)$ has the following universal mapping property: whenever $\varphi : \mathfrak{g} \to A$ is a Lie algebra homomorphism into a complex associative algebra A with identity, then there is a unique homomorphism $\Phi : U(\mathfrak{g}) \to A$ of associative algebras with identity such that $\varphi = \Phi \circ \iota$. Lie algebra homology and cohomology are the theory for the set-up in which the initial underlying rings are $U(\mathfrak{g})$ and $\mathbb{C}$.

In other words, in each of the three applications above, many derived functors of importance pass from the category $\mathcal{C}_R$ for a ring R with identity to the category $\mathcal{C}_S$ for another ring S with identity.

The slight generalization of categories $\mathcal{C}_R$ that we shall use for much of the chapter is as follows: Let R be a ring with identity. A **good category** $\mathcal{C}$ of R modules consists of

(i) some nonempty class of unital left R modules closed under passage to submodules, quotients, and finite direct sums (the **modules** of the category),

(ii) the full sets $\mathrm{Hom}_R(A, B)$ of all R linear homomorphisms from A to B for each A and B as in (i) (the **morphisms**, or **maps**, of the category).

For example the collection of all finitely generated abelian groups, as a subcategory of $\mathcal{C}_{\mathbb{Z}}$, is a good category.[1] So is the collection of all **torsion abelian groups**,

[1] One reason for working with this slight generalization is to emphasize that a certain property of categories $\mathcal{C}_R$, namely that they have "enough projectives" and "enough injectives" in a sense to be made precise below in Section 5, does not necessarily persist for slight variants of $\mathcal{C}_R$.

i.e., abelian groups whose elements all have finite order, as a subcategory of $\mathcal{C}_{\mathbb{Z}}$.

The definition of "good category" specifies *left* R modules that are unital. However, the theory applies equally well to right R modules that are unital, since a unital right R module becomes a unital left module for the opposite ring R^o, i.e., the ring whose underlying abelian group is the same as for R and whose multiplication is given by $a \circ b = ba$.

The special property of a functor $F : \mathcal{C} \to \mathcal{C}'$ used for passing from a complex in one good category to a complex in another good category is that it is **additive**, namely that $F(\varphi_1 + \varphi_2) = F(\varphi_1) + F(\varphi_2)$ whenever φ_1 and φ_2 are in the same $\operatorname{Hom}_R(A, B)$. The initial examples of additive functors are tensor product $M \otimes_R (\,\cdot\,)$, which passes from $\mathcal{C}_R$ to $\mathcal{C}_{\mathbb{Z}}$ if M is a right R module, and Hom in each variable: $\operatorname{Hom}_R(\,\cdot\,, M)$ and $\operatorname{Hom}_R(M, \,\cdot\,)$, both of which pass from $\mathcal{C}_R$ to $\mathcal{C}_{\mathbb{Z}}$ if M is a left R module. In Section 2 we shall consider additive functors in more detail.

The set-up with good categories does not subsume the cohomology of sheaves, nor some other applications of interest, such as the cohomology of vector bundles with a fixed base. The cohomology of sheaves is an important tool in algebraic geometry and several complex variables, and it cannot be ignored. Consequently one ultimately wants the theory to extend to other categories than good categories of modules. In addition, it is quite useful to have the theory work for the categories opposite to two given categories if it works for two given categories, and this feature means that the general theory should not insist that the objects be sets of elements and the morphisms be functions on such elements. Accordingly the abstract theory is carried out for "abelian categories," which will be defined in Section 8. The idea for creating the abstract theory is to take the theory for good categories of modules and rephrase all of the results for all abelian categories. In many instances the proofs will translate easily to the general setting, but in other instances it will be necessary to eliminate individual elements from arguments and obtain new arguments that rely only on complexes, exact sequences, and commutative diagrams. Some of this detail will be carried out in Section 8.

Sections 2–3 establish the framework of homology and cohomology in the context of good categories of modules. Section 2 discusses complexes and exact sequences at length, and Section 3 shows how a short exact sequence of complexes leads to a long exact sequence in homology or cohomology. This is the first main result of the theory and finds multiple uses later in the chapter.

Section 4 contains a discussion of "projectives and injectives" that expands and systematizes Theorem 3.31, which concerned the flexible role of resolutions in computing the cohomology of groups. Once that flexibility is in place in the more general setting of good categories, Sections 5–6 introduce derived functors and some of their properties. The main examples of derived functors at this stage are functors $\operatorname{Ext}(\,\cdot\,, \,\cdot\,)$ and $\operatorname{Tor}(\,\cdot\,, \,\cdot\,)$ obtained from Hom and tensor product; these are examined more closely in Section 7. The example given in Section III.5 and now being used as motivation requires some subtlety to be regarded as a derived functor. That example was the system of functors $H^n(G, \cdot)$ yielding cohomology of the group G with coefficients in the module $(\cdot)$; these were obtained in Section III.5 by applying the functor $\mathrm{Hom}_{\mathbb{Z}G}(\cdot, M)$ to any free resolution of $\mathbb{Z}$ in the category $\mathcal{C}_{\mathbb{Z}G}$. It is seen in examples in Section 5 that the effect of using the free resolution was to compute $H^n(G, M)$ as $\mathrm{Ext}^n_{\mathbb{Z}G}(\cdot, M)$ when the variable is set equal to $\mathbb{Z}$; realizing this result as a derived functor in the M variable requires knowing that one gets the same result from $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \cdot)$ when its variable is set equal to M. This conclusion is part of Theorem 4.31, which is proved in Section 7.

The first seven sections complete the treatment of the rudiments of homological algebra in the setting of good categories. One more central technique beyond that of derived functors is the mechanism of spectral sequences, but we shall omit this topic to save space.[2]

The chapter concludes with some discussion of abelian categories in Section 8. The foundations of homological algebra have to be redone completely when objects are no longer necessarily sets of elements. After this step, one introduces a substitute notion of "member" for elements, establishes its properties, and immediately obtains extensions of much of the theory to all abelian categories. A supplementary argument is needed whenever the theory for good categories uses an element-by-element construction of a homomorphism.

Sheaves are introduced in the last section of text in Chapter X, and their cohomology is mentioned very briefly there.

2. Complexes and Additive Functors

Let $\mathcal{C}$ be a good category of R modules in the sense of Section 1. A **complex** in $\mathcal{C}$ is a finite or infinite sequence of modules and maps in $\mathcal{C}$ such that the consecutive compositions are all 0. There is no harm in assuming that the indexing for the sequence is done by all of $\mathbb{Z}$, since we can always adjoin 0 modules and 0 maps as necessary to fill out the indexing. The indices may be increasing or decreasing, and, as we saw in Section III.5, this distinction is only a formality. However, the distinction is very convenient when it comes to applications, since homology is normally associated with decreasing indices and cohomology is normally associated with increasing indices.

Thus let us be more precise about the indexing. A **chain complex** in $\mathcal{C}$ is a sequence of pairs $X = \{(X_n, \partial_n)\}_{n=-\infty}^{\infty}$ in which each X_n is a module in $\mathcal{C}$,

[2]For the reader who is interested in learning about spectral sequences, this author is partial to the explanation of the topic in Appendix D of the book by Knapp and Vogan in the Selected References. The setting in that appendix is limited to good categories of modules, and some important applications are included.

each ∂_n is a map in $\mathrm{Hom}_R(X_{n+1}, X_n)$, and $\partial_n\partial_{n+1} = 0$ for all n. The maps ∂_n are sometimes called **boundary maps**, or **boundary operators**. We define the **homology** of X, written $H_*(X) = \{H_n(X)\}_{n=-\infty}^{\infty}$ with subscripts, to be the sequence of modules in $\mathcal{C}$ given by

$$H_n(X) = (\ker \partial_{n-1})/(\text{image}\, \partial_n).$$

The members of the space $\ker \partial_{n-1}$ are called n-**cycles**, and the members of the space image ∂_n are called n-**boundaries**.

EXAMPLES OF CHAIN COMPLEXES.

(1) Simplicial homology. Let S be a simplicial complex of dimension N, and number its vertices. For each integer n, the group $C_n(S)$ of simplicial n-chains is the free abelian group on the set of simplices of dimension n. This is 0 for $n < 0$ and $n > N$. In elementary topology one defines the boundary of each n-simplex to be the member of $C_{n-1}(S)$ equal to an integer combination of its faces, the coefficient of the face being $(-1)^i$ if the missing vertex for the face is the i^{th} of the $n+1$ vertices of the given n-simplex. This definition is extended additively to the boundary map $\partial_{n-1} : C_n(S) \to C_{n-1}(S)$, and a combinatorial argument gives $\partial_n\partial_{n-1} = 0$ for all n. Thus $X = \{(C_n(S), \partial_{n-1})\}$ is a complex. The associated homology $H_n(X)$ is the n^{th} (integral simplicial) homology of the simplicial complex S and is usually denoted by $H_n(S)$.

(2) Cubical singular homology. Let S be a topological space. For $n \geq 0$, a **singular** n-**cube** in S is a continuous function $T : I^n \to S$, where I^n denotes the n-fold product of the closed interval $[0, 1]$ with itself. The free abelian group on the set of n-cubes is denoted by $Q_n(S)$. A singular n-cube T is **degenerate** if its values are independent of one of the n variables. The subgroup of $Q_n(S)$ generated by the degenerate singular n-cubes is denoted by $D_n(S)$, and the quotient $C_n(S) = Q_n(S)/D_n(S)$ is the group of **cubical singular** n-**chains**. One defines a boundary operator from $Q_n(S)$ to $Q_{n-1}(S)$ for each n in analogy with the definition in the previous example and shows that it carries $D_n(S)$ into $D_{n-1}(S)$. Consequently the boundary operator descends to a homomorphism of abelian groups $\partial_{n-1} : C_n(S) \to C_{n-1}(S)$. A combinatorial argument shows that $\partial_n\partial_{n-1} = 0$; thus we get a complex. The associated homology is the n^{th} (integral singular) homology of S and is usually denoted by $H_n(S)$.

(3) Free resolution of $\mathbb{Z}$ in $\mathcal{C}_{\mathbb{Z}G}$. Let G be a group. Then the standard resolution of $\mathbb{Z}$ in the category $\mathcal{C}_{\mathbb{Z}G}$, as given in Theorem 3.20, is a chain complex in that category.

Let us make the class of chain complexes for the good category $\mathcal{C}$ into a category. Each chain complex is to be an object. If $X = \{(X_n, \partial_n\})$ and $X' = \{(X'_n, \partial'_n)\}$

are two chain complexes in $\mathcal{C}$, a morphism in $\mathrm{Morph}(X, X')$ is any **chain map** $f = \{f_n\}$, defined as a sequence of maps $f_n \in \mathrm{Hom}_R(X_n, X'_n)$ such that the diagram

$$\begin{array}{ccc} X_n & \xrightarrow{\partial_{n-1}} & X_{n-1} \\ {\scriptstyle f_n}\big\downarrow & & \big\downarrow{\scriptstyle f_{n-1}} \\ X'_n & \xrightarrow{\partial'_{n-1}} & X'_{n-1} \end{array}$$

commutes for all n. Briefly $f\partial = \partial' f$. Since the f_n's are functions, it is customary to use function notation $f : X \to X'$ for chain maps. The system $\{1_{X_n}\}$ of identity maps serves as an identity morphism, and coordinate-by-coordinate composition is associative. Thus the result is a category.

The next step is to observe that homology H_*, as applied to chain maps for the category $\mathcal{C}$, is a covariant functor from the category of chain maps to itself. The effect of the functor on objects is to send X to $H_*(X) = \{(H_n(X), 0)\}$. If $f : X \to X'$ is a chain map, then the formula $\partial'_{n-1}(f_n(x_n)) = f_{n-1}(\partial_{n-1}(x_n))$ shows that $f_n(\ker\partial_{n-1}) \subseteq \ker\partial'_{n-1}$, and the formula $\partial'_n(f_{n+1}(x_{n+1})) = f_n(\partial_n(x_{n+1}))$ shows that $f_n(\mathrm{image}\,\partial_n) \subseteq \mathrm{image}\,\partial'_n$. Therefore f_n descends to the quotient, giving a map $H(f_n) : H_n(X) \to H_n(X')$. The assembled collection of maps $H_*(f) : H_*(X) \to H_*(X')$ is manifestly a chain map. Instead of writing $H(f_n)$ for the map induced by f_n on the n^{th} homology, we shall often write $(f_n)_*$ or $\bar{f}_n$, especially in diagrams, to make the notation less cumbersome. Since the identity chain map yields the identity on $H_*(X)$ and since compositions go to compositions in the same order, homology H_* is a covariant functor.

If $f : X \to X'$ and $g : X \to X'$ are two chain maps, then a **homotopy** h of f to g is a system of maps $h = \{h_n\}$ increasing degrees by 1, i.e., having h_n carry X_n into X'_{n+1}, such that $h_{n-1}\partial_{n-1} + \partial'_n h_n = f_n - g_n$ for all n. Briefly $h\partial + \partial' h = f - g$. When such an h exists, we say that f and g are **homotopic**, and we write $f \simeq g$. This relation is an equivalence relation.

Proposition 4.1. If $f : X \to X'$ and $g : X \to X'$ are homotopic chain maps in the good category $\mathcal{C}$, then f and g induce the *same* maps $H_*(f)$ and $H_*(g)$ on homology, i.e., $H_n(f)$ and $H_n(g)$ are the same map of $H_n(X)$ into $H_n(X')$ for each n.

PROOF. Let h be a homotopy, and suppose that $\partial_{n-1}(x_n) = 0$. Then the computation $f_n(x_n) - g_n(x_n) = h_{n-1}\partial_{n-1}(x_n) + \partial'_n h_n(x_n) = 0 + \partial'_n h_n(x_n)$ shows that the images of x_n under f_n and g_n in X'_n differ by a member of image ∂'_n. $\square$

Briefly let us translate all of these definitions and conclusions into statements when the complexes have increasing indices. A **cochain complex** in $\mathcal{C}$ is a sequence of pairs $X = \{(X_n, d_n)\}_{n=-\infty}^{\infty}$ in which each X_n is a module in $\mathcal{C}$, each d_n is a map in $\mathrm{Hom}_R(X_n, X_{n+1})$, and $d_{n+1}d_n = 0$ for all n. The maps d_n are sometimes called **coboundary maps**, or **coboundary operators**. We define the **cohomology** of X, written $H^*(X) = \{H^n(X)\}_{n=-\infty}^{\infty}$ with superscripts, to be the sequence of modules in $\mathcal{C}$ given by $H^n(X) = (\ker d_n)/(\mathrm{image}\, d_{n-1})$. The members of the space $\ker d_n$ are called n-**cocycles**, and the members of image d_{n-1} are called n-**coboundaries**.

EXAMPLES OF COCHAIN COMPLEXES.

(1) Singular cohomology. Let S be a topological space, let $X = \{(C_n(S), \partial_{n-1})\}$ be its complex of cubical singular n-chains, and let M be any abelian group. If $C^n(S, M) = \mathrm{Hom}_{\mathbb{Z}}(C_n(S), M)$ and if $d_n : C^n(S, M) \to C^{n+1}(S, M)$ is the map $d_n = \mathrm{Hom}(\partial_{n+1}, 1)$, then $Y = \{(C^n(S, M)), d_n)\}$ is a cochain complex, and its cohomology, written $H^*(Y) = \{H^n(S, M)\}$, is the (integral singular) cohomology of S with coefficients in M.

(2) Cohomology of groups. Let G be a group, and let M be an abelian group on which G acts by automorphisms. Let $C^n(G, M)$ be the abelian group of functions from the n-fold product of G with itself into M, the functions being added pointwise. Define $\delta_n : C^n(G, M) \to C^{n+1}(G, M)$ as in Section III.5. Then $X = \{(C^n(G, M), \delta_n)\}$ is a cochain complex, and its cohomology $H^*(X) = \{H^n(G, M)\}$ is the cohomology of G with coefficients in M.

The cochain complexes for the good category $\mathcal{C}$ form a category for which the morphisms from $X = \{(X_n, d_n)\}$ to $X' = \{(X'_n, d'_n)\}$ are **cochain maps** $f = \{f_n\}$; the latter are defined by the conditions that f_n carry X_n to X'_n and $fd = df$, i.e., $f_{n+1}d_n = d_n f_n$ for all n. Cohomology H^*, as applied to cochain maps for the category $\mathcal{C}$, is a covariant functor from the category of cochain maps to itself. The effect of the functor on objects is to send X to $H^*(X) = \{(H^n(X), 0)\}$, and the argument that a cochain map $f : X \to X'$ carries $H^*(X)$ to $H^*(X')$ via a cochain map $H^*(f)$ is the same as for chain maps. Instead of writing $H(f_n)$ for the map induced by f_n on the n^{th} cohomology, we shall often write $(f_n)^*$ or $\bar{f}_n$, especially in diagrams, to make the notation less cumbersome.[3]

If $f : X \to X'$ and $g : X \to X'$ are two cochain maps, then a **homotopy** h of f to g is a system of maps $h = \{h_n\}$ decreasing degrees by 1, i.e., having h_n carry X_n into X'_{n-1}, such that $h_{n+1}d_n + d'_{n-1}h_n = f_n - g_n$ for all n. Briefly $hd + d'h = f - g$. When such an h exists, we say that f and g are **homotopic**, and we write $f \simeq g$. This relation is an equivalence relation.

[3]The notation with the bar is to be avoided when there might be some ambiguity about which of homology and cohomology is involved.

Proposition 4.1′. If $f : X \to X'$ and $g : X \to X'$ are homotopic cochain maps in the good category $\mathcal{C}$, then f and g induce the *same* maps $H^*(f)$ and $H^*(g)$ on cohomology, i.e., $H^n(f)$ and $H^n(g)$ are the same map of $H^n(X)$ into $H^n(X')$ for each n.

PROOF. Let h be a homotopy, and suppose that $d_n(x_n) = 0$. Then the computation $f_n(x_n) - g_n(x_n) = h_{n+1}d_n(x_n) + d'_{n-1}h_n(x_n) = 0 + d'_{n-1}h_n(x_n)$ shows that the images of x_n under f_n and g_n in X'_n differ by a member of image d'_{n-1}. $\square$

A chain or cochain complex written neutrally as $X = \{X(n)\}$ is **exact** at $X(n)$ if the kernel of the outgoing map at $X(n)$ equals the image of the incoming map at $X(n)$ (as opposed to merely containing the image). The complex is **exact**, or is an **exact sequence**, if it is exact at every $X(n)$. A **short exact sequence** is an exact sequence of the form

$$0 \longrightarrow A \overset{\varphi}{\longrightarrow} B \overset{\psi}{\longrightarrow} C \longrightarrow 0,$$

understood to have 0's at all positions beyond each end. The conditions on the 5-term complex above for it to be exact are that φ be one-one, ψ be onto C, and that ψ exhibit C as isomorphic to $B/\operatorname{image}\varphi$. To make the terminology more symmetric, it is customary to introduce a name for the quotient of the range of a homomorphism η by the image of η; this quotient is defined to be the **cokernel** of the homomorphism and is denoted by $\operatorname{coker}\eta$. The conditions for exactness above can then be restated more symmetrically as

$$\ker\varphi = \operatorname{coker}\psi = 0 \qquad \text{and} \qquad \operatorname{image}\varphi = \ker\psi.$$

An exact sequence can always be broken into short exact sequences by stretching each link

$$\cdots \longrightarrow A \overset{\varphi}{\longrightarrow} B \overset{\psi}{\longrightarrow} \cdots$$

into

$$\cdots \longrightarrow A \overset{\varphi}{\longrightarrow} \operatorname{image}\varphi \longrightarrow 0 \longrightarrow 0 \longrightarrow \ker\psi \overset{\text{inc}}{\longrightarrow} B \overset{\psi}{\longrightarrow} \cdots$$

and breaking it between the 0's; here "inc" denotes the inclusion mapping of $\ker\psi$ into B. This stretching process does not take us outside our good category, since good categories are assumed to be closed under passage to submodules and quotients. Conversely if we have two exact sequences

$$\cdots \longrightarrow A \overset{\varphi}{\longrightarrow} C \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow C \overset{i}{\longrightarrow} B \overset{\psi}{\longrightarrow} \cdots,$$

then we can combine them into an exact sequence

$$\cdots \longrightarrow A \xrightarrow{i\varphi} B \xrightarrow{\psi} \cdots .$$

Exactness at A of the merged sequence follows because $\ker(i\varphi) = \ker\varphi$, and exactness at B follows because $\ker\psi = \operatorname{image} i = \operatorname{image}(i\varphi)$.

Any map $\varphi : A \to B$ in our good category can be expressed in terms of an exact sequence by including the kernel and cokernel:

$$0 \longrightarrow \ker\varphi \xrightarrow{i} A \xrightarrow{\varphi} B \xrightarrow{q} \operatorname{coker}\varphi \longrightarrow 0;$$

here $i : \ker\varphi \to A$ is the inclusion, and $q : B \to \operatorname{coker}\varphi$ is the quotient mapping. All the modules and maps in the exact sequence are in the category, since good categories are assumed to be closed under passage to submodules and quotients. We shall use the following special case of this observation in Section 3.

Proposition 4.2. Let $X = \{(X_n, \partial_n)\}_{n=-\infty}^{\infty}$ be a chain complex in a good category with ∂_n in $\operatorname{Hom}_R(X_{n+1}, X_n)$ for each n. Then the boundary operator ∂_{n-1} on X_n descends to the quotient as a mapping $\bar\partial_{n-1} : \operatorname{coker}\partial_n \to \ker\partial_{n-2}$ and yields an exact sequence

$$0 \longrightarrow H_n(X) \xrightarrow{i} \operatorname{coker}\partial_n \xrightarrow{\bar\partial_{n-1}} \ker\partial_{n-2} \xrightarrow{q} H_{n-1}(X) \longrightarrow 0.$$

Here i is the inclusion $i : \ker\partial_{n-1}/\operatorname{image}\partial_n \to X_n/\operatorname{image}\partial_n$, and q is the quotient $q : \ker\partial_{n-2} \to \ker\partial_{n-2}/\operatorname{image}\partial_{n-1}$. This association of a six-term exact sequence to X for each n is functorial in the sense that if $X' = \{(X'_n, \partial'_n)\}_{n=-\infty}^{\infty}$ is a second chain complex and if $f : X \to X'$ is a chain complex, then the diagram

$$\begin{tikzcd}
H_n(X) \arrow[r, "i"] \arrow[d] & \operatorname{coker}\partial_n \arrow[r, "\bar\partial_{n-1}"] \arrow[d] & \ker\partial_{n-2} \arrow[r, "q"] \arrow[d] & H_{n-1}(X) \arrow[d] \\
H_n(X') \arrow[r, "i'"] & \operatorname{coker}\partial'_n \arrow[r, "\bar\partial'_{n-1}"] & \ker\partial'_{n-2} \arrow[r, "q'"] & H_{n-1}(X')
\end{tikzcd}$$

commutes; here the vertical maps are those induced by f_{n-1} and f_n.

REMARKS.

(1) The term "functorial" in the statement has a precise meaning in this and other contexts. Each chain complex is being carried to a 6-term exact sequence for each n. The chain complexes and the 6-term exact sequences both form categories, the morphisms in each case being chain maps. To say that the passage

from the objects of one category to the other is **functorial** is to say that the passage between the categories is actually a functor, i.e., chain maps for the chain complexes are sent to chain maps for the 6-term exact sequences, the identity goes to the identity, and compositions go to compositions. The latter two conditions are evident, and what needs proof is that chain maps are carried to chain maps.[4]

(2) For a cochain complex $X = \{(X_n, d_n)\}_{n=-\infty}^{\infty}$ with d_n in $\operatorname{Hom}_R(X_n, X_{n+1})$, the corresponding exact sequence is

$$0 \longrightarrow H_{n-1}(X) \xrightarrow{\ i\ } \operatorname{coker} d_{n-2} \xrightarrow{\bar{d}_{n-1}} \ker d_n \xrightarrow{\ q\ } H_n(X) \longrightarrow 0,$$

and it is functorial with respect to cochain maps.

PROOF. To see that the map $\bar{\partial}_{n-1}$ carries $\operatorname{coker} \partial_n$ to $\ker \partial_{n-2}$, we write it as a composition

$$\operatorname{coker} \partial_n = X_n / \operatorname{image} \partial_n \to X_n / \ker \partial_{n-1} \cong \operatorname{image} \partial_{n-1} \subseteq \ker \partial_{n-2},$$

with the arrow induced by the inclusion $\operatorname{image} \partial_n \subseteq \ker \partial_{n-1}$ and with the isomorphism induced by applying ∂_{n-1} to X_n and passing to the quotient. Then we have $\ker \bar{\partial}_{n-1} = \ker \partial_{n-1} / \operatorname{image} \partial_n = H_n(X)$ and

$$\begin{aligned} \operatorname{coker} \bar{\partial}_{n-1} &= \ker \partial_{n-2} \big/ \bar{\partial}_{n-1}(X_n / \operatorname{image} \partial_n) = \ker \partial_{n-2} \big/ \partial_{n-1} X_n \\ &= \ker \partial_{n-2} / \operatorname{image} \partial_{n-1} = H_{n-1}(X), \end{aligned}$$

and the exactness of the sequence is a special case of the exactness noted in the paragraph before the proposition.

For the assertion that the association is functorial, the left square commutes because the verticals are both induced by the same map f_n, and the right square commutes because the verticals are both induced by the same map f_{n-1}. For the middle square the commutativity follows from the fact that $f_{n-1}\partial_{n-1} = \partial'_{n-1} f_n$. □

[4] Some authors use the word "natural" instead of the word "functorial" in this situation. Authors who do this may have the notion of "natural transformation" between two functors in mind, or they may not. For those who do not, it seems advisable to use a different term like "functorial" to avoid confusion. For those who do, the allusion to a natural transformation is at best tortured in this instance. A natural transformation refers to two categories $\mathcal{C}$ and $\mathcal{C}'$, and the most intuitive choice for $\mathcal{C}$ here is the category of chain complexes X. There are to be two functors from $\mathcal{C}$ to $\mathcal{C}'$ and the natural transformation relates the values of those functors on X, for each X; no second complex X' enters into matters. To have X' involved in a natural transformation would mean including at least two chain complexes in each object of $\mathcal{C}$. In other instances, however, some additional structure may be present. Then the distinction between "functorial" and "natural" may be one of emphasis concerning the data. The statements of Propositions 4.29 and 4.30 below provide examples.

As was mentioned in Section 1, our interest will be in functors $F : \mathcal{C} \to \mathcal{C}'$ between two good categories, not necessarily involving the same ring, with the property of being **additive**. This means that $F(\varphi_1 + \varphi_2) = F(\varphi_1) + F(\varphi_2)$ when φ_1 and φ_2 are in the same $\mathrm{Hom}_R(A, B)$.

An additive functor sends any 0 map to the corresponding 0 map. Consequently it always sends complexes to complexes. Moreover, since any functor carries the identity map of each $\mathrm{Hom}_R(A, A)$ to an identity map, an additive functor has to send any module A for which the 0 map and the identity coincide to another such module. The 0 module is the unique module A with this property, and thus an additive functor has to send the 0 module to a 0 module.

Moreover, additive functors carry finite direct sums to finite direct sums. (Recall that good categories are closed under finite direct sums.) This fact needs proper formulation, and we need first to express direct sums in terms of modules and maps. From the point of view of category theory, we shall take advantage of the fact that for left R modules, product and coproduct coincide and are given by direct sum. If $C \cong A \oplus B$, then there are thus projections $p_A : C \to A$ and $p_B : C \to B$ and injections $\iota_A : A \to C$ and $\iota_B : B \to C$ such that

$$\begin{aligned} p_A\iota_A &= 1_A \qquad \text{and} \qquad p_B\iota_B = 1_B, \\ p_B\iota_A &= 0 \qquad\ \ \text{and} \qquad p_A\iota_B = 0, \end{aligned}$$

and

$$\iota_A p_A + \iota_B p_B = 1_C.$$

Conversely if we have maps p_A, ι_A, p_B, and ι_B with these properties, then the modules $A = \text{image } p_A$ and $B = \text{image } p_B$ have the property that C is the internal direct sum $C = \iota_A A \oplus \iota_B B$, and ι_A and ι_B are one-one. In fact, the equation $\iota_A p_A + \iota_B p_B = 1_C$ shows that $\iota_A A + \iota_B B = C$. To see that $\iota_A A \cap \iota_B B = 0$, let x be in the intersection. Then $p_B x$ lies in $p_B\iota_A A$, which is 0, and $p_A x$ lies in $p_A\iota_B B$, which is 0. Thus $\iota_A p_A + \iota_B p_B = 1_C$ gives $0 = \iota_A p_A x + \iota_B p_B x = x$. Hence $\iota_A A \cap \iota_B B = 0$ and $C = \iota_A A \oplus \iota_B B$. Finally the equations $p_A\iota_A = 1_A$ and $p_B\iota_B = 1_B$ imply that ι_A and ι_B are one-one.

With direct sum now expressed in terms of modules and maps, let us return to the effect of additive functors on direct sums. Let $C \cong A \oplus B$, and let p_A, p_B, ι_A, and ι_B be as above. Suppose that the additive functor F is covariant. Applying F to the displayed identities in the previous paragraph and using that F is additive, we see that $F(p_A)$, $F(p_B)$, $F(\iota_A)$, and $F(\iota_B)$ have the properties that allow us to recognize a direct sum. Hence

$$F(C) = F(\iota_A)F(A) \oplus F(\iota_B)F(B)$$

with $F(\iota_A)$ and $F(\iota_B)$ one-one. Thus

$$F(C) \cong F(A) \oplus F(B).$$

If instead F is contravariant, then the roles of the projections and the injections get interchanged, but we still obtain $F(C) \cong F(A) \oplus F(B)$.

An additive functor $F : \mathcal{C} \to \mathcal{C}'$ between two good categories is **exact** if it transforms exact sequences into exact sequences. Proposition 4.3 below will show that exact covariant functors preserve kernels, images, cokernels, submodules, quotients, and more. However, exact functors occur only infrequently; we shall see a few examples of them in Section 4. For examples of failures at exactness, it was shown in Section X.6 of *Basic Algebra* that if

$$0 \to M \xrightarrow{\varphi} N \xrightarrow{\psi} P \to 0$$

is a short exact sequence in the category $\mathcal{C}_R$, if E is a unital left R module, and if E' is a unital right R module, then the following sequences in $\mathcal{C}_{\mathbb{Z}}$ are exact:

$$E' \otimes_R M \xrightarrow{1\otimes\varphi} E' \otimes_R N \xrightarrow{1\otimes\psi} E' \otimes_R P \longrightarrow 0,$$

$$0 \longrightarrow \operatorname{Hom}_R(E, M) \xrightarrow{\operatorname{Hom}(1,\varphi)} \operatorname{Hom}_R(E, N) \xrightarrow{\operatorname{Hom}(1,\psi)} \operatorname{Hom}_R(E, P),$$

$$\operatorname{Hom}_R(M, E) \xleftarrow{\operatorname{Hom}(\varphi,1)} \operatorname{Hom}_R(N, E) \xleftarrow{\operatorname{Hom}(\psi,1)} \operatorname{Hom}_R(P, E) \longleftarrow 0;$$

on the other hand, the extensions of these complexes to 5-term complexes by the adjoining of a 0 need not be exact, and thus the functors $E' \otimes_R (\,\cdot\,)$, $\operatorname{Hom}_R(E,\,\cdot\,)$, and $\operatorname{Hom}_R(\,\cdot\,, E)$ are not exact for suitable choices of R, E, and E'.

Proposition 4.3. An additive functor $F : \mathcal{C} \to \mathcal{C}'$ between two good categories is exact if and only if it carries all short exact sequences into short exact sequences.

REMARK. This proposition makes it a little easier to test concrete additive functors for exactness than it would be from the definition.

PROOF. Necessity is obvious. For sufficiency, let

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C$$

be exact, and let the additive functor F be covariant, the contravariant case being completely analogous. Put $A_1 = \ker\varphi$, $B_1 = \ker\psi$, and $C_1 = \operatorname{image}\psi$. Since $\psi\varphi = 0$, we can factor φ as $\varphi = \varphi_2\varphi_1$, where $\varphi_1 : A \to B_1$ is φ with its range space reduced and where $\varphi_2 : B_1 \to B$ is the inclusion. Similarly we can factor ψ as $\psi = \psi_2\psi_1$, where $\psi_1 : B \to C_1$ is ψ with its range space reduced and where $\psi_2 : C_1 \to C$ is the inclusion. Of the sequences

$$0 \longrightarrow A_1 \longrightarrow A \xrightarrow{\varphi_1} B_1 \longrightarrow 0,$$

$$0 \longrightarrow B_1 \xrightarrow{\varphi_2} B \xrightarrow{\psi_1} C_1 \longrightarrow 0,$$

$$0 \longrightarrow C_1 \xrightarrow{\psi_2} C \to C/C_1 \longrightarrow 0,$$

the first and the third are trivially exact, and the second is exact because $\ker\psi_1 = \ker\psi = \operatorname{image}\varphi = \operatorname{image}\varphi_2$. The hypothesis that F carries short exact sequences to short exact sequences thus implies that the three sequences

$$F(A) \xrightarrow{F(\varphi_1)} F(B_1) \longrightarrow 0,$$
$$F(B_1) \xrightarrow{F(\varphi_2)} F(B) \xrightarrow{F(\psi_1)} F(C_1),$$
$$0 \longrightarrow F(C_1) \xrightarrow{F(\psi_2)} F(C)$$

are exact. From these, $\ker F(\psi_1) = \operatorname{image} F(\varphi_2)$. Also, $F(\psi_2)$ is one-one, so that

$$\ker F(\psi_1) = \ker\big(F(\psi_2)F(\psi_1)\big) = \ker F(\psi),$$

and $F(\varphi_1)$ is onto, so that

$$\operatorname{image} F(\varphi_2) = \operatorname{image}\big(F(\varphi_2)F(\varphi_1)\big) = \operatorname{image} F(\varphi).$$

Hence $\ker F(\psi) = \operatorname{image} F(\varphi)$, and

$$F(A) \xrightarrow{F(\varphi)} F(B) \xrightarrow{F(\psi)} F(C)$$

is exact, as required. $\square$

Proposition 4.4. Let $F : \mathcal{C} \to \mathcal{C}'$ be an additive functor between good categories, let X be a complex in $\mathcal{C}$, and let $F(X)$ be the corresponding complex in $\mathcal{C}'$. If F is exact, then F carries the homology or cohomology of X to the homology or cohomology of $F(X)$.

REMARKS. Our convention is to refer to homology when the indexing goes down and cohomology when the indexing goes up. If F is covariant, it preserves the indexing, while if F is contravariant, it reverses it. For the proof we shall use notation A, B, C for modules that is neutral with respect to the indexing. The arguments are qualitatively different in the covariant and contravariant cases, and we shall give both of them.

PROOF IN THE COVARIANT CASE. Let

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C$$

be a given complex, thus having $\psi\varphi = 0$, and form the image complex

$$F(A) \xrightarrow{F(\varphi)} F(B) \xrightarrow{F(\psi)} F(C).$$

We are to exhibit an isomorphism

$$F(\ker\psi/\operatorname{image}\varphi) \cong \ker F(\psi)/\operatorname{image} F(\varphi). \tag{$*$}$$

Let $i : \operatorname{image}\varphi \to \ker\psi$ and $j : \ker\psi \to B$ be the inclusions, and let $q : \ker\psi \to \ker\psi/\operatorname{image}\varphi$ be the quotient map. Applying F to the exact sequence

$$0 \longrightarrow \operatorname{image}\varphi \xrightarrow{\ i\ } \ker\psi \xrightarrow{\ q\ } \ker\psi/\operatorname{image}\varphi \longrightarrow 0$$

and using exactness, we obtain an isomorphism via $F(q)$:

$$F(\ker\psi/\operatorname{image}\varphi) \cong F(\ker\psi)\big/F(i)F(\operatorname{image}\varphi). \tag{$**$}$$

Since j is one-one and F is exact, $F(j)$ is one-one. Thus application of $F(j)$ to the right side of $(**)$ gives

$$F(\ker\psi/\operatorname{image}\varphi) \cong F(j)F(\ker\psi)\big/F(ji)F(\operatorname{image}\varphi). \tag{$\dagger$}$$

If $\overline{\varphi}$ denotes φ with its range reduced to its image, then $\varphi = ji\overline{\varphi}$. Applying F to the two exact sequences

$$\ker\psi \xrightarrow{j} B \xrightarrow{\psi} C,$$
$$A \xrightarrow{\overline{\varphi}} \operatorname{image}\varphi \longrightarrow 0$$

gives us $F(j)F(\ker\psi) = \ker F(\psi)$ and $F(\operatorname{image}\varphi) = F(\overline{\varphi})F(A)$. Applying $F(ji)$ to the second of these and substituting both into the right side of $(\dagger)$ transforms $(\dagger)$ into $(*)$ and gives the required isomorphism. $\square$

PROOF IN THE CONTRAVARIANT CASE. Let

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C$$

be given with $\psi\varphi = 0$, and form the image complex

$$F(A) \xleftarrow{F(\varphi)} F(B) \xleftarrow{F(\psi)} F(C).$$

We are to exhibit an isomorphism

$$F(\ker\psi/\operatorname{image}\varphi) \cong \ker F(\varphi)/\operatorname{image} F(\psi). \tag{$*$}$$

Let $j : \ker\psi \to B$ be the inclusion, let $\bar{j} : \ker\psi/\operatorname{image}\varphi \to B/\operatorname{image}\psi$ be the induced map between quotients, and let q, q', q'' be the quotient maps

$$q : B \to B/\ker\psi,$$
$$q' : B \to B/\operatorname{image}\varphi,$$
$$q'' : B/\operatorname{image}\varphi \to B/\ker\psi.$$

These satisfy $q = q''q'$. Applying F to the exact sequence

$$0 \longrightarrow \ker\psi/\operatorname{image}\varphi \xrightarrow{\bar{j}} B/\operatorname{image}\varphi \xrightarrow{q''} B/\ker\psi \longrightarrow 0$$

and using exactness, we obtain an isomorphism via $F(\bar{j})$:

$$F(\ker\psi/\operatorname{image}\varphi) \cong F(B/\operatorname{image}\varphi)\Big/F(q'')F(B/\ker\psi). \qquad (**)$$

Since q' is onto and F is exact, $F(q')$ is one-one. Thus application of $F(q')$ to the right side of $(**)$ gives

$$F(\ker\psi/\operatorname{image}\varphi) \cong F(q')F(B/\operatorname{image}\varphi)\Big/F(q)F(B/\ker\psi). \qquad (\dagger)$$

Applying F to the three exact sequences

$$A \xrightarrow{\varphi} B \xrightarrow{q'} B/\operatorname{image}\varphi,$$
$$\ker\psi \xrightarrow{j} B \xrightarrow{\psi} C,$$
$$\ker\psi \xrightarrow{j} B \xrightarrow{q} B/\ker\psi$$

gives us $F(q')F(B/\operatorname{image}\varphi) = \ker F(\varphi)$ and $F(q)F(B/\ker\psi) = \ker F(j) = \operatorname{image} F(\psi)$. Substituting both these equalities into the right side of $(\dagger)$ transforms $(\dagger)$ into $(*)$ and gives the required isomorphism. $\square$

We were reminded before Proposition 4.3 that Hom_R and $\otimes_R$ need not yield exact functors. The partial exactness that they exhibit, as opposed to exactness itself, is more typical of additive functors, and we incorporate this behavior into two definitions. We shall define left and right exactness in such a way that Hom_R is left exact in each variable and $\otimes_R$ is right exact. An additive functor F is **left exact** if the exactness of

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$

implies the exactness of

$$0 \longrightarrow F(A) \xrightarrow{F(\varphi)} F(B) \xrightarrow{F(\psi)} F(C) \qquad (F \text{ covariant}),$$
$$0 \longrightarrow F(C) \xrightarrow{F(\psi)} F(B) \xrightarrow{F(\varphi)} F(A) \qquad (F \text{ contravariant}).$$

We say that F is **right exact** if the exactness of the sequence with $0, A, B, C, 0$ above implies the exactness of

$$F(A) \xrightarrow{F(\varphi)} F(B) \xrightarrow{F(\psi)} F(C) \longrightarrow 0 \qquad (F \text{ covariant}),$$

$$F(C) \xrightarrow{F(\psi)} F(B) \xrightarrow{F(\varphi)} F(A) \longrightarrow 0 \qquad (F \text{ contravariant}).$$

The words "left" and "right" refer to the part of the *target* sequence that is exact when the arrows are arranged to point to the right. A consequence (but not the full content) of these definitions in each case is an assertion about one-one or onto maps. For example a left exact covariant F carries one-one maps to one-one maps; we have only to start from a one-one map $\varphi : A \to B$ and set up a short exact sequence with $C = B/\operatorname{image}\varphi$, and the definition shows that $F(\varphi)$ is one-one.

Proposition 4.5. If F is a covariant left exact functor, then F carries an exact sequence

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C$$

into an exact sequence

$$0 \longrightarrow F(A) \xrightarrow{F(\varphi)} F(B) \xrightarrow{F(\psi)} F(C).$$

REMARK. The expected analogs of this result are valid if F is contravariant or if F is right exact or both.

PROOF. Starting from the given exact sequence, let $i : \operatorname{image}\psi \to C$ be the inclusion, and let $\overline{\psi} : B \to \operatorname{image}\psi$ be ψ with its range space reduced. Then $\psi = i\overline{\psi}$, and the sequences

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\overline{\psi}} \operatorname{image}\psi \longrightarrow 0$$

and
$$0 \longrightarrow \operatorname{image}\psi \xrightarrow{i} C \longrightarrow C/\operatorname{image}\psi \longrightarrow 0$$

are exact. Applying F and using its left exactness, we see that

$$0 \longrightarrow F(A) \xrightarrow{F(\varphi)} F(B) \xrightarrow{F(\overline{\psi})} F(\operatorname{image}\psi)$$

and
$$0 \longrightarrow F(\operatorname{image}\psi) \xrightarrow{F(i)} F(C)$$

are exact. Thus $F(i)$ is one-one, and $F(\psi) = F(i\overline{\psi}) = F(i)F(\overline{\psi})$ has the same kernel as $F(\overline{\psi})$. The exactness of the first image complex shows that $\ker F(\overline{\psi}) = \operatorname{image} F(\varphi)$, and the proof of the required exactness is complete. $\square$

3. Long Exact Sequences

As in Section 2, let $\mathcal{C}$ be a good category. We have seen that chain complexes in $\mathcal{C}$ themselves form a category whose morphisms are chain maps. If we have several chain maps in succession, each with an index $n \in \mathbb{Z}$, we can say that they form an "exact sequence" of chain maps if for each n, the sequences of modules and maps having index n form an exact sequence in $\mathcal{C}$. Our objective in this section is to show that any short exact sequence of complexes of this kind yields a "long exact sequence" of modules and maps in $\mathcal{C}$ involving all the indices. More precisely we are able to construct for each n a "connecting homomorphism" relating[5] what happens with each index n to what happens for index $n+1$ or $n-1$ and incorporating modules and maps for all indices into a single exact sequence of infinite length.

By way of preparation for the construction of connecting homomorphisms, let us be more explicit about the discussion in Section 2 of how a chain map carries the homology of one complex to the homology of another complex. Let

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} \\ A' & \xrightarrow{\varphi'} & B' \end{array}$$

be a commutative diagram in the good category $\mathcal{C}$. Let us observe that $\varphi(\ker\alpha) \subseteq \ker\beta$; in fact, any $a \in \ker\alpha$ has $0 = \varphi'\alpha(a) = \beta\varphi(a)$, and thus $\varphi(a)$ is in $\ker\beta$. Let us observe further that $\varphi'(\alpha(A)) = \beta(\varphi(A)) \subseteq \beta(B)$; since φ' carries A' into B', it follows that φ' descends to a mapping $\overline{\varphi}'$ defined on $A'/\alpha(A) = \operatorname{coker}\alpha$ and taking values in $B'/\beta(B) = \operatorname{coker}\beta$. We can summarize these remarks by the inclusions

$$\varphi(\ker\alpha) \subseteq \ker\beta \qquad \text{and} \qquad \overline{\varphi}'(\operatorname{coker}\alpha) \subseteq \operatorname{coker}\beta.$$

Using these remarks, we can now construct a "connecting homomorphism" whenever we have a diagram as in Figure 4.1 below.

[5]For readers familiar with the use of homology in topology, connecting homomorphisms arise when one works with the homology of a topological space, the homology of a subspace, and the relative homology of the space and the subspace; the construction in this section may be regarded as an abstract version of that construction.

$$\begin{tikzcd}
 & A \arrow[r, "\varphi"] \arrow[d, "\alpha"] & B \arrow[r, "\psi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\varphi'"] & B' \arrow[r, "\psi'"] & C' &
\end{tikzcd}$$

FIGURE 4.1. Snake diagram. The rows are assumed exact, and the squares commute. In this situation the Snake Lemma constructs a connecting homomorphism $\omega : \ker\gamma \to \operatorname{coker}\alpha$.

Lemma 4.6 (Snake Lemma). In a good category $\mathcal{C}$, a snake diagram as in Figure 4.1 induces a homomorphism $\omega : \ker\gamma \to \operatorname{coker}\alpha$ with

$$\ker\omega = \psi(\ker\beta) \qquad \text{and} \qquad \operatorname{image}\omega = \varphi'^{-1}(\operatorname{image}\beta)/\operatorname{image}\alpha,$$

and with $\omega(c) = \varphi'^{-1}(\beta(\psi^{-1}(c))) + \operatorname{image}\alpha$ for $c \in \ker\gamma$, and then

$$\ker\alpha \xrightarrow{\overline{\varphi}} \ker\beta \xrightarrow{\overline{\psi}} \ker\gamma \xrightarrow{\omega} \operatorname{coker}\alpha \xrightarrow{\overline{\varphi}'} \operatorname{coker}\beta \xrightarrow{\overline{\psi}'} \operatorname{coker}\gamma$$

is an exact sequence. Here $\overline{\varphi}$ and $\overline{\psi}$ are restrictions of φ and ψ, and $\overline{\varphi}'$ and $\overline{\psi}'$ are descended versions of φ and ψ. If φ is one-one, then $\overline{\varphi}$ is one-one. If ψ' is onto C', then $\overline{\psi}'$ is onto $\operatorname{coker}\gamma$.

REMARKS. The homomorphism ω is called a **connecting homomorphism**. The name "Snake Lemma" comes from the pattern that the six-term exact sequence makes when superimposed on the enlarged version of Figure 4.1 shown in Figure 4.2.

$$\begin{tikzcd}
 & \ker\alpha \arrow[r] \arrow[d] & \ker\beta \arrow[r] \arrow[d] & \ker\gamma \arrow[d] & \\
 & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[d] & \\
 & \operatorname{coker}\alpha \arrow[r] & \operatorname{coker}\beta \arrow[r] & \operatorname{coker}\gamma &
\end{tikzcd}$$

FIGURE 4.2. Enlarged snake diagram.

PROOF. First let us construct ω and see that it is well defined. Let c be in $\ker\gamma$. Since ψ is onto C, write $c = \psi(b)$ for some $b \in B$. The commutativity of the second square in Figure 4.1 gives $0 = \gamma(c) = \gamma\psi(b) = \psi'(\beta b)$. Thus $\beta(b)$ is in $\ker\psi' = \operatorname{image}\varphi'$, and $\beta(b) = \varphi'(a')$ for some $a' \in A'$; the element a' is uniquely determined, since φ' is one-one. Define $\omega(c) = a' + \alpha(A)$.

The only choice in this definition is that of b, and we are to show that any other choice leads to the same member of $\operatorname{coker}\alpha$. If $\bar{b}$ is another choice and if $\beta(\bar{b}) = \varphi'(\bar{a}')$ with $a' \in A'$, then $\psi(\bar{b} - b) = c - c = 0$ shows that $\bar{b} - b = \varphi(a)$ for some $a \in A$. Thus $\varphi'(\bar{a}' - a') = \beta(\bar{b} - b) = \beta\varphi(a) = \varphi'(\alpha(a))$. Since φ' is one-one, $\bar{a}' - a' = \alpha(a)$, and $\bar{a}'$ and a' are exhibited as in the same coset of A' modulo $\alpha(A)$.

Let us compute $\ker\omega$. Suppose that $\omega(c) = 0$, i.e., that $\omega(c)$ is in $\alpha(A)$. Say $\omega(c) = \alpha(a)$. By construction of ω, $\omega(c) = a' + \alpha(A)$ for an element $a' \in A'$ such that $\beta(b) = \varphi'(a')$ and $c = \psi(b)$. In this case, $a' = \alpha(a)$. So $\beta(b) = \varphi'\alpha(a) = \beta\varphi(a)$, and thus $b - \varphi(a)$ is in $\ker\beta$. Consequently $c = \psi(b) = \psi(b) - \psi\varphi(a)$ is in $\psi(\ker\beta)$, and $\ker\omega \subseteq \psi(\ker\beta)$. For the reverse inclusion, if c is in $\psi(\ker\beta)$, choose $b \in \ker\beta$ with $\psi(b) = c$. Then $\gamma(c) = \gamma\psi(b) = \psi'\beta(b) = 0$ shows that $\omega(c)$ is defined. Since $c = \psi(b)$, the construction of ω shows that $\beta(b) = \varphi'(a')$ for some $a' \in A'$. Since b is in $\ker\beta$ and since φ' is one-one, this a' must be 0. Then $\omega(c) = a' + \alpha(A) = 0 + \alpha(A)$, c is in $\ker\omega$, and $\psi(\ker\beta) \subseteq \ker\omega$.

Now we compute $\operatorname{image}\omega$. Our step-by-step definition of ω shows that $\operatorname{image}\omega \subseteq \varphi'^{-1}(\operatorname{image}\beta)/\alpha(A)$. For the reverse inclusion, suppose that $a' \in A'$ is in $\varphi'^{-1}(\operatorname{image}\beta)$, i.e., has $\varphi'(a') = \beta(b)$ for some $b \in B$. Then the element $c = \varphi(b)$ of C has $\gamma(c) = \gamma\psi(b) = \psi'\beta(b) = \psi'\varphi'(a') = 0$, and $\omega(c)$ is therefore defined. Our definition of ω makes $\omega(c) = a' + \alpha(A)$, and thus $\varphi'^{-1}(\operatorname{image}\beta)/\alpha(A) \subseteq \operatorname{image}\omega$.

We are left with establishing the exactness of the displayed sequence of six terms at the four positions other than the ends and with proving the two assertions in the last sentence of the lemma.

The condition of exactness at $\ker\beta$ is that $\varphi(\ker\alpha) = \ker\psi \cap \ker\beta$. The inclusion $\subseteq$ follows from the equalities $0 = \psi\varphi$ and $\beta\varphi(\ker\alpha) = \varphi'\alpha(\ker\alpha) = 0$. For the inclusion $\supseteq$, let $b \in B$ satisfy $\psi(b) = \beta(b) = 0$. Exactness at B gives $b = \varphi(a)$ with $a \in A$. Then $0 = \beta(b) = \beta\varphi(a) = \varphi'\alpha(a)$ with φ' one-one implies that $\alpha(a) = 0$, and a is in $\ker\alpha$. Thus b is in $\varphi(\ker\alpha)$, and exactness at $\ker\beta$ is proved. If φ is one-one, then certainly its restriction $\overline{\varphi}$ is one-one.

The condition of exactness at $\ker\gamma$ is that $\ker\omega = \psi(\ker\beta)$, and this was proved in the third paragraph of the proof.

By the result of the fourth paragraph, the condition of exactness at $\operatorname{coker}\alpha$ is that $\varphi'^{-1}(\beta(B))/\alpha(A)$ equal $\ker\overline{\varphi}'$, where $\overline{\varphi}' : A'/\alpha(A) \to B'/\beta(B)$ is the map induced by φ'. The members of $\ker\overline{\varphi}'$ are those cosets $a' + \alpha(A)$ with

$\varphi'(a' + \alpha(A)) \subseteq \beta(B)$. Since $\varphi'\alpha(A) = \beta\varphi(A) \subseteq \beta(B)$, the condition on $a' + \alpha(A)$ is that $\varphi'(a')$ be in $\beta(B)$, hence that a' be in $\varphi'^{-1}(\beta(B))$, hence that the coset $a' + \alpha(A)$ be in $\varphi'^{-1}(\beta(B))/\alpha(A)$. Thus we have exactness at $\operatorname{coker}\alpha$.

At $\operatorname{coker}\beta$, we know that the descended map $\overline{\varphi}'$ maps $\operatorname{coker}\alpha$ into $\operatorname{coker}\beta$, and we are to show that $\overline{\varphi}'(\operatorname{coker}\alpha) = \ker\overline{\psi}'$. Inclusion $\subseteq$ follows because $\psi'\varphi' = 0$ implies $\overline{\psi}'\overline{\varphi}'(a' + \alpha(A)) = \overline{\psi}'(\varphi'(a') + \beta(B)) = \psi'\varphi'(a') + \gamma(C) = \gamma(C)$. For the reverse inclusion let $b' \in B'$ have $\overline{\psi}'(b' + \beta(B)) = \gamma(C)$. Then $\psi'(b')$ is in $\gamma(C)$. Since $\psi : B \to C$ is onto, we can find $b \in B$ with $\psi'(b') = \gamma\psi(b) = \psi'\beta(b)$. Hence $b' - \beta(b)$ is in $\ker\psi' = \operatorname{image}\varphi'$, and $b' - \beta(b) = \varphi'(a')$ for some $a' \in A'$. Consequently $b' + \beta(B) = \varphi'(a') + \beta(b) + \beta(B) = \varphi'(a') + \beta(B) = (\varphi')_*(a' + \alpha(A))$, and $b' + \beta(B)$ is exhibited as in $(\varphi')_*(a' + \alpha(A))$, i.e., in $(\varphi')_*(\operatorname{coker}\alpha)$. Thus we have exactness at $\operatorname{coker}\beta$. Finally if ψ' is onto C', then certainly its descended map $\overline{\psi}'$ is onto $\operatorname{coker}\gamma$. This completes the proof. □

Theorem 4.7. Let $A = \{(A_n, \alpha_n)\}$, $B = \{(B_n, \beta_n)\}$, and $C = \{(C_n, \gamma_n)\}$ be chain complexes in a good category $\mathcal{C}$, and suppose that $\varphi = \{\varphi_n\} : A \to B$ and $\psi = \{\psi_n\} : B \to C$ are chain maps such that the sequence

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

of chain complexes is exact. Then this exact sequence of chain complexes induces an exact sequence in homology of the form

$$\cdots \longrightarrow H_{n+1}(C) \xrightarrow{\overline{\omega}_n} H_n(A) \xrightarrow{\overline{\varphi}_n} H_n(B) \xrightarrow{\overline{\psi}_n} H_n(C) \xrightarrow{\overline{\omega}_{n-1}} H_{n-1}(A) \longrightarrow \cdots .$$

Here the map $\overline{\omega}_n : H_{n+1}(C) \to H_n(A)$ has descended from the connecting homomorphism ω_n defined on $\ker\gamma_n$ in C_{n+1} and having range $\operatorname{coker}\alpha_n = A_n/\operatorname{image}\alpha_n$.

REMARKS.

(1) The exact sequence in homology is called the **long exact sequence** in homology corresponding to the short exact sequence of chain complexes, and the maps ω_n are called **connecting homomorphisms**. As the proof will show, these connecting homomorphisms arise by *two* applications of the Snake Lemma, not just one.

(2) In more detail the diagram of the short exact sequence of chain complexes is of the form

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{n+1} & \xrightarrow{\varphi_{n+1}} & B_{n+1} & \xrightarrow{\psi_{n+1}} & C_{n+1} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \alpha_n} & & \Big\downarrow{\scriptstyle \beta_n} & & \Big\downarrow{\scriptstyle \gamma_n} & & \\
0 & \longrightarrow & A_n & \xrightarrow{\varphi_n} & B_n & \xrightarrow{\psi_n} & C_n & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \alpha_{n-1}} & & \Big\downarrow{\scriptstyle \beta_{n-1}} & & \Big\downarrow{\scriptstyle \gamma_{n-1}} & & \\
0 & \longrightarrow & A_{n-1} & \xrightarrow{\varphi_{n-1}} & B_{n-1} & \xrightarrow{\psi_{n-1}} & C_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

The rows are exact, the columns are chain complexes, and the squares commute.

(3) The corresponding result for cochain complexes involves the diagram

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A_{n+1} & \xrightarrow{\varphi_{n+1}} & B_{n+1} & \xrightarrow{\psi_{n+1}} & C_{n+1} & \longrightarrow & 0 \\
 & & \Big\uparrow{\scriptstyle \alpha_n} & & \Big\uparrow{\scriptstyle \beta_n} & & \Big\uparrow{\scriptstyle \gamma_n} & & \\
0 & \longrightarrow & A_n & \xrightarrow{\varphi_n} & B_n & \xrightarrow{\psi_n} & C_n & \longrightarrow & 0 \\
 & & \Big\uparrow{\scriptstyle \alpha_{n-1}} & & \Big\uparrow{\scriptstyle \beta_{n-1}} & & \Big\uparrow{\scriptstyle \gamma_{n-1}} & & \\
0 & \longrightarrow & A_{n-1} & \xrightarrow{\varphi_{n-1}} & B_{n-1} & \xrightarrow{\psi_{n-1}} & C_{n-1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

and the corresponding **long exact sequence** in cohomology is

$$\cdots \longrightarrow H^{n-1}(C) \xrightarrow{\overline{\omega}_n} H^n(A) \xrightarrow{\overline{\varphi}_n} H^n(B) \xrightarrow{\overline{\psi}_n} H^n(C) \xrightarrow{\overline{\omega}_{n+1}} H^{n+1}(A) \longrightarrow \cdots .$$

The result for cochain complexes is a consequence of the result for chain complexes and follows by making adjustments in the notation.

PROOF. We regard the top two displayed rows of the diagram in Remark 2 as a snake diagram. Applying the Snake Lemma (Lemma 4.6), we obtain a connecting homomorphism ω_n and an exact sequence

$$\ker\alpha_n \xrightarrow{\overline{\varphi}_{n+1}} \ker\beta_n \xrightarrow{\overline{\psi}_{n+1}} \ker\gamma_n \xrightarrow{\omega_n} \operatorname{coker}\alpha_n \xrightarrow{\overline{\varphi}'_n} \operatorname{coker}\beta_n \xrightarrow{\overline{\psi}'_n} \operatorname{coker}\gamma_n.$$

Using Proposition 4.2 for each of the chain complexes $A = \{(A_n, \alpha_n)\}$, $B = \{(B_n, \beta_n)\}$, and $C = \{(C_n, \gamma_n)\}$, we see that we obtain a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & H_n(A) \arrow[d] & H_n(B) \arrow[d] & H_n(C) \arrow[d] & \\
 & \operatorname{coker}\alpha_n \arrow[r, "\overline{\varphi}_n"] \arrow[d, "\overline{\alpha}_{n-1}"] & \operatorname{coker}\beta_n \arrow[r, "\overline{\psi}_n"] \arrow[d, "\overline{\beta}_{n-1}"] & \operatorname{coker}\gamma_n \arrow[r] \arrow[d, "\overline{\gamma}_{n-1}"] & 0 \\
0 \arrow[r] & \ker\alpha_{n-2} \arrow[r, "\overline{\varphi}_{n-1}"] \arrow[d] & \ker\beta_{n-2} \arrow[r, "\overline{\psi}_{n-1}"] \arrow[d] & \ker\gamma_{n-2} \arrow[d] & \\
 & H_{n-1}(A) \arrow[d] & H_{n-1}(B) \arrow[d] & H_{n-1}(C) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

in which the rows and columns are exact and the squares commute. The third and fourth rows form a snake diagram, and the second and fifth rows identify the kernels and cokernels. Thus the Snake Lemma gives us an exact sequence

$$H_n(A) \xrightarrow{\overline{\varphi}_n} H_n(B) \xrightarrow{\overline{\psi}_n} H_n(C) \xrightarrow{\Omega} H_{n-1}(A) \xrightarrow{\overline{\varphi}'_{n-1}} H_{n-1}(B) \xrightarrow{\overline{\psi}'_{n-1}} H_{n-1}(C)$$

for a suitable connecting homomorphism Ω. Repeating this argument for all n proves exactness at all modules of the long exact sequence.

To complete the proof, we have only to identify Ω. Reference to the statement of the Snake Lemma shows that the formula for Ω is

$$\Omega(\bar{c}) = (\overline{\varphi}'_{n-1})^{-1}(\overline{\beta}_{n-1}(\overline{\psi_n}^{-1}(\bar{c}))) + \operatorname{image}\overline{\alpha}_{n-1}$$

for $\bar{c} \in H_n(C)$. Meanwhile, the connecting homomorphism from the first application of the Snake Lemma is $\omega_{n-1}(c) = (\varphi'_{n-1})^{-1}(\beta_{n-1}(\psi_n^{-1}(c))) + \operatorname{image}\alpha_{n-1}$ for $c \in \ker\gamma_{n-1}$. Thus $\Omega(c + \operatorname{image}\gamma_n) = \omega_{n-1}(c) + \operatorname{image}\alpha_{n-1}$ as asserted. $\square$

Corollary 4.8. If

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

is an exact sequence of chain complexes in a good category and if A is exact, then $H_n(B) \cong H_n(C)$ for all n; if instead C is exact, then $H_n(A) \cong H_n(B)$ for all n. Consequently if any two of the three chain complexes are exact, then the third one is exact.

PROOF. Theorem 4.7 gives the long exact sequence

$$\cdots \longrightarrow H_{n+1}(C) \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \longrightarrow H_{n-1}(A) \longrightarrow \cdots .$$

If $H_n(A) = 0$ and $H_{n-1}(A) = 0$, then we see that $H_n(B) \cong H_n(C)$. If $H_{n+1}(C) = 0$ and $H_n(C) = 0$, then we see that $H_n(A) \cong H_n(B)$.

If two of the three chain complexes are exact, then one of the two is A or C, and the result in the previous paragraph applies. Then the other two complexes (B and C, or A and B) have isomorphic homology. The hypothesis says that one of these two sequences of homology groups is 0. Therefore the other one is 0. $\square$

To conclude the discussion, we shall prove results saying that the exact sequences produced by Lemma 4.6 and Theorem 4.7 are functorial.

Lemma 4.9. In a good category $\mathcal{C}$, the six-term exact sequence that is obtained from a snake diagram as in Figure 4.1 is functorial in the following sense: If there are two horizontal planar snake diagrams, one with tildes ($\sim$) over all modules and maps and the other as is, and if there are vertical maps f_A, etc., in three dimensions from the tilde version of the snake diagram to the original version such that all vertical squares commute, then the squares of the diagram

$$\begin{array}{ccccccccccc}
\ker\widetilde{\alpha} & \xrightarrow{\overline{\widetilde{\varphi}}} & \ker\widetilde{\beta} & \xrightarrow{\overline{\widetilde{\psi}}} & \ker\widetilde{\gamma} & \xrightarrow{\widetilde{\omega}} & \operatorname{coker}\widetilde{\alpha} & \xrightarrow{\overline{\widetilde{\varphi}'}} & \operatorname{coker}\widetilde{\beta} & \xrightarrow{\overline{\widetilde{\psi}'}} & \operatorname{coker}\widetilde{\gamma} \\
\Big\downarrow{\scriptstyle \bar{f}_A} & & \Big\downarrow{\scriptstyle \bar{f}_B} & & \Big\downarrow{\scriptstyle \bar{f}_C} & & \Big\downarrow{\scriptstyle \bar{f}_{A'}} & & \Big\downarrow{\scriptstyle \bar{f}_{B'}} & & \Big\downarrow{\scriptstyle \bar{f}_{C'}} \\
\ker\alpha & \xrightarrow{\overline{\varphi}} & \ker\beta & \xrightarrow{\overline{\psi}} & \ker\gamma & \xrightarrow{\omega} & \operatorname{coker}\alpha & \xrightarrow{\overline{\varphi}'} & \operatorname{coker}\beta & \xrightarrow{\overline{\psi}'} & \operatorname{coker}\gamma
\end{array}$$

all commute.

PROOF. For the first square from the left, the assumed commutativity shows that $f_{A'}\widetilde{\alpha} = \alpha f_A$, and thus $x \in \ker\widetilde{\alpha}$ implies $f_A(x) \in \ker\alpha$; similarly $x \in \ker\widetilde{\beta}$ implies $f_B(x) \in \ker\beta$. Thus the maps of the square are well defined. We are given also that $\varphi f_A = f_B\widetilde{\varphi}$, and this proves that the square commutes. The second square from the left is handled similarly.

For the fourth square from the left, the equation $f_{A'}\widetilde{\alpha} = \alpha f_A$ shows that $y = \widetilde{\alpha}(x)$ implies $f_{A'}(y) = \alpha(f_A(x))$, and thus $y \in \operatorname{image}\widetilde{\alpha}$ implies $f_{A'}(y) \in \operatorname{image}\alpha$; this means that $f_{A'}$ descends to a map $\bar{f}_{A'}$ of $\operatorname{coker}\widetilde{\alpha}$ to $\operatorname{coker}\alpha$. Similarly $f_{B'}$ descends to a map $\bar{f}_{B'}$ of $\operatorname{coker}\widetilde{\beta}$ to $\operatorname{coker}\beta$. Thus the maps of the square are well defined. We are given also that $\varphi' f_{A'} = f_{B'}\widetilde{\varphi}'$, and this proves that the square commutes. The fifth square from the left is handled similarly.

We are left with the third square from the left. The map at the left side of this square was shown to be well defined in the first paragraph of the proof, and the map at the right side of this square was shown to be meaningful in the second paragraph of the proof. We are to prove that the square commutes. Referring to the construction of $\widetilde{\omega}$, let $\widetilde{c}$ be in $\ker\widetilde{\gamma}$, choose $\widetilde{b}$ in $\widetilde{B}$ with $\widetilde{\psi}(\widetilde{b}) = \widetilde{c}$, and write $\widetilde{\beta}(\widetilde{b}) = \widetilde{\varphi}'(\widetilde{a}')$. Then $\widetilde{\omega}(\widetilde{c})$ is defined to be the coset of $\widetilde{a}'$. Using the assumed commutativity, we compute that $\psi f_B(\widetilde{b}) = f_C\widetilde{\psi}(\widetilde{b}) = f_C(\widetilde{c})$ and that

$$\varphi' f_{A'}(\widetilde{a}') = f_{B'}\widetilde{\varphi}'(\widetilde{a}') = f_{B'}\widetilde{\beta}(\widetilde{b}) = \beta f_B(\widetilde{b}).$$

Thus $f_B(\widetilde{b})$ is an element whose image under ψ is $f_C(\widetilde{c})$, and β of this element is $\varphi' f_{A'}(\widetilde{a}')$. Consequently the coset of $\omega(f_C(\widetilde{c}))$ is to be the coset of $f_{A'}(\widetilde{a}') = f_{A'}\widetilde{\omega}(c)$. This proves the desired commutativity. $\square$

Theorem 4.10. In a good category $\mathcal{C}$, the long exact sequence that is obtained from a short exact sequence of chain complexes as in Theorem 4.7 is functorial in the following sense: if there are two short exact sequences of chain complexes as in the theorem, one with tildes ($\sim$) over all modules and maps and the other as is, each viewed as lying in a horizontal plane, and if there are vertical maps f_{A_n}, etc., from the tilde version of the exact sequence of chain complexes to the original version such that all vertical squares commute, then the squares of the diagram

$$\begin{array}{ccccccccccccc}
\longrightarrow & H_{n+1}(\widetilde{C}) & \xrightarrow{\overline{\widetilde{\omega}}_n} & H_n(\widetilde{A}) & \xrightarrow{\overline{\widetilde{\varphi}}_n} & H_n(\widetilde{B}) & \xrightarrow{\overline{\widetilde{\psi}}_n} & H_n(\widetilde{C}) & \xrightarrow{\overline{\widetilde{\omega}}_{n-1}} & H_{n-1}(\widetilde{A}) & \longrightarrow \\
& \big\downarrow{\scriptstyle f_{C_{n+1}}} & & \big\downarrow{\scriptstyle f_{A_n}} & & \big\downarrow{\scriptstyle f_{B_n}} & & \big\downarrow{\scriptstyle f_{C_n}} & & \big\downarrow{\scriptstyle f_{A_{n-1}}} & \\
\longrightarrow & H_{n+1}(C) & \xrightarrow{\overline{\omega}_n} & H_n(A) & \xrightarrow{\overline{\varphi}_n} & H_n(B) & \xrightarrow{\overline{\psi}_n} & H_n(C) & \xrightarrow{\overline{\omega}_{n-1}} & H_{n-1}(A) & \longrightarrow
\end{array}$$

all commute.

PROOF. Theorem 4.7 was proved by three applications of Proposition 4.2, which includes its own assertion of functoriality, and two applications of Lemma 4.6, whose functoriality is addressed in Lemma 4.9. The argument involved only manipulations with diagrams, and functoriality is in place for every step. Hence functoriality is in place for the end result, and passage to the long exact sequence is functorial. $\square$

4. Projectives and Injectives

In Section III.5 we exploited the fact that certain complexes were exact and involved free modules in order to obtain chain maps and homotopies. The hypothesis "free" entered the arguments through Propositions 3.25 and 3.27; in both cases an R homomorphism was to be constructed from a free R module to some other R module, and a computation revealed how the R homomorphism should be defined on free generators. The universal mapping property of free modules allowed the R homomorphism to be extended from the generators to the whole free module. Examination of those arguments shows that it is enough to assume that the domain on which this R homomorphism is to be constructed is a "projective" R module, in the sense to be defined below, and we begin with that notion.

Let $\mathcal{C}$ be a good category of unital left R modules. We say that a module P in this category is **projective** in $\mathcal{C}$ or is a **projective** in $\mathcal{C}$ if whenever a diagram in the category is given as in Figure 4.3 with ψ mapping onto B, then there exists $\sigma : P \to C$ in $\mathcal{C}$ such that the diagram commutes.

$$\begin{tikzcd} & P \arrow[d, "\tau"] \arrow[dr, dotted, "\sigma"] & \\ 0 & B \arrow[l] & C \arrow[l, "\psi"'] \end{tikzcd}$$

FIGURE 4.3. Defining property of a projective.

If P is a free R module in $\mathcal{C}$, then P is projective in $\mathcal{C}$. In fact, for each free generator x of P, we choose an element c_x in C with $\psi(c_x) = \tau(x)$. Then we define $\sigma(x) = c_x$ and extend σ to a homomorphism. We give further examples of projectives shortly. First let us establish in Lemma 4.11 an ostensibly stronger property that projectives automatically satisfy.

Lemma 4.11. If P is projective in the good category $\mathcal{C}$ and if the diagram

$$\begin{tikzcd} & P \arrow[d, "\tau"] \arrow[dr, dotted, "\sigma"] & \\ A' & A \arrow[l, "\varphi"'] & A'' \arrow[l, "\psi"'] \end{tikzcd}$$

in $\mathcal{C}$ has $\ker\varphi = \text{image}\,\psi$ and $\varphi\tau = 0$, then there exists a map $\sigma : P \to A''$ in $\mathcal{C}$ such that the diagram commutes.

PROOF. The hypotheses force $\text{image}\,\tau \subseteq \ker\varphi = \text{image}\,\psi$. Thus if we put $B = \text{image}\,\psi$ and $C = A''$, then the above diagram leads to the diagram in Figure

4.3. The hypothesis "projective" therefore gives us the map σ in Figure 4.3 with $\tau = \psi\sigma$, and the same σ is the required map here. $\square$

EXAMPLES OF PROJECTIVES.

(1) If R is a field F and if $\mathcal{C}$ is the category of all vector spaces over F, then every module is free, hence projective, since every vector space has a basis.

(2) For general R, if $\mathcal{C} = \mathcal{C}_R$ is the category of all unital left R modules, then the projectives are the direct summands of free modules. This fact is easily verified from Figure 4.3 as follows: In one direction if $F = P \oplus P'$ is a free R module and the diagram in Figure 4.3 is given, extend τ to F as 0 on P', find σ from the fact that the free module F is projective, and restrict σ to P. In the other direction if P is projective, find a free R module F mapping onto P by a map ψ, and put $B = P$, $C = F$, and $\tau = 1$ in Figure 4.3. Then the equality $1_P = \tau = \psi\sigma$ forces σ to be one-one, and it follows that $P \cong \operatorname{image}\sigma$. Consequently $F = \operatorname{image}\sigma \oplus \ker\psi$.

(3) For $R = \mathbb{Z}$, the category $\mathcal{C} = \mathcal{C}_{\mathbb{Z}}$ of all unital R modules is the category of all abelian groups. Then the projective modules are the free abelian groups by (2), since any subgroup of a free abelian group is free abelian.

(4) For R equal to any (commutative) principal ideal domain, the projective modules in the category $\mathcal{C}_R$ of all unital R modules are the free modules, by the same argument as in (3) in combination with the Fundamental Theorem of Finitely Generated Modules (Theorem 8.25 of *Basic Algebra*).

(5) For $R = \mathbb{Z}$, two good categories that were listed in Section 2 were the category of all finitely generated abelian groups and the category of all torsion abelian groups. With the first of these, the projectives are the free abelian groups of finite rank, by the same argument as in (3). With the second of these, Problem 1 at the end of the chapter asks for a verification that some module in the category fails to be the image of any projective in the category.

We come to the main result concerning flexibility in setting up chain complexes. This result generalizes Proposition 3.25 through Corollary 3.30 in Section III.5.

Theorem 4.12. Let $X = \{(X_n, \partial_n)\}_{n=-\infty}^{\infty}$ and $X' = \{(X'_n, \partial'_n)\}_{n=-\infty}^{\infty}$ be chain complexes in the good category $\mathcal{C}$, and let r be an integer. Let $\{f_n : X_n \to X'_n\}_{n\le r}$ be a family of maps in $\mathcal{C}$ such that $\partial'_{n-1} f_n = f_{n-1}\partial_{n-1}$ for $n \le r$. If X_n is projective for $n > r$ and X' is exact at each X'_n with $n \ge r$, then $\{f_n : X_n \to X'_n\}_{n\le r}$ extends to a chain map $f : X \to X'$, and f is unique up to homotopy. More precisely any two extensions are homotopic by a homotopy h such that $h_n = 0$ for $n \le r$.

REMARKS. The diagrams in question are

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+1}"] & X_{n+1} \arrow[r, "\partial_n"] \arrow[d, dashed, "f_{n+1}"] & X_n \arrow[r, "\partial_{n-1}"] \arrow[d, "f_n"] & X_{n-1} \arrow[r, "\partial_{n-2}"] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "\partial'_{n+1}"] & X'_{n+1} \arrow[r, "\partial'_n"] & X'_n \arrow[r, "\partial'_{n-1}"] & X'_{n-1} \arrow[r, "\partial'_{n-2}"] & \cdots\cdots
\end{tikzcd}$$

for the construction of the chain map and

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n+2} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+2}"] & X_{n+1} \arrow[r, "\partial_n"] \arrow[d, "f_{n+1}"] \arrow[dl, dashed, "h_{n+1}"] & X_n \arrow[r, "\partial_{n-1}"] \arrow[d, "f_n"] \arrow[dl, "h_n"] & X_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] \arrow[dl, "h_{n-1}"] & \cdots \\
\cdots \arrow[r] & X'_{n+2} \arrow[r, "\partial'_{n+1}"] & X'_{n+1} \arrow[r, "\partial'_n"] & X'_n \arrow[r, "\partial'_{n-1}"] & X'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

for the construction of the homotopy.

PROOF. For the existence of the chain map, it is enough by induction to construct f_{r+1}. Matters are therefore as in the first of the above diagrams with $n = r$. Since X' is exact at X'_r and X_{r+1} is projective, we are in the situation of Lemma 4.11 with $P = X_{r+1}$, $A'' = X'_{r+1}$, $A = X'_r$, $A' = X'_{r-1}$, $\psi = \partial'_r$, $\varphi = \partial'_{r-1}$, and $\tau = f_r\partial_r$. The lemma gives a map $\sigma : P \to A''$ with $\psi\sigma = \tau$. If we take $f_{r+1} = \sigma$, then $\psi\sigma = \tau$ says that $\partial'_r f_{r+1} = f_r\partial_r$, and the inductive construction of the chain map is complete.

For the uniqueness up to homotopy, let $f : X \to X'$ and $g : X \to X'$ be two chain maps such that $f_n = g_n$ for $n \le r$. Define $h_n : X_n \to X'_{n+1}$ to be 0 for $n \le r$, and observe that the system of functions $\{h_n\}_{n\le r}$ satisfies $h_{n-1}\partial_{n-1} + \partial'_n h_n = f_n - g_n$ for $n \le r$ because $f_n = g_n$ for $n \le r$. Proceeding inductively, suppose that $s \ge r$ and that h_n has been constructed for $n \le s$ such that $h_{n-1}\partial_{n-1} + \partial'_n h_n = f_n - g_n$ for $n \le s$. We are to construct $h_{s+1} : X_{s+1} \to X'_{s+2}$. This is the situation of the second diagram above with $n = s$. Since $s \ge r$, X' is exact at X'_{s+1} and X_{s+1} is projective. Thus we are in the situation of Lemma 4.11 with $P = X_{s+1}$, $A'' = X'_{s+2}$, $A = X'_{s+1}$, $A' = X'_s$, $\psi = \partial'_{s+1}$, $\varphi = \partial'_s$, and $\tau = (f_{s+1} - g_{s+1}) - h_s\partial_s$. The lemma gives a map $\sigma : P \to A''$ with $\psi\sigma = \tau$. If we take $h_{s+1} = \sigma$, then $\psi\sigma = \tau$ says that $\partial'_{s+1}h_{s+1} = (f_{s+1} - g_{s+1}) - h_s\partial_s$, and the inductive construction of the homotopy is complete. $\square$

A **resolution** in the category $\mathcal{C}$ is an exact chain complex $X = \{(X_n, \partial_n)\}_{n=-\infty}^{\infty}$ or cochain complex $X = \{(X_n, d_n)\}_{n=-\infty}^{\infty}$ such that $X_n = 0$ for $n \le -2$. We say that the complex is a **resolution of** X_{-1}, and we abbreviate it as

$$X = (X^+ \xrightarrow{\partial_{-1}} X_{-1}) \qquad \text{or} \qquad X = (X^+ \xleftarrow{d_{-1}} X_{-1}),$$

with X^+ referring to

$$X^+ : \qquad \cdots \xrightarrow{\partial_2} X_2 \xrightarrow{\partial_1} X_1 \xrightarrow{\partial_0} X_0$$

or

$$X^+ : \qquad \cdots \xleftarrow{d_2} X_2 \xleftarrow{d_1} X_1 \xleftarrow{d_0} X_0$$

in the respective cases. A chain complex $X = (X^+ \xrightarrow{\varepsilon} M)$ that forms a resolution is called a **free resolution** of M if every X_n for $n \geq 0$ is a free module. It is called a **projective resolution** of M if every X_n for $n \geq 0$ is projective.

Corollary 4.13. Let M be a module in a good category $\mathcal{C}$ and let

$$X = (X^+ \xrightarrow{\varepsilon} M) \qquad \text{and} \qquad X' = (X'^+ \xrightarrow{\varepsilon'} M)$$

be two projective resolutions of M. Then there exist chain maps $f : X \to X'$ and $g : X' \to X$ with $f_{-1} = 1_M$ and $g_{-1} = 1_M$, and any two such chain maps f and g have the property that $gf : X \to X$ is homotopic to 1_X and $fg : X' \to X'$ is homotopic to $1_{X'}$.

PROOF. The existence of f extending $f_{-1} = 1_M$ is immediate by applying the first part of Theorem 4.12 with $r = -1$. The hypotheses apply because X_n is projective for $n > -1$ and X' is exact at X'_n for $n \geq -1$. A similar argument shows the existence of g.

If we have f and g, then $gf : X \to X$ and $1_X : X \to X$ are chain maps that extend the partial chain map given for $n \leq -1$ by 1_M for $n = -1$ and by 0 for $n \leq -2$. Since again X_n is projective for $n > -1$ and X' is exact at X'_n for $n \geq -1$, the second part of the theorem shows that gf and 1_X are homotopic. A similar argument shows that fg and $1_{X'}$ are homotopic. $\square$

There is an analogous sequence of results that ends with resolutions that are cochain maps. They will be equally as useful as the above results when we introduce derived functors in the next section. For the results below, the notion of a projective is replaced by that of an injective. We say that a module I in the good category $\mathcal{C}$ is **injective** in $\mathcal{C}$ or is an **injective** in $\mathcal{C}$ if whenever a diagram in the category is given as in Figure 4.4 with φ mapping one-one from B into C, then there exists $\sigma : B \to I$ in $\mathcal{C}$ such that the diagram commutes.

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & B \arrow[u, "\tau"] \arrow[r, "\varphi"] & C, \arrow[ul, dashed, "\sigma"'] \end{tikzcd}$$

FIGURE 4.4. Defining property of an injective.

We can think of the condition as saying that we can always extend such a τ from B to C, the extension being σ. In any event, we give some examples after proving an analog of Lemma 4.11.

Lemma 4.14. If I is injective in the good category $\mathcal{C}$ and if the diagram

$$\begin{tikzcd} & I & \\ A' \arrow[r, "\psi"] & A \arrow[u, "\tau"] \arrow[r, "\varphi"] & A'', \arrow[ul, dotted, "\sigma"'] \end{tikzcd}$$

in $\mathcal{C}$ has $\ker\varphi = \text{image}\,\psi$ and $\tau\psi = 0$, then there exists a map $\sigma : A'' \to I$ in $\mathcal{C}$ such that the diagram commutes.

PROOF. The hypotheses force $\ker\tau \supseteq \text{image}\,\psi = \ker\varphi$. Thus $\tau : A \to I$ and $\varphi : A \to A''$ descend to maps $\overline{\tau} : A/\ker\varphi \to I$ and $\overline{\varphi} : A/\ker\varphi \to A''$. If we put $B = A/\ker\varphi$ and $C = A''$, then the above diagram leads to Figure 4.4 with $\overline{\tau}$ and $\overline{\varphi}$ in place of τ and φ. The hypothesis "injective" gives us σ in Figure 4.4 with $\overline{\tau} = \sigma\overline{\varphi}$, and the same σ is the required map in the diagram above. $\square$

EXAMPLES OF INJECTIVES.

(1) If R is a field F and if $\mathcal{C}$ is the category of all vector spaces over F, then every module is injective. In fact, in Figure 4.4 we write $C = \text{image}\,\varphi \oplus B'$, and we let $\eta : \text{image}\,\varphi \to B$ be the inverse of $\varphi : B \to \text{image}\,\varphi$. Then we can define σ to be 0 on B' and to be $\tau\eta$ on $\text{image}\,\varphi$.

(2) Let $\mathcal{C}$ be the category of all abelian groups (unital $\mathbb{Z}$ modules). An abelian group G is said to be **divisible** if for each integer $n \neq 0$ and each $x \in G$, there exists $y \in G$ with $ny = x$. Two examples of divisible abelian groups are the additive group of rationals and the additive group of rationals modulo 1. It is easy to see that any quotient of a divisible group is divisible and that direct sums of divisible groups are divisible. Let us see for abelian groups that injective is equivalent to divisible.

The argument that injective implies divisible is easy: Let I be injective. Given $x \in I$ and $n \neq 0$, let $B = C = \mathbb{Z}$, let $\tau : \mathbb{Z} \to I$ have $\tau(k) = kx$, and let $\varphi : \mathbb{Z} \to \mathbb{Z}$ have $\varphi(k) = kn$. Setting up Figure 4.4, we obtain $\sigma : \mathbb{Z} \to I$ with $\tau = \sigma\varphi$. If we put $y = \sigma(1)$ and evaluate both sides at 1, then we obtain $x = \tau(1) = \sigma(\varphi(1)) = \sigma(n) = n\sigma(1) = ny$, as required.

The argument that divisible implies injective uses Zorn's Lemma. Let I be injective, and suppose that B, C, φ, and τ are given as in Figure 4.4. Consider the set $\mathcal{S}$ of abelian-group homomorphisms σ' having domain a subgroup of C containing $\varphi(B)$, having range I, and having $\sigma'\varphi = \tau$. Order $\mathcal{S}$ by inclusion upward of the corresponding sets of ordered pairs. The set $\mathcal{S}$ is nonempty because the homomorphism σ' with domain $\varphi(B)$ and values $\sigma'(\varphi(b)) = \tau(b)$ lies in $\mathcal{S}$; σ' is well defined because φ is assumed one-one. Zorn's Lemma yields a maximal element σ in $\mathcal{S}$, say with domain $\overline{C}$. We show that $\overline{C} = C$. Arguing by contradiction, suppose that $\overline{C}$ is a proper subgroup. Let c be in C but not $\overline{C}$. The set of integers k with kc in $\overline{C}$ is an ideal in $\mathbb{Z}$, and we let n be a generator. Since I is divisible, there exists an element a in I with $na = \sigma(nc)$. Define $\widetilde{\sigma}$ on the subgroup generated by c and $\overline{C}$ by the formula $\widetilde{\sigma}(kc + \bar{c}) = ka + \sigma(\bar{c})$ for $k \in \mathbb{Z}$ and $\bar{c} \in \overline{C}$. We need to check that $\widetilde{\sigma}$ is well defined. If $kc + \bar{c} = k'c + \bar{c}'$, then $(k - k')c = \bar{c}' - \bar{c}$ is in $\overline{C}$, and thus $k - k' = qn$ for some integer q. Hence $\widetilde{\sigma}(kc + \bar{c}) - \widetilde{\sigma}(k'c + \bar{c}) = (k - k')a + \sigma(\bar{c} - \bar{c}') = qna + \sigma(\bar{c} - \bar{c}') = q\sigma(nc) + \sigma(\bar{c} - \bar{c}') = q\sigma(nc) - \sigma((k - k')c) = q\sigma(nc) - q\sigma(nc) = 0$. Therefore $\widetilde{\sigma}$ is a nontrivial additive extension of σ, in contradiction to maximality of σ, and the proof is complete.

(3) For $R = \mathbb{Z}$, two good categories that were listed in Section 2 were the category of all finitely generated abelian groups and the category of all torsion abelian groups. With the first of these, Problem 1 at the end of the chapter asks for a verification that some module in the category fails to be a submodule of any injective. With the second of these, the injectives are the torsion divisible groups.

The next proposition extends Example 2 and its proof to general R. Although the condition in the proposition is not very intuitive for general R, it has a simple interpretation for (commutative) principal ideal domains; see Problem 4 at the end of the chapter.

Proposition 4.15. A unital left R module I is injective for the good category of all unital left R modules if and only if every R homomorphism of a left ideal J of R into I extends to an R homomorphism $R \to I$.

PROOF. The necessity is immediate from Figure 4.4 and the definition of "injective" if we take $B = J$, $C = R$ and write τ for the given R homomorphism of J into I.

For the sufficiency, suppose that I and a diagram as in Figure 4.4 are given. Consider the set $\mathcal{S}$ of R module homomorphisms σ' having domain an R submodule of C containing $\varphi(B)$ and having range I such that $\sigma'\varphi = \tau$, and order $\mathcal{S}$ by inclusion upward of the corresponding sets of ordered pairs. The set $\mathcal{S}$ is nonempty because the homomorphism σ' with domain $\varphi(B)$ and values $\sigma'(\varphi(b)) = \tau(b)$ lies in $\mathcal{S}$; σ' is well defined because φ is assumed one-one. Zorn's Lemma yields a maximal element σ in $\mathcal{S}$, say with domain $\overline{C}$. We show that $\overline{C} = C$. Arguing by contradiction, suppose that $\overline{C}$ is a proper R submodule of C. Let c be in C but not $\overline{C}$. The set of elements $r \in R$ with rc in $\overline{C}$ is a left ideal J in R, and the mapping $\psi(r) = \sigma(rc)$ is a well-defined R homomorphism of J into I. By hypothesis, ψ extends to an R homomorphism

$\Psi : R \to I$. Define $\widetilde{\sigma}$ on the subgroup generated by c and $\overline{C}$ by the formula $\widetilde{\sigma}(rc + \bar{c}) = \Psi(r) + \sigma(\bar{c})$ for $r \in R$ and $\bar{c} \in \overline{C}$. We need to check that $\widetilde{\sigma}$ is well defined. If $rc + \bar{c} = r'c + \bar{c}'$, then $(r - r')c = \bar{c}' - \bar{c}$ is in $\overline{C}$, and thus $r - r'$ is in J. Consequently $\Psi(r) - \Psi(r') = \psi(r - r') = \sigma((r - r')c)$. Hence $\widetilde{\sigma}(rc+\bar{c}) - \widetilde{\sigma}(r'c+\bar{c}) = (\Psi(r) - \Psi(r')) + \sigma(\bar{c} - \bar{c}') = \sigma((r-r')c) + \sigma(\bar{c} - \bar{c}') = \sigma((r - r')c) - \sigma((r - r')c) = 0$. Therefore $\widetilde{\sigma}$ is a nontrivial extension of σ, in contradiction to maximality of σ, and the proof is complete. $\square$

Now we can prove an analog of Theorem 4.12 for cochain complexes. This result had no counterpart in Chapter III.

Theorem 4.16. Let $X = \{(X_n, d_n)\}_{n=-\infty}^{\infty}$ and $X' = \{(X'_n, d'_n)\}_{n=-\infty}^{\infty}$ be cochain complexes in the good category $\mathcal{C}$, and let r be an integer. Let $\{f_n : X_n \to X'_n\}_{n \le r}$ be a family of maps in $\mathcal{C}$ such that $d'_{n-1} f_{n-1} = f_n d_{n-1}$ for $n \le r$. If X is exact at each X_n with $n \ge r$ and X'_n is injective for $n > r$, then $\{f_n : X_n \to X'_n\}_{n \le r}$ extends to a cochain map $f : X \to X'$, and f is unique up to homotopy. More precisely any two extensions are homotopic by a homotopy h such that $h_n = 0$ for $n \le r$.

REMARKS. The diagrams in question are

$$\begin{tikzcd}
\cdots \arrow[r, "d_{n-2}"] & X_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "f_{n-1}"] & X_n \arrow[r, "d_n"] \arrow[d, "f_n"] & X_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, dashed, "f_{n+1}"] & \cdots \\
\cdots \arrow[r, "d'_{n-2}"] & X'_{n-1} \arrow[r, "d'_{n-1}"] & X'_n \arrow[r, "d'_n"] & X'_{n+1} \arrow[r, "d'_{n+1}"] & \cdots
\end{tikzcd}$$

for the construction of the cochain map and

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "f_{n-1}"] & X_n \arrow[r, "d_n"] \arrow[d, "f_n"] \arrow[dl, "h_n"] & X_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] \arrow[dl, "h_{n+1}"] & X_{n+2} \arrow[r] \arrow[d, "f_{n+2}"] \arrow[dl, dashed, "h_{n+2}"] & \cdots \\
\cdots \arrow[r] & X'_{n-1} \arrow[r, "d'_{n-1}"] & X'_n \arrow[r, "d'_n"] & X'_{n+1} \arrow[r, "d'_{n+1}"] & X'_{n+2} \arrow[r] & \cdots
\end{tikzcd}$$

for the construction of the homotopy.

PROOF. For the existence of the cochain map, it is enough by induction to construct f_{r+1}. Matters are therefore as in the first of the above diagrams with $n = r$. Since X is exact at X_r and X'_{r+1} is injective, we are in the situation of Lemma 4.14 with $I = X'_{r+1}$, $A'' = X_{r+1}$, $A = X_r$, $A' = X_{r-1}$, $\psi = d_{r-1}$, $\varphi = d_r$, and $\tau = d'_r f_r$. The lemma gives a map $\sigma : A'' \to I$ with $\sigma\varphi = \tau$. If we take $f_{r+1} = \sigma$, then $\sigma\varphi = \tau$ says that $f_{r+1} d_r = d'_r f_r$, and the inductive construction of the cochain map is complete.

For the uniqueness up to homotopy, let $f : X \to X'$ and $g : X \to X'$ be two cochain maps such that $f_n = g_n$ for $n \le r$. Define $h_n : X_n \to X'_{n-1}$ to be 0 for $n \le r + 1$, and observe that the system of functions $\{h_n\}_{n\le r}$ satisfies $h_{n+1}d_n + d'_{n-1}h_n = f_n - g_n$ for $n \le r$ because $f_n = g_n$ for $n \le r$. Proceeding inductively, suppose that $s \ge r$ and that h_n has been constructed for $n \le s+1$ such that $h_{n+1}d_n + d'_{n-1}h_n = f_n - g_n$ for $n \le s$. We are to construct $h_{s+2} : X_{s+2} \to X'_{s+1}$. This is the situation of the second diagram with $n = s$. Since $s \ge r$, X is exact at X_{s+1} and X'_{s+1} is injective. Thus we are in the situation of Lemma 4.14 with $I = X'_{s+1}$, $A'' = X_{s+2}$, $A = X_{s+1}$, $A' = X_s$, $\psi = d_s$, $\varphi = d_{s+1}$, and $\tau = (f_{s+1} - g_{s+1}) - d'_s h_{s+1}$. The lemma gives a map $\sigma : A'' \to I$ with $\sigma\varphi = \tau$. If we take $h_{s+2} = \sigma$, then $\sigma\varphi = \tau$ says that $h_{s+2}d_{s+1} = (f_{s+1} - g_{s+1}) - d'_s h_{s+1}$, and the inductive construction of the homotopy is complete. □

A cochain complex $X = (X^+ \xleftarrow{\varepsilon} M)$ that forms a resolution is called an **injective resolution** of M if every X_n for $n \ge 0$ is an injective.

Corollary 4.17. Let M be a module in a good category $\mathcal{C}$ and let

$$X = (X^+ \xleftarrow{\varepsilon} M) \qquad \text{and} \qquad X' = (X'^+ \xleftarrow{\varepsilon'} M)$$

be two injective resolutions of M. Then there exist cochain maps $f : X \to X'$ and $g : X' \to X$ with $f_{-1} = 1_M$ and $g_{-1} = 1_M$, and any two such cochain maps f and g have the property that $gf : X \to X$ is homotopic to 1_X and $fg : X' \to X'$ is homotopic to $1_{X'}$.

Proof. The existence of f extending $f_{-1} = 1_M$ is immediate by applying the first part of Theorem 4.16 with $r = -1$. The hypotheses apply because X is exact at X_n for $n \ge -1$ and X'_n is injective for $n > -1$. A similar argument shows the existence of g.

If we have f and g, then $gf : X \to X$ and $1_X : X \to X$ are cochain maps that extend the partial cochain map given for $n \le -1$ by 1_M for $n = -1$ and by 0 for $n \le -2$. Since again X is exact at X_n for $n \ge -1$ and X'_n is injective for $n > -1$, the second part of the theorem shows that gf and 1_X are homotopic. A similar argument shows that fg and $1_{X'}$ are homotopic. □

We conclude with elementary characterizations of projectives and injectives that will turn out to be quite useful in the next two sections. We begin with a lemma[6] that will be useful now and will be helpful as motivation in the next section.

[6] The lemma is a slight variant of Problem 5 at the end of Chapter X of *Basic Algebra*.

Lemma 4.18. Let $\mathcal{C}$ be a good category of unital left R modules, and let

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

be an exact sequence in $\mathcal{C}$. Then the following conditions are equivalent:

(a) B is a direct sum $B = B' \oplus \ker \psi$ of modules in $\mathcal{C}$,
(b) there exists an R homomorphism $\sigma : C \to B$ such that $\psi\sigma = 1_C$,
(c) there exists an R homomorphism $\tau : B \to A$ such that $\tau\varphi = 1_A$.

REMARK. When the equivalent conditions of this lemma are satisfied, one says that the exact sequence is **split**.

PROOF. If (a) holds, then $\psi\big|_{B'}$ is one-one from B' onto C. Let σ be its inverse. Then $\sigma : C \to B'$ is one-one with $\psi\sigma = 1_C$. So (b) holds.

If (b) holds, then any b in B has the property that $b-\sigma\psi(b)$ has $\psi(b-\sigma\psi(b)) = \psi(b) - 1_C\psi(b) = 0$ and is therefore in image φ. Write $b - \sigma\psi(b) = \varphi(a)$ for some a depending on b; a is unique because φ is one-one. If $\tau : B \to A$ is defined by $\tau(b) = a$, then τ is an R homomorphism by the uniqueness of a. Consider $\tau(\varphi(a))$ for a in A. The element $b = \varphi(a)$ has $b - \sigma\psi(b) = \varphi(a) - \sigma\psi\varphi(a) = \varphi(a) - \sigma(0) = \varphi(a)$, and the definition of τ therefore says that $\tau(\varphi(a)) = a$. Hence $\tau\varphi = 1_A$, and (c) holds.

If (c) holds, then $B' = \ker \tau$ is an R submodule of B. If b is in $B' \cap \operatorname{image} \varphi$, then $b = \varphi(a)$ for some $a \in A$ and also $0 = \tau(b) = \tau\varphi(a) = 1_A(a) = a$. So $b = 0$, and $B' \cap \operatorname{image} \varphi = 0$. If $b \in B$ is given, write $b = (b - \varphi\tau(b)) + \varphi\tau(b)$. Then $\varphi\tau(b)$ is certainly in image φ, and $\tau(b-\varphi\tau(b)) = \tau(b) - 1_A\tau(b) = 0$ shows that $b - \varphi\tau(b)$ is in B'. Therefore $B = B' \oplus \operatorname{image} \varphi$. Since image $\varphi = \ker \psi$, we see that $B = B' \oplus \ker \psi$ and that (a) holds. $\square$

Proposition 4.19. If $\mathcal{C}$ is a good category of unital left R modules, then

(a) a module P in $\mathcal{C}$ is projective if and only if $\operatorname{Hom}_R(P, \,\cdot\,)$ is an exact functor from $\mathcal{C}$ into $\mathcal{C}_{\mathbb{Z}}$, if and only if every exact sequence

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

in $\mathcal{C}$ splits when its third nonzero member C equals P, and

(b) a module I in $\mathcal{C}$ is injective if and only if $\operatorname{Hom}_R(\,\cdot\,, I)$ is an exact functor from $\mathcal{C}$ into $\mathcal{C}_{\mathbb{Z}}$, if and only if every exact sequence

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

in $\mathcal{C}$ splits when its first nonzero member A equals I.

PROOF. For (a), suppose that P is given. The functor $\operatorname{Hom}_R(P,\ \cdot\ )$ is covariant and left exact, no matter what P is. Proposition 4.3 shows it is exact if and only if it carries short exact sequences into short exact sequences, and the left exactness means that the functor is exact if and only if it carries onto maps from B to C to onto maps from $\operatorname{Hom}_R(P, B)$ to $\operatorname{Hom}_R(P, C)$. If $\psi : B \to C$ is given, then $\operatorname{Hom}(1, \psi) : \operatorname{Hom}_R(P, B) \to \operatorname{Hom}_R(P, C)$ operates on a map σ in $\operatorname{Hom}_R(P, B)$ by $\operatorname{Hom}(1, \psi)(\sigma) = \psi\sigma$. The statement that the equation $\psi\sigma = \tau$ is solvable for σ for each τ in $\operatorname{Hom}_R(P, C)$ whenever ψ is onto is precisely the statement that Figure 4.3 is solvable for σ for all possible τ's whenever $B \longrightarrow C \longrightarrow 0$ is exact, and thus P is projective if and only if the functor is exact.

If P is projective and an exact sequence with $C = P$ is given, take $\tau = 1_P$ in Figure 4.3. The projective property yields a map $\sigma : P \to B$ with $\psi\sigma = 1_P$, and Lemma 4.18b shows that the exact sequence splits.

Conversely suppose that every short exact sequence with P as its third nonzero member splits. Suppose that a diagram as in Figure 4.3 is given with $\psi : C \to B$ onto and with τ mapping P into B. Let $S = C \oplus P$, and let T be the R submodule $\{(c, x) \in C \oplus P \mid \psi(c) = \tau(x)\}$ of S. Denote the projections of S to C and P by p_C and p_P, and let $j : T \to S$ be the inclusion. The map[7] $p_P j$ carries T onto P; in fact, if $x \in P$ is given, then $\psi : C \to B$ onto implies that there exists $c_x \in C$ with $\psi(c_x) = \tau(x)$. Then (c_x, x) lies in T, and $p_P j(c_x, x) = p_P(c_x, x) = x$. Consequently we have a 5-term exact sequence with terms 0, $\ker(p_P j)$, T, P, 0, and this must split by hypothesis. Thus there exists a map $q : P \to T$ with $p_P jq = 1_P$. Define $\sigma = p_C jq$. For $x \in P$, $jq(x)$ is some member of S of the form (c, x) with $\psi(c) = \tau(x)$. Hence $\psi\sigma(x) = \psi p_C jq(x) = \psi p_C(c, x) = \psi(c) = \tau(x)$. Thus $\psi\sigma = \tau$, and $\sigma : P \to C$ is the required map that exhibits P as projective.

For (b), suppose that I is given. The functor $\operatorname{Hom}_R(\ \cdot\ , I)$ is contravariant and left exact, no matter what I is. It is exact if and only if it carries one-one maps from A to B to onto maps from $\operatorname{Hom}_R(B, I)$ to $\operatorname{Hom}_R(A, I)$. If $\varphi : A \to B$ is given, then $\operatorname{Hom}(\varphi, 1) : \operatorname{Hom}_R(B, I) \to \operatorname{Hom}_R(A, I)$ operates on a map σ in $\operatorname{Hom}_R(B, I)$ by $\operatorname{Hom}(\varphi, 1)(\sigma) = \sigma\varphi$. The statement that the equation $\sigma\varphi = \tau$ is solvable for σ for each τ in $\operatorname{Hom}_R(A, I)$ whenever φ is one-one is precisely the statement that Figure 4.4 is solvable for σ for all possible τ's whenever $0 \longrightarrow A \longrightarrow B$ is exact, and thus I is injective if and only if the functor is exact.

If I is injective and an exact sequence with $A = I$ is given, take $\tau = 1_I$ in Figure 4.4. The injective property yields a map $\sigma : B \to I$ with $\sigma\varphi = 1_I$, and Lemma 4.18c shows that the exact sequence splits.

Conversely suppose that every short exact sequence with I as its first nonzero member splits. Suppose that a diagram as in Figure 4.4 is given with $\varphi : A \to B$ one-one and with τ mapping A into I. Let $S = B \oplus I$, and let T be the quotient of

[7]The pair $(p_C j, p_P j)$ is called the **pullback** of (τ, ψ). See Problem 35 at the end of the chapter.

S by the R submodule $\{(\varphi(a), -\tau(a)) \mid a \in A\}$. Denote the inclusions of B and I into S by i_B and i_I, and let $k : S \to T$ be the quotient mapping. The composition[8] ki_I is one-one from I into T. In fact, if $ki_I(x) = 0$ for some $x \in I$, then $(0, x)$ is a member of S of the form $(\varphi(a), -\tau(a))$ for some $a \in A$; thus $\varphi(a) = 0$, and the fact that φ is one-one implies that $a = 0$ and hence that $x = -\tau(a) = 0$. Consequently we have a 5-term exact sequence with terms 0, I, T, T/I, 0, and this must split by hypothesis. Thus there exists a map $r : T \to I$ with $rki_I = 1_I$. Define $\sigma = rki_B$. For $a \in A$, $i_B\varphi(a) - i_I\tau(a) = (\varphi(a), -\tau(a))$ is in $\ker k$. Thus $ki_B\varphi(a) = ki_I\tau(a)$, and $\sigma\varphi(a) = rki_B\varphi(a) = rki_I\tau(a) = 1_I\tau(a) = \tau(a)$ for $a \in A$. Therefore $\sigma\varphi = \tau$, and $\sigma : A \to I$ is the required map that exhibits I as injective. $\square$

5. Derived Functors

Now we shall undertake the main construction of the chapter, that of "derived functors." Let $\mathcal{C}$ be a good category of unital left R modules. Arranging for derived functors to be defined on every module in $\mathcal{C}$ requires that each module M in $\mathcal{C}$ have either a projective resolution or an injective resolution, and thus $\mathcal{C}$ must have either many projectives or many injectives in a suitable sense. Let us make the condition precise.

We say that $\mathcal{C}$ has **enough projectives** if every module in $\mathcal{C}$ is a quotient of a projective in $\mathcal{C}$. Suppose that this condition is satisfied. Let M be a module in $\mathcal{C}$, and let X_0 be a projective that maps onto M, say by a map ε. Then $\ker\varepsilon$ is in $\mathcal{C}$, since good categories are closed under the passage to submodules, and we let X_1 be a projective in $\mathcal{C}$ that maps onto $\ker\varepsilon$, say by a map ∂_0. Similarly let X_2 be a projective that maps onto $\ker\partial_0$ in X_1, say by a map ∂_1, and so on. The result is that we obtain a projective resolution of the form $X^+ \xrightarrow{\varepsilon} M$ with X^+ given by

$$X^+ : \qquad \cdots \longrightarrow X_2 \xrightarrow{\partial_1} X_1 \xrightarrow{\partial_0} X_0.$$

Consequently the condition "enough projectives" implies that every module in $\mathcal{C}$ has a projective resolution in $\mathcal{C}$.

Similarly we say that $\mathcal{C}$ has **enough injectives** if every module in $\mathcal{C}$ is a submodule of an injective in $\mathcal{C}$. Suppose that this condition is satisfied. Let M be a module in $\mathcal{C}$, and let X_0 be an injective into which M embeds, say by a map ε. Then $X_0/\operatorname{image}\varepsilon$ is in $\mathcal{C}$, since good categories are closed under the passage to quotient modules, and we let X_1 be an injective into which $X_0/\operatorname{image}\varepsilon$ embeds, say by a map $d_0^\#$. Let d_0 be the composition of the quotient map from X_0 to $X_0/\operatorname{image}\varepsilon$, followed by $d_0^\#$; then d_0 maps X_0 into X_1 with $\ker d_0 = \operatorname{image}\varepsilon$.

[8] The pair (ki_B, ki_I) is called the **pushout** of (τ, φ). See Problem 35 at the end of the chapter.

We let X_2 be an injective into which $X_1/\operatorname{image} d_0$ embeds, say by $d_1^\#$, and we let d_1 be the composition of the quotient map from X_1 to $X_1/\operatorname{image} d_0$, followed by $d_1^\#$; then d_1 maps X_1 into X_2 with $\ker d_1 = \operatorname{image} d_0$. Continuing in this way, we obtain an injective resolution of the form $X^+ \xleftarrow{\varepsilon} M$ with X^+ given by

$$X^+ : \qquad \cdots \xleftarrow{d_2} X_2 \xleftarrow{d_1} X_1 \xleftarrow{d_0} X_0.$$

Consequently the condition "enough injectives" implies that every module in $\mathcal{C}$ has an injective resolution in $\mathcal{C}$.

The category $\mathcal{C}_R$ of all unital left R modules certainly has enough projectives. In fact, every module in $\mathcal{C}_R$ is the quotient of a free R module, and free R modules are projective in $\mathcal{C}_R$. It is less trivial but still true that $\mathcal{C}_R$ has enough injectives. Let us pause for a moment to prove this result in Proposition 4.20 below.

As is shown in Problems 1–2 at the end of the chapter, other good categories of unital left R modules may or may not have enough projectives or enough injectives, and a good category may have the one without the other.

Proposition 4.20. If R is any ring with identity, then the category of all unital left R modules has enough injectives.

PROOF. We treat first the case that $R = \mathbb{Z}$. In view of Example 2 of injectives, we are to exhibit an arbitrary abelian group A as isomorphic to a subgroup of a divisible group. We know that A is isomorphic to a quotient of some free abelian group. Write $A \cong F/S$ with F a direct sum of copies of $\mathbb{Z}$ and S equal to some subgroup of F. Taking a $\mathbb{Z}$ basis for F and forming a $\mathbb{Q}$ vector space with that same basis, we can regard F as a subgroup of the additive group D of a rational vector space. The group D is divisible, and A is isomorphic to a subgroup of D/S. Any quotient of a divisible group is divisible, and thus D/S is divisible.

Now we allow R to be any ring with identity. We shall make use of various results from Chapter X of *Basic Algebra*. If M is any unital left R module, let us denote by $\mathcal{F}M$ the underlying abelian group[9] of M. If we regard R as an $(\mathbb{Z}, R)$ bimodule, then Proposition 10.17 makes $\operatorname{Hom}_{\mathbb{Z}}(R, \mathcal{F}M)$ into a left R module, with $r\varphi(r') = \varphi(r'r)$ for r and r' in R. The mapping $m \mapsto \varphi_m$ with $\varphi_m(r) = rm$ is a one-one R homomorphism of M into $\operatorname{Hom}_{\mathbb{Z}}(R, \mathcal{F}M)$. From the previous paragraph we can find a divisible abelian group with $\mathcal{F}M \subseteq D$, and we can then regard the left R module $\operatorname{Hom}_{\mathbb{Z}}(R, \mathcal{F}M)$ as an R submodule of $\operatorname{Hom}_{\mathbb{Z}}(R, D)$. Consequently we can regard M as an R submodule of $\operatorname{Hom}_{\mathbb{Z}}(R, D)$. We are going to prove that $I = \operatorname{Hom}_{\mathbb{Z}}(R, D)$ is injective in $\mathcal{C}_R$.

We digress for a moment to make a side calculation. With D fixed and N equal to any unital left R module, we make use of the isomorphism

$$\operatorname{Hom}_R(N, \operatorname{Hom}_{\mathbb{Z}}(R, D)) \cong \operatorname{Hom}_{\mathbb{Z}}(R \otimes_R N, D)$$

[9] $\mathcal{F}$ is called the **forgetful functor** from $\mathcal{C}_R$ to $\mathcal{C}_{\mathbb{Z}}$.

given in Proposition 10.23 of *Basic Algebra*; in the expression $R \otimes_R N$, the left factor of R is to be regarded as a right R module (and not also a left R module), and then $R \otimes_R N$ is really $\mathcal{F}(R \otimes_R N)$ in the sense that the tensor product retains only the structure of an abelian group. Meanwhile, Corollary 10.19a gives us

$$\operatorname{Hom}_{\mathbb{Z}}(R \otimes_R N, D) \cong \operatorname{Hom}_{\mathbb{Z}}(N, D);$$

here the R on the left is an (R, R) bimodule, and the isomorphism is one of left R modules. However, there is no harm in applying $\mathcal{F}$ to both sides and obtaining

$$\operatorname{Hom}_{\mathbb{Z}}(\mathcal{F}(R \otimes_R N, D)) \cong \operatorname{Hom}_{\mathbb{Z}}(\mathcal{F}N, D).$$

Thus

$$\operatorname{Hom}_R(N, \operatorname{Hom}_{\mathbb{Z}}(R, D)) \cong \operatorname{Hom}_{\mathbb{Z}}(\mathcal{F}N, D). \tag{$*$}$$

If we track down the isomorphisms in the results of Chapter X, we see that the map from left to right sends $\varphi \in \operatorname{Hom}_R(N, \operatorname{Hom}_{\mathbb{Z}}(R, D))$ to the map $\Phi \in \operatorname{Hom}_{\mathbb{Z}}(\mathcal{F}N, D)$ with $\Phi(x) = \varphi(x)(1)$ for $x \in N$, and the inverse sends Φ to φ with $\varphi(x)(r) = \Phi(rn)$.

Now we return to $I = \operatorname{Hom}_{\mathbb{Z}}(R, D)$. By Proposition 4.19b, I will be injective if and only if $\operatorname{Hom}_R(\,\cdot\,, I)$ is an exact functor. Since this functor is contravariant and left exact, it is enough to prove that if $0 \longrightarrow A \xrightarrow{\psi} B$ is exact in $\mathcal{C}_R$, then

$$\operatorname{Hom}_R(B, I) \xrightarrow{\operatorname{Hom}(\psi,1)} \operatorname{Hom}_R(A, I) \longrightarrow 0 \tag{$**$}$$

is exact in $\mathcal{C}_{\mathbb{Z}}$. Let us reinterpret $(**)$ in the light of the isomorphism $(*)$ when $N = B$ and $N = A$. If φ is in $\operatorname{Hom}_R(B, \operatorname{Hom}_{\mathbb{Z}}(R, D))$, then $\operatorname{Hom}(\psi, 1)(\varphi)$ is the member $\varphi\psi$ of $\operatorname{Hom}_R(A, \operatorname{Hom}_{\mathbb{Z}}(R, D))$. The corresponding members of $\operatorname{Hom}_{\mathbb{Z}}(\mathcal{F}B, D)$ and $\operatorname{Hom}_{\mathbb{Z}}(\mathcal{F}A, D)$ are Φ with $\Phi(b) = \varphi(b)(1)$ and a member Φ' of $\operatorname{Hom}_{\mathbb{Z}}(\mathcal{F}A, D)$ with $\Phi'(a) = \varphi\psi(a)(1)$. Thus $\Phi' = \Phi(\mathcal{F}\psi)$, and the mapping $\operatorname{Hom}(\psi, 1)$ in $(**)$ translates under the isomorphisms $(*)$ into the mapping $\operatorname{Hom}(\mathcal{F}\psi, 1)$ of $\operatorname{Hom}_{\mathbb{Z}}(\mathcal{F}B, D)$ into $\operatorname{Hom}_{\mathbb{Z}}(\mathcal{F}A, D)$. The group D is divisible, hence injective in $\mathcal{C}_{\mathbb{Z}}$. Since $\mathcal{F}\psi : \mathcal{F}A \to \mathcal{F}B$ is one-one and D is injective in $\mathcal{C}_{\mathbb{Z}}$, Proposition 4.19b shows that $\operatorname{Hom}(\mathcal{F}\psi, 1)$ carries $\operatorname{Hom}_{\mathbb{Z}}(\mathcal{F}B, D)$ onto $\operatorname{Hom}_{\mathbb{Z}}(\mathcal{F}A, D)$. Therefore $(**)$ is exact, and we conclude that I is injective in $\mathcal{C}_R$. $\square$

Derived functors of an additive functor F from one good category to another will be useful when F is left exact or right exact, and there will be one derived functor for each integer $n \geq 0$. The value of the n^{th} derived functor on a module M is obtained by taking a projective or injective resolution of M according to the rule in Figure 4.5, applying F to the resolution, dropping the term $F(M)$

that occurs in degree -1, and forming the n^{th} homology or cohomology of the resulting complex. The full traditional notation for the derived functor in question appears in Figure 4.5, along with an abbreviated notation that we shall tend to use.

The choice of projective or injective resolution at the start is made in such a way that the 0^{th} derived functor is naturally isomorphic to F; this condition will be clarified in Proposition 4.21 below. If a projective resolution is to be used, one makes the assumption that the domain category has enough projectives; if an injective resolution is to be used, one makes the assumption that the domain category has enough injectives.

If the resulting complex obtained by applying F to the resolution is a chain complex, the abbreviated notation is F_n for the n^{th} derived functor; otherwise it is F^n. The full traditional notation involves using an L or R in front of F to denote the one-sided exactness, left or right, that F is *not* assumed to have, and the subscript or superscript n is moved from F to the L or R.

Exactness	—variant	Resolution	—ology	Notation	Example
right	co—	projective	hom—	$F_n,\ L_nF$	$M \otimes_R (\,\cdot\,)$
right	contra—	injective	hom—	$F_n,\ L_nF$	$M \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(\,\cdot\,, I)$, I injective
left	co—	injective	cohom—	$F^n,\ R^nF$	$\mathrm{Hom}_R(M,\ \cdot\,)$
left	contra—	projective	cohom—	$F^n,\ R^nF$	$\mathrm{Hom}_R(\,\cdot\,, M)$

FIGURE 4.5. Formation of derived functors.

There are several things that need elaboration in this definition, and we take them up right away.

First there is the fact that $F_n(M)$ or $F^n(M)$ is well defined. Suppose that we start with two resolutions X and X' of M (projective or injective by the rules in Figure 4.5). Corollary 4.13 or 4.17 gives us chain or cochain maps $f : X \to X'$ and $g : X' \to X$ with $f_{-1} = 1_M$ and $g_{-1} = 1_M$ and shows that $gf : X \to X$ is homotopic to 1_X and that $fg : X' \to X'$ is homotopic to $1_{X'}$. For definiteness let us suppose that F is covariant and right exact; then chain maps are involved and the derived functors of F are to be denoted by F_n. Applying F to our chain maps, we obtain chain maps $F(f) : F(X) \to F(X')$, $F(g) : F(X') \to F(X)$, $F(gf) : F(X) \to F(X)$, and $F(fg) : F(X') \to F(X')$. The last two of these are homotopic to $1_{F(X)} : F(X) \to F(X)$ and to $1_{F(X')} : F(X') \to F(X')$, respectively, by F of the respective homotopies. Proposition 4.1 shows that $F(g)F(f) = F(gf)$ induces the identity on $H_*(F(X))$ and that $F(f)F(g) = F(fg)$ induces the identity on $H_*(F(X'))$. Consequently the mappings induced on homology by $F(f)$ and $F(g)$ are two-sided inverses of one another. Thus $F_n(M)$ as computed from X is isomorphic to $F_n(M)$ as computed from X'.

Moreover, this isomorphism is canonical. If $f' : X \to X'$ is another chain map, then the same calculation shows that $F(f')$ and $F(g)$ induce two-sided inverses of each other on homology, and hence $F(f) = F(f')$ on homology. Thus $F_n(M)$ is well defined up to canonical isomorphism when F is covariant and right exact. The other three situations in Figure 4.5 are handled in similar fashion and lead to analogous conclusions.

Next we make F_n or F^n into a functor. To do do, let $\varphi : M \to M'$ be given. For definiteness, again let us suppose that F is covariant and right exact. Let X and X' be projective resolutions of M and M', respectively, and apply Theorem 4.12 to produce a chain map $\Phi : X \to X'$ with $\Phi_{-1} = \varphi$. Then $F(\Phi) : F(X) \to F(X')$ is a chain map and induces maps on homology that we denote by $F_n(\varphi)$. Here $F_n(\varphi)$ maps $F_n(M)$ into $F_n(M')$.

Let us see that $F_n(\varphi)$ is well defined. If X is replaced by $\overline{X}$, Corollary 4.13 produces chain maps $f : X \to \overline{X}$ and $g : \overline{X} \to X$ with $f_{-1} = 1_M$ and $g_{-1} = 1_M$, and Theorem 4.12 produces a chain map $\overline{\Phi} : \overline{X} \to X'$ with $\overline{\Phi}_{-1} = \varphi$. Since $\overline{\Phi} \circ f$ and Φ are both chain maps from X to X' that equal φ in degree -1, Theorem 4.12 shows that $\overline{\Phi} \circ f$ is homotopic to Φ. Similarly $\Phi \circ g$ and $\overline{\Phi}$ are chain maps from $\overline{X}$ to X' and are homotopic. By Proposition 4.1, $F(\overline{\Phi} \circ f) = F(\Phi)$ on homology, and $F(\Phi \circ g) = F(\overline{\Phi})$ on homology. Thus on homology $F(\overline{\Phi})$ corresponds to $F(\Phi)$ under the canonical isomorphism $F(f)$, whose inverse on homology is $F(g)$. In short, $F_n(\varphi)$ is well defined up to the previously obtained canonical isomorphisms. The other three situations in Figure 4.5 are handled in similar fashion and lead to analogous conclusions.

Tracing through the definition of how derived functors affect maps, we see that the map 1 goes to the map 1 and that compositions go to compositions, in the same order as for F. Thus the derived functors are indeed functors. The derived functors of a covariant functor are covariant, and the derived functors of a contravariant functor are contravariant.

We need to check that the derived functors are additive. If $\varphi : M \to M'$ and $\varphi' : M \to M'$ are given, then we can proceed as above and use a single resolution of M and a single resolution of M' to investigate φ, φ', and $\varphi + \varphi'$. Then it is apparent that the chain or cochain maps built from maps of M to M' add in the same way as the maps, and the result is that each F_n or F^n is additive with particular choices of the resolutions in place. Allowing the resolutions to vary means that we have to take canonical isomorphisms into account, and after doing so, we still get additivity.

If two functors F and G from $\mathcal{C}$ to $\mathcal{C}'$ of the same type in Figure 4.5 are naturally isomorphic, then F_n and G_n (or else F^n and G^n) are naturally isomorphic for all n. In fact, if T is the natural isomorphism, then T associates a member T_A

of $\operatorname{Hom}(F(A), G(A))$ to each module A in $\mathcal{C}$. Take a projective or injective resolution $X = \{X_n\}$ of A, as appropriate, and form the two complexes $F(X)$ and $G(X)$. The system $\{T_{X_n}\}$ is then a chain map from $F(X)$ to $G(X)$, with inverse $\{T_{X_n}^{-1}\}$, and the homology or cohomology of $F(X)$ is exhibited as isomorphic to the homology or cohomology of $G(X)$. This much shows that $F_n(A) \cong G_n(A)$ (or $F^n(A) \cong G^n(A)$) for all n. We omit the details of verifying the naturality of this isomorphism in the A variable for each n.

Proposition 4.21. In the four situations of derived functors in Figure 4.5, under the assumption that the domain category for F has enough projectives or enough injectives as appropriate, the 0^{th} derived functor of F is naturally isomorphic to F.

PROOF IF F IS COVARIANT AND RIGHT EXACT. Let

$$X_1 \xrightarrow{\partial_0} X_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

be the terms in degree $1, 0, -1, -2$ of a projective resolution of M. By Proposition 4.5 and its remark, the right exactness and covariance of F imply that

$$F(X_1) \xrightarrow{F(\partial_0)} F(X_0) \xrightarrow{F(\varepsilon)} F(M) \longrightarrow 0$$

is exact. The derived-functor module $F_0(M)$ is computed as the 0^{th} homology of

$$F(X_1) \xrightarrow{F(\partial_0)} F(X_0) \longrightarrow 0.$$

Thus

$$F_0(M) = F(X_0)/\operatorname{image} F(\partial_0) = F(X_0)/\ker F(\varepsilon).$$

Since $F(\varepsilon)$ is onto $F(M)$, the right side here is $\cong F(M)$ via $F(\varepsilon)$.

This establishes the isomorphism. Let us prove that it is natural in the variable M. If $\varphi : M \to M'$ is given, we are to prove that the diagram

$$\begin{tikzcd} F_0(M) \arrow[r, "\text{via } F(\varepsilon)"] \arrow[d, "F_0(\varphi)"'] & F(M) \arrow[d, "F(\varphi)"] \\ F_0(M') \arrow[r, "\text{via } F(\varepsilon')"] & F(M') \end{tikzcd} \qquad (*)$$

commutes. Using Theorem 4.12, we form the part of a chain map that is indicated:

$$\begin{tikzcd} X_1 \arrow[r, "\partial_0"] \arrow[d, "f_1"'] & X_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"'] & M \arrow[r] \arrow[d, "\varphi"'] & 0 \\ X_1' \arrow[r, "\partial_0'"] & X_0' \arrow[r, "\varepsilon'"] & M' \arrow[r] & 0 \end{tikzcd}$$

Application of F gives a commutative diagram

$$\begin{array}{ccc} F(X_0) & \xrightarrow{F(\varepsilon)} & F(M) \\ {\scriptstyle F(f_0)}\big\downarrow & & {\scriptstyle F(\varphi)}\big\downarrow \\ F(X_0') & \xrightarrow{F(\varepsilon')} & F(M') \end{array}$$

and this becomes $(*)$ upon passage to the quotients $F(X_0)/\ker F(\varepsilon)$ and $F(X_0')/\ker F(\varepsilon')$. This completes the proof. $\square$

EXAMPLES.

(1) The invariants functor $F(M) = M^G$ for a group G. Suppose that a group G acts on an abelian group M by automorphisms. This situation is completely equivalent to considering M as a unital left $\mathbb{Z}G$ module, where $\mathbb{Z}G$ is the integer group ring of G. The subgroup of **invariants** of M is

$$M^G = \{m \in M \mid gm = m \text{ for all } g \in G\}.$$

The formulas $F(M) = M^G$ for such a module M and $F(h) = h\big|_{M^G}$ for h in $\mathrm{Hom}_{\mathbb{Z}G}(M, M')$ define a covariant additive functor called the **invariants functor**; we can think of F as carrying $\mathcal{C}_{\mathbb{Z}G}$ into itself, but it is preferable to think of it as carrying $\mathcal{C}_{\mathbb{Z}G}$ into the category $\mathcal{C}_{\mathbb{Z}}$ of abelian groups. The functor F is naturally isomorphic to the functor $H = \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, \,\cdot\,)$, where $\mathbb{Z}$ is made into a $\mathbb{Z}G$ module with trivial G action; as with F, we consider H as a functor from $\mathcal{C}_{\mathbb{Z}G}$ to $\mathcal{C}_{\mathbb{Z}}$. To see the isomorphism, we associate to each module M the abelian-group homomorphism $T_M : M^G \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, M)$ defined by $T_M(m) = \varphi_m$ with $\varphi_m(k) = m$ for all $k \in \mathbb{Z}$. If h is in $\mathrm{Hom}_{\mathbb{Z}G}(M, M')$, then the two maps $T_{M'} \circ F(h)$ and $H(h) \circ T_M$ of $F(M)$ into $H(M')$ are equal, since at each $m \in M^G$ we have

$$H(h)T_M(m) = H(h)(\varphi_m) = \mathrm{Hom}(1, h)(\varphi_m) = h\varphi_m = \varphi_{h(m)} = T_{M'}F(h)(m).$$

This identity means that $\{T_M\}$ is a natural transformation; we readily check for each M that T_M carries M^G one-one onto $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, M)$, and thus $\{T_M\}$ is a natural isomorphism.

Because of this natural isomorphism, the invariants functor is covariant and left exact. Its derived functors F^n or H^n are obtained by using an injective resolution $I \leftarrow M \leftarrow 0$, applying the functor $(\,\cdot\,)^G$ or $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, \,\cdot\,)$, dropping the term in degree -1, and forming cohomology. Briefly

$$F^n(M) \cong H^n(I^G) \cong H^n(\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, I))$$

for an injective resolution $I \leftarrow M \leftarrow 0$.

It turns out that the result is given also by the cohomology-of-groups functors $H^n(G, M)$ even though this was not the procedure by which we obtained group cohomology in Section III.5. In fact, what Section III.5 said to do was to start from a free resolution (a projective resolution would have been good enough) such as $P \longrightarrow M \longrightarrow 0$ of $\mathbb{Z}$ in $\mathcal{C}_{\mathbb{Z}}$, apply the contravariant left exact functor $\mathrm{Hom}_{\mathbb{Z}G}(\,\cdot\,, M)$, drop the term in degree -1, and form cohomology. Briefly then, Section III.5 said that

$$H^n(G, M) \cong H^n(\mathrm{Hom}_{\mathbb{Z}G}(P, M)) \quad \text{for a projective resolution } P \to \mathbb{Z} \to 0.$$

The fact that $H^n(G, M)$ can be computed in either of these ways is not particularly obvious from what we have done so far, but it will be a special case of the natural isomorphism of functors Ext^n and ext^n that is proved as Theorem 4.31 in Section 7. With either formula for $H^n(G, M)$, we obtain $H^0(G, M) \cong M^G$ in agreement with Proposition 4.21.

(2) The co-invariants functor $F(M) = M_G$ for a group G. In the same setting as in Example 1, the subgroup of **co-invariants** of M is

$$M_G = M\big/(\text{subgroup generated by all } gm - m \text{ for } g \in G,\ m \in M).$$

The functor F can be seen to be naturally isomorphic to the functor H with $H(M) = \mathbb{Z} \otimes_{\mathbb{Z}G} M$. It is therefore covariant and right exact. Its derived functors are given by

$$F_n(M) \cong H_n(P_G) \cong H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} P) \quad \text{for a projective resolution } P \to M \to 0.$$

These are by definition the **homology-of-groups** functors $H_n(G, M)$. Although the equality is not particularly obvious, $H_n(G, M)$ can be computed also from

$$H_n(G, M) \cong H_n(P \otimes_{\mathbb{Z}G} M) \quad \text{for a projective resolution } P \to \mathbb{Z} \to 0.$$

This isomorphism is a special case of the natural isomorphism of functors Tor_n and tor_n that is mentioned just before Proposition 4.29 in Section 7; the proof is completely analogous to the proof of Theorem 4.31. With either formula for $H_n(G, M)$, we obtain $H_0(G, M) \cong M_G$ in agreement with Proposition 4.21.

(3) Derived functors with $R = \mathbb{Z}$. For the ring $\mathbb{Z}$ and the category $\mathcal{C}_{\mathbb{Z}}$ (or more generally for $\mathcal{C}_R$ for any principal ideal domain R), projective resolutions and injective resolutions can be fairly short, and derived functors in degree ≥ 2 are all 0. Let M be a given unital $\mathbb{Z}$ module, i.e., an abelian group. We know that M is the quotient of some free abelian group X_0, say with a quotient map ε, and then $X_1 = \ker \varepsilon$ is a subgroup of a free abelian group and hence is free abelian. Thus a projective resolution of M is

$$0 \longrightarrow X_1 \xrightarrow{\text{inc}} X_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

The kinds of derived functors that make use of projective resolutions are the covariant right exact ones and the contravariant left exact ones. If F is such a functor, then we are led to the complexes

$$0 \longrightarrow F(X_1) \xrightarrow{F(\mathrm{inc})} F(X_0) \xrightarrow{F(\varepsilon)} 0$$

and
$$0 \longleftarrow F(X_1) \xleftarrow{F(\mathrm{inc})} F(X_0) \xleftarrow{F(\varepsilon)} 0$$

in the two cases. Thus the values of the derived functors are $F_0(M) \cong M$ and $F_1(M) = \ker F(\varepsilon)$ in the first case, and $F^0(M) \cong M$ and $F^1(M) = \operatorname{coker} F(\varepsilon)$ in the second case. Higher derived functors are 0. Similar remarks apply to injective resolutions and the remaining two cases for derived functors in Figure 4.5. Every abelian group is a subgroup of a divisible group, which is injective in $\mathcal{C}_{\mathbb{Z}}$, and the quotient of the divisible group by the given abelian group is divisible, hence injective. Thus we can arrange for all terms of an injective resolution to be 0 beyond the X_1 term, and an analysis of the results similar to the one above is possible.

6. Long Exact Sequences of Derived Functors

The first four theorems of this section say that a short exact sequence of modules leads to a long exact sequence of derived functor modules and that it does so in a functorial way. Let us suppose that $F : \mathcal{C} \to \mathcal{C}'$ is an additive functor between good categories. For the first of the theorems, suppose further that $\mathcal{C}$ has enough projectives and that F is one of the types of functors in Figure 4.5 making use of projective resolutions in the definition of its derived functors. The last of these conditions means that F is to be covariant right exact or contravariant left exact.

To prove such a theorem, we shall want to apply Theorem 4.7, which produces a long exact sequence from a short exact sequence of complexes. To each of the modules in the given short exact sequence, we attach a projective resolution. If these projective resolutions can somehow be related by chain maps so as to give a short exact sequence of projectives in each degree, then we can apply F to the entire diagram, invoke Theorem 4.7, and obtain the desired long exact sequence. Application of Theorem 4.10, in combination with some further checking, will show that the passage from the given short exact sequence of modules to the long exact sequence of derived functor modules is functorial in the modules of the short exact sequence.

Thus the problem is to obtain the compatible projective resolutions. Proposition 4.19a gives us a clue about what to look for: any short exact sequence of projectives has to be split. Here is the statement of the first theorem.

Theorem 4.22. Let $F : \mathcal{C} \to \mathcal{C}'$ be an additive functor between two good categories. Suppose that F either is covariant right exact or is contravariant left exact, and suppose that $\mathcal{C}$ has enough projectives. Whenever there are three modules and two maps in $\mathcal{C}$ forming a short exact sequence

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0,$$

then the derived functors of F on the three modules form a long exact sequence in $\mathcal{C}'$ as follows:

(a) If F is covariant and right exact, then the long exact sequence is

$$\begin{aligned} 0 \longleftarrow F(C) \longleftarrow F(B) \longleftarrow F(A) \longleftarrow F_1(C) \longleftarrow F_1(B) \longleftarrow F_1(A) \\ \longleftarrow F_2(C) \longleftarrow F_2(B) \longleftarrow F_2(A) \longleftarrow F_3(C) \longleftarrow \cdots . \end{aligned}$$

(b) If F is contravariant and left exact, then the long exact sequence is

$$\begin{aligned} 0 \longrightarrow F(C) \longrightarrow F(B) \longrightarrow F(A) \longrightarrow F^1(C) \longrightarrow F^1(B) \longrightarrow F^1(A) \\ \longrightarrow F^2(C) \longrightarrow F^2(B) \longrightarrow F^2(A) \longrightarrow F^3(C) \longrightarrow \cdots . \end{aligned}$$

We begin with a lemma.

Lemma 4.23. In the good category $\mathcal{C}$, suppose that the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 & A \arrow[l] \arrow[d, "\varphi"'] & P_A \arrow[l, "\varepsilon_A"'] \arrow[d, "i_A"'] & M_A \arrow[l, "\psi_A"'] \arrow[d, dashed, "\varphi_1"'] & 0 \arrow[l] \\
0 & B \arrow[l] \arrow[d, "\psi"'] & P_A \oplus P_C \arrow[l, dashed, "\varepsilon_B"'] \arrow[d, "p_C"'] & M_B \arrow[l, dashed, "\psi_B"'] \arrow[d, dashed, "\psi_1"'] & 0 \arrow[l, dashed] \\
0 & C \arrow[l] \arrow[d] & P_C \arrow[l, "\varepsilon_C"'] \arrow[d] & M_C \arrow[l, "\psi_C"'] \arrow[d] & 0 \arrow[l] \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

has the first two columns and the two rows with solid arrows exact and has P_A and P_C projective. Here i_A is the inclusion into the first component of $P_A \oplus P_C$, and p_C is the projection onto the second component. Then there exist a module M_B and maps ε_B, ψ_B, φ_1, and ψ_1 such that the whole diagram, including the dashed arrows, has exact rows and columns and has all squares commuting.

PROOF. The module $P_A \oplus P_C$ is in $\mathcal{C}$ because $\mathcal{C}$ is good, and it is easy to see that $P_A \oplus P_C$ is projective. Let us define ε_B. Since P_C is projective, there exists $h : P_C \to B$ such that $\psi h = \varepsilon_C$, and we put $\varepsilon_B(x_A, x_C) = \varphi\varepsilon_A x_A + h x_C$. Then the equation

$$\varphi\varepsilon_A x_A = \varepsilon_B(x_A, 0) = \varepsilon_B i_A x_A$$

says that the upper left square commutes, and the equation

$$\psi\varepsilon_B(x_A, x_C) = \psi\varphi\varepsilon_A x_A + \psi h x_C = 0 + \varepsilon_C x_C = \varepsilon_C p_C(x_A, x_C)$$

says that the lower left square commutes.

To see that ε_B is onto B, let $b \in B$ be given. Since p_C and ε_C are onto, so is $\varepsilon_C p_C = \psi\varepsilon_B$. Thus we can choose (x_A, x_C) in $P_A \oplus P_C$ with $\psi(b) = \psi\varepsilon_B(x_A, x_C)$. Hence $b - \varepsilon_B(x_a, x_C)$ lies in $\ker\psi = \operatorname{image}\varphi$, and we can write

$$b - \varepsilon_B(x_A, x_C) = \varphi(a) = \varphi\varepsilon_A(x'_A) = \varepsilon_B i_A(x'_A) = \varepsilon_B(x'_A, 0)$$

for some $x'_A \in P_A$. Then $b = \varepsilon_B(x_A + x'_A, x_C)$, and ε_B is onto.

Let $M_B = \ker\varepsilon_B$, and let $\psi_B : M_B \to P_A \oplus P_C$ be the inclusion. For m_A in M_A, let $\varphi_1(m_A) = (\psi_A m_A, 0)$. Then $\varphi_1(m_A)$ is in M_B because

$$\varepsilon_B(\psi_A m_A, 0) = \varphi\varepsilon_A\psi_A m_A + h0 = \varphi 0 + h0 = 0.$$

Moreover, this definition of φ_1 makes the upper right square commute.

To define ψ_1, let (x_A, x_C) be in M_B, so that $\varepsilon_B(x_A, x_C) = 0$. Then $0 = \psi\varepsilon_B(x_A, x_C) = \varepsilon_C p_C(x_A, x_C) = \varepsilon_C(x_C)$, x_C lies in $\ker\varepsilon_C = \operatorname{image}\psi_C$, and $x_C = \psi_C(m_C)$ for a unique m_C in M_C. We put $\psi_1(x_A, x_C) = m_C$. Then the equation

$$\psi_C\psi_1(x_A, x_C) = \psi_C(m_C) = x_C = p_C(x_A, x_C) = p_C\psi_B(x_A, x_C)$$

shows that the lower right square commutes.

Now all the squares commute, and all the rows and columns are exact except possibly the third column. Corollary 4.8 allows us to conclude that the third column is exact, and the proof of the lemma is complete. $\square$

PROOF OF THEOREM 4.22. The main step is to construct projective resolutions of A, B, and C by an inductive process in such a way that the three resolutions together form an exact sequence of chain complexes. We start by forming projective resolutions

$$0 \longleftarrow A \xleftarrow{\varepsilon_A} X_0 \xleftarrow{\alpha_0} X_1 \xleftarrow{\alpha_1} \cdots$$

and

$$0 \longleftarrow C \xleftarrow{\varepsilon_C} Z_0 \xleftarrow{\gamma_0} Z_1 \xleftarrow{\gamma_1} \cdots .$$

Replacing X_1 by $M_A = \ker\alpha_0$ and Z_1 by $M_C = \ker\gamma_0$, we are led to the starting diagram in Lemma 4.23. Application of the lemma produces a short exact sequence

$$0 \longleftarrow B \xleftarrow{\varepsilon_B} X_0 \oplus Z_0 \xleftarrow{\text{inc}} M_B \longleftarrow 0$$

and the vertical maps φ_1 and ψ_1 that make the squares commute in the lemma. Next we move everything one step to the right, applying the lemma to a diagram as in the lemma with first and third rows

$$0 \longleftarrow \ker\varepsilon_A \xleftarrow{\alpha_0} X_1 \xleftarrow{\text{inc}} \ker\alpha_0 \longleftarrow 0$$

and $$0 \longleftarrow \ker\varepsilon_C \xleftarrow{\gamma_0} Z_1 \xleftarrow{\text{inc}} \ker\gamma_0 \longleftarrow 0$$

and with an exact sequence in the first column involving the maps φ_1 and ψ_1. Application of the lemma produces a short exact sequence

$$0 \longleftarrow \ker\varepsilon_B \xleftarrow{\beta_0} X_1 \oplus Z_1 \xleftarrow{\text{inc}} \ker\beta_0 \longleftarrow 0$$

and the vertical maps φ_2 and ψ_2 that make the squares commute in the lemma. We can put these steps together to form the following diagram with exact rows and columns and with commuting squares:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 & A \arrow[l] \arrow[d, "\varphi"'] & X_0 \arrow[l, "\varepsilon_A"'] \arrow[d, "i_{X_0}"'] & X_1 \arrow[l, "\alpha_0"'] \arrow[d, "i_{X_1}"'] & \ker\alpha_1 \arrow[l, "\text{inc}"'] \arrow[d, "\varphi_2"'] & 0 \arrow[l] \\
0 & B \arrow[l] \arrow[d, "\psi"'] & X_0 \oplus Z_0 \arrow[l, "\varepsilon_B"'] \arrow[d, "p_{Z_0}"'] & X_1 \oplus Z_1 \arrow[l, "\beta_0"'] \arrow[d, "p_{Z_1}"'] & \ker\beta_1 \arrow[l, "\text{inc}"'] \arrow[d, "\psi_2"'] & 0 \arrow[l] \\
0 & C \arrow[l] \arrow[d] & Z_0 \arrow[l, "\varepsilon_C"'] \arrow[d] & Z_1 \arrow[l, "\gamma_0"'] \arrow[d] & \ker\gamma_1 \arrow[l, "\text{inc}"'] \arrow[d] & 0 \arrow[l] \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

We can repeat the use of Lemma 4.23, starting from the last column of the above diagram and more of the projective resolutions of A and C, and then we can merge the new result with the diagram above to obtain a diagram with one additional column. Continuing in this way, we arrive at three projective resolutions and vertical maps that together form an exact sequence of chain complexes.

To obtain a long exact sequence for our derived functors, we apply the functor F to the final diagram above, except that we drop the left column of 0's and the column containing A, B, C. After the application of F, the remaining columns are still exact because the columns in $\mathcal{C}$ are split and because F sends split exact sequences to split exact sequences.[10] Then we apply Theorem 4.7, taking Proposition 4.21 into account, and the long exact sequence results except for the one detail of the 0 at the end. In other words, we still have to prove exactness at $F(C)$. But exactness at this point is immediate from the assumed one-sided exactness of F. This completes the proof. $\square$

Before addressing the functoriality of the association in Theorem 4.22, let us record the corresponding result when the derived functor makes use of injective resolutions.

Theorem 4.24. Let $F : \mathcal{C} \to \mathcal{C}'$ be an additive functor between two good categories. Suppose that F either is contravariant right exact or is covariant left exact, and suppose that $\mathcal{C}$ has enough injectives. Whenever there are three modules and two maps in $\mathcal{C}$ forming a short exact sequence

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0,$$

then the derived functors of F on the three modules form a long exact sequence in $\mathcal{C}'$ as follows:

(a) If F is contravariant and right exact, then the long exact sequence is

$$\begin{aligned} 0 \longleftarrow F(A) \longleftarrow F(B) \longleftarrow F(C) \longleftarrow F_1(A) \longleftarrow F_1(B) \longleftarrow F_1(C) \\ \longleftarrow F_2(A) \longleftarrow F_2(B) \longleftarrow F_2(C) \longleftarrow F_3(A) \longleftarrow \cdots . \end{aligned}$$

(b) If F is covariant and left exact, then the long exact sequence is

$$\begin{aligned} 0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow F^1(A) \longrightarrow F^1(B) \longrightarrow F^1(C) \\ \longrightarrow F^2(A) \longrightarrow F^2(B) \longrightarrow F^2(C) \longrightarrow F^3(A) \longrightarrow \cdots . \end{aligned}$$

PROOF. The necessary modifications to the proof of Theorem 4.22 are fairly straightforward, but some comments are in order concerning how Lemma 4.23 is to be modified. In the diagram in the statement of Lemma 4.23, all the horizontal arrows are to be reversed, the projectives P_A and P_C are to be replaced by injectives

[10] A split exact sequence is the union of two four-term exact sequences from each end, and F is exact on each of these. In addition, we saw in Section 2 that F respects direct sums. It follows that F carries split exact sequences to split exact sequences.

I_A and I_C, and M_A and M_C are the quotients $M_A = I_A/\varepsilon_A(A)$ and $M_C = I_C/\varepsilon_C(C)$. Let us define ε_B. Since I_A is injective, choose $h : B \to I_A$ with $h\varphi = \varepsilon_A$, and put $\varepsilon_B(b) = (h(b), \varepsilon_C\psi(b))$. Then the equation

$$\varepsilon_B\varphi(a) = (h\varphi a, \varepsilon_C\psi\varphi a) = (\varepsilon_A(a), 0) = i_A\varepsilon_A(a)$$

says that the upper left square commutes, and the equation

$$\varepsilon_C\psi(b) = p_C(h(b), \varepsilon_C\psi(b)) = p_C\varepsilon_B(b)$$

says that the lower left square commutes.

To see that ε_B is one-one, let $\varepsilon_B(b) = 0$. Then $0 = p_C\varepsilon_B(b) = \varepsilon_C\psi(b)$. Since ε_C is one-one, $\psi(b) = 0$, b lies in $\ker\psi = \text{image}\,\varphi$, and $b = \varphi(a)$. Then $0 = \varepsilon_B(b) = \varepsilon_B\varphi(a) = i_A\varepsilon_A(a)$, and $a = 0$ because i_A and ε_A are one-one. Hence $b = \varphi(a) = 0$, and ε_B is one-one.

Let $M_B = (I_A \oplus I_C)/\varepsilon_B(B)$, and let $\psi_B : I_A \oplus I_C \to M_B$ be the quotient map. To define φ_1, we let $\varphi_1(m_A) = \psi_B(x_A, 0)$ if $m_A = \psi_A x_A$ with $x_A \in I_A$. If x'_A is another preimage of m_A under ψ_A^{-1}, then $x'_A - x_A = \varepsilon_A(a)$ for some $a \in A$, and $\psi_B(x_A, 0) - \psi_B(x'_A, 0) = \psi_B i_A\varepsilon_A(a) = \psi_B\varepsilon_B\varphi(a) = 0$; hence φ_1 is well defined. Since $\psi_B i_A x_A = \psi_B(x_A, 0) = \varphi_1 m_A = \varphi_1\psi_A x_A$, the upper right square commutes. To define ψ_1, let $m_B \in M_B$ be $\psi_B(x_A, x_C)$, and define $\psi_1(m_B) = \psi_C(x_C)$. If (x'_A, x'_C) is another preimage of m_B under ψ_B^{-1}, then $(x'_A, x'_C) - (x_A, x_C) = \varepsilon_B(b)$ for some $b \in B$, and $\psi_C(x'_C) - \psi_C(x_C) = \psi_C p_C(x'_A, x'_C) - \psi_C p_C(x_A, x_C) = \psi_C p_C\varepsilon_B(b) = \psi_C\varepsilon_C\psi(b) = 0$; hence ψ_1 is well defined. Since $\psi_C p_C(x_A, x_C) = \psi_C(x_C) = \psi_1(m_B) = \psi_1\psi_B(x_A, x_C)$, the lower right square commutes.

Now all the squares commute, and all the rows and columns are exact except possibly the third column. Corollary 4.8 allows us to conclude that the third column is exact, and the proof of the analog of Lemma 4.23 for injectives is complete. Theorem 4.24 then follows routinely. $\square$

Theorem 4.25. Let $F : \mathcal{C} \to \mathcal{C}'$ be an additive functor between two good categories. Suppose that F either is covariant right exact or is contravariant left exact, and suppose that $\mathcal{C}$ has enough projectives. Then the passage as in Theorem 4.22 from short exact sequences in $\mathcal{C}$ to long exact sequences of derived functor modules in $\mathcal{C}'$ is functorial in the following sense: whenever

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{A} & \xrightarrow{\widetilde{\varphi}} & \widetilde{B} & \xrightarrow{\widetilde{\psi}} & \widetilde{C} & \longrightarrow & 0 \\
 & & {\scriptstyle f_A}\big\downarrow & & {\scriptstyle f_B}\big\downarrow & & {\scriptstyle f_C}\big\downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C & \longrightarrow & 0
\end{array}$$

is a diagram in $\mathcal{C}$ with exact rows and commuting squares, then the long exact sequences of derived functors of F on $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ and A, B, C make commutative squares with the maps induced by the derived functors on f_A, f_B, f_C.

PROOF. The proof of Theorem 4.22 involved constructing a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & A & \xleftarrow{\varepsilon_A} & X_0 & \xleftarrow{\alpha_0} & X_1 & \xleftarrow{\alpha_1} & X_2 & \longleftarrow & \cdots \\
 & & {\scriptstyle\varphi}\downarrow & & {\scriptstyle i_{X_0}}\downarrow & & {\scriptstyle i_{X_1}}\downarrow & & {\scriptstyle i_{X_2}}\downarrow & & \\
0 & \longleftarrow & B & \xleftarrow{\varepsilon_B} & X_0 \oplus Z_0 & \xleftarrow{\beta_0} & X_1 \oplus Z_1 & \xleftarrow{\beta_1} & X_2 \oplus Z_2 & \longleftarrow & \cdots \\
 & & {\scriptstyle\psi}\downarrow & & {\scriptstyle p_{Z_0}}\downarrow & & {\scriptstyle p_{Z_1}}\downarrow & & {\scriptstyle p_{Z_2}}\downarrow & & \\
0 & \longleftarrow & C & \xleftarrow{\varepsilon_C} & Z_0 & \xleftarrow{\gamma_0} & Z_1 & \xleftarrow{\gamma_1} & Z_2 & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and commuting squares in which each X_n and Z_n is projective, and a similar diagram corresponds to the given short exact sequence with tildes on it. The present theorem will follow from the functoriality in Theorem 4.10 if we can arrange that these two diagrams can be embedded in a 3-dimensional diagram with each of these diagrams in a horizontal plane and with vertical maps from the one diagram to the other such that all vertical squares commute.

We are given vertical maps f_A, f_B, and f_C, which we can regard as extending from the diagram with tildes to the other diagram. In addition, Theorem 4.12 gives us chain maps $\{f_{X_n}\}$ and $\{f_{Z_n}\}$ with $f_{X_{-1}} = f_A$ and $f_{X_{-1}} = f_C$, and all the completed vertical squares in the 3-dimensional diagram commute. To complete the proof, we construct by induction for $n \geq 0$ a map $f_n : \widetilde{X}_n \oplus \widetilde{Z}_n \to X_n \oplus Z_n$ such that

$$p_{Z_n} f_n = f_{Z_n} p_{\widetilde{Z}_n}, \qquad f_n i_{\widetilde{X}_n} = i_{X_n} f_{X_n}, \qquad \beta_{n-1} f_n = f_{n-1} \widetilde{\beta}_{n-1}, \qquad (*)$$

with the understanding that $\beta_{-1} = \varepsilon_B$. To make it possible for the inductive step to include the starting step of the induction, let us write $X_{-1} = A$, $Z_{-1} = B$, $i_{X_{-1}} = \varphi$, $p_{Z_{-1}} = \psi$, $\alpha_{-1} = \varepsilon_A$, $\gamma_{-1} = \varepsilon_C$, and $f_{-1} = f_B$. Also, let us understand any module or map with subscript -2 to be 0.

We shall construct f_n. For $\widetilde{z} \in \widetilde{Z}_n$, we apply $p_{Z_{n-1}}$ to the difference $\beta_{n-1}(0, f_{Z_n}\widetilde{z}) - f_{n-1}\widetilde{\beta}_{n-1}(0, \widetilde{z})$ and get

$$\begin{aligned} p_{Z_{n-1}}\beta_{n-1}(0, f_{Z_n}\widetilde{z}) &- p_{Z_{n-1}} f_{n-1}\widetilde{\beta}_{n-1}(0, \widetilde{z}) \\ &= \gamma_{n-1} p_{Z_n}(0, f_{Z_n}\widetilde{z}) - f_{Z_{n-1}} p_{\widetilde{Z}_{n-1}}\widetilde{\beta}_{n-1}(0, \widetilde{z}) \\ &= \gamma_{n-1} f_{Z_n}\widetilde{z} - f_{Z_{n-1}}\widetilde{\gamma}_{n-1} p_{\widetilde{Z}_n}(0, \widetilde{z}) \\ &= f_{Z_{n-1}}\widetilde{\gamma}_{n-1}\widetilde{z} - f_{Z_{n-1}}\widetilde{\gamma}_{n-1}\widetilde{z} = 0. \end{aligned}$$

Thus $\beta_{n-1}(0, f_{Z_n}\widetilde{z}) - f_{n-1}\widetilde{\beta}_{n-1}(0, \widetilde{z}) = i_{X_{n-1}}(x)$ for a unique $x \in X_{n-1}$, and we define $\tau : \widetilde{Z}_n \to X_{n-1}$ by saying that $\tau(\widetilde{z})$ should be this x. This makes

$$i_{X_{n-1}}\tau(\widetilde{z}) = \beta_{n-1}(0, f_{Z_n}\widetilde{z}) - f_{n-1}\widetilde{\beta}_{n-1}(0, \widetilde{z}).$$

Setting up the diagram

$$\begin{tikzcd} & \widetilde{Z}_n \arrow[d, "\tau"] \arrow[dr, dashed, "\sigma"] & \\ X_{n-2} & X_{n-1} \arrow[l, "\alpha_{n-2}"'] & X_n \arrow[l, "\alpha_{n-1}"'] \end{tikzcd}$$

we prepare to invoke Lemma 4.11. We have

$$\begin{aligned} i_{X_{n-2}}\alpha_{n-2}\tau(\widetilde{z}) = \beta_{n-2} i_{X_{n-1}}\tau(\widetilde{z}) &= \beta_{n-2}\beta_{n-1}(0, f_{Z_n}\widetilde{z}) - \beta_{n-2} f_{n-1}\widetilde{\beta}_{n-1}(0, \widetilde{z}) \\ &= 0 - f_{n-2}\widetilde{\beta}_{n-2}\widetilde{\beta}_{n-1}(0, \widetilde{z}) = 0. \end{aligned}$$

Since $i_{X_{n-2}}$ is one-one, $\alpha_{n-2}\tau = 0$, and Lemma 4.11 applies. Thus we obtain $\sigma : \widetilde{Z}_n \to X_n$ with $\alpha_{n-1}\sigma = \tau$, and σ satisfies

$$i_{X_{n-1}}\alpha_{n-1}\sigma(\widetilde{z}) = \beta_{n-1}(0, f_{Z_n}\widetilde{z}) - f_{n-1}\widetilde{\beta}_{n-1}(0, \widetilde{z}). \qquad (**)$$

Define

$$f_n(\widetilde{x}, \widetilde{z}) = (f_{X_n}(\widetilde{x}) - \sigma(\widetilde{z}), f_{Z_n}(\widetilde{z})). \qquad (\dagger)$$

With f_n defined, we are to prove the three formulas $(*)$. For the first formula in $(*)$, we apply p_{Z_n} to both sides of $(\dagger)$ and obtain $p_{Z_n} f_n(\widetilde{x}, \widetilde{z}) = f_{Z_n}(\widetilde{z}) = f_{Z_n} p_{\widetilde{Z}_n}(\widetilde{x}, \widetilde{z})$, which is the desired formula. The second formula in $(*)$ at $\widetilde{x}$ is just $(\dagger)$ with $\widetilde{z} = 0$.

We are left with proving the third formula in $(*)$. Using the second formula in $(*)$, we have

$$\begin{aligned} \beta_{n-1} f_n(\widetilde{x}, 0) &= \beta_{n-1} f_n i_{\widetilde{X}_n}(\widetilde{x}) = \beta_{n-1} i_{X_n} f_{X_n}(\widetilde{x}) \\ &= i_{X_{n-1}}\alpha_{n-1} f_{X_n}(\widetilde{x}) = i_{X_{n-1}} f_{X_{n-1}}\widetilde{\alpha}_{n-1}(\widetilde{x}) \\ &= f_{n-1} i_{\widetilde{X}_{n-1}}\widetilde{\alpha}_{n-1}(\widetilde{x}) = f_{n-1}\widetilde{\beta}_{n-1} i_{\widetilde{X}_n}(\widetilde{x}) \\ &= f_{n-1}\widetilde{\beta}_{n-1}(\widetilde{x}, 0). \end{aligned} \qquad (\dagger\dagger)$$

Also,

$$\begin{aligned}
\beta_{n-1} f_n(0, \widetilde{z}) &= -\beta_{n-1} i_{X_n} \sigma(\widetilde{z}) + \beta_{n-1}(0,\, f_{Z_n}(\widetilde{z})) && \text{by } (\dagger) \\
&= -i_{X_{n-1}} \alpha_{n-1} \sigma(\widetilde{z}) + \beta_{n-1}(0,\, f_{Z_n}(\widetilde{z})) && \text{by commutativity} \\
&= f_{n-1} \widetilde{\beta}_{n-1}(0, \widetilde{z}) && \text{by } ({*}{*}).
\end{aligned}$$

Adding this equality and ($\dagger\dagger$), we obtain the third formula of ($*$). This completes the proof. $\square$

The version of Theorem 4.25 appropriate for Theorem 4.24 is the following, and its proof is similar.

Theorem 4.26. Let $F : \mathcal{C} \to \mathcal{C}'$ be an additive functor between two good categories. Suppose that F either is contravariant right exact or is covariant left exact, and suppose that $\mathcal{C}$ has enough injectives. Then the passage as in Theorem 4.24 from short exact sequences in $\mathcal{C}$ to long exact sequences of derived functor modules in $\mathcal{C}'$ is functorial in the following sense: whenever

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{A} & \xrightarrow{\widetilde{\varphi}} & \widetilde{B} & \xrightarrow{\widetilde{\psi}} & \widetilde{C} & \longrightarrow & 0 \\
 & & {\scriptstyle f_A}\big\downarrow & & {\scriptstyle f_B}\big\downarrow & & {\scriptstyle f_C}\big\downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C & \longrightarrow & 0
\end{array}$$

is a diagram in $\mathcal{C}$ with exact rows and commuting squares, then the long exact sequences of derived functors of F on $\widetilde{A}, \widetilde{B}, \widetilde{C}$ and A, B, C make commutative squares with the maps induced by the derived functors on f_A, f_B, f_C.

We come to an important application of the long exact sequences in Theorems 4.22 and 4.24. Projective and injective resolutions make it easy to work with derived functors theoretically, but in practice any computations with them are likely to be difficult. It is therefore convenient to be able to compute derived functors from other resolutions than projective and injective ones.[11] For definiteness let us work with the case of a covariant *left* exact functor in a good category with

[11] The case of sheaf cohomology illustrates this point well. The present theory extends from good categories of modules to arbitrary abelian categories along the lines of Section 8 below, and the cohomology theory of sheaves fits into this more general framework. One additive functor of interest with sheaves is the "global-sections" functor. Its derived functors can be formed with injective resolutions, built from "flabby" sheaves, but flabby sheaves as a practical matter are too big to be useful in computations. In the theory of several complex variables for example, one approach is to substitute "fine" sheaves in resolutions; these permit computations and fall under the abelian-category generalization of Theorem 4.27 below.

enough injectives; this is the most important case in applications, and the other three cases in Figure 4.5 can be handled in similar fashion. Let $F : \mathcal{C} \to \mathcal{C}'$ be an additive functor between good categories that is covariant left exact. A module M in $\mathcal{C}$ is said to be F-acyclic if $F^n(M) = 0$ for all $n \geq 1$. Every module M that is injective in $\mathcal{C}$ is F-acyclic, since $0 \longrightarrow M \longrightarrow M \longrightarrow 0$ is an injective resolution of M from which we can see that $F^n(M) = 0$ for $n \geq 1$. An F-**acyclic resolution** of a module A in $\mathcal{C}$ is a resolution $X = (A \longrightarrow X^+)$ in which X_n is an F-acyclic module for all $n \geq 0$.

Theorem 4.27. Let $\mathcal{C}$ and $\mathcal{C}'$ be two good categories, let F be an additive functor from $\mathcal{C}$ to $\mathcal{C}'$ that is covariant and left exact, and suppose that $\mathcal{C}$ has enough injectives. If a module A in $\mathcal{C}$ has an F-acyclic resolution $X = (A \longrightarrow X^+)$ and if $I = (A \longrightarrow I^+)$ is any injective resolution of A, then any cochain map $f : X \to I$ with $f_{-1} = 1_A$ induces an isomorphism $F^n(A) \cong H^n(F(X))$ for each $n \geq 0$.

REMARKS. Such a cochain map always exists and is unique up to homotopy, according to Theorem 4.16. Theorem 4.27 says that the derived functors of F on any module A can be computed from any F-acyclic resolution of A; it is not necessary to work only with injective resolutions. The same result as in the theorem holds with $F_n(A) \cong H_n(F(A))$ if F is contravariant and right exact. If F is covariant right exact or contravariant left exact and if $\mathcal{C}$ has enough projectives, then any chain map from a projective resolution of A to an F-acyclic resolution[12] induces an isomorphism of the derived functors of A with the homology or cohomology of F of the F-acyclic resolution.

PROOF. The injective resolution is at our disposal, according to Corollary 4.17. Using the hypothesis that $\mathcal{C}$ has enough injectives, choose for each n an injective J_n containing X_n, let $g_n : X_n \to J_n$ be the inclusion, and make $\{J_n\}$ into an injective resolution of 0 with coboundary maps 0. Then replace I in the assumptions by $I \oplus J$ and f by (f, g). The result is that we have reduced the theorem to the case that f is one-one. Changing notation, we may assume from the outset that the injective resolution is $I = (A \longrightarrow I^+)$ and that the chain map $f : X \to I$ is one-one in each degree.

Put $Y_n = I_n/f_n(X_n) = \operatorname{coker} f_n$. The sequence

$$0 \longrightarrow X_n \xrightarrow{f_n} I_n \longrightarrow Y_n \longrightarrow 0 \qquad (*)$$

is exact, and Theorem 4.24a shows that the sequence

$$F^k(I_n) \longrightarrow F^k(Y_n) \longrightarrow F^{k+1}(X_n)$$

[12]For this situation, F-acyclic resolutions are understood to be chain complexes rather than cochain complexes.

is exact for every $k \geq 0$. Since I_n and X_n are F-acyclic for $n \geq 0$, the end terms are 0 for all $k \geq 1$. Consequently Y_n is F-acyclic for all $n \geq 0$.

Referring to $(*)$ for n and for $n+1$, we see that the coboundary map from I_n to I_{n+1} induces a compatible coboundary map from Y_n to Y_{n+1}. Thus we may consider $Y = (0 \longrightarrow Y^+)$ as a cochain complex with $Y^+ = \{Y_n\}_{n \geq 0}$. Then the equations $(*)$ for all $n \geq 0$, together with the coboundary maps, make

$$0 \longrightarrow X \xrightarrow{f} I \longrightarrow Y \longrightarrow 0 \tag{$**$}$$

into a short exact sequence of complexes. Since X and I are exact, Corollary 4.8 shows that Y is exact.

If we apply F to the short exact sequence of complexes $(**)$, we obtain a planar diagram

$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(I) \longrightarrow F(Y) \longrightarrow 0 \tag{$\dagger$}$$

whose rows are the result of applying F to $(*)$, whose columns are complexes, and whose squares commutes. As usual we drop the row for $n = -1$, replacing it with a row of 0's. Let us prove that $(\dagger)$ is in fact a short exact sequence of complexes. In fact, the result of applying F to $(*)$ is the long exact sequence that begins

$$0 \longrightarrow F(X_n) \longrightarrow F(I_n) \longrightarrow F(Y_n) \longrightarrow F^1(X_n).$$

For $n \geq 0$, X_n is F-acyclic. Thus $F^1(X_n) = 0$, and the exactness for $n \geq 0$ follows. For $n \leq -1$, the rows of the diagram $(\dagger)$ are 0 and hence are exact. Thus $(\dagger)$ is a short exact sequence of complexes.

We shall now prove that $F(Y) = (0 \longrightarrow F(Y^+))$ is exact. Combining this fact with the exactness of the rows of $(\dagger)$ and applying Corollary 4.8 will then yield $H^n(F(X)) \cong H^n(F(I))$ for all $n \geq 0$. Since $H^n(F(I)) = F^n(A)$, this step will complete the proof.

To prove that $F(Y) = (0 \longrightarrow F(Y^+))$ is exact, define $Z_0 = Y_0$ and $Z_n = \operatorname{coker}(Y_{n-1} \to Y_n)$ for $n \geq 1$. Let $d_n : Y_n \to Y_{n+1}$ be the coboundary map. For each $n \geq 0$, the complex

$$0 \longrightarrow Y_n/\ker d_n \longrightarrow Y_{n+1} \longrightarrow Z_{n+1} \longrightarrow 0$$

is exact. Since $\ker d_n = \operatorname{image} d_{n-1}$ by exactness of Y, we have $Y_n/\ker d_n = Y_n/\operatorname{image} d_{n-1} = Z_n$, and thus

$$0 \longrightarrow Z_n \longrightarrow Y_{n+1} \longrightarrow Z_{n+1} \longrightarrow 0 \tag{$\dagger\dagger$}$$

is exact for all $n \geq 0$.

Let us use $(\dagger\dagger)$ to prove the preliminary result that Z_n is F-acyclic for all $n \geq 0$. For $n = 0$, $Z_0 = Y_0$, and Y_0 is known to be F-acyclic. Proceeding inductively, suppose that Z_n is known to be F-acyclic. Applying Theorem 4.24a to (††), we see that

$$F^k(Y_{n+1}) \longrightarrow F^k(Z_{n+1}) \longrightarrow F^{k+1}(Z_n)$$

is exact for all $n \geq 0$ and all $k \geq 0$. For $n \geq 0$ and $k \geq 1$, the left end is 0 because Y_{n+1} is F-acyclic, and the right end is 0 because Z_n is F-acyclic by the inductive hypothesis. Therefore the middle term is 0, Z_{n+1} is F-acyclic, and the induction is complete.

Theorem 4.24a when applied to (††) shows that

$$0 \longrightarrow F(Z_n) \longrightarrow F(Y_{n+1}) \longrightarrow F(Z_{n+1}) \longrightarrow F^1(Z_n)$$

is exact for all $n \geq 0$, and we now know that the term at the right end is 0. Therefore

$$0 \longrightarrow F(Z_n) \longrightarrow F(Y_{n+1}) \longrightarrow F(Z_{n+1}) \longrightarrow 0 \tag{‡}$$

is exact for all $n \geq 0$.

Now we can prove that the complex

$$0 \longrightarrow F(Y_0) \longrightarrow F(Y_1) \longrightarrow F(Y_2) \longrightarrow F(Y_3) \longrightarrow \cdots \tag{‡‡}$$

is exact at each module $F(Y_n)$. We know from Section 2 that we can merge two exact sequences

$$\cdots \to F(Y_{n+1}) \to F(Z_{n+1}) \to 0 \quad \text{and} \quad 0 \to F(Z_{n+1}) \to F(Y_{n+2}) \to \cdots$$

into a single exact sequence

$$\cdots \longrightarrow F(Y_{n+1}) \longrightarrow F(Y_{n+2}) \longrightarrow \cdots .$$

Consequently inductive application of (‡) shows that the sequence

$$0 \to F(Z_0) \longrightarrow F(Y_1) \longrightarrow F(Y_2) \longrightarrow \cdots \longrightarrow F(Y_{n+1}) \longrightarrow F(Z_{n+1}) \to 0$$

is exact for each $n \geq 0$. In addition, we know that $Z_0 = Y_0$ by definition. Therefore (‡‡) is exact at $F(Y_n)$ for each $n \geq 0$, and the proof is complete. $\square$

Theorems 4.22 and 4.24 produce a long exact sequence from one additive functor and a short exact sequence of modules. Although it may at first seem odd to do so, we can obtain a different long exact sequence by varying the functor and fixing the module. This result, given as Proposition 4.28 below, will be used in the next section in analyzing the Ext and Tor functors.

Let $\mathcal{C}$ and $\mathcal{C}'$ be two good categories, and let F, G, H be three additive functors from $\mathcal{C}$ to $\mathcal{C}'$. For definiteness, suppose that F, G, H are covariant and right exact. Suppose that there is a natural transformation S of F into G and there is a natural transformation T of G into H. We say that the sequence

$$F \xrightarrow{S} G \xrightarrow{T} H$$

is **exact on projectives** if for every projective P in $\mathcal{C}$, the sequence

$$0 \longrightarrow F(P) \xrightarrow{S_P} G(P) \xrightarrow{T_P} H(P) \longrightarrow 0$$

is exact. Analogous definitions are to be made with projectives or injectives for the three other kinds of derived functors as in Figure 4.5.

Proposition 4.28. Let $\mathcal{C}$ and $\mathcal{C}'$ be two good categories, let F, G, H be three additive functors from $\mathcal{C}$ to $\mathcal{C}'$, suppose that F, G, H are covariant and right exact, and suppose that $\mathcal{C}$ has enough projectives. If there are natural transformations $S : F \to G$ and $T : G \to H$ such that the sequence $F \xrightarrow{S} G \xrightarrow{T} H$ is exact on projectives, then the derived functors of F, G, H on each module A in $\mathcal{C}$ form a long exact sequence

$$\begin{aligned} 0 \longleftarrow H(A) \longleftarrow G(A) \longleftarrow F(A) \longleftarrow H_1(A) \longleftarrow G_1(A) \longleftarrow F_1(A) \\ \longleftarrow H_2(A) \longleftarrow G_2(A) \longleftarrow F_2(A) \longleftarrow H_3(A) \longleftarrow \cdots . \end{aligned}$$

The passage from A to the long exact sequence is functorial in A.

REMARKS. The same long exact sequence and functoriality hold with the arrows reversed and F and H interchanged if the three functors are contravariant and left exact. If F, G, H are contravariant and right exact or are covariant and left exact, then analogous conclusions are valid provided $\mathcal{C}$ has enough injectives and the natural transformations S and T are exact on injectives.

PROOF. If $P = (P^+ \longrightarrow A)$ is a projective resolution of A, then the natural transformations S and T give us a planar diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F(P_0) & \xrightarrow{S_{P_0}} & G(P_0) & \xrightarrow{T_{P_0}} & H(P_0) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F(P_1) & \xrightarrow{S_{P_1}} & G(P_1) & \xrightarrow{T_{P_1}} & H(P_1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

in which the columns are complexes, the rows are exact because the sequence $F \xrightarrow{S} G \xrightarrow{T} H$ is exact on projectives, and the squares commute because S and T are natural transformations. The construction of the long exact sequence then follows from Theorem 4.7.

For the functoriality, suppose that $\varphi : A \to A'$ is a map between two modules of $\mathcal{C}$. Let $P = (P^+ \longrightarrow A)$ and $P' = (P'^+ \longrightarrow A)$ be projective resolutions of A and A', and use Theorem 4.12 to extend φ to a chain map $\{\varphi_n\}$ of P to P'. Then the planar diagrams as above for P and P' can be embedded in a 3-dimensional diagram in such a way that the various maps $F(\varphi_n)$, $G(\varphi_n)$, and $H(\varphi_n)$ connecting the diagram for P to the diagram for P' make all squares commute. The functoriality now follows immediately from Theorem 4.10. $\square$

7. Ext **and** Tor

In this section we study the derived functors of Hom and tensor product. Although we shall treat each as carrying unital left R modules, where R is a ring with identity, to abelian groups, the theory applies also to more complicated versions of Hom and tensor product, such as when one of the R modules in question is actually a bimodule for the rings R and S and the result of Hom or tensor product is an S module. Problems 9–11 at the end of the chapter address the situation with bimodules.

We know that $\mathrm{Hom}_R(A, B)$ is a contravariant left exact functor of the A variable and a left exact covariant functor of the B variable. Thus we have two initial choices for inserting resolutions and creating derived functors, namely

$$\mathrm{Ext}^n_R(A, B) = H^n\big(\mathrm{Hom}_R(P, B)\big), \qquad \text{with } P = (A \leftarrow P^+) \text{ projective},$$

and

$$\mathrm{ext}^n_R(A, B) = H^n\big(\mathrm{Hom}_R(A, I)\big), \qquad \text{with } I = (B \to I^+) \text{ injective}.$$

Existence of the first one depends on having enough projectives in the category of the A variable, and existence of the second one depends on having enough injectives in the category of the B variable. Each of these, just as with Hom, depends on two variables, one in contravariant fashion and the other in covariant fashion. Thus Ext and ext are not functors of two variables in the strict sense of our definitions. Instead, they are examples of "bifunctors," of which $\mathrm{Hom}_R(\,\cdot\,,\,\cdot\,)$ is the prototype, and the main result, Theorem 4.31 below, in essence says that Ext and ext are naturally isomorphic as bifunctors, provided the first domain category has enough projectives *and* the second has enough injectives. Among other things this natural isomorphism will justify and explain how we were able to define cohomology of groups in more than one way.[13]

In the case of tensor product $A \otimes_R B$, similar remarks apply. Here A is a unital right R module, and B is a unital left R module. The module A in a natural way is a unital left R^o module, where R^o is the opposite ring of R, and thus tensor product is to be regarded as defined on the product of two categories of left modules just as Hom is. We can regard tensor product as an actual functor in either variable, and the functor is covariant right exact in both cases. Again we have two initial choices for inserting resolutions and creating derived functors, namely

$$\mathrm{Tor}_n^R(A, B) = H^n\big(P \otimes_R B\big), \qquad \text{with } P = (A \leftarrow P^+) \text{ projective},$$

and

$$\mathrm{tor}_n^R(A, B) = H^n\big(A \otimes_R P\big), \qquad \text{with } P' = (B \leftarrow P'^+) \text{ projective}.$$

These exist if the domain categories have enough projectives. Both Tor and tor can be considered as covariant functors of two variables, or else as "bifunctors," and one can show in the same way as for Ext and ext that Tor and tor are naturally isomorphic. There is no need to write out the details. It is customary to write Tor for the common value.

Proposition 4.29. Let $\mathcal{C}$ and $\mathcal{C}'$ be good categories of unital left R modules, and suppose that $\mathcal{C}$ has enough projectives. Then the contravariant left exact functors $\mathrm{Hom}_R(\,\cdot\,, B)$ from $\mathcal{C}$ to $\mathcal{C}_{\mathbb{Z}}$ and their derived functors $\mathrm{Ext}_R^n(\,\cdot\,, B)$ have the following properties:

(a) Whenever $0 \to A' \to A \to A'' \to 0$ is a short exact sequence in $\mathcal{C}$, then there is a corresponding long exact sequence

$$\begin{aligned}
0 &\longrightarrow \mathrm{Hom}_R(A'', B) \longrightarrow \mathrm{Hom}_R(A, B) \longrightarrow \mathrm{Hom}_R(A', B)\\
&\longrightarrow \mathrm{Ext}_R^1(A'', B) \longrightarrow \mathrm{Ext}_R^1(A, B) \longrightarrow \mathrm{Ext}_R^1(A', B)\\
&\longrightarrow \mathrm{Ext}_R^2(A'', B) \longrightarrow \mathrm{Ext}_R^2(A, B) \longrightarrow \mathrm{Ext}_R^2(A', B) \to \mathrm{Ext}_R^3(A'', B) \to \cdots
\end{aligned}$$

in $\mathcal{C}_{\mathbb{Z}}$ for each module B in $\mathcal{C}'$. The passage from short exact sequences in $\mathcal{C}$ to long exact sequences of derived functor modules in $\mathcal{C}_{\mathbb{Z}}$ is functorial in its dependence on the exact sequence in the first variable in the sense of Theorem 4.25 and is natural in the second variable in the sense that if a map $\eta : \widetilde{B} \to B$ is given, then $\mathrm{Hom}(1, \eta)$ defines a chain map from the long exact sequence for $\widetilde{B}$ to the long exact sequence for B.

[13]It would add only definitions to our discussion to say precisely what a general bifunctor is and what a general natural transformation between bifunctors is, and we shall skip that detail, in effect incorporating the definitions into the theorem.

(b) If P is a projective in $\mathcal{C}$ and I is an injective in $\mathcal{C}'$, then $\mathrm{Ext}^n_R(P, B) = 0 = \mathrm{Ext}^n_R(A, I)$ for all $n \geq 1$ and all modules A in $\mathcal{C}$ and B in $\mathcal{C}'$.

(c) Whenever $0 \to B' \to B \to B'' \to 0$ is a short exact sequence in $\mathcal{C}'$, then there is a corresponding long exact sequence

$$\begin{aligned}
0 &\longrightarrow \mathrm{Hom}_R(A, B') \longrightarrow \mathrm{Hom}_R(A, B) \longrightarrow \mathrm{Hom}_R(A, B'') \\
&\longrightarrow \mathrm{Ext}^1_R(A, B') \longrightarrow \mathrm{Ext}^1_R(A, B) \longrightarrow \mathrm{Ext}^1_R(A, B'') \\
&\longrightarrow \mathrm{Ext}^2_R(A, B') \longrightarrow \mathrm{Ext}^2_R(A, B) \longrightarrow \mathrm{Ext}^2_R(A, B'') \to \mathrm{Ext}^3_R(A, B') \to \cdots
\end{aligned}$$

in $\mathcal{C}_{\mathbb{Z}}$ for each module A in $\mathcal{C}$. The passage from short exact sequences in $\mathcal{C}'$ to long exact sequences of derived functor modules in $\mathcal{C}_{\mathbb{Z}}$ is functorial in the exact sequence in the second variable and is natural in the first variable in the sense that if a map $\eta : \widetilde{A} \to A$ is given, then $\mathrm{Hom}(\eta, 1)$ defines a chain map from the long exact sequence for A to the long exact sequence for $\widetilde{A}$.

REMARKS. The naturality in the B parameter of the construction of the long exact sequence in (a) implies that Ext^n_R is a covariant functor of the second variable for fixed argument of the first variable. It implies also that all maps $\mathrm{Ext}^n_R(\alpha, 1)$ commute with all maps $\mathrm{Ext}^n_R(1, \beta)$.

PROOF. For (a), Theorem 4.22b gives the exact sequence, and Theorem 4.25 proves the functoriality in the first variable. For the naturality in the second variable, let $\eta : \widetilde{B} \to B$ be given. The proof of Theorem 4.22 produces a short exact sequence of projective resolutions of A', A, A'' to which the functor in that theorem is then applied. We now have two such functors $\mathrm{Hom}_R(\,\cdot\,, \widetilde{B})$ and $\mathrm{Hom}_R(\,\cdot\,, B)$, and the maps within each image diagram are all of the form $\mathrm{Hom}(\alpha, 1)$. The two diagrams fit into a 3-dimensional diagram, and the maps between the two diagrams are of the form $\mathrm{Hom}(1, \eta)$. Since all maps $\mathrm{Hom}(\alpha, 1)$ commute with all maps $\mathrm{Hom}(1, \beta)$, the 3-dimensional diagram is commutative. The corresponding long exact sequences are then related by a cochain map according to Theorem 4.10.

For (b), $0 \leftarrow P \leftarrow P \leftarrow 0$ is a projective resolution of P, and hence any derived functor that is defined by projective resolutions is 0 in degree ≥ 1. In addition, Proposition 4.19b shows that $\mathrm{Hom}_R(\,\cdot\,, I)$ is an exact functor, and hence its derived functors are 0 in degree ≥ 1.

For (c), we shall apply Proposition 4.28 in its version for contravariant left exact functors. Let $\varphi : B' \to B$ and $\psi : B \to B''$ be the maps in the given short exact sequence, and let F, G, H be the functors with $F(A) = \mathrm{Hom}_R(A, B')$, $G(A) = \mathrm{Hom}_R(A, B)$, $H(A) = \mathrm{Hom}_R(A, B'')$. Then we have a natural transformation S of F into G given by $S_A = \mathrm{Hom}(1, \varphi)$ and a natural transformation T of G into H given by $T_A = \mathrm{Hom}(1, \psi)$. Since

$$0 \longrightarrow \mathrm{Hom}_R(P, B') \xrightarrow{S_P} \mathrm{Hom}_R(P, B) \xrightarrow{T_P} \mathrm{Hom}_R(P, B'') \longrightarrow 0$$

is exact by Proposition 4.19a, the sequence

$$F \xrightarrow{S} G \xrightarrow{T} H$$

is exact on projectives. Proposition 4.28 in its version for contravariant left exact functors then says that there is a long exact sequence

$$\begin{aligned} 0 \longrightarrow F(A) \longrightarrow G(A) \longrightarrow H(A) \longrightarrow F_1(A) \longrightarrow G_1(A) \longrightarrow H_1(A) \\ \longrightarrow F_2(A) \longrightarrow G_2(A) \longrightarrow H_2(A) \longrightarrow F_3(A) \longrightarrow \cdots \end{aligned}$$

and that the passage to this long exact sequence is functorial in A. This much establishes the long exact sequence in (c) and the naturality in the A variable. For the behavior in the second variable with A fixed, suppose that we have a second exact sequence $0 \to \widetilde{B}' \to \widetilde{B} \to \widetilde{B}'' \to 0$ that maps to the given one by a chain map f. Let F', G', H' be the functors $\operatorname{Hom}_R(\,\cdot\,, \widetilde{B}'), \operatorname{Hom}_R(\,\cdot\,, \widetilde{B}), \operatorname{Hom}_R(\,\cdot\,, \widetilde{B}'')$. We then get two horizontal planar diagrams of the kind in the proof of Proposition 4.28, one for F', G', H' and one for F, G, H. The maps within each of the two diagrams are maps in the A variable. The two diagrams embed in a 3-dimensional diagram with vertical maps $\operatorname{Hom}_R(1, f)$, and the 3-dimensional diagram is commutative because all maps $\operatorname{Hom}(\alpha, 1)$ commute with all maps $\operatorname{Hom}(1, \beta)$. Application of Theorem 4.10 then completes the proof of functoriality in the exact sequence in the second variable. $\square$

Proposition 4.30. Let $\mathcal{C}$ and $\mathcal{C}'$ be good categories of unital left R modules, and suppose that $\mathcal{C}'$ has enough injectives. Then the covariant left exact functors $\operatorname{Hom}_R(A, \,\cdot\,)$ from $\mathcal{C}'$ to $\mathcal{C}_{\mathbb{Z}}$ and their derived functors $\operatorname{ext}_R^n(A, \,\cdot\,)$ have the following properties:

(a) Whenever $0 \to A' \to A \to A'' \to 0$ is a short exact sequence in $\mathcal{C}$, then there is a corresponding long exact sequence

$$\begin{aligned} 0 &\longrightarrow \operatorname{Hom}_R(A'', B) \longrightarrow \operatorname{Hom}_R(A, B) \longrightarrow \operatorname{Hom}_R(A', B) \\ &\longrightarrow \operatorname{ext}_R^1(A'', B) \longrightarrow \operatorname{ext}_R^1(A, B) \longrightarrow \operatorname{ext}_R^1(A', B) \\ &\longrightarrow \operatorname{ext}_R^2(A'', B) \longrightarrow \operatorname{ext}_R^2(A, B) \longrightarrow \operatorname{ext}_R^2(A', B) \to \operatorname{ext}_R^3(A'', B) \to \cdots \end{aligned}$$

in $\mathcal{C}_{\mathbb{Z}}$ for each module B in $\mathcal{C}'$. The passage from short exact sequences in $\mathcal{C}$ to long exact sequences of derived functor modules in $\mathcal{C}_{\mathbb{Z}}$ is functorial in its dependence on the exact sequence in the first variable and is natural in the second variable in the sense that if a map $\eta : \widetilde{B} \to B$ is given, then $\operatorname{Hom}(1, \eta)$ defines a chain map from the long exact sequence for $\widetilde{B}$ to the long exact sequence for B.

(b) If P is a projective in $\mathcal{C}$ and I is an injective in $\mathcal{C}'$, then $\mathrm{ext}^n_R(P, B) = 0 = \mathrm{ext}^n_R(A, I)$ for all $n \geq 1$ and all modules A in $\mathcal{C}$ and B in $\mathcal{C}'$.

(c) Whenever $0 \to B' \to B \to B'' \to 0$ is a short exact sequence in $\mathcal{C}'$, then there is a corresponding long exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_R(A, B') \longrightarrow \mathrm{Hom}_R(A, B) \longrightarrow \mathrm{Hom}_R(A, B'') \\ &\longrightarrow \mathrm{ext}^1_R(A, B') \longrightarrow \mathrm{ext}^1_R(A, B) \longrightarrow \mathrm{ext}^1_R(A, B'') \\ &\longrightarrow \mathrm{ext}^2_R(A, B') \longrightarrow \mathrm{ext}^2_R(A, B) \longrightarrow \mathrm{ext}^2_R(A, B'') \to \mathrm{ext}^3_R(A, B') \to \cdots \end{aligned}$$

in $\mathcal{C}_{\mathbb{Z}}$ for each module A in $\mathcal{C}$. The passage from short exact sequences in $\mathcal{C}'$ to long exact sequences of derived functor modules in $\mathcal{C}_{\mathbb{Z}}$ is functorial in the exact sequence in the second variable and is natural in the first variable in the sense that if a map $\eta : \widetilde{A} \to A$ is given, then $\mathrm{Hom}(\eta, 1)$ defines a chain map from the long exact sequence for A to the long exact sequence for $\widetilde{A}$.

REMARKS. The naturality in the A parameter of the construction of the long exact sequence in (c) implies that ext^n_R is a contravariant functor of the first variable for fixed argument of the second variable. It implies also that all maps $\mathrm{ext}^n_R(\alpha, 1)$ commute with all maps $\mathrm{ext}^n_R(1, \beta)$.

PROOF. The proof of (c) is a simple variant of the proof of Proposition 4.29a, the proof of (b) is a simple variant of the proof of Proposition 4.29b, and the proof of (a) is a simple variant of the proof of Proposition 4.29c. □

Propositions 4.29 and 4.30 show that Ext and ext, as functors of the first variable and as functors of the second variable, generate the same long exact sequences, the first under the assumption that $\mathcal{C}$ has enough projectives and the second under the assumption that $\mathcal{C}'$ has enough injectives. Theorem 4.31 will show that Ext and ext may be treated as equal if both assumptions are satisfied. It is customary therefore to use Ext as the notation in both cases; thus Ext exists if either $\mathcal{C}$ has enough projectives or $\mathcal{C}'$ has enough injectives. In both cases, Ext has a long exact sequence in the first variable and another long exact sequence in the second variable.

Theorem 4.31. Let $\mathcal{C}$ and $\mathcal{C}'$ be good categories of unital left R modules, and suppose that $\mathcal{C}$ has enough projectives and $\mathcal{C}'$ has enough injectives. Then $\mathrm{Ext}^n_R(\,\cdot\,,\,\cdot\,)$ and $\mathrm{ext}^n_R(\,\cdot\,,\,\cdot\,)$ are naturally isomorphic from $\mathcal{C} \times \mathcal{C}'$ to $\mathcal{C}_{\mathbb{Z}}$ in the sense that for each $n \geq 0$ and each pair of modules (A, B) in $\mathcal{C} \times \mathcal{C}'$, there exists an isomorphism $T_{(n,A,B)}$ in $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Ext}^n_R(A, B), \mathrm{ext}^n_R(A, B))$ such that if φ is in

$\operatorname{Hom}_R(A, A')$ and ψ is in $\operatorname{Hom}_R(B, B')$, then the diagrams

$$\begin{array}{ccc} \operatorname{Ext}^n_R(A, B) & \xrightarrow{T_{(n,A,B)}} & \operatorname{ext}^n_R(A, B) \\ {\scriptstyle \operatorname{Ext}^n(\varphi,1)} \big\uparrow & & \big\uparrow {\scriptstyle \operatorname{ext}^n(\varphi,1)} \\ \operatorname{Ext}^n_R(A', B) & \xrightarrow{T_{(n,A',B)}} & \operatorname{ext}^n_R(A', B) \end{array}$$

and

$$\begin{array}{ccc} \operatorname{Ext}^n_R(A, B) & \xrightarrow{T_{(n,A,B)}} & \operatorname{ext}^n_R(A, B) \\ {\scriptstyle \operatorname{Ext}^n(1,\psi)} \big\downarrow & & \big\downarrow {\scriptstyle \operatorname{ext}^n(1,\psi)} \\ \operatorname{Ext}^n_R(A, B') & \xrightarrow{T_{(n,A,B')}} & \operatorname{ext}^n_R(A, B') \end{array}$$

commute.

REMARKS. The reader will be able to observe that a certain part of this proof amounts to showing that 3-dimensional diagrams in the shape of a cube having 5 faces equal to commuting squares and having suitable hypotheses on the maps automatically have their sixth face equal to a commuting square. The hypotheses concerning the faces and the maps come from Propositions 4.29 and 4.30, as well as induction. We shall not try to abstract a general result of this kind, however.

PROOF. We induct on n for $n \geq 0$. Several steps are involved in the proof, and we complete all of them for a particular n before going on to $n+1$. The steps for a particular n are

(i) to define $T_{(n,A,B)}$ in the presence of an injective I and a one-one map $\mu : B \to I$ and to observe that $T_{(n,A,B)}$ is an isomorphism,

(ii) to show that the same $T_{(n,A,B)}$ results independently of the choice of I,

(iii) to prove the commutativity of the second diagram in the statement of the theorem, and

(iv) to prove the commutativity of the first diagram in the statement of the theorem.

The first base case of the induction is $n = 0$, for which we take $T_{(0,A,B)}$ to be the identity on $\operatorname{Hom}_R(A, B)$. Then (i) through (iv) are immediate.

The other base case of the induction is $n = 1$. Let (A, B) be given. An injective I and a one-one map $\mu : B \to I$ exist as in (i) because $\mathcal{C}'$ has enough injectives. Then we have an exact sequence

$$0 \longrightarrow B \xrightarrow{\mu} I \xrightarrow{\nu} C \longrightarrow 0 \qquad (*)$$

in which $C = I/\mu(B)$ and ν is the quotient map. We know from Propositions 4.29b and 4.30b that $\operatorname{Ext}^1_R(A, I) = 0 = \operatorname{ext}^1_R(A, I)$. Therefore Propositions

4.29c and 4.30c give us exact sequences

$$\mathrm{Hom}_R(A, I) \xrightarrow{\mathrm{Hom}(1,\nu)} \mathrm{Hom}_R(A, C) \xrightarrow{\omega_{E,0}} \mathrm{Ext}^1_R(A, B) \longrightarrow 0$$

and

$$\mathrm{Hom}_R(A, I) \xrightarrow{\mathrm{Hom}(1,\nu)} \mathrm{Hom}_R(A, C) \xrightarrow{\omega_{e,0}} \mathrm{ext}^1_R(A, B) \longrightarrow 0$$

in which $\omega_{E,0}$ and $\omega_{e,0}$ are suitable connecting homomorphisms. We define $T_{(1,A,B)} = \omega_{e,0}(\omega_{E,0})^{-1}$. This definition is meaningful, since the exactness of the two sequences gives

$$(\omega_{E,0})^{-1}(0) = \ker \omega_{E,0} = \mathrm{Hom}(1, \nu)(\mathrm{Hom}_R(A, I)) = \ker \omega_{e,0};$$

by an analogous computation, $\omega_{E,0}(\omega_{e,0})^{-1}$ is a well-defined function, and it is evidently a two-sided inverse. Thus $T_{(1,A,B)}$ is an isomorphism. This completes step (i).

In order to be able to handle steps (ii) and (iii) without being repetitive, let a map $\psi : B \to B'$ be given. For (ii), B' will be B, and ψ will be the identity on B. For (iii), B' and ψ will be general. Given ψ and one-one maps $\mu : B \to I$ and $\mu' : B' \to I'$, we can form the exact rows and the first column of the diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "\mu"] \arrow[d, "\psi"'] & I \arrow[r, "\nu"] \arrow[d, "f"'] & C \arrow[r] \arrow[d, "\bar f"'] & 0 \\ 0 \arrow[r] & B' \arrow[r, "\mu'"] & I' \arrow[r, "\nu'"] & C' \arrow[r] & 0. \end{tikzcd} \qquad (**)$$

If we think of I and I' as extended to injective resolutions, Theorem 4.16 allows us to fill in a cochain map from the one extension to the other, and the first new step of that cochain map is f. If we define $\bar f = \nu' f \nu^{-1}$, then $\bar f$ is well defined because

$$\begin{aligned} \nu' f \nu^{-1}(0) &= \nu' f \ker \nu = \nu' f \text{ image } \mu \\ &= \nu' f \mu(B) = \nu' \mu' \psi(B) = 0(\psi(B)) = 0, \end{aligned}$$

and the squares of the diagram $(**)$ now commute. Continuing with the effort to cut down on repetitive arguments, let $k \geq 1$ be an integer that will be 1 when $n = 1$ and will be different later in the proof. Applying Proposition 4.29c to $(**)$ gives us a commuting square

$$\begin{tikzcd} \mathrm{Ext}^{k-1}_R(A, C) \arrow[r, "\omega_{E,k-1}"] \arrow[d, "\mathrm{Ext}^{k-1}(1,\bar f)"'] & \mathrm{Ext}^k_R(A, B) \arrow[d, "\mathrm{Ext}^k(1,\psi)"] \\ \mathrm{Ext}^{k-1}_R(A, C') \arrow[r, "\omega'_{E,k-1}"] & \mathrm{Ext}^k_R(A, B') \end{tikzcd} \qquad (\dagger)$$

for $k \geq 1$, and Proposition 4.30c gives us a similar commuting square for ext for $k \geq 1$.

For each module in the diagram with Ext when $k = 1$, there is a map to the corresponding module in the diagram with ext. These maps are $T_{(k-1,A,C)}$ for the upper left and $T_{(k-1,A,C')}$ for the lower left. The maps for the upper right and lower right depend on the step of the argument.

For step (ii), we are taking $B' = B$, and the maps at the right are the two versions of $T_{(k,A,B)}$, one for the injective I and one for the injective I'. Let us call them $T_{(k,A,B)}$ and $T'_{(k,A,B)}$. We are to prove that $T'_{(k,A,B)} \operatorname{Ext}^k(1, \psi) = \operatorname{ext}^k(1, \psi) T_{(k,A,B)}$ for $\psi = 1$. The relevant definitions are

$$T_{(k,A,B)} = \omega_{(e,k-1)} T_{(k-1,A,C)} \omega^{-1}_{(E,k-1)}$$

and
$$T'_{(k,A,B)} = \omega'_{(e,k-1)} T_{(k-1,A,C')} (\omega'_{(E,k-1)})^{-1},$$

or equivalently

$$T_{(k,A,B)} \omega_{(E,k-1)} = \omega_{(e,k-1)} T_{(k-1,A,C)}$$

and
$$T'_{(k,A,B)} \omega'_{(E,k-1)} = \omega'_{(e,k-1)} T_{(k-1,A,C')}.$$

Since $T_{(k-1,A,C)}$ and $T_{(k-1,A,C')}$ are known inductively to be well defined and to satisfy (iii), we have $\operatorname{ext}^{k-1}(1, \bar{f}) T_{(k-1,A,C)} = T_{(k-1,A,C')} \operatorname{Ext}^{k-1}(1, \bar{f})$. Thus

$$\begin{aligned}
\operatorname{ext}^k(1, \psi) T_{(k,A,B)} \omega_{(E,k-1)} &= \operatorname{ext}^k(1, \psi) \omega_{(e,k-1)} T_{(k-1,A,C)} \\
&= \omega'_{(e,k-1)} \operatorname{ext}^{k-1}(1, \bar{f}) T_{(k-1,A,C)} = \omega'_{(e,k-1)} T_{(k-1,A,C')} \operatorname{Ext}^{k-1}(1, \bar{f}) \\
&= T'_{(k,A,B)} \omega'_{(E,k-1)} \operatorname{Ext}^{k-1}(1, \bar{f}) = T'_{(k,A,B)} \operatorname{Ext}^k(1, \psi) \omega_{(E,k-1)}.
\end{aligned}$$

Since $\operatorname{Ext}^k(1, \psi) = 1$ and $\operatorname{ext}^k(1, \psi) = 1$ when $\psi = 1$, step (ii) follows for $n = 1$, i.e., $T_{(k,A,B)}$ is well defined.

For step (iii), we are allowing general B', and the maps at the right between the two versions of (†) are the well-defined isomorphisms $T_{(k,A,B)}$ and $T_{(k,A,B')}$. We are to prove that $T_{(k,A,B')} \operatorname{Ext}^k(1, \psi) = \operatorname{ext}^k(1, \psi) T_{(k,A,B)}$. The argument in the previous paragraph applies if we change $T'_{(k,A,B)}$ systematically to $T_{(k,A,B')}$ and take into account that $\omega_{(E,k-1)}$ is onto, and step (iii) follows for $n = 1$.

For step (iv), let $\varphi : A \to A'$ be given. The conclusion of Proposition 4.29c that the dependence is natural in the first variable gives us a commuting square

$$\begin{array}{ccc}
\operatorname{Ext}_R^{k-1}(A, C) & \xrightarrow{\omega_{E,k-1}} & \operatorname{Ext}_R^k(A, B) \\
{\scriptstyle \operatorname{Ext}^{k-1}(\varphi,1)} \uparrow & & \uparrow {\scriptstyle \operatorname{Ext}^k(\varphi,1)} \\
\operatorname{Ext}_R^{k-1}(A', C) & \xrightarrow{\omega'_{E,k-1}} & \operatorname{Ext}_R^k(A', B)
\end{array} \qquad (††)$$

for $k \geq 1$ and for suitable connecting homomorphisms $\omega_{E,k-1}$ and $\omega'_{E,k-1}$, and Proposition 4.30c gives a similar commuting square for ext for $k \geq 1$. For each module in the diagram with Ext when $k = 1$, there is a map to the corresponding module in the diagram with ext. These maps are $T_{(k-1,A,C)}$ for the upper left, $T_{(k-1,A',C)}$ for the lower left, $T_{(k,A,B)}$ for the upper right, and $T_{(k,A',B)}$ for the lower right. We are to prove that $T_{(k,A,B)} \operatorname{Ext}^k(\varphi, 1) = \operatorname{ext}^k(\varphi, 1) T_{(k,A',B)}$. The relevant definitions are

$$T_{(k,A,B)}\omega_{(E,k-1)} = \omega_{(e,k-1)}T_{(k-1,A,C)}$$

and

$$T_{(k,A',B)}\omega'_{(E,k-1)} = \omega'_{(e,k-1)}T_{(k-1,A',C)}.$$

Since $T_{(k-1,A,C)}$ and $T_{(k-1,A',C)}$ are known inductively to satisfy (iv), we have $\operatorname{ext}^{k-1}(\varphi, 1)T_{(k-1,A',C)} = T_{(k-1,A,C)} \operatorname{Ext}^{k-1}(\varphi, 1)$. Thus

$$\begin{aligned}
\operatorname{ext}^k(\varphi, 1)T_{(k,A',B)}\omega'_{(E,k-1)} &= \operatorname{ext}^k(\varphi, 1)\omega'_{(e,k-1)}T_{(k-1,A',C)} \\
&= \omega_{(e,k-1)} \operatorname{ext}^{k-1}(\varphi, 1)T_{(k-1,A',C)} = \omega_{(e,k-1)}T_{(k-1,A,C)} \operatorname{Ext}^{k-1}(\varphi, 1) \\
&= T_{(k,A,B)}\omega_{(E,k-1)} \operatorname{Ext}^{k-1}(\varphi, 1) = T_{(k,A,B)} \operatorname{Ext}^k(\varphi, 1)\omega'_{(E,k-1)}.
\end{aligned}$$

Since $\omega'_{(E,k-1)}$ is onto, step (iv) follows for $n = 1$. This completes the proof for $n = 1$.

For the inductive step, suppose that steps (i) through (iv) have been carried out for some $n \geq 1$. Let us carry out step (i) for stage $n+1$. For a given B, we know from Propositions 4.29b and 4.30b that $\operatorname{Ext}^n_R(A, I) = 0 = \operatorname{ext}^n_R(A, I)$. Hence Propositions 4.29c and 4.30c give us exact sequences

$$0 \longrightarrow \operatorname{Ext}^n_R(A, C) \xrightarrow{\omega_{E,n}} \operatorname{Ext}^{n+1}_R(A, B) \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{ext}^n_R(A, C) \xrightarrow{\omega_{e,n}} \operatorname{ext}^{n+1}_R(A, B) \longrightarrow 0.$$

In other words, $\omega_{E,n}$ and $\omega_{e,n}$ are isomorphisms. If we put

$$T_{(n+1,A,B)} = \omega_{e,n}T_{(n,A,C)}\omega^{-1}_{E,n},$$

then $T_{(n+1,A,B)}$ is an isomorphism of $\operatorname{Ext}^{n+1}_R(A, B)$ onto $\operatorname{ext}^{n+1}_R(A, B)$. This completes step (i) for stage $n + 1$.

We now refer back to our argument for $n = 1$ and put $k = n + 1$ throughout. Tracing matters through, we see that the argument carries out steps (ii) through (iv) for stage $n + 1$. This completes the induction and the proof. □

8. Abelian Categories

Not all situations in which one wants to apply homological algebra are limited to good categories of unital left R modules for some ring R. We have mentioned sheaves as one example, and we shall develop some properties of sheaves in Chapter X. Implicitly we have carried along a second example: all chain complexes within a good category, with chain maps as morphisms, form a category in which short exact sequences have remarkable properties, such as those in Theorems 4.7 and 4.10.

A setting to which one can generalize well such basic parts of homological algebra is that of "abelian categories," which we define in this section. It is advisable not to require that the objects in an abelian category actually be sets of individual elements; otherwise there is little chance that the notion of abelian category could be self dual. The morphisms of the category are then effectively all we have to work with, since a morphism already determines its "domain" and "range." If X and Y are objects, then a morphism in $\operatorname{Morph}(X, Y)$ need not be a function, but at least $\operatorname{Morph}(X, Y)$ is a set with elements to it. Since objects no longer have elements, books usually suppress the objects in the discussion to the point of referring to things like kernels and cokernels as morphisms rather than objects. It is perhaps more comfortable to think of a kernel as a pair, consisting of an object and a morphism into another object, rather than just as the embedding morphism, and we shall follow the more comfortable convention temporarily.

We introduce the notion of "abelian category" in stages. We begin with some definitions and remarks that make sense in a general category. First of all, let us have names for X and Y when referring to morphisms in $\operatorname{Morph}(X, Y)$ that do not require us to think in terms of functions. The convention is that if u is in $\operatorname{Morph}(X, Y)$, then X is the **domain** of u and Y is the **codomain**. We allow ourselves to write compositions of morphisms as gf or as $g \circ f$.

Next, it is possible to generalize usefully the notions of "one-one" and "onto" to make them applicable in any category. The definitions are in terms of cancellation laws. In the category $\mathcal{C}$, a morphism $u \in \operatorname{Morph}(X, Y)$ is a **monomorphism**[14] if for any f and g in the same set $\operatorname{Morph}(W, X)$ such that $uf = ug$, it follows that $f = g$. Any isomorphism is certainly a monomorphism. The composition of two monomorphisms is a monomorphism. In fact, if u and v are monomorphisms with $vuf = vug$, then $uf = ug$ because v is a monomorphism, and $f = g$ because u is a monomorphism. If m is a monomorphism in $\operatorname{Morph}(X, Y)$ and u is any morphism in $\operatorname{Morph}(Y, X)$ such that $mu = 1_Y$, then m is an isomorphism. In fact, $mu = 1_Y$ implies $mum = 1_Y m = m$, which implies $um = 1_X$, since m is a monomorphism; therefore u is a two-sided inverse to m.

[14]Some authors use the word "monic" or the word "mono" as an adjectival form of this noun.

The morphism $u \in \mathrm{Morph}(X, Y)$ is an **epimorphism**[15] if for any f' and g' in the same set $\mathrm{Morph}(Y, Z)$ such that $f'u = g'u$, it follows that $f' = g'$. Any isomorphism is an epimorphism. The composition of two epimorphisms is an epimorphism. If e is an epimorphism in $\mathrm{Morph}(X, Y)$ and u is any morphism in $\mathrm{Morph}(Y, X)$ such that $ue = 1_X$, then e is an isomorphism.

Finally a **zero object** 0 in a category $\mathcal{C}$ is an object such that for each X in $\mathrm{Obj}(\mathcal{C})$, each of $\mathrm{Morph}(X, 0)$ and $\mathrm{Morph}(0, X)$ has exactly one member. It is immediate that any two zero objects are isomorphic: if 0 and $0'$ are zero objects, then $\mathrm{Morph}(0, 0)$ and $\mathrm{Morph}(0', 0')$ each have just one member, which must be 1_0 and $1_{0'}$ in the two cases; the composition of the member of $\mathrm{Morph}(0, 0')$ followed by the member of $\mathrm{Morph}(0', 0)$ must be 1_0, and the composition in the other order must be $1_{0'}$, and the isomorphism of 0 with $0'$ has been exhibited.

Suppose that a zero object exists. Since the composition law for morphisms in $\mathcal{C}$ insists that the composite of a member of $\mathrm{Morph}(X, 0)$ and a member of $\mathrm{Morph}(0, Y)$ be in $\mathrm{Morph}(X, Y)$, it follows that $\mathrm{Morph}(X, Y)$ has a distinguished member, which we denote by 0_{XY}. This is called the **zero morphism** of $\mathrm{Morph}(X, Y)$. By associativity it satisfies $f0_{XY} = 0_{XZ}$ for all $f \in \mathrm{Morph}(Y, Z)$ and $0_{XY}g = 0_{WY}$ for all $g \in \mathrm{Hom}(W, X)$. Since $\mathrm{Morph}(0, 0)$ has just one element, we have $0_{00} = 1_0$. If X is any other object such that $\mathrm{Morph}(X, X)$ has $0_{XX} = 1_X$, then X is a zero object; in fact, the equalities $0_{X0}0_{0X} = 0_{00} = 1_0$ and $0_{0X}0_{X0} = 0_{XX} = 1_X$ show that X and 0 are isomorphic.

An **additive category** $\mathcal{C}$ is a category with the following three properties:

(i) $\mathcal{C}$ has a zero object,

(ii) the product and the coproduct[16] of any two objects in $\mathcal{C}$ exists in $\mathcal{C}$,

(iii) each set $\mathrm{Morph}(X, Y)$ is an abelian group with the property that the operation is $\mathbb{Z}$ bilinear in the sense that if the operation is $+$ and if f, f' are arbitrary in $\mathrm{Morph}(X, Y)$ and g, g' are arbitrary in $\mathrm{Morph}(Y, Z)$, then

$$(g + g') \circ (f + f') = g \circ f + g' \circ f + g \circ f' + g' \circ f'$$

and

$$g \circ (-f) = (-g) \circ f = -(g \circ f).$$

If $\mathcal{C}$ is an additive category, then so is the opposite category $\mathcal{C}^{\mathrm{opp}}$; this fact will enable us to use duality arguments occasionally. We shall henceforth write $\mathrm{Hom}(X, Y)$ in place of $\mathrm{Morph}(X, Y)$ for additive categories.

The zero morphism 0_{XY} of $\mathrm{Hom}(X, Y)$ is the additive identity 0 of the abelian group $\mathrm{Hom}(X, Y)$. In fact, 0_{0Y} is the additive identity of $\mathrm{Hom}(0, Y)$, since $\mathrm{Hom}(0, Y)$ has just one element. Therefore $0_{XY} = 0_{0Y}0_{X0} = (0_{0Y} + 0_{0Y})0_{X0} = 0_{0Y}0_{X0} + 0_{0Y}0_{X0} = 0_{XY} + 0_{XY}$, and we obtain $0 = 0_{XY}$.

[15] Some authors use the word "epi" as an adjectival form of this noun.

[16] These are defined in Section IV.11 of *Basic Algebra*. They are always unique up to canonical isomorphism when they exist.

In an additive category a morphism u in $\mathrm{Hom}(X, Y)$ is a monomorphism if whenever $uf = 0$ with f in some $\mathrm{Hom}(W, X)$, then $f = 0$; a morphism u in $\mathrm{Hom}(X, Y)$ is an epimorphism if whenever $f'u = 0$ with f' in some $\mathrm{Hom}(Y, Z)$, then $f' = 0$.

This much structure forces products and coproducts to amount to the same thing in an additive category. The precise result is as follows.

Proposition 4.32. In an additive category, let (C, p_A, p_B) be a product of two objects A and B. Then there exist unique $i_A \in \mathrm{Hom}(A, C)$ and $i_B \in \mathrm{Hom}(B, C)$ such that

$$p_A i_A = 1_A, \quad p_B i_B = 1_B, \quad i_A p_A + i_B p_B = 1_C.$$

These satisfy $p_A i_B = 0$ and $p_B i_A = 0$, and (C, i_A, i_B) is a coproduct of A and B.

REMARKS.

(1) Since the defining properties of an additive category are self dual, any coproduct has a similar structure and becomes a product. The proof in effect will show more—that whenever there are data $A, B, C, i_A, i_B, p_A, p_B$ satisfying the displayed identities, then (C, p_A, p_B) is a product of A and B, and (C, i_A, i_B) is a coproduct. Thus a product/coproduct can be recognized without reference to other objects in the category.

(2) To emphasize the analogy with modules or vector spaces, we write $A \oplus B$ for a product or coproduct of A and B in $\mathcal{C}$ and call it the **direct sum** of A and B. The notation is understood to carry the morphisms i_A, i_B, p_A, p_B along with it. The direct sum is unique up to an isomorphism that carries the one set of morphisms i_A, i_B, p_A, p_B to the other.

PROOF. To the pair $1_A \in \mathrm{Hom}(A, A)$ and $0 \in \mathrm{Hom}(A, B)$, the product C associates a unique $i_A \in \mathrm{Hom}(A, C)$ with $p_A i_A = 1_A$ and $p_B i_A = 0$. Similarly the coproduct associates a unique $i_B \in \mathrm{Hom}(B, C)$ with $p_A i_B = 0$ and $p_B i_B = 1_B$. Computing with the aid of the $\mathbb{Z}$ bilinearity and associativity, we have

$$p_A(i_A p_A + i_B p_B) = 1_A p_A + 0 p_B = p_A$$

and

$$p_B(i_A p_A + i_B p_B) = 0 p_A + 1_B p_B = p_B.$$

Therefore $h = i_A p_A + i_B p_B$ is a member of $\mathrm{Hom}(C, C)$ with the property that $p_A h = p_A$ and $p_B h = p_B$. Since 1_C is another member of $\mathrm{Hom}(C, C)$ with this property, the assumed uniqueness shows that $h = 1_C$. This proves the displayed formulas in the proposition and the formulas $p_A i_B = 0$ and $p_B i_A = 0$.

For uniqueness of i_A and i_B, suppose that i'_A and i'_B satisfy $i'_A p_A + i'_B p_B = 1_C$. Right multiplication by i_A gives $i_A = 1_C i_A = (i'_A p_A + i'_B p_B) i_A = i'_A 1_A + i'_B 0 = i'_A$, and similarly $i_B = i'_B$.

To see that (C, i_A, i_B) is a coproduct of A and B, let $f \in \operatorname{Hom}(A, X)$ and $g \in \operatorname{Hom}(B, X)$ be given, and define $h = fp_A + gp_B$. This is in $\operatorname{Hom}(C, X)$, has $hi_A = fp_Ai_A + gp_Bi_A = f1_A = f$, and similarly has $hi_B = g$. For uniqueness suppose that k is in $\operatorname{Hom}(C, X)$ with $ki_A = f$ and $ki_B = g$. Then $ki_Ap_A = fp_A$ and $ki_Bp_B = gp_B$. Addition gives

$$k = k1_C = k(i_Ap_A + i_Bp_B) = fp_A + gp_B = h,$$

and uniqueness is proved. $\square$

For an additive category $\mathcal{C}$, the notions of the kernel and cokernel of a morphism are defined by universal mapping properties. Problems 18–22 at the end of Chapter VI of *Basic Algebra* discussed universal mapping properties abstractly, saying what they are in a general context. For current purposes it is enough to know that what a universal mapping property produces (if it produces anything at all) is a pair consisting of an object and a morphism, and moreover the pair is automatically unique (if it exists) up to canonical isomorphism.

We allow ourselves to write morphisms as arrows in any of the customary ways for functions. Thus a member u of $\operatorname{Hom}(A, B)$ may be written as $A \xrightarrow{u} B$, and a composition of u followed by a morphism $v \in \operatorname{Hom}(B, C)$, which has been written as $v \circ u$ or as vu, may be written as $A \xrightarrow{u} B \xrightarrow{v} C$.

If $A \xrightarrow{u} B$ is a morphism in the additive category $\mathcal{C}$, then the **kernel** of u, denoted by $\ker u$, is a pair (K, i) with $i \in \operatorname{Hom}(K, A)$ such that the composition $K \xrightarrow{i} A \xrightarrow{u} B$ has $ui = 0$ and such that for any pair (K', i') with i' in $\operatorname{Hom}(K', A)$ for which $ui' = 0$, there exists a unique $\varphi \in \operatorname{Hom}(K', K)$ with $i\varphi = i'$. See Figure 4.6. It is customary to drop all mention of K in the definition of kernel, saying that the kernel is i, since any mention of i carries along K as the domain of i; we shall adopt this abbreviated terminology shortly but shall refer to the pair (K, i) as the kernel for the time being.

$$\begin{tikzcd}
K \arrow[r, "i"] & A \arrow[r, "u"] & B \\
K' \arrow[u, dashed, "\varphi"] \arrow[ur, "i'"'] & &
\end{tikzcd}$$

FIGURE 4.6. Universal mapping property of a kernel (K, i) of u.

The brief form of the definition of kernel is that $u \circ (\ker u) = 0$ and

$$ui' = 0 \qquad \text{implies} \qquad i' = (\ker u) \circ \varphi \quad \text{uniquely.}$$

The kernel of u is determined only up to an isomorphism applied to K; that is, i is determined only up to right multiplication by an isomorphism. The condition for (K, i) to be a kernel is equivalent to the exactness of the sequence of abelian groups

$$0 \longrightarrow \operatorname{Hom}(K', K) \xrightarrow{i\circ(\,\cdot\,)} \operatorname{Hom}(K', A) \xrightarrow{u\circ(\,\cdot\,)} \operatorname{Hom}(K', B).$$

In fact, $ui = 0$ makes the sequence a complex, the existence of φ produces exactness at $\operatorname{Hom}(K', A)$, and the uniqueness of φ produces exactness at $\operatorname{Hom}(K', K)$.

Similarly the **cokernel** of u, denoted by $\operatorname{coker} u$, is a pair (C, p) with p in $\operatorname{Hom}(B, C)$ such that the composition $A \xrightarrow{u} B \xrightarrow{p} C$ has $pu = 0$ and such that for any pair (C', p') with p' in $\operatorname{Hom}(B, C')$ for which $p'u = 0$, there exists a unique $\psi \in \operatorname{Hom}(C, C')$ with $\psi p = p'$. See Figure 4.7. It is customary to drop all mention of the object C in the definition of cokernel, saying that the cokernel is p, since any mention of p carries along C as the *co*domain of p; we shall adopt this abbreviated terminology shortly but shall refer to the pair (C, p) as the cokernel for the time being.

$$\begin{tikzcd} C \arrow[d, dashed, "\psi"'] & B \arrow[l, "p"'] \arrow[dl, "p'"] & A \arrow[l, "u"'] \\ C' & & \end{tikzcd}$$

FIGURE 4.7. Universal mapping property of a cokernel (C, p) of u.

The brief form of the definition of cokernel is that $(\operatorname{coker} u) \circ u = 0$ and

$$p'u = 0 \qquad \text{implies} \qquad p' = \psi \circ (\operatorname{coker} u) \quad \text{uniquely}.$$

The cokernel of u is determined only up to an isomorphism applied to C; that is, p is determined only up to left multiplication by an isomorphism. The condition for (C, p) to be a cokernel is equivalent to the exactness of the sequence of abelian groups

$$0 \longrightarrow \operatorname{Hom}(C, C') \xrightarrow{(\,\cdot\,)\circ p} \operatorname{Hom}(B, C') \xrightarrow{(\,\cdot\,)\circ u} \operatorname{Hom}(A, C').$$

In fact, $pu = 0$ makes the sequence a complex, the existence of ψ produces exactness at $\operatorname{Hom}(B, C')$, and the uniqueness of ψ produces exactness at $\operatorname{Hom}(C, C')$.

Proposition 4.33. Let $\mathcal{C}$ be an additive category. If an element u of $\operatorname{Hom}(A, B)$ has a kernel (K, i) and if $m \in \operatorname{Hom}(B, B')$ is a monomorphism, then (K, i) is also a kernel of mu. If u has a cokernel (C, p) and if $e \in \operatorname{Hom}(A', A)$ is an epimorphism, then (C, p) is also a cokernel of ue. Briefly

$$\ker(mu) = \ker u \qquad \text{and} \qquad \operatorname{coker}(ue) = \operatorname{coker} u.$$

REMARK. We can safely omit the proof of any dual statement about additive categories, since the dual follows by expressing the original argument as a diagram, reversing all the arrows, and writing down the argument that the new diagram represents.

PROOF. We test whether $i = \ker u$ is a kernel of mu. We know that $(mu)i = m(ui) = 0$. Suppose that $mui' = 0$ with $i' \in \operatorname{Morph}(K', A)$. Since m is a monomorphism, $ui' = 0$. Because i is a kernel of u, we obtain $i' = i\varphi$ for a unique $\varphi \in \operatorname{Morph}(K', K)$. Hence i is a kernel of mu. The statement about cokernels is dual. □

Proposition 4.34. Let $\mathcal{C}$ be an additive category. If an element u of $\operatorname{Hom}(A, B)$ has a kernel (K, i), then i is a monomorphism. Dually if u has a cokernel (C, p), then p is an epimorphism.

PROOF. Suppose that u has a kernel (K, i). For any object K', the zero morphism $i' = 0$ of $\operatorname{Hom}(K', A)$ has the property that $ui' = 0$. The uniqueness property of the kernel says that the φ in $\operatorname{Hom}(K', K)$ with $i\varphi = i'$ is unique. Evidently $\varphi = 0$ is one such choice and hence is the only such choice. Thus if f in $\operatorname{Hom}(K', K)$ has $if = 0$, then $f = 0$. Therefore i is a monomorphism. □

Propositions 4.33 and 4.34 give a first hint that the notation (K, i) for the kernel, which we know is redundant, may also be inconvenient; it would be far simpler to refer to the kernel as i, and analogously for cokernels. Then Proposition 4.33 could truly be stated as the displayed formulas in its statement, and Proposition 4.34 would have the tidier statement that every kernel is a monomorphism and every cokernel is an epimorphism. Let us therefore now allow ourselves to regard kernels and cokernels as morphisms, rather than pairs consisting of an object and a morphism. With this convention in place, we always have $u \circ (\ker u) = 0$ and $(\operatorname{coker} u) \circ u = 0$.

Proposition 4.35. Let $\mathcal{C}$ be an additive category, and let u be in $\operatorname{Hom}(A, B)$. If u has a kernel and $\ker u$ has a cokernel, then $\operatorname{coker}(\ker u)$ is a kernel of u. Briefly

$$\ker(\operatorname{coker}(\ker u)) = \ker u.$$

Dually if u has a cokernel and $\operatorname{coker} u$ has a kernel, then

$$\operatorname{coker}(\ker(\operatorname{coker} u)) = \operatorname{coker} u.$$

PROOF. Let (K, i) be a kernel of u, and let (C, p) be a cokernel of i. We are to show that i is a kernel of p. For the existence step, suppose that i' in $\operatorname{Hom}(K', A)$ has $pi' = 0$. We are to show that i' factors as $i' = i\varphi$ for some unique φ in $\operatorname{Hom}(K', K)$. We know that $ui = 0$. Since $p = \operatorname{coker} i$, u factors as $u = \psi p$ for some ψ in $\operatorname{Hom}(C, B)$. Then $ui' = (\psi p)i' = \psi(pi') = 0$. Since $i = \ker u$, i' factors as $i' = i\varphi$ as required. This proves existence of φ.

For the uniqueness step, suppose that $pi' = 0$ for some i' in some $\operatorname{Hom}(K', A)$. If i' were to have two distinct factorizations, say as $i' = i\varphi = i\overline{\varphi}$, then i could not be a monomorphism, in contradiction to Proposition 4.34 and the fact that $i = \ker u$. This proves uniqueness of φ. □

An **abelian category** $\mathcal{C}$ is an additive category with the following two properties:

(iv) every morphism has a kernel and a cokernel,
(v) every monomorphism is a kernel, and every epimorphism is a cokernel.

It is evident that the opposite category of any abelian category is abelian. Thus we can continue to use duality arguments.

Property (iv) is certainly desirable if one wants to have a theory involving homology and cohomology. Property (v) may be viewed as a converse to Proposition 4.34; some other authors use a different but equivalent formulation of this axiom. The objective is to have a generalization of the kind of factorization that one has with homomorphisms of abelian groups: any homomorphism factors canonically as the product of the canonical passage to the quotient by the kernel, followed by an isomorphism of this quotient onto the image of the homomorphism, followed by the inclusion of the image into the range.

Proposition 4.36. In any abelian category, every morphism that is both a monomorphism and an epimorphism is an isomorphism.

PROOF. If $f \in \operatorname{Hom}(K, A)$ is a monomorphism, then $f = \ker g$ for some g in some $\operatorname{Hom}(A, B)$ by (v). This fact implies that $gf = g \circ (\ker g) = 0$. If f is also an epimorphism, then the equality $gf = 0$ implies that $g = 0$. Hence $f = \ker 0_{AB}$. Taking $K' = A$ and $i' = 1_A$ in Figure 4.6, we have $0i' = 0$ and thus have $1_A = f\varphi$ for some φ in $\operatorname{Hom}(A, K)$. Thus the monomorphism f has a right inverse and must be an isomorphism. $\square$

Lemma 4.37. In an abelian category $\mathcal{C}$, every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel.

PROOF. If m is a monomorphism, then (v) says that $m = \ker u$ for some u. Substituting into the first conclusion of Proposition 4.35, we obtain $\ker(\operatorname{coker} m) = m$. If e is an epimorphism, then (v) says that $e = \operatorname{coker} u$ for some u. Substituting into the second conclusion of Proposition 4.35, we obtain $\operatorname{coker}(\ker e) = e$. $\square$

Proposition 4.38. In an abelian category $\mathcal{C}$, any morphism f factors as $f = me$ for a monomorphism m and an epimorphism e. Here one such factorization is given by

$$m = \ker(\operatorname{coker} f) \qquad \text{and} \qquad e = \operatorname{coker}(\ker f).$$

Any other such factorization $f = m'e'$ has the property that there is some isomorphism x with $e' = xe$ and $m'x = m$.

PROOF. Put $m = \ker(\operatorname{coker} f)$. Since $(\operatorname{coker} f)f = 0$, the brief form of the definition of kernel gives $f = me$ for some e. We are going to prove that e is an epimorphism. Thus suppose that $re = 0$ for some morphism r. The brief form of the definition of kernel shows that $e = (\ker r)e'$ for some morphism e'. Then we have

$$f = me = m(\ker r)e' = m'e', \qquad \text{where } m' = m \ker r.$$

Being a kernel, $\ker r$ is a monomorphism. As the composition of two monomorphisms, m' is a monomorphism. Lemma 4.37 shows that $m' = \ker p'$, where $p' = \operatorname{coker} m'$.

Put $p = \operatorname{coker} m$. The definition of m and the second identity of Proposition 4.35 gives $p = \operatorname{coker}(\ker(\operatorname{coker} f)) = \operatorname{coker} f$. Since $m' = \ker p'$, we have $p'm' = 0$. Hence $p'f = p'm'e' = 0$. Since $p = \operatorname{coker} f$, the brief form of the definition of cokernel shows that $p' = sp$ for some s. Thus $p'm = spm = 0$, the latter equality holding because $p = \operatorname{coker} m$. Since $m' = \ker p'$, the brief form of the definition of kernel gives $m = m't$ for some t.

Resubstituting for m' gives $m = m't = m(\ker r)t$. Since m is a monomorphism, we can cancel and obtain $1_X = (\ker r)t$, where X is the codomain of $\ker r$. In other words, $\ker r$ has a right inverse. Being a monomorphism, it must be an isomorphism. Since any morphism v has $v \ker v = 0$, we obtain $r \ker r = 0$ and conclude that $r = 0$. Therefore e is an epimorphism, as asserted.

Since e is an epimorphism, Lemma 4.37 gives $e = \operatorname{coker}(\ker e)$, and Proposition 4.33 gives $\ker e = \ker(me) = \ker f$. Therefore $e = \operatorname{coker}(\ker f)$. This completes the proof of existence of the decomposition.

For uniqueness, suppose that $f = m'e'$ for a monomorphism m' and an epimorphism e'. Proposition 4.33 gives $\ker f = \ker(m'e') = \ker e'$, as well as $\ker f = \ker(me) = \ker e$, the understanding being that these equalities hold up to an isomorphism on the right. Set $u = \ker e$ and $u' = \ker e'$; then $u = u'w$ for some isomorphism w. Since e and e' are epimorphisms, Lemma 4.37 gives $e = \operatorname{coker} u$ and $e' = \operatorname{coker} u'$. Since m' is a monomorphism, the equality $0 = f(\ker f) = fu = m'e'u$ implies that $e'u = 0$; by the brief form of the definition of $\operatorname{coker} u$ as a cokernel, e' factors as $e' = xe$ for a unique x. Similarly the equality $0 = f \ker f = fu' = meu$ implies that $eu = 0$; by the brief form of the definition of $\operatorname{coker} u'$ as a cokernel, e factors as $e = x'e'$ for a unique x'. Then $e = x'e' = x'xe$; since e is an epimorphism, $x'x$ is the identity on its domain. Similarly $e' = xe = xx'e'$, and it follows that xx' is the identity on its domain. Consequently x is an isomorphism. Multiplying $e' = xe$ by m' on the left gives $me = f = m'e' = m'xe$; since e is an epimorphism, $m = m'x$. This completes the proof. $\square$

With this canonical factorization in hand, we introduce two terms that will simplify the definition of "exact sequence." We define the **image** and **coimage** of $f = me$ in $\mathrm{Hom}(A, B)$ by

$$m = \text{image}\, f \qquad \text{and} \qquad e = \text{coimage}\, f.$$

In words, the image of any morphism is its monomorphism factor, and the coimage is its epimorphism factor; in particular, a monomorphism is its own image, and an epimorphism is its own coimage.[17] Let us see what the factorization and these formulas say in terms of diagrams. We write (K, i) for the kernel of f and (C, p) for the cokernel of f. Let I be the codomain of e, which equals the domain of m. In terms of a diagram, the situation for f is then given by

$$K \xrightarrow[=\ker f]{i=\ker e} A \xrightarrow[=\text{coimage}\, f]{e=\text{coker}\, i} I \xrightarrow[=\text{image}\, f]{m=\ker p} B \xrightarrow[=\text{coker}\, f]{p=\text{coker}\, m} C.$$

The top row of labels explains the relationships among i, e, m, p, and the bottom row of labels relates i, e, m, p to f. The morphism f itself is the composition of the two morphisms in the center.

In a good category of modules, we can interpret this diagram in terms of the two short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{i} & A & \xrightarrow{e} & A/\,\text{image}\, i & \longrightarrow & 0, \\ 0 & \longrightarrow & A/\,\text{image}\, i & \xrightarrow{m} & B & \xrightarrow{p} & C & \longrightarrow & 0, \end{array}$$

which we can merge into a single 6-term exact sequence

$$0 \longrightarrow K \xrightarrow{i} A \xrightarrow{me=f} B \xrightarrow{p} C \longrightarrow 0.$$

Now we can define complexes and exact sequences for abelian categories, and we can readily check that the new definitions are consistent with the definitions for good categories of modules. A **chain complex** is a doubly infinite sequence of morphisms with decreasing indexing such that the consecutive compositions are defined and are 0. If $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, C)$ are given morphisms, then the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is **exact** at B if image $f = \ker g$, or equivalently if coker $f =$ coimage g. As usual in the subject of abelian categories, the equality sign here means "can be taken as." In more detail if f and g decompose as $f = me$ and $g = m'e'$, image f is defined to be m, and $\ker g$ equals $\ker e'$. Thus the condition for exactness is

[17] The term "coimage" is not really needed for recognizing exact sequences, but it makes any implementation of duality more symmetric.

that m be a kernel of e'. Since $u(\ker u) = 0$ for any morphism u, exactness at B implies that $e'm = 0$. Then $gf = m'e'me = 0$, and we see that the given sequence (when extended by 0's at each end) is a complex.

Exactness of any finite or infinite sequence of morphisms whose consecutive compositions are defined means exactness at every object X in the sequence for which there is an incoming morphism in some $\mathrm{Hom}(W, X)$ and there is an outgoing morphism in some $\mathrm{Hom}(X, Y)$. With the kind of indexing used for a chain complex, a sequence

$$\cdots \longrightarrow X_{n+1} \xrightarrow{m_n e_n} X_n \xrightarrow{m_{n-1}e_{n-1}} X_{n-1} \longrightarrow \cdots$$

is exact if $m_n = \ker e_{n-1}$, or equivalently if $e_{n-1} = \operatorname{coker} m_n$, for all n.

For a sequence of four morphisms of the form

$$0 \longrightarrow K \xrightarrow{m} A \xrightarrow{e} C \longrightarrow 0,$$

exactness means exactness at K, A, and C. The conditions are that m is a monomorphism, e is an epimorphism, and $m = \ker e$ (or equivalently that $e = \operatorname{coker} m$). In this case the sequence is called a **short exact sequence**.

One can now proceed to define **projectives** and **injectives** for any abelian category as certain objects in the same way as in Figures 4.3 and 4.4, and extend all the results of earlier sections of this chapter to all abelian categories. We shall not carry out this detail.[18]

Instead, we shall indicate an approach to carrying out this detail that takes most of the difficulty out of translating results from the context of good categories to the context of abelian categories. It is to use the notion of "members." The word "members" in the present setting refers to something that substitutes for elements in situations in which objects need not necessarily be sets of elements. The idea is to recast elements, when they exist, in terms of morphisms and then to generalize the resulting definition. For orientation, consider the category $\mathcal{C}_R$ of all unital left R modules, R being a ring with identity. Let us write R_0 for the left R module R. The elements of a unital left R module X are then in one-one correspondence with the R homomorphisms of R_0 into X, the element x corresponding to the homomorphism that carries r to rx. Thus the category $\mathcal{C}_R$ has a distinguished object R_0 such that the elements of any object X are in one-one correspondence with $\mathrm{Hom}(R_0, X)$. Hence any argument about elements for this category immediately translates into an argument about morphisms.

The trouble is that a general abelian category has no distinguished object to play the role of R_0. The idea for getting around this difficulty is to take all possible

[18]The entire theory for abelian categories is carried out in detail in Freyd's book *Abelian Categories: An Introduction to the Theory of Functors*.

objects X_0 in place of R_0, consider the union on X_0 of *all* sets $\operatorname{Hom}(X_0, X)$, introduce an equivalence relation, and hope for the best.

The definition is as follows. Let $\mathcal{C}$ be an abelian category, fix X in $\operatorname{Obj}(\mathcal{C})$, and consider all morphisms with codomain X. Two such morphisms x and y are said to be **equivalent** morphisms for current purposes, written $x \equiv y$, if there exist epimorphisms u and v such that $xu = yv$. It is evident that "equivalent" is reflexive and symmetric. Transitivity requires proof, and we return to this matter in a moment. Once $\equiv$ has been shown to be an equivalence relation, an equivalence class of such morphisms is called a **member** of X. We write $x \in_m X$ to indicate that x is a morphism with codomain X, hence to indicate that x is a morphism whose equivalence class is a member of X. To avoid clumsy wording when there is really no possibility of confusion, we often simply say that x is a member of X. The question arises whether this definition presents any set-theoretic difficulties. As usual in category theory, one can answer the question painlessly by working when necessary only with subcategories for which the objects actually form a set; in this case, the union over all objects X and Y in the subcategory of all the groups $\operatorname{Hom}(X, Y)$ of morphisms is a set, and there is no problem. Let us return to a special case of our example.

EXAMPLE OF MEMBERS. Let $\mathcal{C} = \mathcal{C}_{\mathbb{Z}}$ be the category of all abelian groups, and fix an abelian group X. If x is an abelian-group homomorphism with codomain X, let us use Proposition 4.38 to write $x = me$ for a monomorphism m and an epimorphism e. Then $x \equiv m$, and thus we might just as well consider only one-one homomorphisms into X. If H is the image of x, then we can view x as a composition $x = i_H y$ of a homomorphism y carrying the domain of x onto H, followed by the inclusion $i_H : H \to X$. The homomorphism y is an isomorphism, hence is an epimorphism. Thus $x \equiv i_H$. It is apparent that no two inclusions of subgroups of X into X are equivalent morphisms. Since every inclusion of a subgroup of X into X yields a member of X, the members of X are exactly the subgroups of X. Thus for example the set of members of $\mathbb{Z}$ corresponds to the set of integers ≥ 0, in which addition is lost, and does not correspond exactly to the set of elements of $\mathbb{Z}$. This fact is a little discouraging, but it turns out not to be as bad an omen as one might expect.

Returning to the setting of a general abelian category, we work toward a proof that $\equiv$ is an equivalence relation. We need the notion of the "pullback" of two morphisms, which we define by a universal mapping property momentarily. The appropriate construction establishing existence appears in the next proposition. Then we prove a proposition for using pullback as a tool, and afterward we prove the transitivity.

In an abelian category $\mathcal{C}$, let X, Y, Z be objects, and let $f \in \operatorname{Hom}(Y, Z)$ and $g \in \operatorname{Hom}(X, Z)$ be morphisms. A **pullback** of the pair (f, g) is a triple

$(W, \widetilde{f}, \widetilde{g})$ in which W is an object in $\mathcal{C}$, in which $\widetilde{f}$ and $\widetilde{g}$ are morphisms with $\widetilde{f} \in \operatorname{Hom}(W, Y)$ and $\widetilde{g} \in \operatorname{Hom}(W, X)$, and in which the following universal mapping property holds: whenever $(W', \widetilde{f}', \widetilde{g}')$ is a triple such that W' is an object in $\mathcal{C}$ and $\widetilde{f}'$ and $\widetilde{g}'$ are morphisms with $\widetilde{f}' \in \operatorname{Hom}(W', Y)$ and $\widetilde{g}' \in \operatorname{Hom}(W', X)$ and with $f\widetilde{g}' = g\widetilde{f}'$, then there exists a unique $\varphi \in \operatorname{Hom}(W', W)$ such that $\widetilde{f}' = \widetilde{f}\varphi$ and $\widetilde{g}' = \widetilde{g}\varphi$. See Figure 4.8.

$$\begin{tikzcd}
W \arrow[r, dashed, "\widetilde{f}"] \arrow[d, dashed, "\widetilde{g}"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & Z
\end{tikzcd}$$

FIGURE 4.8. The pullback of a pair (f, g) of morphisms.

Proposition 4.39. In an abelian category $\mathcal{C}$, let X, Y, Z be objects, and let $f \in \operatorname{Hom}(X, Z)$ and $g \in \operatorname{Hom}(Y, Z)$ be morphisms. Let $X \oplus Y$ be the direct sum, let p_X and p_Y be the projections on the two factors, define $h = fp_X - gp_Y$ in $\operatorname{Hom}(X \oplus Y, Z)$, and let $m = \ker h$. Then a pullback $(W, \widetilde{f}, \widetilde{g})$ of (f, g) is given by $W = \operatorname{domain} m$, $\widetilde{f} = p_Y m$, and $\widetilde{g} = p_X m$.

REMARKS. The dual statement asserts the existence of a **pushout** of a pair of morphisms, and it is a consequence of Proposition 4.39. Problem 35 at the end of the chapter points out that the proof of Proposition 4.19a made use of a concretely constructed pullback, while the proof of Proposition 4.19b made use of a concretely constructed pushout.

PROOF. From $hm = h \ker h = 0$, we obtain $0 = fp_X m - gp_Y m = f\widetilde{g} - g\widetilde{f}$, and thus $f\widetilde{g} = g\widetilde{f}$. Now suppose that W', $\widetilde{f}'$, and $\widetilde{g}'$ are given with $f\widetilde{g}' = g\widetilde{f}'$. Then $m' = (\widetilde{g}', \widetilde{f}')$ is a morphism in $\operatorname{Hom}(W', X \oplus Y)$ such that $hm' = fp_X m' - gp_Y m' = f\widetilde{g}' - g\widetilde{f}' = 0$. Therefore m' factors through $m = \ker h$ as $(\widetilde{g}', \widetilde{f}') = m\varphi$ for a unique $\varphi \in \operatorname{Hom}(W', W)$. Application of p_X and p_Y to this equality gives $\widetilde{g}' = p_X m\varphi = \widetilde{g}\varphi$ and $\widetilde{f}' = p_Y m\varphi = \widetilde{f}\varphi$. $\square$

Proposition 4.40. In the notation of Figure 4.8 and Proposition 4.39 if f is a monomorphism, then so is $\widetilde{f}$. If f is an epimorphism, then so is $\widetilde{f}$; in the case of an epimorphism, $\ker f$ factors as $\ker f = \widetilde{g}(\ker \widetilde{f}\,)$.

PROOF. Throughout the proof let i_X and i_Y be the injections associated with the direct sum $X \oplus Y$. Suppose that f is a monomorphism, and suppose that $\widetilde{f}w = 0$ for some morphism with codomain W. Since $\widetilde{f} = p_Y m$, $p_Y m w = 0$. Then $0 = (fp_X - gp_Y)mw = fp_X mw - 0 = fp_X mw$. Since f is a monomorphism, $p_X mw = 0$. Since also $\widetilde{f}w = p_Y mw = 0, mw = (i_X p_X + i_Y p_Y)mw = 0$. But m is a monomorphism, and therefore $w = 0$. Consequently $\widetilde{f}$ is a monomorphism.

For the remainder of the proof, assume that f is an epimorphism. Let us see that $h = fp_X - gp_Y$ is an epimorphism. In fact, if $zh = 0$, then $0 = z(fp_X - gp_Y)i_X = zfp_Xi_X = zf$. Since f is an epimorphism, $z = 0$. Thus h is an epimorphism.

It follows from Lemma 4.37 that $h = \operatorname{coker}(\ker h) = \operatorname{coker} m$. To prove that $\widetilde{f}$ is an epimorphism, suppose that $v\widetilde{f} = 0$ for some morphism v with domain Y. This means that $vp_Ym = 0$. Since h is the cokernel of m, vp_Y factors as $vp_Y = v'h$ for some morphism v'. Applying i_X on the right end of both sides gives $0 = vp_Yi_X = v'hi_X = v'(fp_X - gp_Y)i_X = v'fp_Xi_X = v'f$. Since f is an epimorphism, $v' = 0$. Hence $vp_Y = v'h = 0$. Since p_Y is an epimorphism, $v = 0$. Therefore $\widetilde{f}$ is an epimorphism.

Now set $k = \ker f$, and let K be its domain. The morphisms $k \in \operatorname{Hom}(K, X)$ and $0 \in \operatorname{Hom}(K, Y)$ have $fk = 0 = g0$. If we set $W' = K$, $\widetilde{f}' = 0$, and $\widetilde{g}' = k$, then $f\widetilde{g}' = g\widetilde{f}'$, and Proposition 4.39 produces a unique φ in $\operatorname{Hom}(K, W)$ with $0 = \widetilde{f}\varphi$ and $k = \widetilde{g}\varphi$. We shall show that φ is a kernel of $\widetilde{f}$, and then the equation $k = \widetilde{g}\varphi$ completes the proof.

We know that $\widetilde{f}\varphi = 0$. Thus suppose that $\widetilde{f}v = 0$ for some morphism v in some $\operatorname{Hom}(K', W)$. Since $f\widetilde{g} = g\widetilde{f}$, we have $f\widetilde{g}v = g\widetilde{f}v = 0$. Thus $\widetilde{g}v$ factors through $k = \ker f$ as $\widetilde{g}v = kv'$ for some v' in $\operatorname{Hom}(K', K)$.

Put $\Phi = v - \varphi v'$. Then $\widetilde{f}\Phi = \widetilde{f}v - \widetilde{f}\varphi v' = 0 - 0 = 0$, and $\widetilde{g}\Phi = \widetilde{g}v - \widetilde{g}\varphi v' = kv' - kv' = 0$. Consequently if we put $W'' = K'$, $\widetilde{f}'' = 0$, and $\widetilde{g}'' = 0$, then Φ and 0 are two morphisms in $\operatorname{Hom}(K', W)$ with $\widetilde{f}'' = \widetilde{f}\Phi = \widetilde{f}0$ and $\widetilde{g}'' = \widetilde{g}\Phi = \widetilde{f}0$. By uniqueness of the morphism in the universal mapping property for pullbacks, $\Phi = 0$. Therefore $v = \varphi v'$, and v has been exhibited as factoring through φ.

If v factors through φ also as $v = \varphi v''$, then $0 = \varphi(v' - v'')$, and we have $k(v' - v'') = \widetilde{g}\varphi(v' - v'') = 0$. Since $k = \ker f$ is a monomorphism, $v' = v''$. Thus the factorization of v through φ is unique, and φ is a kernel of $\widetilde{f}$. This completes the proof. $\square$

Proposition 4.41. Let $\mathcal{C}$ be an abelian category, let X be an object in $\mathcal{C}$, and define $x \equiv y$ for two morphisms x and y with codomain X if there exist epimorphisms u and v with $xu = yv$. Then the relation $\equiv$ on the morphisms with codomain X is transitive and hence is an equivalence relation.

REMARK. A nontrivial special case is that the obvious equivalences $xu \equiv x$ and $x \equiv xv$ imply the nonobvious equivalence $xu \equiv xv$ when u and v are epimorphisms.

PROOF. Assuming that $x \equiv y$ and $y \equiv z$, write $xu = yv$ and $yr = zs$ for epimorphisms u, v, r, s. Since v and r have the same codomain, namely domain(y), the pullback $(\widetilde{v}, \widetilde{r})$ of (v, r) as in Proposition 4.39 is well defined, and Proposition 4.40 shows that $\widetilde{v}$ and $\widetilde{r}$ are epimorphisms. Since $r\widetilde{v} = v\widetilde{r}$, we obtain $x u\widetilde{r} = y v\widetilde{r} = y r\widetilde{v} = z s\widetilde{v}$. The morphisms $u\widetilde{r}$ and $s\widetilde{v}$ are epimorphisms as compositions of epimorphisms, and therefore $x \equiv z$. $\square$

Fix an object X. Then 0_{0X} is a member of X called the **zero member**, denoted by 0. Every zero morphism 0_{YX} with codomain X is equivalent to 0_{0X}; in fact, $0_{YX} = 0_{0X}0_{Y0}$. The morphism 0_{Y0} is an epimorphism because if $f \in \mathrm{Hom}(0, Z)$ has $f0_{Y0} = 0_{YZ}$, then f is the unique element 0_{0Z} of $\mathrm{Hom}(0, Z)$. Conversely any nonzero morphism r in $\mathrm{Hom}(Y, X)$ is inequivalent to 0_{YX}. In fact, an equality $ru = 0_{YX}v$ for epimorphisms u and v would imply that $r = 0_{YX}$, since we can cancel in the equality $ru = 0_{YX}v = 0_{YX}u$.

Each $x \in_m X$ has a "negative," namely the class of the negative of the representative x of the member; i.e., taking the negative of a morphism is respected in passing to classes. We write $-x \in_m X$ for the negative. (*Warning:* As the example with the category of abelian groups shows, one should use care in inferring any relationship between "negatives" and zero members.)

If f is a morphism in $\mathrm{Hom}(X, Y)$, then each member $x \in_m X$ yields by composition a well-defined member $fx \in_m Y$. To see that this notion is indeed well defined, suppose that $x \equiv x'$, and choose epimorphisms u and v with $xu = x'v$. Then $(fx)u = f(xu) = f(x'v) = (fx')v$ shows that $fx \equiv fx'$.

The main result is Theorem 4.42 below, which gives a calculus for diagram chases using members in general abelian categories. After the proof we shall be content with one example of how the theorem allows all the diagram chases in earlier sections of this chapter to be extended to general abelian categories. The example is the proof of the part of the Snake Lemma that involves an explicit construction.[19] More examples appear in Problems 34–35 at the end of the chapter.

Theorem 4.42. The members of an abelian category satisfy the following properties:

(a) a morphism $f \in \mathrm{Hom}(X, Y)$ is a monomorphism if and only if every $x \in_m X$ with $fx \equiv 0$ has $x \equiv 0$,
(b) a morphism $f \in \mathrm{Hom}(X, Y)$ is a monomorphism if and only if every pair of members $x \in_m X$ and $x' \in_m X$ with $fx \equiv fx'$ has $x \equiv x'$,
(c) a morphism $g \in \mathrm{Hom}(X, Y)$ is an epimorphism if and only if for each $y \in_m Y$, there exists some $x \in_m X$ with $gx \equiv y$,
(d) a morphism $h \in \mathrm{Hom}(X, Y)$ is the 0 morphism if and only if every $x \in_m X$ has $hx \equiv 0$,
(e) a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact at Y if and only if $gf = 0$ and also each $y \in_m Y$ with $gy \equiv 0$ has some $x \in_m X$ with $fx \equiv y$,

[19]For more detail about this example and for further examples, see Mac Lane's *Categories for the Working Mathematician.*

(f) whenever x, y, z are members of an object X and $x \equiv yu + zv$ for some epimorphisms u and v, then $xu' - yv' \equiv z$ for some epimorphisms u' and v'.

REMARKS.

(1) The interpretations of (a) through (e) are straightforward enough and already give an indication that the notion of a member may be of some help in translating proofs for good categories into proofs for abelian categories. Application of (d) to the difference $f_1 - f_2$ of two morphisms in $\mathrm{Hom}(X, Y)$ shows that $f_1 x \equiv f_2 x$ for all $x \in_m X$ implies $f_1 = f_2$.

(2) The interpretation of (f) is more subtle. As the example with the Snake Lemma below will show, conclusion (f) makes it possible to mirror in the theory of members the kind of subtraction that takes place with elements of a module to get their difference to be in the kernel of some homomorphism.

PROOF. For (a) and (b), if f is a monomorphism and $fx \equiv fx'$, then $fxu = fx'v$ for suitable epimorphisms u and v, and cancellation yields $xu = x'v$ and hence $x \equiv x'$. Conversely suppose $fx \equiv 0$ only for $x \equiv 0$. If f has $fx' = 0_{AY}$ for some x' in some $\mathrm{Hom}(A, X)$, then $fx' \equiv 0$ and so $x' \equiv 0$ by hypothesis. In this case, $x' = 0_{AX}$ because we know that nonzero morphisms are not equivalent to 0.

For (c), suppose that g is an epimorphism. If $y \in_m Y$ is given, let y be in $\mathrm{Hom}(X', Y)$, and let $(\widetilde{g}, \widetilde{y})$ be the pullback of (g, y), satisfying $y\widetilde{g} = g\widetilde{y}$. Proposition 4.40 shows that $\widetilde{g}$ is an epimorphism, and then $y \equiv gx$ for $x = \widetilde{y}$. Conversely if g fails to be an epimorphism, then there exists $r \neq 0$ in some $\mathrm{Hom}(Y, Z)$ with $rg = 0_{XZ}$. If there is some x in some $\mathrm{Hom}(A, X)$ with $gx \equiv 1_Y$, we can compose with r on the left of both sides and obtain $rgx \equiv r1_Y = r$. Since the left side equals 0_{AZ}, which is equivalent to 0_{YZ}, we obtain $0_{YZ} \equiv 0_{AZ} \equiv r$, which we know not to be true for nonzero members r of $\mathrm{Hom}(Y, Z)$.

For (d), if $h = 0_{XY}$ and if x is in $\mathrm{Hom}(Z, X)$, then $hx = 0_{XY}x = 0_{ZY} \equiv 0_{0Y}$. Conversely if every x in every $\mathrm{Hom}(Z, X)$ has $hx \equiv 0_{0Y}$, we take $Z = X$ and $x = 1_X$. Then $hu = hxu = 0_{0Y}v$ for some epimorphisms $u \in \mathrm{Hom}(A, X)$ and $v \in \mathrm{Hom}(A, 0)$. This says that $hu = 0_{AY} = 0_{XY}u$. Since u is an epimorphism, $h = 0_{XY}$.

For (e), let $f = me$ be the decomposition of f as in Proposition 4.38. Then $m = \operatorname{image} f$, and we define $k = \ker g$. If the sequence is exact at Y, then $gf = 0$ as part of the definition. Suppose $y \in_m Y$ has $gy \equiv 0$, i.e., $gy = 0$. Since $m = \ker g$ by exactness, the equality $gy = 0$ and the definition of kernel together imply that $y = my'$ for some y'. Using Proposition 4.39, let (e, y') have $(\widetilde{e}, \widetilde{y}')$ as pullback, satisfying $e\widetilde{y}' = y'\widetilde{e}$. Since e by construction is an epimorphism, Proposition 4.40 shows that $\widetilde{e}$ is an epimorphism. From the computation $f\widetilde{y}' = me\widetilde{y}' = my'\widetilde{e} = y\widetilde{e}$, we obtain $f\widetilde{y}' \equiv y$. Then $x = \widetilde{y}'$ has $x \in_m X$ and $fx \equiv y$.

Conversely suppose that $gf = 0$ and that the other condition holds. Since e is an epimorphism, the equality $gf = 0$ implies that $gm = 0$. The definition of $k = \ker g$ thus gives $m = k\varphi$ for some morphism φ. Meanwhile, the morphism $k = \ker g$ has $k \in_m Y$ and $gk = 0$. Thus $gk \equiv 0$. The hypothesis produces $x \in_m X$ with $fx \equiv k$, i.e., with $mexu = kv$ for suitable epimorphisms u and v. Write $ex = m'e'$ according to Proposition 4.38. Then $mm'e'u = kv$, and the uniqueness in Proposition 4.38 shows that $k = mm'\psi$ for some isomorphism ψ. Putting the results together gives $m = k\varphi = mm'\psi\varphi$ and $k = mm'\psi = k\varphi m'\psi$. Since m and k are monomorphisms, $1 = m'\psi\varphi$ and $1 = \varphi m'\psi$. These show that φ has a left inverse and a right inverse, hence is an isomorphism. Then m' too is an isomorphism, and $k = m$ except for a factor of an isomorphism on the right side. This means that we can take $\ker g = \text{image } f$ and that the given sequence is exact at Y.

For (f), let $x \equiv yu + zv$. Then $xu_1 = (yu + zv)v_1$, and $xu_1 - y(uv_1) = zvv_1$. Consequently $xu_1 - y(uv_1) \equiv zvv_1 \equiv z$, and (f) follows with $u' = u_1$ and $v' = uv_1$. □

Theorem 4.42 enables us to use members to verify properties of morphisms in diagrams, but it does not by itself construct any morphisms. That is, just because we know what the equivalence class of fx should be for every $x \in_m X$ does not mean that we have a construction of f; it means only that we know how to work with f once f is known to exist. Specifically we know from Remark 1 with the theorem that there cannot be a different morphism g with $fx \equiv gx$ for all $x \in_m X$. Some tools that we have for constructing morphisms for a general abelian category are the existence of kernels and cokernels via Axiom (iv), Proposition 4.39 asserting the existence of pullbacks of pairs of morphisms, and the dual of Proposition 4.39 asserting the existence of pushouts of pairs of morphisms. For particular categories of interest, the hypotheses "enough projectives" and "enough injectives" provide additional constructions of morphisms.

The most complicated example of a constructed mapping that we encountered in the theory for good categories was the connecting homomorphism in the Snake Lemma. In the generalization to abelian categories, the construction of the connecting morphism has to go outside the usual diagram given in Figure 4.2. Problem 33 at the end of the chapter will compare the actual construction and Figure 4.2 for the chain map of exact sequences of abelian groups given below and observe that the two diagrams are different:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 8} & \mathbb{Z} & \xrightarrow{1 \mapsto 1 \bmod 8} & \mathbb{Z}/8\mathbb{Z} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \times 4} & & \big\downarrow{\scriptstyle \times 2} & & \big\downarrow{\scriptstyle 1 \bmod 8 \mapsto 2 \bmod 4} & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 4} & \mathbb{Z} & \xrightarrow{1 \mapsto 1 \bmod 4} & \mathbb{Z}/4\mathbb{Z} & \longrightarrow & 0
\end{array}$$

The domain of the connecting homomorphism for this situation is the set of even members of $\mathbb{Z}/8\mathbb{Z}$, and the mapping carries $2 + 8\mathbb{Z}$ to $1 + 4\mathbb{Z}$ in $\mathbb{Z}/4\mathbb{Z}$.

EXAMPLE OF DIAGRAM CHASE. In the setting of the Snake Lemma (Lemma 4.6), we shall construct the connecting morphism ω and verify that its value on each member of its domain corresponds to what we expect on the basis of Lemma 4.6. The given snake diagram, partially enlarged toward Figure 4.2, is

$$\begin{tikzcd}
 & & & C_0 \arrow[d, "k"] & \\
 & A \arrow[r, "\varphi"] \arrow[d, "\alpha"] & B \arrow[r, "\psi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\varphi'"] \arrow[d, "p"'] & B' \arrow[r, "\psi'"] & C' & \\
 & A'_0 & & &
\end{tikzcd} \tag{$*$}$$

with the rows exact and the squares commuting. The added parts at the top and bottom are the kernel (C_0, k) of γ and the cokernel (A'_0, p) of α. Once the connecting homomorphism has been constructed, the proof of exactness will involve a diagram chase that makes rather straightforward use of Theorem 4.42, including conclusion (f). By contrast, the initial construction will involve a different sort of diagram, namely

$$\begin{tikzcd}
 & & B_0 \arrow[r, dashed, "\widetilde{\psi}"] \arrow[d, dashed, "\widetilde{k}"] & C_0 \arrow[d, "k"] & \\
0 \arrow[r, dashed] & \overline{A} \arrow[r, "\overline{\varphi}"] \arrow[d, "\overline{\alpha}"] \arrow[ur, dashed, "\widetilde{\varphi}"] & B \arrow[r, "\psi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\overline{\gamma}"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\varphi'"] \arrow[d, "p"] & B' \arrow[r, "\overline{\psi}'"] \arrow[d, dashed, "\widetilde{p}"] & \overline{C}' \arrow[r, dashed] & 0 \\
 & A'_0 \arrow[r, dashed, "\widetilde{\varphi}'"] & B'_0 \arrow[ur, dashed, "\widetilde{\psi}'"'] & &
\end{tikzcd}$$

In the construction we adjust the first row of $(*)$ to make it exact when a 0 is included at the left end. To do so, we factor φ according to Proposition 4.38 as $\varphi = me$, we let $\overline{A} = \operatorname{domain} m = \operatorname{codomain} e$, and we write $\overline{\varphi}$ for m. The commutativity of the left square of $(*)$ implies that $\varphi'\alpha(\ker\varphi) = \beta\varphi(\ker\varphi) = 0$. Since φ' is a monomorphism, $\alpha(\ker\varphi) = 0$. Then the fact $e = \operatorname{coker}(\ker\varphi)$ implies that α factors through e as $\alpha = \overline{\alpha}e$ for some $\overline{\alpha}$ with domain $\overline{A}$. Consequently the left square in the adjusted diagram commutes, and the first row is exact with the 0 inserted at the left. Since e is an epimorphism, $p = \operatorname{coker}\alpha = \operatorname{coker}(\overline{\alpha}e) = \operatorname{coker}\overline{\alpha}$, and the vertical line at the left is exact.

By a dual argument starting from a factorization of ψ', we can replace the triple (C', ψ', γ) in similar fashion by $(\overline{C}', \overline{\psi}', \overline{\gamma})$, see that $k = \ker\overline{\gamma}$, and add a 0 at the end of the second row to obtain an exact sequence.

Next, let $(B_0, \widetilde{\psi}, \widetilde{k})$ be a pullback of (ψ, k). Proposition 4.40 shows that $\widetilde{\psi}$ is an epimorphism and that $\ker\psi = \widetilde{k}\ker\widetilde{\psi}$. Since the first row is a short exact sequence, we know that $\overline{\varphi} = \ker\psi$, and the condition $\ker\psi = \widetilde{k}\ker\widetilde{\psi}$ shows that $\widetilde{\varphi} = \ker\widetilde{\psi}$ satisfies $\overline{\varphi} = \widetilde{k}\widetilde{\varphi}$. This completes the dashed arrows in the top part of the diagram. By a dual argument using $p = \operatorname{coker}\overline{\alpha}$, we complete the dashed arrows in the bottom part of the diagram, deducing from $\overline{\psi}' = \operatorname{coker}\varphi'$ the fact that $\widetilde{\psi}' = \operatorname{coker}\widetilde{\varphi}'$ satisfies $\overline{\psi}' = \widetilde{\psi}'\widetilde{p}$.

Lemma 4.37 shows from $\widetilde{\varphi} = \ker\widetilde{\psi}$ that $\widetilde{\psi} = \operatorname{coker}\widetilde{\varphi}$, and it shows from $\widetilde{\psi}' = \operatorname{coker}\widetilde{\varphi}'$ that $\widetilde{\varphi}' = \ker\widetilde{\psi}'$. With these formulas in hand, we can construct the connecting homomorphism. Define $\omega_0 = \widetilde{p}\beta\widetilde{k}$ in $\operatorname{Hom}(B_0, B_0')$ to be the composition down the center. Then $\omega_0\widetilde{\varphi} = \widetilde{p}\beta\widetilde{k}\widetilde{\varphi} = \widetilde{\varphi}'p\overline{\alpha} = 0$, the last equality holding because $p\overline{\alpha} = 0$. Therefore ω_0 factors through $\widetilde{\psi} = \operatorname{coker}\widetilde{\varphi}$ as $\omega_0 = \omega_1\widetilde{\psi}$ for some $\omega_1 \in \operatorname{Hom}(C_0, B_0')$. The morphism ω_1 satisfies $\widetilde{\psi}'\omega_1\widetilde{\psi} = \widetilde{\psi}'\widetilde{p}\beta\widetilde{k} = \overline{\gamma}k\widetilde{\psi} = 0$, the last equality holding because $\overline{\gamma}k = 0$. Since $\widetilde{\psi}$ is an epimorphism, we can cancel it, obtaining $\widetilde{\psi}'\omega_1 = 0$. Therefore ω_1 factors through $\widetilde{\varphi}' = \ker\widetilde{\psi}'$ as $\omega_1 = \widetilde{\varphi}'\omega$ for some morphism $\omega \in \operatorname{Hom}(C_0, A_0')$.

The construction of ω is now complete, and the assertion is that the value of ω on members corresponds to what we expect from the proof of Lemma 4.6. Since equivalences $\omega x \equiv \omega' x$ for some other candidate ω' for the connecting morphism and for all $x \in_m C_0$ would imply that $\omega = \omega'$, the argument will show that we have found the unique morphism taking the prescribed values on members.

During the verification we refer to $(*)$ to do the diagram chase. The member of C corresponding to $x \in_m C_0$ is $kx \in_m C$. Since ψ is an epimorphism, Theorem 4.42c produces $b \in_m B$ with $\psi b \equiv kx$. Then $\psi'\beta b \equiv \gamma\psi b \equiv \gamma kx \equiv 0$, since $\gamma k = 0$. Theorem 4.42e and exactness at B' imply that $\varphi' a' \equiv \beta b$ for some $a' \in A'$, and the class of a' is unique (for the b under consideration) by Theorem 4.42b because φ' is a monomorphism. We shall verify that $\omega x \equiv pa'$, and then the class of ωx matches what we expect from the proof of Lemma 4.6.

First let us show that a different choice of b, say b_1, leads to the same class pa'. We are given that $\psi b \equiv \psi b_1$. Let a' and a'_1 be the corresponding members of A' with $\varphi' a' \equiv \beta b$ and $\varphi' a'_1 \equiv \beta b_1$. We shall make repeated use of Theorem 4.42f, letting subscripted u's and v's denote suitable epimorphisms. From $\psi b \equiv \psi b_1$, Theorem 4.42f gives $\psi b u_1 - \psi b_1 v_1 \equiv 0$, i.e., $\psi(bu_1 - b_1 v_1) \equiv 0$. By Theorem 4.42e and exactness at B, $bu_1 - b_1 v_1 \equiv \varphi a$ for some $a \in_m A$. Hence $\beta b u_1 - \beta b_1 v_1 \equiv \beta\varphi a \equiv \varphi' \alpha a$. Two applications of Theorem 4.42f starting from $\beta b u_1 - \beta b_1 v_1 \equiv \varphi' \alpha a$ give

$$\varphi' a' \equiv \beta b \equiv \varphi' \alpha a u_2 + \beta b_1 v_2,$$

and then
$$\varphi' a' u_3 - \varphi' \alpha a v_3 \equiv \beta b_1 \equiv \varphi' a'_1.$$

Since φ' is a monomorphism, Theorem 4.42b says that

$$a' u_3 - \alpha a v_3 \equiv a'_1.$$

Applying p, we obtain $pa' u_3 - p\alpha a v_3 \equiv pa'_1$. Since $p\alpha = 0$, we can drop the term $p\alpha v_3$, and we conclude that $pa' \equiv pa' u_3 \equiv pa'_1$.

We can now return to the verification that $\omega x \equiv pa'$, making use of the adjusted diagram as necessary.[20] Since $\widetilde{\psi}$ is an epimorphism, Theorem 4.42c produces $b_0 \in_m B_0$ with $\widetilde{\psi} b_0 \equiv x$. Then $\widetilde{k} b_0 \in_m B$ has $\psi \widetilde{k} b_0 \equiv k \widetilde{\psi} b_0 \equiv kx$. Hence $\widetilde{k} b_0$ is a member of B like b and b_1 in the previous paragraph. The above argument shows that $\beta \widetilde{k} b_0 \in_m B'$ has $\beta \widetilde{k} b_0 \equiv \varphi' a'$ for some $a' \in_m A'$ and that $pa' \in_m A'_0$ is what we should hope for as the value of ωx. So we compute that

$$\widetilde{\varphi}' \omega x \equiv \omega_1 x \equiv \omega_1 \widetilde{\psi} b_0 \equiv \omega_0 b_0 \equiv \widetilde{p} \beta \widetilde{k} b_0 \equiv \widetilde{p} \varphi' a' \equiv \widetilde{\varphi}' pa'.$$

Since $\widetilde{\varphi}'$ is a monomorphism by the dual of Proposition 4.40, Theorem 4.42b shows that $\omega x \equiv \varphi' a'$, which is the formula we were seeking.

9. Problems

1. (a) Prove that the good category of all finitely generated abelian groups has enough projectives but not enough injectives.
   (b) Prove that the good category of all torsion abelian groups has enough injectives but not enough projectives.
2. Let $\mathcal{C}_{\mathbb{Z}}$ be the category of all abelian groups. Give an example of a nonzero good category $\mathcal{C}$ of abelian groups that has enough projectives and enough injectives but for which no nonzero projective for $\mathcal{C}_{\mathbb{Z}}$ lies in $\mathcal{C}$ and no nonzero injective for $\mathcal{C}$ lies in $\mathcal{C}_{\mathbb{Z}}$.

[20] *Warning:* The construction of ω involves B_0 and B'_0, which are in the adjusted diagram but are not in $(*)$. These objects do not necessarily coincide with the domain of $\ker\beta$ and the codomain of $\operatorname{coker}\beta$.

3. Let R be a semisimple ring in the sense of Chapter II, and let $\mathcal{C}_R$ be the category of all unital left R modules. Prove that every module in $\mathcal{C}_R$ is projective and injective.

4. Let R be a (commutative) principal ideal domain, and let $\mathcal{C}_R$ be the category of all unital R modules. A module M in $\mathcal{C}_R$ is **divisible** if for each $a \neq 0$ in R and $x \in M$, there exists $y \in M$ with $ay = x$.
   (a) Referring to Example 2 of injectives in Section 4, prove that injective for $\mathcal{C}_R$ implies divisible.
   (b) Deduce from Proposition 4.15 that divisible implies injective for $\mathcal{C}_R$.

5. Let R be a (commutative) principal ideal domain, and let $\mathcal{C}_R$ be the category of all unital R modules. Prove that every module M in $\mathcal{C}_R$ has an injective resolution of the form $0 \to M \to I_0 \to I_1 \to 0$ with I_0 and I_1 injective.

6. Let $\mathcal{C}, \mathcal{C}', \mathcal{C}''$ be good categories of modules with enough projectives and enough injectives, let $G : \mathcal{C} \to \mathcal{C}'$ be a one-sided exact functor with derived functors G_n or G^n, and let $F : \mathcal{C}' \to \mathcal{C}''$ be an exact functor.
   (a) Prove that if F is covariant, then $F \circ G$ is one-sided exact, and its derived functors satisfy $(F \circ G)_n = F \circ G_n$ or $(F \circ G)^n = F \circ G^n$.
   (b) Prove that if F is contravariant, then $F \circ G$ is one-sided exact, and its derived functors satisfy $(F \circ G)^n = F \circ G_n$ or $(F \circ G)^n = F \circ G^n$.

7. Let $\mathcal{C}, \mathcal{C}', \mathcal{C}''$ be good categories of modules with enough projectives and enough injectives, let $F : \mathcal{C} \to \mathcal{C}'$ be an exact functor, and let $G : \mathcal{C}' \to \mathcal{C}''$ be a one-sided exact functor with derived functors G_n or G^n.
   (a) Suppose that F is covariant, that G_n or G^n is defined from projective resolutions, and that F carries projectives to projectives. Prove that $G \circ F$ is one-sided exact and that its derived functors satisfy $(G \circ F)_n = G_n \circ F$ or $(G \circ F)^n = G^n \circ F$.
   (b) Suppose that F is covariant, that G_n or G^n is defined from injective resolutions, and that F carries injectives to injectives. Prove that $G \circ F$ is one-sided exact and that its derived functors satisfy $(G \circ F)_n = G_n \circ F$ or $(G \circ F)^n = G^n \circ F$.
   (c) Suppose that F is contravariant, that G_n or G^n is defined from projective resolutions, and that F carries injectives to projectives. Prove that $G \circ F$ is one-sided exact and that its derived functors satisfy $(G \circ F)^n = G^n \circ F$ or $(G \circ F)_n = G_n \circ F$.
   (d) Suppose that F is contravariant, that G_n or G^n is defined from injective resolutions, and that F carries projectives to injectives. Prove that $G \circ F$ is one-sided exact and that its derived functors satisfy $(G \circ F)^n = G^n \circ F$ or $(G \circ F)_n = G_n \circ F$.

8. Let G be a group, and let $F = (F^+ \to \mathbb{Z})$ be a free resolution of the trivial $\mathbb{Z}G$ module $\mathbb{Z}$ in the category $\mathbb{Z}G$. If M is an abelian group on which G acts by automorphisms, then we know that the cohomology $H^n(G, M)$ is defined to be the n^{th} cohomology of the cochain complex $\operatorname{Hom}_{\mathbb{Z}G}(F^+, M)$ and the homology $H_n(G, M)$ is defined to be the n^{th} homology of the chain complex $F^+ \otimes_{\mathbb{Z}G} M$. Take for granted the result of Proposition 3.32 that if G is a finite cyclic group with generator s, then

$$\cdots \xrightarrow{T} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{T} \cdots \xrightarrow{N} \mathbb{Z}G \xrightarrow{T} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is a free resolution of $\mathbb{Z}G$, where T and N are the left $\mathbb{Z}G$ module homomorphisms defined by

$$T = \text{multiplication by } (s) - (1),$$
$$N = \text{multiplication by } (1) + (s) + \cdots + (s^{n-1}).$$

Prove that $H^n(G, M) \cong H^{n+2}(G, M)$ and $H_n(G, M) \cong H_{n+2}(G, M)$ for all $n \geq 1$ and all M when G is a finite cyclic group.

Problems 9–11 concern changes of rings. Fix a homomorphism $\rho : R \to S$ of rings with identity. This homomorphism determines three functors of interest, denoted by $\mathcal{F}_S^R : \mathcal{C}_S \to \mathcal{C}_R$, $P_R^S : \mathcal{C}_R \to \mathcal{C}_S$, and $I_R^S : \mathcal{C}_R \to \mathcal{C}_S$. The first takes an S module M and makes it into an R module $\mathcal{F}_S^R(M)$ by the definition $rm = \rho(r)m$ for $r \in R$ and $m \in M$; the effect on an S homomorphism is to leave the function unchanged and to regard it as an R homomorphism; this functor is manifestly exact. For the second, regard S as an (S, R) bimodule with right R action given by $sr = s\rho(r)$, and define $P_R^S(M) = S \otimes_R M$ for M in $\operatorname{Obj}(\mathcal{C}_R)$ and $P_R^S(\varphi) = 1_S \otimes \varphi$ for φ in $\operatorname{Hom}_R(M, N)$; this functor is covariant and right exact. For the third, regard S as an (R, S) bimodule with left R action given by $rs = \rho(r)s$, and define $I_R^S(M) = \operatorname{Hom}_R(S, M)$ for M in $\operatorname{Obj}(\mathcal{C}_R)$ and $I_R^S(\varphi) = \operatorname{Hom}(1_S, \varphi)$ for φ in $\operatorname{Hom}_R(M, N)$; this functor is covariant and left exact.

9. If $\mathcal{C}$ and $\mathcal{D}$ are good categories of modules and if $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are covariant additive functors such that there exist isomorphisms of abelian groups

$$\operatorname{Hom}(F(A), B) \cong \operatorname{Hom}(A, G(B))$$

natural for A in $\operatorname{Obj}(\mathcal{C})$ and for B in $\operatorname{Obj}(\mathcal{D})$, then F is said to be **left adjoint** to G and G is said to be **right adjoint** to F.

(a) Prove that if G carries onto maps in $\mathcal{D}$ to onto maps in $\mathcal{C}$, then F carries projectives in $\mathcal{C}$ to projectives in $\mathcal{D}$.

(b) Prove that if F carries one-one maps in $\mathcal{C}$ to one-one maps in $\mathcal{D}$, then G carries injectives in $\mathcal{D}$ to injectives in $\mathcal{C}$. (Educational note: The conclusions in this problem extend to any abelian categories $\mathcal{C}$ and $\mathcal{D}$, and in this enlarged setting, (b) follows from (a) by duality.)

10. (a) Prove that P_R^S is left adjoint to $\mathcal{F}_S^R$.
 (b) Deduce from the previous problem that P_R^S sends projectives in $\mathcal{C}_R$ to projectives in $\mathcal{C}_S$.
 (c) Prove that if the right R module S is projective, then P_R^S is exact. (Educational note: In the subject of Lie algebra homology and cohomology, this hypothesis is satisfied when S is the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$, R is the universal enveloping algebra of a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and $\rho : R \to S$ is the inclusion. It is satisfied also in the subject of homology and cohomology of groups if S is the group algebra $\mathbb{K}G$ of a group G over a field $\mathbb{K}$ and if R is the group algebra $\mathbb{K}H$ of a subgroup H. See Problem 13c below.)
 (d) Using Problem 7, prove that if the right R module S is projective, then $\mathrm{Ext}_S^k(P_R^S M, N) \cong \mathrm{Ext}_R^k(M, \mathcal{F}_S^R N)$ naturally in each variable (M being in $\mathrm{Obj}(\mathcal{C}_R)$ and N being in $\mathrm{Obj}(\mathcal{C}_S)$).
 (e) Even without the assumption that the right R module S is projective, let $X = (X^+ \to M)$ be a projective resolution of a module M in $\mathcal{C}_R$, and let $Y = (Y^+ \to P_R^S M)$ be a projective resolution of $P_R^S M$ in $\mathcal{C}_S$. Construct a chain map from $P_R^S X$ to Y extending the identity map on $P_R^S M$, and use it to obtain the associated homomorphism $\mathrm{Ext}_S^k(P_R^S M, N) \to \mathrm{Ext}_R^k(M, \mathcal{F}_S^R N)$ natural in each variable.

11. (a) Prove that I_R^S is right adjoint to $\mathcal{F}_S^R$.
 (b) Deduce from Problem 9 that I_R^S sends injectives in $\mathcal{C}_R$ to injectives in $\mathcal{C}_S$.
 (c) Prove that if the right R module S is projective, then I_R^S is exact.
 (d) Using Problem 7, prove that if the right R module S is projective, then $\mathrm{Ext}_S^k(M, I_R^S N) \cong \mathrm{Ext}_R^k(\mathcal{F}_S^R M, N)$ naturally in each variable (M being in $\mathrm{Obj}(\mathcal{C}_S)$ and N being in $\mathrm{Obj}(\mathcal{C}_R)$).
 (e) Even without the assumption that the right R module S is projective, let $X = (X^+ \to N)$ be an injective resolution of a module N in $\mathcal{C}_R$, and let $Y = (Y^+ \to I_R^S N)$ be an injective resolution of $I_R^S N$ in $\mathcal{C}_S$. Construct a chain map from Y to $I_R^S N$ extending the identity map on $I_R^S N$, and use it to obtain the associated homomorphism $\mathrm{Ext}_S^k(M, I_R^S N) \to \mathrm{Ext}_R^k(\mathcal{F}_S^R M, N)$ natural in each variable.

Problems 12–13 concern the effect on cohomology of groups of changing the group. The main result is the exactness of the "inflation-restriction sequence"; this is applied particularly in algebraic number theory to relate Brauer groups (see Chapter III) for different field extensions. Let J and K be groups, and let $\rho : J \to K$ be a group homomorphism. By the universal mapping property of group rings, ρ extends to a ring homomorphism, also denoted by ρ, from $\mathbb{Z}J$ into $\mathbb{Z}K$. For any group G, we make use of the standard free resolution $F(G) = (F(G)^+ \xrightarrow{\varepsilon} \mathbb{Z})$ of $\mathbb{Z}$ in the category $\mathcal{C}_{\mathbb{Z}G}$, as described before Theorem 3.20. A $\mathbb{Z}$ basis of $F_n(G)$ consists of all tuples $(g_0, \ldots, g_n)$, and a $\mathbb{Z}G$ basis consists of those members of the $\mathbb{Z}$ basis with $g_0 = 1$. In the context of the groups J and K, any $\mathbb{Z}K$ module M becomes a $\mathbb{Z}J$ module by the formula $xm = \rho(x)m$ for $x \in \mathbb{Z}J$ and $m \in M$. In particular, each free $\mathbb{Z}K$ module $F_n(K)$ can be regarded as a $\mathbb{Z}J$ module. Meanwhile, the homomorphism $\rho : J \to K$ induces a function from the $\mathbb{Z}J$ basis of $F_n(J)$ into $F_n(K)$ by the formula $\rho(1, j_1, \ldots, j_n) = (1, \rho(j_1), \ldots, \rho(j_n))$ for $j_1, \ldots, j_n \in J$, and this extends to a $\mathbb{Z}J$ homomorphism, still called ρ, of $F_n(J)$ into $F_n(K)$. A look at the formula for the boundary operators ∂_J and ∂_K in Section III.5 shows that ρ is a chain map in the sense that $\partial_K \rho = \rho \partial_J$. If M is any unital left $\mathbb{Z}K$ module, then it follows that $\operatorname{Hom}(\rho, 1) : \operatorname{Hom}(F(K), M) \to \operatorname{Hom}(F(J), M)$ is a cochain map. Consequently we get maps on cohomology for each n of the form $H^n(\rho) : H^n(K, M) \to H^n(J, M)$. There are two cases of special interest:

(i) If $\rho : H \to G$ is the inclusion of a subgroup into a group, then the mapping on cohomology is called the **restriction homomorphism**

$$\operatorname{Res} : H^n(G, M) \to H^n(H, M).$$

(ii) If H is a normal subgroup of G, let $\rho : G \to G/H$ be the quotient homomorphism. For any $\mathbb{Z}G$ module M, let M^H be the subgroup of H invariants. Then G/H acts on M^H. The above construction is applicable to the module M^H for the group ring $\mathbb{Z}(G/H)$ of G/H, and we form the mapping on cohomology $H^n(G/H, M^H) \to H^n(G, M^H)$. The inclusion of the $\mathbb{Z}G$ module M^H in M induces a mapping $H^n(G, M^H) \to H^n(G, M)$, and the composition is called the **inflation homomorphism**

$$\operatorname{Inf} : H^n(G/H, M^H) \to H^n(G, M).$$

When H is a normal subgroup of G and M is a $\mathbb{Z}G$ module and $q \geq 1$ is an integer such that $H^k(H, M) = 0$ for $1 \leq k \leq q - 1$, the **inflation-restriction sequence** is the sequence of abelian groups and homomorphisms

$$0 \longrightarrow H^q(G/H, M^H) \xrightarrow{\operatorname{Inf}} H^q(G, M) \xrightarrow{\operatorname{Res}} H^q(H, M).$$

12. For $q = 1$, use direct arguments to prove the exactness of the inflation-restriction sequence by carrying out the following steps:
   (a) By sorting out the isomorphism $\Phi_q : \operatorname{Hom}_{\mathbb{Z}G}(F_q, M) \to C^q(G, M)$ of Section III.5, show that the effect of a homomorphism $\rho : G \to G'$ on $C^q(G', M)$ is given by $(\rho^* f)(g_1, \ldots, g_q) = f(\rho(g_1), \ldots, \rho(g_q))$.
   (b) Verify that $\operatorname{Res} \circ \operatorname{Inf} = 0$ by looking at cocycles.
   (c) Show that Inf is one-one on $H^q(G/H, M^H)$ by showing that any cocycle $f : G/H \to M^H$ that is a coboundary when viewed as a function on G is itself a coboundary for G/H.

(d) Show that every member of ker(Res) lies in image(Inf) by showing that any cocycle $f : G \to M$ whose restriction to H is a coboundary may be adjusted to be 0 on H and that an examination of the equation $f(st) = f(s) + sf(t)$ in this case shows f to be a cocycle of G/H with values in M^H.

13. Assume inductively that $q > 1$, that $H^k(H, M) = 0$ for $1 \le k \le q-1$, and that the inflation-restriction sequence is exact for all N for degree $q-1$ whenever $H^k(H, N) = 0$ for $1 \le k < q-1$. Form $B = I_{\mathbb{Z}}^{\mathbb{Z}G} \mathcal{F}_{\mathbb{Z}G}^{\mathbb{Z}} M = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}G, M)$ as in Problems 9–11. Elements of B can be identified with functions φ on G with values in M, and G acts by $(g_0\varphi)(g) = \varphi(gg_0)$.
   (a) For $m \in M$, show that the function $\varphi_m(t) = tm$ is a one-one $\mathbb{Z}G$ homomorphism of M into B. If $N = B/M$, then the sequence $0 \to M \to B \to N \to 0$ is therefore exact in $\mathcal{C}_{\mathbb{Z}G}$.
   (b) Use Problem 11 to verify that $H^k(G, B) \cong \operatorname{Ext}_{\mathbb{Z}}^k(\mathbb{Z}, \mathcal{F}_{\mathbb{Z}G}^{\mathbb{Z}} M)$, and deduce that $H^k(G, B) = 0$ for $k \ge 1$.
   (c) Verify the equality of right $\mathbb{Z}H$ modules $\mathbb{Z}G = A \otimes_{\mathbb{Z}} \mathbb{Z}H$ for some free abelian group A.
   (d) Using (c), show that $\mathcal{F}_{\mathbb{Z}G}^{\mathbb{Z}H} B \cong \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}H, \operatorname{Hom}_{\mathbb{Z}}(A, M))$, and deduce that $H^k(H, B) = 0$ for $k \ge 1$.
   (e) Using the hypothesis that $H^1(H, M) = 0$ and a long exact sequence associated to the short exact sequence in (a), show that $0 \to M^H \to B^H \to N^H \to 0$ is exact.
   (f) Prove that $\mathbb{Z} \otimes_{\mathbb{Z}H} \mathbb{Z}G \cong \mathbb{Z}(G/H)$ as right $\mathbb{Z}G$ modules, where $\mathbb{Z}(G/H)$ is the integral group ring of G/H.
   (g) Show that $B^H = I_{\mathbb{Z}}^{\mathbb{Z}(G/H)} M$, and deduce that $H^k(G/H, B^H) = 0$ for $k \ge 1$.
   (h) Using the long exact sequences for G and for H associated to the short exact sequence of (a), as well as the long exact sequence for G/H associated to the short exact sequence of (e), establish isomorphisms of abelian groups

$$H^{q-1}(G/H, N^H) \cong H^q(G/H, M^H),$$
$$H^{q-1}(G, N) \cong H^q(G, M),$$
$$H^{q-1}(H, N) \cong H^q(H, M).$$

   (i) Set up the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & H^{q-1}(G/H, N^H) & \longrightarrow & H^{q-1}(G, N) & \longrightarrow & H^{q-1}(H, N) \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & H^q(G/H, M^H) & \longrightarrow & H^q(G, M) & \longrightarrow & H^q(H, M)
\end{array}$$

   show that it is commutative, and deduce from the foregoing that the inflation-restriction sequence is exact for M in degree q. (Educational note:

For an application to Brauer groups, let $F \subseteq K \subseteq L$ be fields, and assume that K/F, L/F, and L/K are all finite Galois extensions. The groups in question are $G = \mathrm{Gal}(L/F)$, $H = \mathrm{Gal}(L/K)$, and $G/H = \mathrm{Gal}(K/F)$, and the modules in question are $M = L^\times$ and $M^H = K^\times$. The index q is to be 2, and the vanishing of H^1 is by Hilbert's Theorem 90. The conclusion is that the sequence $0 \to \mathcal{B}(K/F) \to \mathcal{B}(L/F) \to \mathcal{B}(L/K)$ is exact.)

Problems 14–16 introduce the cup product in the cohomology of groups. This is a construction having applications to topology and algebraic number theory. Let G be a group, and form the standard free resolution $F = (F^+ \xrightarrow{\varepsilon} \mathbb{Z})$ of $\mathbb{Z}$ in the category $\mathcal{C}_{\mathbb{Z}G}$, as described before Theorem 3.20. A $\mathbb{Z}$ basis of F_n consists of all tuples $(g_0, \dots, g_n)$, and a $\mathbb{Z}G$ basis consists of those members of the $\mathbb{Z}$ basis with $g_0 = 1$. Let ∂ denote the boundary operator, with the subscript dropped that indicates the degree. Define $\varphi_{p,q} : F_{p+q} \to F_p \otimes_{\mathbb{Z}} F_q$ by

$$\varphi_{p,q}(g_0, \dots, g_{p+q}) = (g_0, \dots, g_p) \otimes (g_p, \dots, g_q).$$

14. Check that $(\varepsilon \otimes \varepsilon) \circ \varphi_{0,0} = \varepsilon$ and that each $\varphi_{p,q}$ with $p \geq 0$ and $q \geq 0$ is a $\mathbb{Z}G$ homomorphism satisfying

$$\varphi_{p,q} \circ \partial = (\partial \otimes 1) \circ \varphi_{p+1,q} + (-1)^p (1 \otimes \partial) \circ \varphi_{p,q+1}.$$

15. If A and B are abelian groups on which G acts by automorphisms, show that G acts by automorphisms on $A \otimes_{\mathbb{Z}} B$ in such a way that $g(a \otimes b) = ga \otimes gb$ for all $a \in A$, $b \in B$, $g \in G$. Thus whenever A and B are unital left $\mathbb{Z}G$ modules, then so is $A \otimes_{\mathbb{Z}} B$.

16. For any unital left $\mathbb{Z}G$ module M, we work with $\mathrm{Hom}_{\mathbb{Z}G}(F_n, M)$ as the space of n-cochains. (Here it is not necessary to unravel the isomorphism given in Section III.5 that relates $\mathrm{Hom}_{\mathbb{Z}G}(F_n, M)$ to the space $C^n(G, M)$ of cochains defined in Chapter VII of *Basic Algebra*.) Define the coboundary operator on the complex $\mathrm{Hom}_{\mathbb{Z}G}(F^+, M)$ to be $d = \mathrm{Hom}(\partial, 1)$. For any unital left $\mathbb{Z}G$ modules A and B, let $f \in \mathrm{Hom}(F_p, A)$ and $g \in \mathrm{Hom}(F_q, B)$ be given. The product cochain $f \cdot g$ is the member of $\mathrm{Hom}_{\mathbb{Z}G}(F_{p+q}, A \otimes_{\mathbb{Z}} B)$ given by $f \cdot g = (f \otimes g) \circ \varphi_{p,q}$.
    (a) Check that $f \cdot g = (df) \cdot g + (-1)^p f \cdot (dg)$.
    (b) How does it follow that this product descends to a homomorphism of abelian groups $a \otimes b \mapsto a \cup b$ carrying the space $H^p(G, A) \otimes_{\mathbb{Z}} H^q(G, B)$ to $H^{p+q}(G, A \otimes_{\mathbb{Z}} B)$? The descended mapping is called the **cup product**.
    (c) Explain why the cup product is functorial in each variable A and B.
    (d) Explain why the cup product for $p = 0$ and $q = 0$ may be identified with the mapping on invariants given by $A^G \otimes B^G \to (A \otimes_{\mathbb{Z}} B)^G$.

Problems 17–20 introduce flat R modules, R being a ring with identity. These modules are of interest in topology and algebraic geometry. Let R^o be the opposite ring of R; right R modules may be identified with left R^o modules. Let $\mathcal{C}_R$ be the category of all unital left R modules; tensor product over R can be regarded as a functor in the second variable, carrying $\mathcal{C}_R$ to $\mathcal{C}_{\mathbb{Z}}$, or as a functor in the first variable, carrying $\mathcal{C}_{R^o}$ to $\mathcal{C}_{\mathbb{Z}}$. A unital right R module M (i.e., a unital left R^o module) is called **flat** if $M \otimes_R (\,\cdot\,)$ is an exact functor from $\mathcal{C}_R$ to $\mathcal{C}_{\mathbb{Z}}$. Since this functor is anyway right exact, M is flat if and only if tensoring with M carries one-one maps to one-one maps, i.e., if and only if whenever $f : A \to B$ is one-one, then $1_M \otimes f : M \otimes_R A \to M \otimes_R B$ is one-one. Take as known the analog for the functor Tor of all the facts about Ext proved in Section 7.

17. Prove for unital right R modules that
    (a) the right R module R is flat,
    (b) a direct sum $F = \bigoplus_{s \in S} F_s$ is flat if and only if each F_s is flat,
    (c) any projective in $\mathcal{C}_{R^o}$ is flat.
18. Let M be a unital right R module. For each finite subset F of M, let M_F be the right R submodule of M generated by the members of F. Prove that M is flat if and only if each M_F is flat.
19. Let B be in $\mathcal{C}_R$, write B as the R homomorphic image of a free left R module F, and form the exact sequence $0 \to K \to F \to B \to 0$ in which K is the kernel of $F \to B$. Prove for each unital right R module A that the sequence
$$0 \to \operatorname{Tor}_1^R(A, B) \to A \otimes_R K \to A \otimes_R F \to A \otimes_R B \to 0$$
is exact. Deduce that A is flat if and only if $\operatorname{Tor}_1^R(A, B) = 0$ for all B.
20. Suppose that R is a (commutative) principal ideal domain, so that in particular $R = R^o$. The **torsion submodule** $T(M)$ of a module M in $\mathcal{C}_R$ consists of all $m \in M$ with $rm = 0$ for some $r \neq 0$ in R.
    (a) Suppose that M is of the form $M = F \oplus T(M)$, where F is a free R module. Using the exact sequence $0 \to F \to M \to T(M) \to 0$, prove that $\operatorname{Tor}_1^R(M, B) = \operatorname{Tor}_1^R(T(M), B)$ for all modules B in $\mathcal{C}_R$.
    (b) Deduce from (a) and Problem 18 that a module M in $\mathcal{C}_R$ is flat if and only if $T(M)$ is flat. (Note that M is not assumed to be of the form $F \oplus T(M)$.)
    (c) By comparing the one-one inclusion $(a) \subseteq R$ for a nonzero $a \in R$ with the induced map from $(a) \otimes_R M$ to $R \otimes_R M$, prove that $T(M) \neq 0$ implies M not flat.
    (d) Deduce that a module M in $\mathcal{C}_R$ is flat if and only if M has 0 torsion, i.e., if and only if M is torsion free. (Educational note: In combination with the result of Problem 19, this condition explains the use of the notation "Tor" for the first derived functor of tensor product.)

Problems 21–25 deal with double chain complexes of abelian groups. A **double chain complex** is a system $\{E_{p,q}\}$ of abelian groups defined for all integers p and q and having boundary homomorphisms $\partial'_p : E_{p,q} \to E_{p-1,q}$ and $\partial''_q : E_{p,q} \to E_{p,q-1}$

such that $\partial'_{p-1,q}\partial'_{p,q} = 0$, $\partial''_{p,q-1}\partial''_{p,q} = 0$, and $\partial'_{p,q-1}\partial''_{p,q} + \partial''_{p-1,q}\partial'_{p,q} = 0$. This set of problems will assume that $E_{p,q} = 0$ if either p or q is sufficiently negative.

21. Let $\{E_{p,q}\}$ be a double complex of abelian groups with boundary homomorphisms as above, let $E_n = \bigoplus_{p+q=n} E_{p,q}$, and define $\partial_n : E_n \to E_{n-1}$ by $\partial_n\big|_{E_{p,q}} = \partial'_{p,q} + \partial''_{p,q}$. Show that the maps ∂_n make the system $\{E_n\}$ into a chain complex. (Note: The indexing on the boundary maps has been changed by 1 from earlier in the chapter in order to simplify the notation that occurs later in these problems.)

22. Let $\mathcal{C}_l$ be a good category of unital left R modules, and let $\mathcal{C}_r$ be a good category of unital left R^o modules; the latter modules are to be regarded as unital right R modules. Let $C = \{C_p\}_{p\geq-\infty}$ and $D = \{D_q\}_{q\geq-\infty}$ be chain complexes with boundary maps $\alpha_p : C_p \to C_{p-1}$ in $\mathcal{C}_r$ and $\beta_q : D_q \to D_{q-1}$ in $\mathcal{C}_l$. It is assumed that $C_p = 0$ for p sufficiently negative and that $D_q = 0$ for q sufficiently negative. Define $E_{p,q} = C_p \otimes_R D_q$, $\partial'_{p,q} = \alpha_p \otimes 1$, and $\partial''_{p,q} = (-1)^p(1 \otimes \beta_q)$. Prove that $\{E_{p,q}\}$ with these mappings is a double complex of abelian groups. (Educational note: Therefore the previous problem creates a chain complex $\{E_n\}$ with boundary maps $\partial_n : E_n \to E_{n-1}$ from this set of data. One writes $E = C \otimes_R D$ for this chain complex and calls it the **tensor product** of the two chain complexes.)

23. In the notation of the previous problem, suppose that $C_p = 0$ if $p < 0$ and $D_q = 0$ if $q < 0$. Let $Z_p = \ker \alpha_p$ and $\overline{Z}_q = \ker \beta_q$. Prove that if c is in Z_p and d is in $\overline{Z}_q$, then $c \otimes d$ is in the subgroup $\ker(\partial'_{p,q} + \partial''_{p,q})$ of $E_{p,q}$ and that as a consequence, there is a canonical homomorphism of $H^p(C) \otimes_R H^q(D)$ into $H^{p+q}(C \otimes_R D)$.

24. Suppose that a double complex E_{pq} of abelian groups has $E_{pq} = 0$ if $p < -1$ or $q < -1$ or $p = q = -1$. Suppose further that $E_{\cdot,q}$ is exact for each $q \geq 0$ and $E_{p,\cdot}$ is exact for each $p \geq 0$. Prove that the r^{th} homology of $E_{-1,q}$ as q varies matches the r^{th} homology of $E_{p,-1}$ as p varies. To do so, start from a cycle a under ∂'' in $E_{-1,k}$ with $k \geq 0$. It is mapped to 0 by ∂', hence has a preimage a' under ∂' in $E_{0,k}$. The element $\partial''a'$ in $E_{0,k-1}$ is mapped to 0 by ∂', hence has a preimage a'' in $E_{1,k-1}$. Continue in this way, and arrive at a cycle in $E_{k,0}$. Then sort out the details.

25. With notation as in Problem 22, let A be in $\mathcal{C}_r$, and let B be in $\mathcal{C}_l$. Let $C = (C^+ \to A)$ be a projective resolution of A, and let $D = (D^+ \to B)$ be a projective resolution of B. Form $E = C \otimes_R D$ as in Problem 22, and apply Problem 24 to give a direct proof (without the machinery of Section 7) that one gets the same result for $\operatorname{Tor}_n^R(A, B)$ by using a projective resolution in the first variable as by using a projective resolution in the second variable.

Problems 26–31 concern the **Künneth Theorem** for homology and the Universal Coefficient Theorem for homology. Both these results have applications to topology.

It will be assumed throughout that R is a (commutative) principal ideal domain.

STATEMENT OF KÜNNETH THEOREM. Let C and D be chain complexes over the principal ideal domain R, and assume that all modules in negative degrees are 0 and that C is flat. Then there is a natural short exact sequence

$$0 \to \bigoplus_{p+q=n} \big(H_p(C) \otimes_R H_q(D)\big) \xrightarrow{\alpha_n} H_n(C \otimes_R D)$$
$$\xrightarrow{\beta_{n-1}} \bigoplus_{p+q=n-1} \operatorname{Tor}_1^R(H_p(C), H_q(D)) \to 0.$$

Moreover, the exact sequence splits, but not naturally.

The point of the theorem is to give circumstances under which the homology of each of two chain complexes C and D determines the homology of the tensor product $E = C \otimes_R D$, the tensor product complex being defined as in Problem 22. Problem 26 below shows that some further hypothesis is needed beyond the limitation on R. A sufficient condition is that one of C and D, say C, be **flat** in the sense that all the modules in it satisfy the condition of flatness defined in Problems 17–20. The problems in the set carry out some of the steps in proving the Künneth Theorem, and then they derive the Universal Coefficient Theorem for homology as a consequence. To keep the ideas in focus, the problems will suppress certain isomorphisms, writing them as equalities.

26. With $R = \mathbb{Z}$, let $C = D$ be the chain complex with $C_0 = \mathbb{Z}/2\mathbb{Z}$ and with $C_p = 0$ for $p \neq 0$. Let C' be the chain complex with $C'_0 = \mathbb{Z}$, with $C'_1 = \mathbb{Z}$, and with $C'_p = 0$ for $p > 1$ and for $p < 0$. Let the boundary map from C'_1 to C'_0 be $\times 2$. Compute the homology of C, C', D, $C \otimes_{\mathbb{Z}} D$, and $C' \otimes_{\mathbb{Z}} D$, and justify the conclusion that the homology of each of two chain complexes does not determine the homology of their tensor product.

27. Let ∂' be the boundary map for C. Show how to set up an exact sequence

$$0 \longrightarrow Z \xrightarrow{\iota} C \xrightarrow{\partial'} B' \longrightarrow 0$$

of complexes in which each module in Z is the submodule of cycles of the corresponding module in C, ι is the inclusion, B is the complex of boundaries, and B' is B with its indices shifted by 1. Why does it follow from the fact that C is flat that Z, B, and B' are flat?

28. Explain why

$$0 \longrightarrow Z \otimes_R D \xrightarrow{\iota\otimes 1} C \otimes_R D \xrightarrow{\partial'\otimes 1} B' \otimes_R D \longrightarrow 0$$

is exact even though D is not assumed to be flat.

29. The long exact sequence in homology corresponding to the short exact sequence in the previous problem has segments of the form

$$H_{n+1}(B' \otimes_R D) \xrightarrow{\omega_n} H_n(Z \otimes_R D) \xrightarrow{\iota_n \otimes 1} H_n(C \otimes_R D) \xrightarrow{\partial'_n \otimes 1} H_n(B' \otimes_R D) \xrightarrow{\omega_{n-1}} H_{n-1}(Z \otimes_R D).$$

Let ∂'' be the boundary map for D, and let $\overline{Z}$, $\overline{B}$, and $\overline{B}'$ be the counterparts for D of the complexes Z, B, and B' for C. Show that
(a) the boundary map in $B' \otimes_R D$ may be regarded as $1 \otimes \partial''$ because the boundary map in B' is 0.
(b) $\ker(1 \otimes \partial'')_n = (B' \otimes_R \overline{Z})_n$ and $\operatorname{image}(1 \otimes \partial'')_{n+1} = (B' \otimes_R \overline{B})_n$ because B' is flat.
(c) $H_n(B' \otimes_R D) \cong (B \otimes_R H(D))_{n-1}$ because B' is flat. (This isomorphism will be treated as an equality below.)
(d) similarly $H_n(Z \otimes_R D) \cong (Z \otimes_R H(D))_n$. (This isomorphism will be treated as an equality below.)

30. Form an exact sequence

$$0 \longrightarrow B \longrightarrow Z \longrightarrow H(C) \longrightarrow 0$$

of complexes, form the low-degree part of the long exact sequence corresponding to applying the functor $(\,\cdot\,) \otimes_R H(D)$, namely

$$0 \to \operatorname{Tor}_1^R(H(C), H(D))_n \to (B \otimes_R H(D))_n \to (Z \otimes_R H(D))_n \to (H(C) \otimes_R H(D))_n \to 0,$$

and rewrite it by (c) and (d) of Problem 29 as

$$0 \to \operatorname{Tor}_1^R(H(C), H(D))_n \xrightarrow{\beta'_n} H_{n+1}(B' \otimes_R D) \xrightarrow{\omega_{n-1}} H_n(Z \otimes_R D) \xrightarrow{\alpha'_n} (H(C) \otimes_R H(D))_n \to 0.$$

(a) Why is the term $\operatorname{Tor}_1^R(Z, H(D))$ in the long exact sequence equal to 0?
(b) In the 5-term exact sequence of Problem 29, rewrite the part of the sequence centered at the map $\partial'_n \otimes 1$ in such a way that two exact sequences

$$\xrightarrow{\iota_n \otimes 1} H_n(C \otimes_R D) \xrightarrow{q} \operatorname{coker}(\iota_n \otimes 1) \longrightarrow 0$$

and

$$0 \longrightarrow \ker \omega_{n-1} \xrightarrow{i} H_n(B' \otimes_R D) \xrightarrow{\omega_{n-1}} H_{n-1}(Z \otimes_R D)$$

result. Why can the group $\ker \omega_{n-1}$ and the homomorphism i be taken to be $\operatorname{Tor}_1^R(H(C), H(D))_{n-1}$ and β'_{n-1}?
(c) Why in (b) can $\operatorname{coker}(\iota_n \otimes 1)$ and q be taken to be $\operatorname{Tor}_1^R(H(C), H(D))_{n-1}$ and some one-one homomorphism β_{n-1} such that $\beta'_{n-1}\beta_{n-1} = \partial'_n \otimes 1$?

(d) Arguing similarly with the map $\iota_n \otimes 1$ in Problem 29, obtain a factorization $\iota_n \otimes 1 = \alpha_n \alpha'_n$ in which $\alpha'_n : (Z \otimes_R H(D))_n \to (H(C) \otimes_R H(D))_n$ is onto and $\alpha_n : (H(C) \otimes_R H(D))_n \to H_n(C \otimes_R D)$ is one-one.

(e) The maps α_n and β_{n-1} having now been defined in the sequence in the statement of the Künneth Theorem, prove that the sequence is exact.

31. **(Universal Coefficient Theorem)** By specializing D in the statement of the Künneth Formula to a chain complex that is a module M in dimension 0 and is 0 in all other dimensions, obtain the natural short exact sequence

$$0 \longrightarrow H_n(C) \otimes_R M \longrightarrow H_n(C \otimes_R M) \longrightarrow \operatorname{Tor}_1^R(H_{n-1}(C), M) \longrightarrow 0,$$

valid whenever R is a principal ideal domain and C is a chain complex whose modules are all 0 in dimension < 0. (Educational note: The exact sequence splits, but not naturally.)

Problems 32–35 concern abelian categories.

32. Let $\mathcal{C}$ be an abelian category. Let $\mathcal{D}$ be the category for which $\operatorname{Obj}(\mathcal{D})$ consists of all chain complexes of objects and morphisms in $\mathcal{C}$ and for which $\operatorname{Morph}(X, Y)$ for any two objects X and Y in $\mathcal{D}$ consists of all chain maps from X to Y. Prove that $\mathcal{D}$ is an abelian category.

33. Consider the snake diagram in the category of all abelian groups consisting of the four rightmost groups in the first row and the four leftmost groups in the second row of the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 8"] \arrow[d, "\times 4"] & \mathbb{Z} \arrow[r, "1\mapsto 1\,\mathrm{mod}\,8"] \arrow[d, "\times 2"] & \mathbb{Z}/8\mathbb{Z} \arrow[r] \arrow[d, "{1\,\mathrm{mod}\,8 \mapsto 2\,\mathrm{mod}\,4}"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 4"] & \mathbb{Z} \arrow[r, "1\mapsto 1\,\mathrm{mod}\,4"] & \mathbb{Z}/4\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Adjoin the 0's to make the diagram become what is displayed. Following the steps in the example of a diagram chase in Section 8, extend this diagram to the auxiliary diagram that appears in that discussion, and show that $(B_0, \widetilde{k})$ for the extended diagram is not a kernel of β.

34. For a general abelian category $\mathcal{C}$ and any M in $\operatorname{Obj}(\mathcal{C})$, verify that $\operatorname{Hom}(\,\cdot\,, M)$ is a left exact contravariant functor from $\mathcal{C}$ to $\mathcal{C}_{\mathbb{Z}}$ and $\operatorname{Hom}(M, \,\cdot\,)$ is a left exact covariant functor from $\mathcal{C}$ to $\mathcal{C}_{\mathbb{Z}}$.

35. Proposition 4.19 shows for any good category $\mathcal{C}$ of unital left R modules that a module P in $\mathcal{C}$ is projective for $\mathcal{C}$ if and only if $\operatorname{Hom}(P, \,\cdot\,)$ is an exact functor, if and only if every short exact sequence $0 \to X \to Y \to P \to 0$ splits. Rewrite this proof in such a way that it applies to arbitrary abelian categories $\mathcal{C}$. For the step in the argument that the splitting of every short exact sequence $0 \to X \to Y \to P \to 0$ implies that P is projective, use the notion of pullback that is developed in Section 8.