Chapter 6
Gaussian Curvature Extrinsically Defined

PROBLEM 6.1. Gaussian Curvature, Extrinsic Definition

a.

Cylinder: The normal curvature is zero in the direction of the vertical generators because these generators are extrinsically straight. In the direction of the generating circles the normal curvature is $1/r$ where r is the radius of the cylinder (and therefore of the generating circles). These are the principal directions and curvatures. In other directions, the normal curvature (equal the extrinsic curvature) of the helix in that direction is (by Problem **2.5.b**) $0 < \frac{4\pi^2 r}{h^2+(2\pi r)^2} = \frac{1}{\frac{h^2}{4\pi^2 r}+r} < \frac{1}{r}$.

Cone: The minimum principal curvature is in the direction of the generators and is zero because the generators are extrinsically straight. The principal direction of maximum normal curvature is orthogonal to this and the principal curvature depends on the cone angle and the distance from the cone point. Consider p to be a point at a distance d from the cone point of a cone with cone angle α. (See Figure 6.A.)

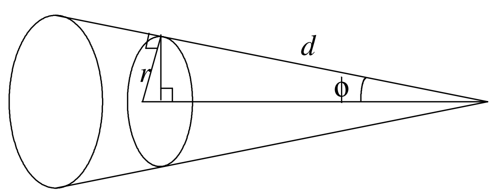

Figure 6.A. Principal curvature on a cone.

The normal curvature of the circle with intrinsic radius d and center at the cone point is $r = d\tan\phi$ and $2\pi d\sin\phi = d\alpha$, and, thus,

$$\text{normal curvature} = 1/r = (1/d)\cot(\arcsin(\tfrac{\alpha}{2\pi})) = (1/d)((\tfrac{2\pi}{\alpha})^2 - 1)^{\frac{1}{2}} \ .$$

Sphere: The normal curvature is all directions is $1/R$ and, thus, every direction is a principal direction and the principal curvature are $1/R$.

b.

From Problem **4.7.a**, we know that $\boldsymbol{\kappa}_{\mathbf{n}}(\gamma) = \langle\, \gamma\,'(0), -\gamma\,'(0)\mathbf{n}\,\rangle\, \mathbf{n}$; but, since $\gamma\,'(0)$ is also a unit tangent vector for γ^*, **4.7.a** also tells us that $\boldsymbol{\kappa}_{\mathbf{n}}(\gamma^*) = \boldsymbol{\kappa}_{\mathbf{n}}(\gamma) = \langle\, \gamma'(0), -\gamma'(0)\mathbf{n}\,\rangle\, \mathbf{n}$. Now, clearly γ^* is a planar curve and, thus, $\boldsymbol{\kappa}(\gamma^*)$ is in the plane that also contains $\boldsymbol{\kappa}_{\mathbf{n}}(\gamma^*)$. Since $\boldsymbol{\kappa}_{\mathbf{n}}(\gamma^*)$ is the projection of $\boldsymbol{\kappa}(\gamma^*)$ onto the normal to the surface $\mathbf{n}(\gamma(0))$ which is in the same plane, then

$$\boldsymbol{\kappa}(\gamma^*) = \boldsymbol{\kappa}_{\mathbf{n}}(\gamma^*) = \boldsymbol{\kappa}_{\mathbf{n}}(\gamma) = \langle\, \gamma'(0), -\gamma'(0)\mathbf{n}\,\rangle\, \mathbf{n}.$$

Examining the Figure 6.1 in the text we see that, if the curves curve away from the normal, then $\gamma'(0)\mathbf{n}$ is in the direction of $\gamma'(0)$ and, thus, $\langle\, \gamma'(0), -\gamma'(0)\mathbf{n}\,\rangle$ is negative.

***c.**

Each direction at (0,0,0) is designated by a constant value of θ, thus the normal curvature in the direction of θ_o is the normal curvature of the curve (at $r = 0$), which is the graph of the function $z = g(r) = f(\theta_o)\, r^2$. From the Theorem before Problem **2.4**, the curvature of this curve is

$$|\boldsymbol{\kappa}(a)| = \frac{|g''(a)|}{[1+(g'(a))^2]^{3/2}} = \frac{2f(\theta)}{\left[1+(2f(\theta)\,0)^2\right]^{3/2}} = 2f(\theta).$$

Thus, for $f(\theta) = (1 - \cos 4\theta)$, the minimum principal directions are when $f(\theta) = 0$ (and $\theta = 0, \pi/2, \pi, 3\pi/2$) and the maximum principal directions when $f(\theta) = 2$ (and $\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$). Thus, there is an angle of a multiple of $\pi/4$ between the minimum and maximum directions. Computer Exercise 6.1 will allow the reader to display and view these surfaces. When there is a C^2 local coordinate patch it is true that the principal directions are orthogonal, as you shall see in the next problem. Thus, it must be that this surface must not have any C^2 coordinate patch.

***d.**

Start with a sphere that contains the surface in its interior and then gradually shrink the sphere until it first touches the surface. At this point of first touching the sphere and the surface share a tangent plane and thus share an inward pointing normal. Since none of the surface is on the exterior of the sphere, every curve on the surface through the common point of tangency must curve in the direction of the normal and thus all normal curvatures are positive.

Problem 6.2. Second Fundamental Form

a.

We just expand using the bilinear properties of the Riemannian metric:

$$\kappa_{\mathbf{n}}(a\mathbf{V}_p+b\mathbf{W}_p) = \langle(a\mathbf{V}_p+b\mathbf{W}_p),(-a\mathbf{V}_p+b\mathbf{W}_p)\mathbf{n}\rangle = \langle(a\mathbf{V}_p+b\mathbf{W}_p),(-a\mathbf{V}_p\mathbf{n}+b\mathbf{W}_p\mathbf{n})\rangle =$$

$$= \langle a\mathbf{V}_p,-a\mathbf{V}_p\mathbf{n}\rangle + \langle b\mathbf{W}_p,-b\mathbf{W}_p\mathbf{n}\rangle + \langle a\mathbf{V}_p,-b\mathbf{W}_p\mathbf{n}\rangle + \langle b\mathbf{W}_p,-a\mathbf{V}_p\mathbf{n}\rangle =$$

$$= a^2\langle \mathbf{V}_p,-\mathbf{V}_p\mathbf{n}\rangle + b^2\langle \mathbf{W}_p,-\mathbf{W}_p\mathbf{n}\rangle + ab\,\langle \mathbf{V}_p,-\mathbf{W}_p\mathbf{n}\rangle + ba\,\langle \mathbf{W}_p,-\mathbf{V}_p\mathbf{n}\rangle =$$

$$= a^2\,\kappa_{\mathbf{n}}(\mathbf{V}_p) + b^2\,\kappa_{\mathbf{n}}(\mathbf{W}_p) + ab\,\langle \mathbf{V}_p,-\mathbf{W}_p\mathbf{n}\rangle + ab\,\langle \mathbf{W}_p,-\mathbf{V}_p\mathbf{n}\rangle.$$

Thus, it is important to look at the quantities such as $\langle \mathbf{V}_p,-\mathbf{W}_p\mathbf{n}\rangle$ with $\mathbf{V}_p \neq \mathbf{W}_p$.

b.

This follows directly from the fact that the Riemannian metric is bilinear and that directional derivative operator is a linear operator.

c.

Since $\mathbf{x}_1$ and $\mathbf{x}_2$ are tangent vectors, $\langle \mathbf{x}_1,\mathbf{n}\rangle = 0 = \langle \mathbf{x}_2,\mathbf{n}\rangle$. Thus, $0 = \mathbf{x}_2\langle \mathbf{x}_1,\mathbf{n}\rangle = \langle \mathbf{x}_2\mathbf{x}_1,\mathbf{n}\rangle + \langle \mathbf{x}_1,\mathbf{x}_2\mathbf{n}\rangle$ and $0 = \mathbf{x}_1\langle \mathbf{x}_2,\mathbf{n}\rangle = \langle \mathbf{x}_1\mathbf{x}_2,\mathbf{n}\rangle + \langle \mathbf{x}_2,\mathbf{x}_1\mathbf{n}\rangle$. Therefore, $\mathrm{II}(\mathbf{x}_1,\mathbf{x}_2) = \langle \mathbf{x}_1,-\mathbf{x}_2\mathbf{n}\rangle = -\langle \mathbf{x}_1,\mathbf{x}_2\mathbf{n}\rangle = \langle \mathbf{x}_{21},\mathbf{n}\rangle = \langle \mathbf{x}_{12},\mathbf{n}\rangle = -\langle \mathbf{x}_2,\mathbf{x}_1\mathbf{n}\rangle = \langle \mathbf{x}_2,-\mathbf{x}_1\mathbf{n}\rangle = \mathrm{II}(\mathbf{x}_2,\mathbf{x}_1)$. Now, using the local coordinates $\mathbf{x}(u^1,u^2)$, we can write $\mathbf{X}_p = \Sigma\, X^i\mathbf{x}_i(a,b)$ and $\mathbf{Y}_p = \Sigma\, Y^j\mathbf{x}_j(a,b)$.

Thus, $\mathrm{II}(\mathbf{X}_p,\mathbf{Y}_p) = \langle \Sigma_i\, X^i\mathbf{x}_i(a,b),-(\Sigma_j\, Y^j\mathbf{x}_j(a,b))\mathbf{n}\rangle = \langle \Sigma_i X^i\mathbf{x}_i(a,b),-\Sigma_j\,(Y^j\mathbf{x}_j(a,b)\mathbf{n})\rangle =$

$$= \Sigma_i\Sigma_j\langle X^i\mathbf{x}_i(a,b),-(Y^j\mathbf{x}_j(a,b)\mathbf{n})\rangle = \Sigma_i\Sigma_j X^iY^j\langle \mathbf{x}_i(a,b),-(\mathbf{x}_j(a,b)\mathbf{n})\rangle =$$

$$= \Sigma_i\Sigma_j X^iY^j\mathrm{II}(\mathbf{x}_i,\mathbf{x}_j) = \Sigma_j\Sigma_i\, Y^jX^i\mathrm{II}(\mathbf{x}_j,\mathbf{x}_i) = \mathrm{II}(\mathbf{Y}_p,\mathbf{X}_p).$$

This is the only place that the assumption C^2 is used in a crucial way in all its power (that is, C^2 requires that all first and second partial derivatives exists and are continuous *and* that the mixed partials are equal).

d.

Part **d** may be solved in at least three ways:

1. Analysis. Using the theory of Lagrange multipliers, we wish to maximize/minimize II(**X**,**X**) subject to the constraint that $\langle \mathbf{X},\mathbf{X}\rangle = 1$. Write the variable vector $\mathbf{X} = X^1\mathbf{x}_1+X^2\mathbf{x}_2$, then the maximum/minimum exists when, for $i = 1,2$, (using Part **c**)

$$0 = \frac{\partial}{\partial X^i}[\langle (X^1\mathbf{x}_1 + X^2\mathbf{x}_2), -(X^1\mathbf{x}_1 + X^2\mathbf{x}_2)\mathbf{n}\rangle + \lambda(\langle X^1\mathbf{x}_1 + X^2\mathbf{x}_2, X^1\mathbf{x}_1 + X^2\mathbf{x}_2\rangle - 1)] =$$

.

The only way that a tangent vector can be perpendicular to both $\mathbf{x}_1$ and $\mathbf{x}_2$ is for the tangent vector to be equal to **0**. Thus, the minimum/maximum **T** are when $-\mathbf{Tn}+\lambda\mathbf{T}= \mathbf{0}$ or $-\mathbf{Tn}= \lambda\mathbf{T}$.

2. Linear algebra. Find local coordinates, $\mathbf{x}(U^1,U^2)$, such that at $p = \mathbf{x}(0,0)$ we have that the coordinate vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthonormal. (This can be done, for example, by constructing geodesic rectangular coordinates starting at p.) With these coordinates the Riemannian metric at p is just the dot product and thus, expressing II(**X**,**X**) in terms of these coordinates,

$$\mathrm{II}(\mathbf{X},\mathbf{X}) = \langle X^1\mathbf{x}_1 + X^2\mathbf{x}_2, -(X^1\mathbf{x}_1 + X^2\mathbf{x}_2)\mathbf{n}\rangle = (X^1, X^2)\begin{pmatrix} \mathrm{II}(\mathbf{x}_1,\mathbf{x}_1) & \mathrm{II}(\mathbf{x}_1,\mathbf{x}_2) \\ \mathrm{II}(\mathbf{x}_2,\mathbf{x}_1) & \mathrm{II}(\mathbf{x}_2,\mathbf{x}_2) \end{pmatrix}\begin{pmatrix} X^1 \\ X^2 \end{pmatrix}.$$

The matrix is symmetric (by Part **c**) and is just the matrix of the linear transformation $f(\mathbf{X}) = -\mathbf{Xn}$. From the theory of eigenvalues and eigenvectors of symmetric matrices, the maximum and minimum values of II(**X**,**X**) are in the directions of the eigenvectors of the linear transformation f. But an eigenvector **T** of f satisfies $f(\mathbf{T}) = -\mathbf{Tn} = \lambda\mathbf{T}$.

3. Geometry. We are minimizing/maximizing the function II(**T**,**T**) on the unit circle and thus, II(**T**,**T**) is an extremum if the directional derivative in a direction tangent to the circle is zero. The direction tangent to the circle at the unit vector **T** can be represented by $\mathbf{T}^\perp$, a unit vector which is perpendicular to **T**. Thus, $0 = \frac{d}{dh}\mathrm{II}(\mathbf{T}+h\mathbf{T}^\perp, \mathbf{T}+h\mathbf{T}^\perp)|_{h=0} = \frac{d}{dh}\langle \mathbf{T} + h\mathbf{T}^\perp, -(\mathbf{T} + h\mathbf{T}^\perp)\mathbf{n}\rangle|_{h=0} =$

.

But this can be zero only if $-\mathbf{Tn}$ is parallel to **T**, that is $-\mathbf{Tn} = \lambda\mathbf{T}$.

e.

Consider $\mathrm{II}(\mathbf{T}_1, \mathbf{T}_2) = \langle \mathbf{T}_1,-\mathbf{T}_2\mathbf{n}\rangle = \langle \mathbf{T}_1,\lambda_2\mathbf{T}_2\rangle = \lambda_2\langle \mathbf{T}_1,\mathbf{T}_2\rangle$, and $\mathrm{II}(\mathbf{T}_1, \mathbf{T}_2) = \mathrm{II}(\mathbf{T}_2, \mathbf{T}_1) = \langle \mathbf{T}_2,-\mathbf{T}_1\mathbf{n}\rangle = \langle \mathbf{T}_2,\lambda_1\mathbf{T}_1\rangle = \lambda_1\langle \mathbf{T}_2,\mathbf{T}_1\rangle = \lambda_1\langle \mathbf{T}_1,\mathbf{T}_2\rangle$. Thus, we have shown that $\lambda_1\langle \mathbf{T}_1,\mathbf{T}_2\rangle = \lambda_2\langle \mathbf{T}_1,\mathbf{T}_2\rangle$. This implies that either $\lambda_1 = \lambda_2$ (in which case II(**T**,**T**) is constant) or $\langle \mathbf{T}_1,\mathbf{T}_2\rangle = 0$ (in which case $\mathbf{T}_1$ is perpendicular to $\mathbf{T}_2$). In the later case, there can be no other directions with $-\mathbf{Tn} = \lambda\mathbf{T}$ except for $-\mathbf{T}_1$ and $-\mathbf{T}_2$ and these do not give additional values since $\mathrm{II}(-\mathbf{T},-\mathbf{T}) = \mathrm{II}(\mathbf{T},\mathbf{T})$; thus, $\mathbf{T}_1$ and $\mathbf{T}_2$ must be the principal directions.

f.

It is clear that in the z-direction, $-\mathbf{Tn} = \lambda\mathbf{T}$, because the curves of constant θ are planar curves with the normal **n** contained in the plane. Thus, from **6.2.d**, at each point, the principal directions are the z-direction and the θ-direction.

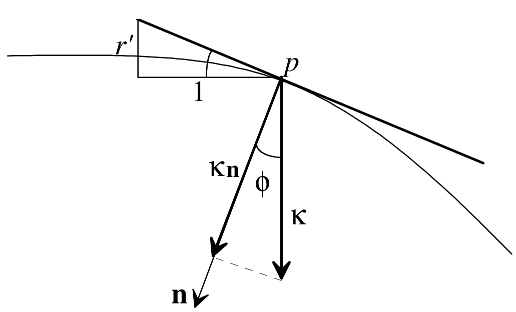

Figure 6.B. Principal curvatures on a surface of revolution.

Now use the section *Curvature of the Graph of a Function*, just before Problem **2.4** to calculate the (extrinsic) curvatures of the generating curves. In the z-direction we get $\kappa = \dfrac{-r''(z)}{[1+(r'(z))^2]^{\frac{3}{2}}}$. But this extrinsic curvature is in the direction normal to the surface and so it is also the normal curvature. Then in the θ-direction, the curve is a circle of radius $r(z)$ and thus, the extrinsic curvature is $1/r(z)$. Examining Figure 6.B we see that the normal curvature (which is the projection onto the normal direction) is $\kappa_2 = \dfrac{1}{r(z)}\cos\phi = \dfrac{1}{r(z)\sqrt{1+(r'(z))^2}}$.

To summarize the above discussion: The directions, $\mathbf{T}_1$, $\mathbf{T}_2$, in which the maximum and minimum of $\mathrm{II}(\mathbf{T},\mathbf{T})$ occur are called the ***principal directions at*** p and the values of $\mathrm{II}(\mathbf{T},\mathbf{T})$ in these directions, κ_1, κ_2, are called ***the principal curvatures at*** p. Note that, κ_1, κ_2, are (by Problem **5.1**) the normal curvatures of unit speed curves in the principal directions. The product $\kappa_1\kappa_2$ is called the ***Gaussian curvature at*** p.

Problem 6.3. The Gauss Map

a.

The spherical image of a cylinder is always a great circle. The spherical image of any sphere is the whole sphere.

The spherical image of a cone with cone angle α will be the latitude circle which is tangent to the cone if you place the cone over the sphere of radius 1. The circumference of this latitude circle is $2\pi\sin\phi$ and looking at the cone is also $\alpha\tan\phi$. Thus, the angle of this latitude measured from the North Pole is $\cos^{-1}(\alpha/2\pi)$, see Figure 6.C.

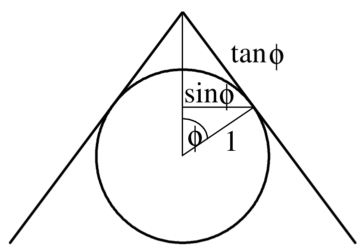

Figure 6.C. Spherical image of a cone.

On a strake, a helix makes an angle with the vertical equal to $\arctan(2\pi r/h)$. The normal has a direction perpendicular to the direction of the helix and thus, has angle to the vertical of $\arctan(h/(2\pi r))$. Thus, one turn of the strake with height h and inner radius r and outer radius $r+\delta$ is the annular region between the latitudes of angles (measured from the pole) $\arctan\frac{h}{2\pi r}$ and $\arctan\frac{h}{2\pi(r+\delta)}$.

On the torus, each of the regions congruent to either A or B are mapped by the Gauss map onto a quarter of the sphere. The union of all A's is mapped onto the whole sphere and the union of all four B's is also mapped onto the whole sphere but in the opposite sense.

b.

If $\mathbf{V}$ is a vector tangent to the surface at $\gamma(s)$ then $\mathbf{V}$ is also tangent to the sphere at $\mathbf{n}(\gamma(s))$, because it is perpendicular in each case to the normal. Thus, $\mathbf{P}(s)$ is a (tangent) vector field along $\mathbf{n}(\gamma(s))$. Now, since $\mathbf{P}(s)$ is a parallel vector field along γ we know that the rate of change of $\mathbf{P}(s)$ is in the direction of the normal to the surface at $\gamma(s)$. Along $\mathbf{n}(\gamma)$, s is not arclength. If we let t represent the arclength along $\mathbf{n}(\gamma)$, then $t(s) = \int_0^s |(\mathbf{n}\circ\gamma)'(s)|\,ds$ and $\frac{dt}{ds} = |(\mathbf{n}\circ\gamma)'(s)|$ and thus $\frac{d}{dt}\mathbf{P}(t) = \frac{d}{ds}\mathbf{P}(s)\frac{ds}{dt} = \frac{d}{ds}\mathbf{P}(s)/|(\mathbf{n}\circ\gamma)'(s)|$, which has the same direction as $\frac{d}{ds}\mathbf{P}(s)$, which is in the direction of the normal to the sphere at $\mathbf{n}(\gamma(s))$; and thus $\mathbf{P}(s)$ is also parallel along $\mathbf{n}(\gamma)$. If $(\mathbf{n}\circ\gamma)'(s) = 0$, then the normal curvature $(\langle\gamma', \gamma'\mathbf{n}\rangle)$ at $\gamma(s)$ is zero.

c.

If γ is simple and small enough that its interior region on the surface is mapped by the Gauss map into an open hemisphere, then, when we parallel transport a vector around γ, the angle between the vector and its parallel transport will be the same on the surface as on the sphere, which means that the measure of the angle (which is the holonomy) is the same up to $2\pi k$ (k an integer) on the sphere and the surface. Now, the same will be true as we shrink (homotopy) the surface curve in its interior. Eventually we can shrink the curve into a very small region on the surface that is very close to being planar and in this region the holonomy must be near zero and thus, for this very small curve $k = 0$. But, we shrunk the curve continuously so k must vary continuously and since k is an integer this means that k must be constant (in this case the constant 0).

d.

Using Problem **6.2.d** we see that $\frac{d}{ds}\mathbf{n}(\gamma(s)) = |\lambda'(s)|\mathbf{Tn}(s) = |\lambda'(s)|(-\kappa_1\mathbf{T}) = -\kappa_1\lambda'(s)$.

PROBLEM 6.4. Gauss-Bonnet and Intrinsic Curvature

a.

We use the same notion of "small" as in Problem **6.3.c**. We have $\text{Area}(\mathbf{n}(R)) = \iint_{\mathbf{n}(R)} dA$. We now will make a change of variables using the Gauss map **n**. In a neighborhood of the point p in a small region R in M choose an infinitesimal rectangle at p with sides dx and dy chosen so they are in the principal directions at p. Then the image of this rectangle on the sphere will have sides in the same directions with lengths $\kappa_1 dx$ and $\kappa_2 dy$, according to Problem **6.3.d**. Then $\text{Area}(\mathbf{n}(R)) = \iint_{\mathbf{n}(R)} dA = \iint_R \kappa_1\kappa_2\, dA$.

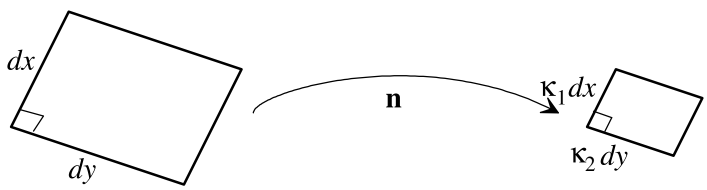

Figure 6.D. The Gauss map transforms an element of area.

b.

Use the same notion of "small" region as before. We know from Problems **5.3** and **5.4.d** that, looking at $\mathbf{n}(\gamma)$, $\mathcal{H}(\mathbf{n}(R)) = 2\pi - \int_{\mathbf{n}(\gamma)} \kappa_g\, ds - \Sigma\alpha_i = \text{Area}(\mathbf{n}(R)) = \iint_R \kappa_1\kappa_2\, dA$. Now, Problem **6.3.c** gives us $\mathcal{H}(R) = \mathcal{H}(\mathbf{n}(R))$ and Problem **5.4.d** gives us $\mathcal{H}(R) = 2\pi - \int_\gamma \kappa_g\, ds - \Sigma\alpha_i$. Note also that the angular measures ($\int_\gamma \kappa_g\, ds$ and α_i) along γ on the surface are the same as the measures ($\int_{\mathbf{n}(\gamma)} \kappa_g\, ds$ and α_i) along $\mathbf{n}(\gamma)$ on the sphere. It is possible that $\mathbf{n}(\gamma)$ is not a simple curve (for example, if γ is around a point with zero curvature on the torus -- Try it!) but this is OK because part of the area enclosed by $\mathbf{n}(\gamma)$ will be negative and part will be positive.

c.

We calculate $K(p) = \lim_{n\to\infty} \mathcal{H}(R_n) / A(R_n) = \lim_{n\to\infty} \dfrac{\iint_{R_n} \kappa_1\kappa_2\, dA}{\iint_{R_n} dA} = \kappa_1\kappa_2$.

d.

Since the Gaussian curvature is equal to the intrinsic curvature it is intrinsic and thus, does not depend on the embedding.

PROBLEM 6.5. 2nd Fundamental Form in Coordinates

a.

This follows directly from the bilinearity of the second fundamental form and Problem **6.2.c**.

b.

The off-diagonal entries are $\mathrm{II}(\mathbf{T}_1,\mathbf{T}_2) = \mathrm{II}(\mathbf{T}_2,\mathbf{T}_1)$, which are equal to 0 by the proof of **6.2.e**.

We see that $\mathbf{T}(\theta) = \cos\theta\ \mathbf{T}_1 + \sin\theta\ \mathbf{T}_2$ and thus, we can use Parts **a** and **b** to calculate $\kappa_{\mathbf{n}}(\mathbf{T}(\theta)) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$. Note on a sphere the normal curvature is the same in all directions and thus, any orthogonal local coordinates on the sphere will have its Second Fundamental Form matrix be a diagonal matrix. This is also true for the standard local coordinates on the cylinder and cone. However, it is not true for the standard local coordinates on the strake.

***c.**

Note that the tangent vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are *not* partial derivatives of f. See Problem **4.4** and its solution for the expressions of $\mathbf{x}_i$ and $\mathbf{n}$. We now calculate

$$\mathbf{x}_{11}(a,b) = (0,0,f_{xx}(a,b)),\ \ \mathbf{x}_{22}(a,b) = (0,0,f_{yy}(a,b)),\ \mathbf{x}_{21}(a,b) = (0,0,f_{yx}(a,b)) = (0,0,f_{xy}(a,b)) = \mathbf{x}_{12}(a,b).$$

Thus, the matrix of the second fundamental form is

$$\begin{pmatrix} \langle \mathbf{x}_{11}(a,b), \mathbf{n}(a,b)\rangle & \langle \mathbf{x}_{12}(a,b), \mathbf{n}(a,b)\rangle \\ \langle \mathbf{x}_{21}(a,b), \mathbf{n}(a,b)\rangle & \langle \mathbf{x}_{22}(a,b), \mathbf{n}(a,b)\rangle \end{pmatrix} = \frac{1}{\sqrt{1+(f_x)^2+(f_y)^2}} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}.$$

*Problem 6.6. Mean Curvature and Minimal Surfaces

a.

The (extrinsic) curvature of the helix is towards the central axis and is tangent to the surface, and thus the normal curvature is zero. The horizontal lines on the strake are extrinsically straight, and thus have no normal curvature. Since we know that the strake is not locally isometric to the plane, it is not possible that all the maximum or minimum normal curvatures are zero. If κ_1 and κ_2 are the principal curvatures and the principal directions are at angles θ and $\theta-\pi/2$ from the horizontal lines, then (since 0 can not be the minimum nor maximum) we can suppose that $\kappa_1 < 0 < \kappa_2$. By Problem **6.5.b** we have

$$\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta = 0 = \kappa_1 \cos^2(\theta-\pi/2) + \kappa_2 \sin^2(\theta-\pi/2) = \kappa_1 \sin^2\theta + \kappa_2 \cos^2\theta$$

and thus, $\left|\frac{\kappa_1}{\kappa_2}\right| = \tan^2\theta = \cot^2\theta$. This can only happen if $\theta = \pi/4$ and $\kappa_1 = -\kappa_2$. Thus, the strake has mean curvature zero.

b.

Use local orthonormal coordinates (x, y) in the principal directions. In each of the principal directions draw a picture of the osculating circle with radius of 1/(normal curvature); then we have the picture in Figure 6.3 in the text, an we can calculate the derivative

$$\tfrac{d}{dh} l_h = \lim_{h\to 0} \frac{l_h - l_0}{h} = \lim_{h\to 0} \frac{\frac{(1/\kappa)-h}{1/\kappa}dx - dx}{h} = \lim_{h\to 0} \frac{(1-\kappa h)dx - dx}{h} = -\kappa\, dx\ .$$

Then set $dA = dx\ dy$ and let $A(h)$ be the area after dA is pushed a distance h in the direction of $\mathbf{n}$. Calculate

$$\tfrac{d}{dh} A = \lim_{h\to 0} \frac{A(h) - dA}{h} = \lim_{h\to 0} \frac{(1-\kappa_1 h)dx\,(1-\kappa_2 h)dy - dxdy}{h} =$$
$$= \lim_{h\to 0} \frac{(-\kappa_1 h - \kappa_2 h + \kappa_1\kappa_2 h^2)dxdy}{h} = (-\kappa_1 - \kappa_2)dxdy = -2H\,dA.$$

c.

We calculate using Problem **6.2.f** with $r(z) = (1/a)\cosh(az + b)$,

$$\kappa_1 = \frac{-r''(z)}{[1+(r'(z))^2]^{\frac{3}{2}}} = \frac{-a\cosh(az+b)}{[1+\sinh^2(az+b)]^{\frac{3}{2}}} = \frac{-a\cosh(az+b)}{[\cosh^2(az+b)]^{\frac{3}{2}}} = \frac{-a}{\cosh^2(az+b)}\ ,$$

$$\kappa_2 = \frac{1}{r(z)\ \sqrt{1+(r'(z))^2}} = \frac{a}{\cosh(az+b)\sqrt{1+\sinh^2(az+b)}} = \frac{a}{\cosh^2(az+b)}\ .$$

Thus, $H = 0$ and the catenoid is a minimal surface.

d.

In order for this surface of revolution to be a minimal surface the principal curvatures must be equal in magnitude and opposite in sign. Thus, we must have $\dfrac{r''(z)}{[1+(r'(z))^2]^{\frac{3}{2}}} = \dfrac{1}{r(z)\ \sqrt{1+(r'(z))^2}}$, or, simplifying, $r''(z)\,r(z) = [1+(r'(z))^2]$. This second order, non-linear differential equation has a unique solution for given initial conditions, $r(0)=c$ and $r'(0)=d$. In order for it to be a surface of revolution of the above form it must be that $c > 0$. It is easy to check that $r(z) = (1/a)\cosh(az + b)$ is a solution for all b and all $a > 0$, and for this $r(z)$, $c = r(0) = \frac{1}{a}\cosh(b)$ and $d = r'(0) = \sinh(b)$. Thus, $b = \sinh^{-1}(d)$, which is defined for all d; and $a = (1/c)\cosh(\sinh^{-1}(d))$, which is also defined for all d and all $c > 0$.

e.

First we express both the catenoid and the helicoid in geodesic rectangular coordinates. For the catenoid, set $b = 0$ and let the base curve be $z = 0$. The base curve, parametrized by arclength is $w \to (\frac{1}{a}\cos aw, \frac{1}{a}\sin az, 0)$. The second coordinate curve is $z \to (\frac{1}{a}\cosh az,\ z)$, which we must now parametrize by arclength. First we find the element of arclength

$$ds = \sqrt{dz^2 + d(\tfrac{1}{a}\cosh az)^2} = \sqrt{dz^2 + (\sinh^2 az)dz^2} = \sqrt{1+\sinh^2 az}\ dz = \cosh az\ dz.$$

Thus, the arclength parameter is $s = \int_0^z \cosh az\ dz = \frac{1}{a}\sinh az$ or $z = \frac{1}{a}\sinh^{-1} as$. Note that

$$\cosh\theta = \sqrt{1+\sinh^2\theta} \quad \text{and, thus,} \quad \cosh(\sinh^{-1} as) = \sqrt{1+(as)^2}\,.$$

Thus the geodesic rectangular coordinates for the catenoid are

$$\mathbf{x}(w,s) = \left(\tfrac{1}{a}\sqrt{1+(as)^2}\ \cos aw, \tfrac{1}{a}\sqrt{1+(as)^2}\ \sin aw, \tfrac{1}{a}\sinh^{-1} as\right).$$

Now calculating the Riemannian metric for the catenoid:

$$\mathbf{x}_1(w,s) = \left(-\sqrt{1+(as)^2}\ \sin aw, \sqrt{1+(as)^2}\ \cos aw, 0\right)$$
$$\mathbf{x}_2(w,s) = \left(\tfrac{as}{\sqrt{1+(as)^2}}\cos aw, \tfrac{as}{\sqrt{1+(as)^2}}\sin aw, \tfrac{1}{\sqrt{1+(as)^2}}\right)$$
$$g_{11}(w,s) = 1+(as)^2,\ g_{12}(w,s) = g_{21}(w,s) = 0,\ g_{22}(w,s) = 1$$

Now we give the geodesic rectangular coordinates for the helicoid with the center line as the base curve: $\mathbf{y}(s,r) = (r\cos\frac{s}{h}, r\sin\frac{s}{h}, s)$. Then calculating the Riemannian metric we get

$$\mathbf{y}_1(s,r) = (-\tfrac{r}{h}\sin\tfrac{s}{h}, \tfrac{r}{h}\cos\tfrac{s}{h}, 1), \quad \mathbf{y}_2(s,r) = (\cos\tfrac{s}{h}, \sin\tfrac{s}{h}, 0)$$
$$g_{11}(s,r) = 1+(\tfrac{r}{h})^2,\ g_{12}(s,r) = g_{21}(s,r) = 0,\ g_{22}(s,r) = 1.$$

Thus, the catenoid and the helicoid are locally isometric if $a = 1/h$ and we map $\mathbf{x}(w,s)$ to $\mathbf{y}(w,s)$.