Chapter 2

Extrinsic Curves

Introduction

The starting point of our extrinsic investigations is *views of space*. As is our ordinary experience of space, only a certain bounded region of space is within our ***field of view*** (***f.o.v.***) and within this field of view there are details too small to be ***distinguished***. This makes sense in our experience of physical space, in computer graphics images, in fixed-point and floating point arithmetic, and also applies to the spaces of our imagination.

Borrowing from the idea of zooms in photography and computer graphics, when we want to investigate more detail of a figure we may ***zoom in on a point***. Then less of the extent of space is included in our field of view but more details are now distinguishable.

We call a figure in our f.o.v. a ***point*** if it does not have two parts which are distinguishable from each other. [Note the connection between this notion and Euclid's definition "A point is that which has no parts".] We say that two figures in the f.o.v. are indistinguishable if each point of the first figure is indistinguishable from some point of the second figure, and each point of the second figure is indistinguishable from some point of the first figure.

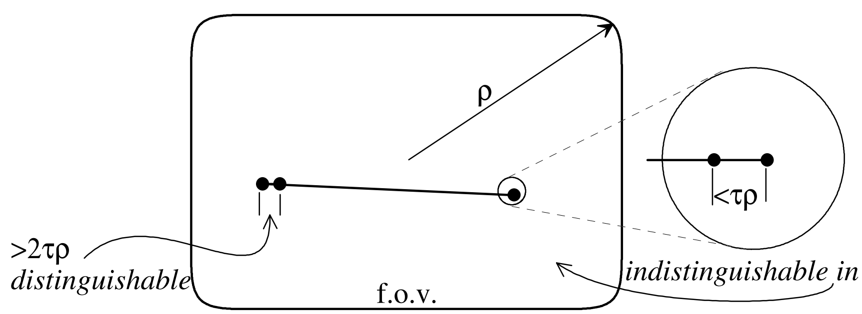


Figure 2.1. Tolerance τ and radius ρ in a f.o.v.

For simplicity we shall assume that we see all parts of this field of view with equal clarity. (That is, we ignore the so-called peripheral vision, which is a region at the edge of our field of view where we can see less detail than in the center of the field of view.) Two distinguishable points in a field of view determine a line segment. In general we can subdivide this segment into 2 parts, 3 parts, 4 parts, et cetera, until each part becomes so small that it is indistinguishable from a point. We can quantify the ***tolerance*** of our vision in a field of view as the ratio $\tau > 0$ such that (see Figure 2.1):

- Every segment (in the field of view) is ***indistinguishable from a point*** if it has length less than $\tau\rho$, where ρ is the radius of the f.o.v.
- Every segment (in the field of view) is ***distinguishable from a point*** if it has length greater than $2\tau\rho$, where ρ is the radius of the f.o.v.

This indeterminate range between $\tau\rho$ and $2\tau\rho$ is convenient because in many situations the border line between distinguishable and indistinguishable is fuzzy. Also, in pixel graphics, we need to take into account the fact that the centers of square pixels are further from the adjacent pixels on the diagonal than from the pixels that are adjacent vertically or horizontally.

We can define ***limits in a f.o.v.*** by asserting that the sequence $\{x_n\}$ converges to y if eventually x_n is indistinguishable from y.

Problem 2.1. Give Examples of F.O.V.'s

a. *Find examples of f.o.v.'s from your experiences. Use them to illustrate the ideas in the* ***Introduction*** *section above.*

b. *Can you describe a f.o.v. that contains all three of* \$0, \$1, *and the national debt as distinguishable points? Can the quantity*

$$\{[\text{national debt}] + \$1\}$$

be distinguished from

$$[\text{national debt}]?$$

Answer the same questions with "\$1" *replaced by* "a grain of sand" *and* "national debt" *replaced by a* "truck load of sand."

c. *Look at the real numbers in the f.o.v. of 4-digit arithmetic (fixed point).* That is, if r is a real number, then in the f.o.v. we see it as $[r]$ which is either a 4-digit non-negative integer or ∞. For example,

$$[0.2] = 0,\ [1.5] = 2,\ [2.499999] = 2,$$
$$[2.500001] = 3,\ [9999] = 9999,\ [10{,}001] = \infty.$$

If $\{p_n\}$ *is a sequence of reals converging to* q*, then can we be sure that eventually* $[p_n]$ *is equal to* $[q]$*? What about* $[p_n - q]$*? How should we define the tolerance in this f.o.v. so that*

$$\{x_n\} \to y \textit{ implies that } \{[x_n]\} \to [y]?$$

Warning: It is not in general true that $p + q = r$ implies $[p] + [q] = [r]$.

***d**† *How does* **c** *change if we use the f.o.v. of 4-digit floating point arithmetic?*

***e.** *It is meaningful for me to say "my house is one kilometer from White Hall." Is that the same as saying "my house is 1000 meters from White Hall"? Explain.*

For a discussion of related issues that come up in real number computations, see Peter R. Turner, "Will the 'Real' Real Arithmetic Please Stand Up?" [**RN**: Turner] and the book [**RN**: Moore] which exposits *interval analysis*, "an approach to computing that treats an interval as a new kind of number."

Archimedian Property

We further assume that the possible f.o.v.'s are restricted so that if v and u are any two f.o.v.'s then there is a third f.o.v. w in which both v and u are finite zooms; that is, the diameters and tolerances of

† Problems and Sections preceded by an asterisk (*) are not essential later in this book.

both v and u are finitely related to the diameter and tolerance of w. (Two lengths, or ratios, $a < b$ are ***finitely related*** if there is a positive integer N such that

$$Na \leq b < (N+1)a \quad \text{or} \quad b/(N+1) < a \leq b/N.$$

This property can be called the ***Archimedian Property***, and we call the space with its f.o.v.'s an ***Archimedian space***. The Archimedian Property effectively rules out infinitesimal and infinite lengths and ratios. The standard real numbers usually studied in mathematics classes satisfy this property. However there are several "non-standard reals" that do not. See, for example, [**RN**: Conway]; [**RN**: Laugwitz]; and [**RN**: Simpson].

In an Archimedian space we can define the notion of a limit:

Definition: If y and x_n (n = 1,2,...) are in a f.o.v. (not necessarily distinguishable – this is merely the requirement that y and the x_n are not infinite), then

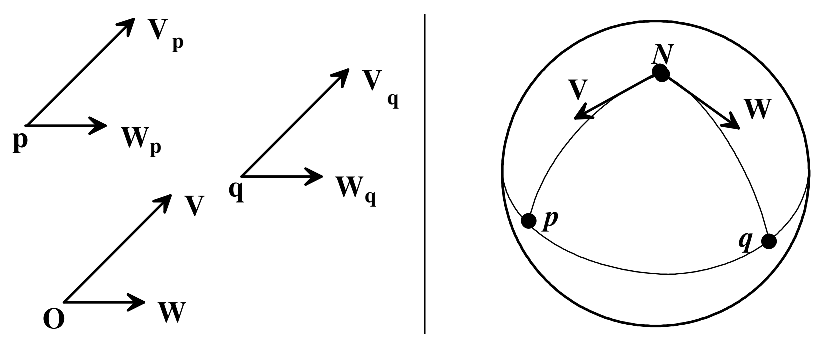
$$\lim_{n\to\infty} x_n = y$$

if x_n is eventually indistinguishable from y in every f.o.v. containing y.

This definition is equivalent to the standard analytic definition as the interested reader can check.

Vectors and Affine Linear Space

It is assumed, for now, that our discussions take place within an Archimedian ***Euclidean space*** wherein any two distinguishable points, **p** and **q**, determine a unique straight line segment. These straight line segments (we often drop the word "straight") are distinguished geometrically by local symmetries in their neighborhoods as discussed in Problem **1.1**. If **p** is any point, then the collection of directed line segments with **p** as the initial end point are the vectors of a ***vector space*** with **p** as its origin, which we call the ***tangent space at*** **p**, $T_{\mathbf{p}}$. When the linear algebraic properties of this Euclidean space are being emphasized, it is usually called an ***geometric affine space***. For a detailed discussion of affine linear spaces see [**LA**: Dodson] and the *Appendix A: Linear Algebra from a Geometric Point of View*. When we pick one point as the origin **O**, then the tangent space $T_{\mathbf{O}}$ is usually considered as the vector space $\mathbf{R}^n$, which is the domain of normal linear algebra.



Global parallelism on plane. **How can you pick Vp, Vq, Wp, Wq?**

Figure 2.2.

In our Euclidean space there is a global notion of parallelism, so we are able to make the following definition:

If **V** is a vector in $\mathbf{R}^n$ (= $T_{\mathbf{O}}$), then we denote by $\mathbf{V}_{\mathbf{p}}$ the unique vector in $T_{\mathbf{p}}$ which is parallel to **V** (two vectors are said to be parallel if they have the same length and direction). Often, we do not even write the subscript and just call them both **V**. (See Figure 2.2.)

It is precisely this global notion of parallelism that is missing on almost all surfaces, except for a few so-called "parallelizable" surfaces, such as a cylinder. To get an experience of this phenomenon the reader should look at the North Pole and two points 90° apart on the equator. However, in Chapter 5 we will define the notion of ***parallel transport along a curve***, which makes sense on (almost) any surface.

Problem 2.2. Smoothness and Tangent Directions

An ***infinitesimally straight curve*** in a Euclidean space is a subset in which for every point **p** on the curve there is a tangent line $T_{\mathbf{p}}$ such that, for every tolerance if you zoom in on **p**, the (orthogonal) projection of the curve onto the tangent line moves points indistinguishably. We shall call it a ***smooth curve*** if the amount of zooming necessary is *uniform* in the sense that, for each tolerance τ, there is a radius ρ such that, if we center a f.o.v. of radius less than ρ and tolerance τ at any point **p** on the curve, then within that f.o.v. the projection of the curve to the tangent line at **p** moves points indistinguishably.

It is possible for a smooth curve to have high enough curvature at **p** so that it does not look smooth at **p** in a particular f.o.v. Conversely, a physical curve may *look smooth* in a f.o.v. but is never ***infinitesimally straight*** because if you zoom in enough the smoothness will disappear.

a. *Look at the curves that are the graphs of the following functions:*

$$\sqrt{10^{-30}+x^2}\ ;\ \frac{x}{\sqrt{10^{-30}+x^2}}\ ;\ |x|\ ;\ x+10^{-15}\frac{x}{|x|}.$$

At which points are they indistinguishable from a straight line in the f.o.v. with tolerance 10^{-4} *and bounds* $-1 < x,y < 1$ (*or in the f.o.v. of a computer graphing program*)*? What about in other f.o.v.'s?*

[Hint: Actually view the graphs of these functions on a computer graphing program such as *Analyzer**©.]

We are thinking of a curve as a geometric object. However, it is sometimes useful to study the geometric curve by representing the curve analytically. There are two main ways of analytically representing a curve — as the graph of a function or as a parametrized curve. In first-year calculus a curve in the x-y plane is normally represented (either implicitly or explicitly) as ***the graph of a function*** $y = f(x)$. Then the derivative $f'(a)$ gives the slope of the line tangent to the graph at the point $(a, f(a))$.

b. *Prove that if the function f is differentiable at p, then the graph* $(x, f(x))$ *is infinitesimally straight at the point* $x = p$.

[Hint: If

$$t(x) = f(\mathrm{p}) + f'(p)(x - p)$$

is the equation of the line tangent to the curve $(x, f(x))$ at the point $(p, f(p))$, then

$$f(x) - t(x) = \{\ [f(x) - f(p)]/(x - p) - f'(p)\ \}(x - p).$$

The curve is infinitesimally straight if, for given tolerance ε, there is a

$$\delta > 0 \text{ (the radius of the zoom window)}$$

such that, for all f.o.v.'s with radius $< \delta$,

$$|x - p| < \delta \ (x \text{ within the zoom window}),$$

it is true that

$$|f(x) - t(x)| < \varepsilon\delta \ (f(x) \text{ is indistinguishable from } t(x)).$$

In general, the value of δ might depend on p as well as on ε.]

c. *Prove that if the graph* $(x, f(x))$ *is infinitesimally straight at the point* $x=p$ *with non-vertical tangent line, then the function f is differentiable at p. Why is the restriction on the tangent line necessary?*

[Hint: Express the tangent line as

$$t(x) = f(\mathrm{p}) + r(x - p)$$

(r is a real number) and pick a f.o.v. with radius equal to $x - p$.]

We now investigate the geometric difference between differentiable (infinitesimally straight) and continuously differentiable.

***d.** *Show that the function*

$$y = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable everywhere but that the derivative is not continuous at the origin. Also, show that its graph is a curve that is infinitesimally straight but not uniformly so in every neighborhood of 0.

[Hint: Investigate this graph with the help of some function graphing program, such as *Analyzer**©.]

Part **d** suggests the following result:

***e.** *Prove that a function is continuously differentiable (on closed finite intervals) if and only if its graph has no vertical tangents and is smooth in the sense that, for each tolerance* $\varepsilon > 0$, *there is one* $\delta > 0$ *that works for all p.*

We can also show as a corollary that

A smooth curve can always locally be written as a graph of a function.

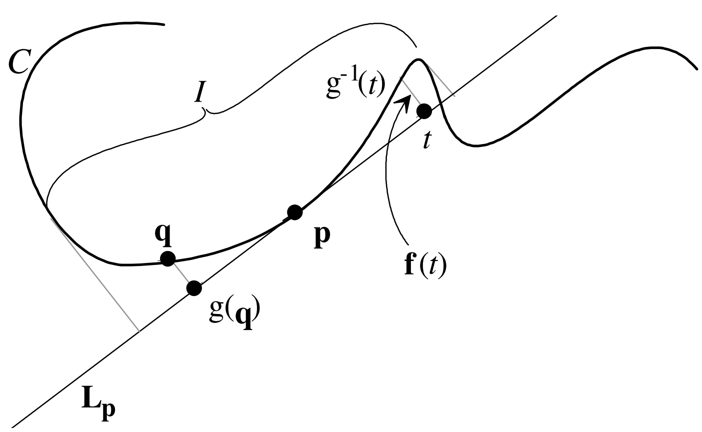


Figure 2.3. Smooth Curve as Graph of a Function.

We now give a proof of this fact in n-space and picture it in the plane:

Let C be a smooth curve in the plane and let $\mathbf{p}$ be a point (not an end point) on C. Let g be the function that assigns to each point on C the orthogonal projection of that point onto $\mathbf{L_p}$, the line tangent to

C at $\mathbf{p}$. (A line is said to be tangent to a curve at a point if it contains the tangent vector at that point.) Then, in any interval $I \subset C$ around $\mathbf{p}$ in which the tangent directions are not perpendicular to the tangent direction at $\mathbf{p}$, the function g is one-to-one and thus has an inverse g^{-1}. Then we can define $\mathbf{f}(t)$ to be the vector from t to $g^{-1}(t)$. Then I is the graph of $\mathbf{f}$ if we coordinatize n-space with $\mathbf{L_p}$ as the first coordinate axis and with the other axes being perpendicular to $\mathbf{L_p}$. (See Figure 2.3.) It follows from **2.2.e** that $\mathbf{f}$ is continuously differentiable.

For geometric investigations it is often best to ***parametrize*** the curve. Think of a particle (or bug) moving along the curve, then a ***parametrization***, $\mathbf{p}(t)$, is a (point-in-space valued) function that specifies the location of the particle at time t. Of course, there are many possible different motions along the curve and thus many possible parametrizations. In the x-y plane $\mathbf{p}(t) = (x(t), y(t))$, and in 3-space with coordinates x-y-z we have $\mathbf{p}(t) = (x(t), y(t), z(t))$; but for geometric purposes it is best to think of $\mathbf{p}(t)$ as a point in space without specific coordinates.

The ***derivative*** of $\mathbf{p}(t)$ with respect to t or ***velocity of the parametrization*** is

$$\mathbf{p}'(t) = (x'(t), y'(t)) \text{ or } \mathbf{p}'(t) = (x'(t), y'(t), z'(t))$$

or, free of coordinates,

$$\mathbf{p}'(t) = \lim_{h\to 0}(1/h)(\mathbf{p}(t+h) - \mathbf{p}(t)).$$

Now, the graph of a function $y = f(x)$ has a natural parametrization $\mathbf{p}(x) = (x, f(x))$ and the derivative of this parametrization is the velocity vector $(1, f'(x))$. Notice that the velocity vector of this parametrization of the graph is never zero. We can also consider the graph of any parametrization. For example,

$$\mathbf{p}(t) = (r\cos t, r\sin t)$$

parametrizes a circle in the plane, and in 3-space its graph

$$\mathbf{q}(t) = (t, r\cos t, r\sin t)$$

represents a helix.

Geometrically, if we again imagine a particle or bug moving along the curve at a constant speed, we can talk about the ***tangent direction*** at a point on a curve as the direction of the motion at that point, and this makes sense without any reference to units. The tangent direction at $\mathbf{p}$ is the direction of the tangent vector at $\mathbf{p}$. In the plane, direction can be designated by the slope, but in 3-space it is more convenient in linear algebra (and more directly relevant to the geometry of a curve) to use unit vectors. The ***unit tangent vector*** is defined to be

$$\mathbf{T}(t) = \frac{\mathbf{p}'(t)}{|\mathbf{p}'(t)|}.$$

In general a vector has direction and magnitude (or length); in a unit vector we factor out the length in order to leave only direction.

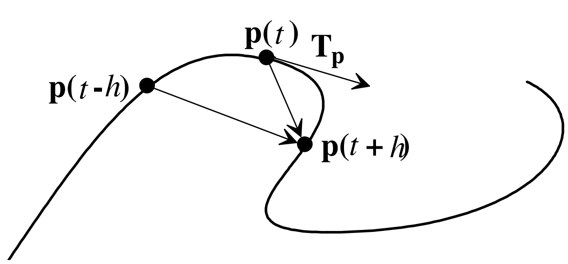


Figure 2.4. Approximating the tangent direction.

If the velocity $\mathbf{p}'(t)$ is non-zero and h is small, then the direction from $\mathbf{p}(t)$ to $\mathbf{p}(t+h)$ approximates the tangent direction at $\mathbf{p}(t)$. Even better, the direction from $\mathbf{p}(t+h)$ to $\mathbf{p}(t-h)$ is almost always orders of magnitude better as an approximation to the tangent direction at $\mathbf{p}(t)$ than the direction from $\mathbf{p}(t+h)$ to $\mathbf{p}(t)$. This is clear from a picture, as in Figure 2.4.

f. *Let $\mathbf{x}(t)$ be a parametrization of a curve in $\mathbf{R}^n$. What is the relationship between the derivative (velocity) existing and the curve being infinitesimally straight at $\mathbf{p} = \mathbf{x}(t)$? Under what conditions does the existence of a velocity vector imply that the curve is smooth? Why?*

Suggestions

Computer Exercise 2.2 allows you to use the computer to draw a curve with its tangent vectors displayed at specified points.

Start by looking at curves in the plane that are the graphs of functions. Then look at circles and other parametrized curves in the plane. Think of the parametrization as describing a motion along the curve. If the parametrization describes the motion of a small bug who is walking along the curve, then in what direction is the bug looking at any given time? If you walk along a path with a sharp corner in it, what happens to the parametrization of your motion when you get to the corner? Is it possible to have a smooth curve which has a parametrization that is not differentiable? Remember the difference between geometric curve, parametrized curve and the graph of a function.

The most natural geometric parametrization of a curve is a ***parametrization with constant speed***, which means that the magnitude of the velocity vector is a constant (and nonzero). When there is a unit of distance then we can have a ***parametrization with respect to arc length***, that is, we label points according to how far they are (along the curve) from some reference point. It is usual in differential geometry texts to use the letter s to denote a parameter with respect to arc length. For example,

$$\mathbf{p}(s) = (\, r\cos(s/r),\, r\sin(s/r)\,)$$

is a parametrization of the circle with respect to arc length. (**Be sure you see why!**) For the helix

$$\mathbf{q}(t) = (\, bt,\, r\cos t,\, r\sin t\,)$$

the velocity vector is

$$\mathbf{v}(t) = \dot{\mathbf{q}}(t) = (b,\ -r\sin t,\ r\cos t)$$

and the speed is

$$|\mathbf{v}(t)| = \tfrac{ds}{dt} = \sqrt{b^2 + r^2}$$

and thus

$$s = t\sqrt{b^2 + r^2}$$

and therefore

$$s \to \left(\frac{bs}{\sqrt{b^2+r^2}},\ r\cos\frac{s}{\sqrt{b^2+r^2}},\ r\sin\frac{s}{\sqrt{b^2+r^2}} \right)$$

is the same helix parametrized by arc length. Note that b is not the height of one turn of the helix.

Notice that if a curve is parametrized with respect to arc length or is parametrized with constant (nonzero) speed, then the curve is infinitesimally straight whenever the velocity vector exists. Moreover, we can prove that

A constant speed curve is smooth if and only if the velocity vector exists and is continuous at each point.

PROBLEM 2.3. Curvature of a Curve in Space

a. *What ways can you think of to quantify the curvature at a point on non-straight smooth curves? Find a method of quantifying curvature that you can conveniently apply to find the curvature at a point on a physical curve.*

Think of the curvature at a point both in terms of how much the curve is turning at the point and also in which direction it is curving. As examples, use circles, helixes, and physical curves that you make out of wire. How is curvature affected by different fields of view?

Since the curvature at a point **p** has both magnitude and direction it is often represented as a vector and denoted by $\boldsymbol{\kappa}_\mathbf{p}$ or $\boldsymbol{\kappa}(\mathbf{p})$ or (when **p** is understood) simply as $\boldsymbol{\kappa}$.

b. *Compare your method(s) with the following classical definition: The curvature is the rate of change at* **p** *of the unit tangent vector* **T** *with respect to arc length, that is,*

$$\boldsymbol{\kappa} = \tfrac{d\mathbf{T}}{ds},\ or\ \boldsymbol{\kappa}(p) = \lim_{h\to 0} \frac{\mathbf{T}(p+h) - \mathbf{T}(p-h)}{2h}.$$

c. *Why is the curvature vector always perpendicular to the tangent direction?*

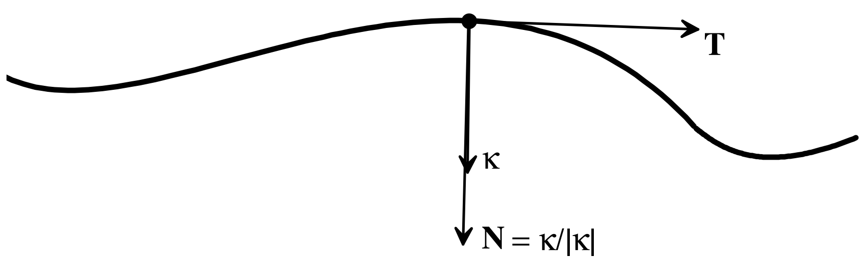


Figure 2.5. The normal vector to a curve.

In the plane there are two directions perpendicular to the tangent direction and, in higher dimensions, infinitely many such perpendicular directions. When the curvature vector is defined and nonzero, then we can define **N** (called the ***normal vector*** to the curve) as the unit vector in the direction of the curvature vector, that is

$$\mathbf{N} = \boldsymbol{\kappa}/|\boldsymbol{\kappa}|. \text{ (See Figure 2.5.).}$$

Note that there are in general many vectors perpendicular to a curve in space, but this definition of **N** picks out one of them to call the normal vector.

Suggestions

You can use Computer Exercise 2.3 to have the computer draw a curve with its curvature vectors displayed at certain points.

Again it may be helpful to think of a bug crawling along the curve. If its motion is parametrized by arc length, how is it crawling? Play with a piece of wire.

Implicit in Problem **2.3** is the Lemma:

The derivative of a unit vector (a geometric direction) is always in a direction perpendicular to the original direction.

If the unit vector were to change in a direction that is not perpendicular to its own direction, then the length of the vector would change. It is important at this point to be able to see the truth of this statement geometrically. But it can also be useful to understand the linear algebra proof that goes like this:

If $\mathbf{V}(s)$ is a vector-valued function of the real variable s such that $|\mathbf{V}(s)|$ = constant, then writing

$$|\mathbf{V}(s)|^2 = \mathbf{V}(s)\cdot\mathbf{V}(s)$$

and differentiating with respect to s and using the product rule, we obtain

$$\mathbf{V}'(s)\cdot\mathbf{V}(s) + \mathbf{V}(s)\cdot\mathbf{V}'(s) = 0$$

and thus $\mathbf{V}'(s)$ is perpendicular to $\mathbf{V}(s)$, since the dot product commutes.

Note that the curvature need not vary continuously even when the curve is infinitesimally straight everywhere. For example, join two circular arcs of different radii or join straight segments to a circular arc such as in Figure 2.6.

If the curve on the right is the cross-section of the edge of a table top, then the discontinuities in the curvature can actually be felt with one's finger. If it is the cross-section of a boat, then the discontinuities in the curvature cause water turbulence and friction. It is also difficult to drive at high speeds on a road with discontinuities in the curvature, because at the discontinuity the steering wheel must be instantaneously turned.

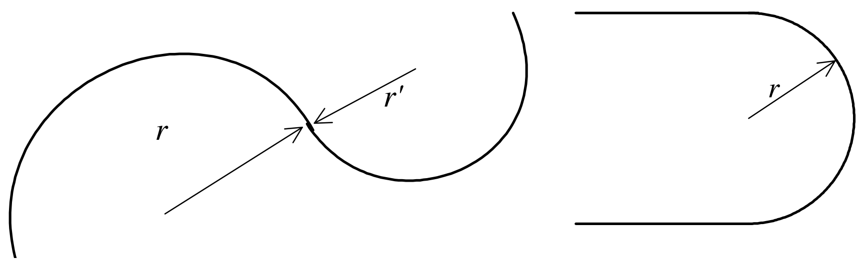


Figure 2.6. Curvature need not vary continuously.

When we say that a curve has ***constant curvature*** we mean the scalar curvature $\kappa = |\boldsymbol{\kappa}|$. This is the only meaning we could have because:

If the curvature is not zero then the curvature vector is never constant (because its direction must be changing).

Be sure you see why this is true.

Curvature of the Graph of a Function

Theorem. *If $(x, f(x))$ is the graph of a twice differentiable function, then at the point $(a, f(a))$ we have*

$$\mathbf{T}(a) = \frac{(1, f'(a))}{\sqrt{1+(f'(a))^2}}$$

and

$$|\boldsymbol{\kappa}(a)| = \frac{|f''(a)|}{[1+(f'(a))^2]^{3/2}}\ .$$

Proof. The graph is a curve $\gamma(x) = (x, f(x))$. Note that the velocity vector

$$\gamma'(x) = (1, f'(a))$$

is never zero and thus, $\mathbf{T}(a)$ is the unit vector in the direction of the velocity vector:

$$\mathbf{T}(a) = \frac{\gamma'(a)}{|\gamma'(a)|} = \frac{(1, f'(a))}{\sqrt{1+(f'(a))^2}}.$$

Now the curvature is the rate of change of **T** with respect to arc-length

$$s = \int_0^a |\gamma'(x)|dx = \int_0^a \sqrt{1+(f'(x))^2}\, dx \ ,$$

thus,

$$\frac{ds}{dx} = \sqrt{1+(f'(x))^2}\ .$$

A geometric interpretation of this last formula is shown in Figure 2.7. From the figure we see that, in a f.o.v. where the graph is indistinguishable from the tangent line,

$$(ds)^2 = (dx)^2 + (dy)^2 = (dx)^2 + (f'dx)^2 = (1+(f')^2)(dx)^2,$$

from which the desired formula follows.

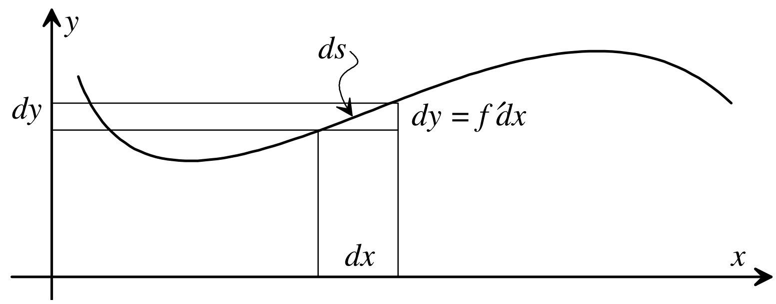


Figure 2.7. Element of arclength.

We can now calculate the curvature vector:

$$\boldsymbol{\kappa}(a) = \frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dx}\frac{dx}{ds} = \frac{d}{dx}\left[\frac{(1,f'(x))}{\sqrt{1+(f'(x))^2}}\right]_{x=a}\left(\frac{ds}{dx}\right)^{-1}_{x=a} =$$
$$= \left[\frac{f''(a)}{[1+(f'(a))^2]^{3/2}}(-f'(a),1)\right]\left(\frac{1}{\sqrt{1+(f'(a))^2}}\right)$$

and thus the theorem follows. Note that at a point at which $f'(a) = 0$,

$$\boldsymbol{\kappa}(a) = f''(a)\ (0,1).$$

PROBLEM 2.4. Osculating Circle

If γ is a smooth curve with nonzero curvature $\boldsymbol{\kappa}_\mathbf{p}$ at the point **p**, then the ***osculating circle at*** **p** is the circle $\boldsymbol{C}_\mathbf{p}$ through **p** which has the same curvature vector and unit tangent vector as γ. (See Figure 2.8.)

a. *Show that the osculating circle lies in the plane determined by the curvature $\boldsymbol{\kappa}_\mathbf{p}$ and unit tangent vector $\mathbf{T}_\mathbf{p}$ and that its radius is $r = 1/|\boldsymbol{\kappa}_\mathbf{p}|$.*

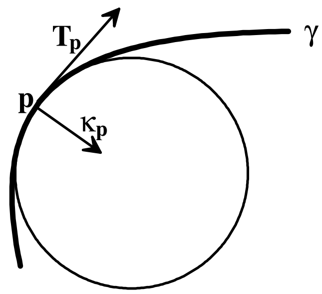


Figure 2.8. Osculating circle.

The plane of the osculating circle is called the ***osculating plane at*** **p**. You may find it helpful at this point to try Computer Exercise 2.4, which allow you to display a curve with its osculating planes (or osculating circles) displayed at various points. The radius, r, of the osculating circle is called the ***radius of curvature at*** **p** and the curvature vector has magnitude (length) $1/r$.

b. *If γ is a smooth planar curve parametrized by arclength and $\theta(s)$ is the angle between* $\mathbf{T_p}$ *and some fixed direction, for every point* $\mathbf{p} = \gamma(s)$*, then show that*

$$|\boldsymbol{\kappa}| = \left|\frac{d\theta}{ds}\right| = \left|\frac{d\mathbf{N}}{ds}\right|.$$

Find a counterexample to show why "planar" is important?

[Hint: What are the possible unit vectors in a plane? Also, use the parametrization of the helix (with respect to arc length),

$$\mathbf{p}(s) = \left(\frac{hs}{\sqrt{h^2+(2\pi r)^2}},\ r\cos\frac{2\pi s}{\sqrt{h^2+(2\pi r)^2}},\ r\sin\frac{2\pi s}{\sqrt{h^2+(2\pi r)^2}}\right),$$

where h is the height of one turn and r is the radius of the helix, and calculate the curvature vector and the derivative of the normal vector.]

In Chapter 5, we will use try to mimic this idea when we investigate the intrinsic curvature of a curve on a surface.

If the curvature is zero at a point, then the tangent line approximates the curve better than any circle and there is no osculating circle or osculating plane. If the curve has zero curvature at every point along an interval of the curve, then that interval is straight (there is no turning), but if the curvature is merely zero at a point, then the curve is not straight at that point (examples include inflection points and the vertex of the graph of $y = x^4$.)

***c.** *Consider a segment of a curve that is indistinguishable from the osculating circle. If we move this segment along the normal vector at each point, then its arclength l will change at the rate $-|\boldsymbol{\kappa}|\,l$.*

[Hint: Look at Figure 2.9. You can show that the derivative $\frac{d}{dh} l_h = -|\boldsymbol{\kappa}|\, l$.]

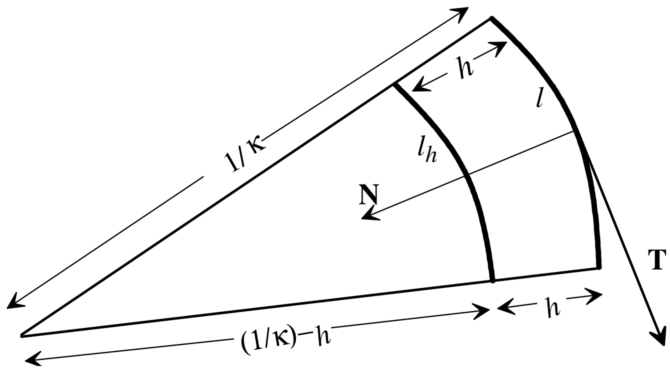


Figure 2.9. Rate of change of arc length.

Problem 2.5. Strakes

To help us understand the idea of curvature of curves and, later, the curvature of surface, let us look at the following example.† To give structural support to large metal cylinders, such as large smoke stacks, engineers sometimes attach a spiraling strip called a ***strake***. See Figure 2.10.

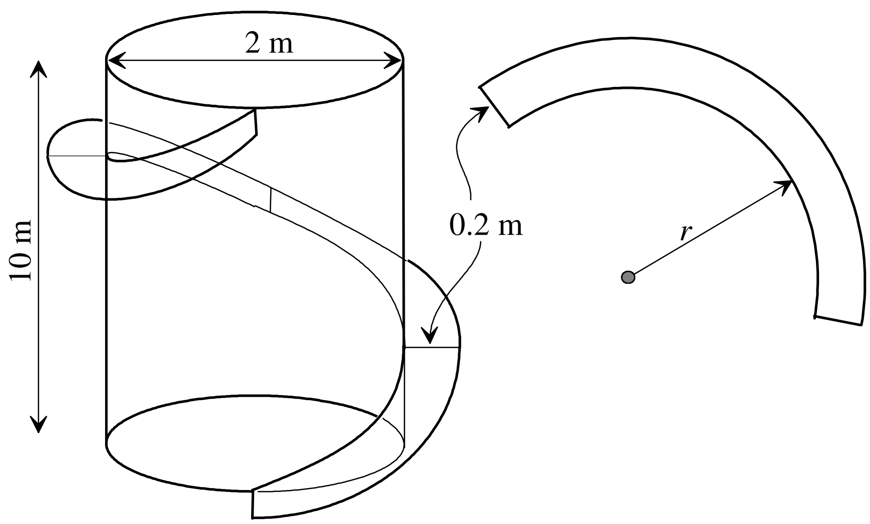


Figure 2.10. A strake.

To produce the strake it is convenient to cut annular pieces from a flat sheet of steel as illustrated in Figure 2.10. These annular pieces are then bent along the helix to form the strake.

a. *Convince yourself that the way to compute the ideal value for the radius r is to require that the helix on the cylinder and the inner curve of the annulus have the same curvature.*

[Hint: What does "curvature" mean?]

b. *Compute this ideal value for r.*

[Hint: Use the formulas given in Problem **2.3** and the dimensions in Figure 2.10.]

c. *Can the flat annulus exactly fit a piece of the strake?*

[Hint: Clearly the strake is not planar as it stands. But can the annular piece of steel be bent without stretching in order to produce the strake? We are asking if the strake is ***locally isometric*** to the plane, or,

† This example is inspired by an example in [**DG**: Morgan, pp.6-10].

in other words, if the local *intrinsic* geometry of the strake is the same as the local geometry on a plane. Cut a paper annulus with the ideal radius and try forming it into a strake. Consider the inner and outer edges of the strake and the annulus. (Notice that both the outer and inner edges of the strake are helixes.) You might compute the curvatures and lengths of the inner and outer edges of the annulus and the corresponding inner and outer edges of the helical strake. Do they agree? If not, how much are they off? What do you conclude? What happens if we make the strake very wide compared to the diameter of the cylinder— such as happens in an auger (see Figure 2.11)?]

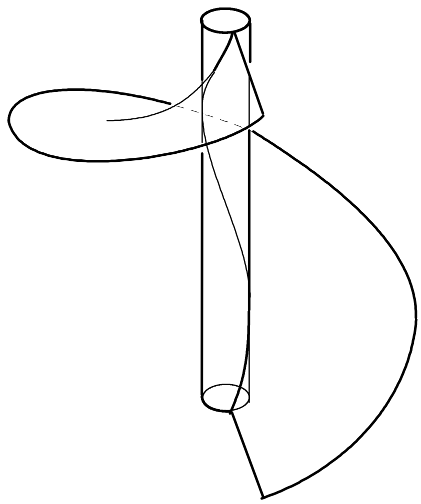


Figure 2.11. Auger.

Problem 2.6. When a Curve Does Not Lie in a Plane

Suppose we have a smooth curve that does not lie in a plane and that has a well-defined (nonzero) curvature vector at each point. Then at some point on the curve the osculating plane (the plane of the osculating circle) must be changing. Let us make a picture of how the osculating plane could be changing at such a point **p**. If the osculating plane is changing then there must be points, **p–** and **p+**, close on either side of **p** with different osculating planes. Let us use three points to draw approximations of these osculating planes [**q**, **p+**, **p–** determine the plane containing **a**, and **p+**, **p–**, **r** determine the plane with **b**] as in Figure 2.12.

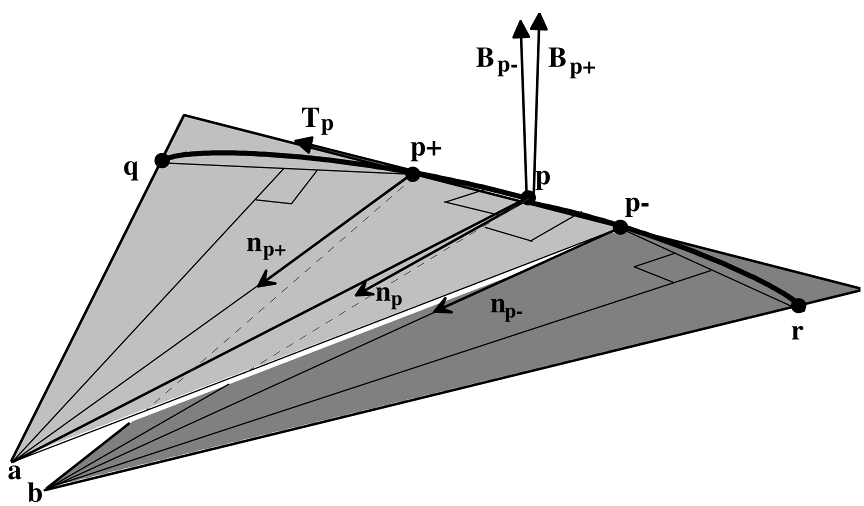


Figure 2.12. Rotating osculating planes.

Notice that the two approximate osculating planes intersect on the cord from **p–** to **p+** and that the three points, **p**, **a**, **b**, lie in a plane that is perpendicular to this chord, which closely approximates $\mathbf{T_p}$.

a. *In the situation described in the above two paragraphs, argue that at* **p** *the osculating planes must be pivoting around the tangent line* $\mathbf{T_p}$ *and that the centers of curvature are changing. Also show that they are changing in the plane perpendicular to* $\mathbf{T_p}$.

[Hint: Study Figure 2.12. If you are having trouble seeing what is happening, then make a model using two sheets of paper and tape.]

A unit vector perpendicular to the osculating plane at **p** is called the ***binormal***, $\mathbf{B_p}$. Note that the unit tangent vector, normal vector, and binormal vector $\{\mathbf{T_p}, \mathbf{N_p}, \mathbf{B_p}\}$ are all unit vectors, which are mutually orthogonal and thus form an orthonormal basis (see Appendix A.3) for $\mathbf{R}^3$. We pick the direction of $\mathbf{B_p}$ by specifying that $\{\mathbf{T_p}, \mathbf{N_p}, \mathbf{B_p}\}$ be ***right handed***, that is, if you curl the fingers of your right hand from $\mathbf{T_p}$ to $\mathbf{N_p}$, then your thumb will point in the direction of $\mathbf{B_p}$. In terms of the cross product, $\mathbf{B_p} = \mathbf{T_p} \times \mathbf{N_p}$. The three unit vectors, $\mathbf{T_p}$, $\mathbf{N_p}$, $\mathbf{B_p}$, are called the ***Frenét frame*** at the point **p** and, of course, they vary from point to point; but, at each point, they form an orthonormal basis for $\mathbf{R}^3$. You may find it useful at this point to use Computer Exercise 2.6 to display a curve with its Frenét frames displayed at various points.

Most books define the ***torsion*** (vector) of a curve to be the rate of change (with respect to arc length) of the binormal, in symbols $\boldsymbol{\tau}_\mathbf{p} = \mathbf{B_p}'$. So we can:

b. *Conclude that a curve with well-defined (nonzero) curvature is planar if and only if the binormal is constant (or, its torsion is everywhere zero).*

Note that $\boldsymbol{\tau}_\mathbf{p} = 0$ implies that the osculating planes are parallel; but why are they the same plane? To see that "nonzero" is necessary: Take two C-shaped curves with zero curvature at their endpoints, then put these two curves together to form an S-shaped smooth plane curve. This resulting S-shaped curve has different binormals on the two different pieces.

You may use part **a** to show part **b** or you may find it helpful to use some version of the Mean Value Theorem from first semester calculus. However, be warned that the Mean Value Theorem applies, in general, only to differentiable real-valued functions of one real variable or to differentiable curves in the plane. Also, if the binormal is constant then all osculating planes are perpendicular to the binormal and are thus parallel *but* not necessarily the same.

For example, one way of stating a ***Mean Value Theorem for Planar Curves*** is:

Given two points **p** *and* **q** *on a differentiable curve in the plane, there is some point* **r** *on the curve between* **p** *and* **q** *such that the tangent vector* $\mathbf{T_r}$ *at* **r** *is parallel to* $\mathbf{p} - \mathbf{q}$. (See Figure 2.13.)

This theorem applies to real-valued functions of one real variable if you consider the graph of the function as a curve in the plane. Note that this result is not true, in general, for curves that do not lie in a plane; for example, the reader can easily find two points on a helix for which it does not hold. (See Problem **4.2**.)

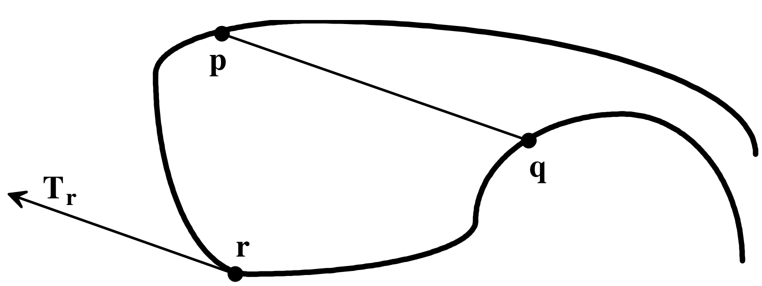


Figure 2.13. Mean Value Theorem for Plane Curves.

When you compress a helical spring so that it becomes more nearly planar, the curvature stays constant (and thus the diameter increases), but the torsion decreases.

From Figure 2.12 we can see that the binormal is changing and changing in a direction which is parallel to the normal since the change is perpendicular to both the tangent line and the binormal (because the binormal is a unit vector). This discussion leads to equations that express the connections between the rate of change (with respect to arc length) of the three unit vectors, **T**, **N**, **B**. These equations, called the Frenét-Serret Equations, were independently found by Fréderic-Jean Frenét (1847, published in 1852) and Joseph Serret (1851):

c.** *Prove the following* ***Frenét-Serret Equations*:*

$$\boldsymbol{\kappa} \equiv \mathbf{T}'(s) = \kappa(s)\mathbf{N}(s),$$
$$\mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s),$$
$$\boldsymbol{\tau} \equiv \mathbf{B}'(s) = -\tau(s)\mathbf{N}(s),$$

where $\kappa(s) = |\mathbf{T}'(s)|$ *is the* ***scalar curvature*** *and*

$$\tau(s) = -[\mathbf{B}'(s)\cdot\mathbf{N}(s)]$$

is the ***scalar torsion****.*

The definition of

$$\tau(s) = -\mathbf{B}'(s)\cdot\mathbf{N}(s)$$

is a sign convention (the most commonly used one), but other conventions are possible and are used in some books.

You may find it easiest initially to prove the first and third equations by using Problem **2.3** and the drawing above. These two equations can then be combined to obtain the second equation by either looking geometrically at the three mutually perpendicular unit vectors or by differentiating the cross product $\mathbf{N} = \mathbf{B} \times \mathbf{T}$. Note that from the right-hand rule we can conclude that, if $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, then $\mathbf{T} = \mathbf{N} \times \mathbf{B}$ and $\mathbf{N} = \mathbf{B} \times \mathbf{T}$.

We will not use the Frenét-Serret Equations in this text, but they are a tool commonly used to study curves in 3-space. See, for example, [**DG**: Millman/Parker, Chapter 2]. In that book is proved [Theorem 5.2]:

> *Any smooth curve with nonzero curvature is completely determined, up to position, by its curvature and torsion.*

d. *Calculate the binormal and torsion of a helix. Show that its scalar torsion is constant.*

In higher dimensions the above discussion of **T**, $\boldsymbol{\kappa}$, and **N** hold without change. In addition, three (non-collinear) points determine a circle and a plane in any dimension, and thus the notion of osculating circle and osculating plane makes sense in all dimensions. However, in higher dimensions, planes are not determined by a normal vector, and thus the binormal vector does not make sense and, when needed, must be replaced by tensors or forms.