

# Preface

In mathematics, as in any scientific research, we find two tendencies present. On the one hand, the tendency toward *abstraction* seeks to crystallize the *logical* relations inherent in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency toward *intuitive understanding* fosters a more immediate grasp of the objects one studies, a live *rapport* with them, so to speak, which stresses the concrete meaning of their relations.

As to geometry, in particular, the abstract tendency has here led to the magnificent systematic theories of Algebraic Geometry, of Riemannian Geometry, and of Topology; these theories make extensive use of abstract reasoning and symbolic calculation in the sense of algebra. Notwithstanding this, it is still as true today as it ever was that intuitive understanding plays a major role in geometry. And such concrete intuition is of great value not only for the research worker, but also for anyone who wishes to study and appreciate the results of research in geometry.

— David Hilbert [SE: Hilbert]

These words, written in 1934 by David Hilbert, the “father of Formalism,” are from the Preface to *Geometry and the Imagination*.

The formalisms of differential geometry are considered by many to be among the most complicated and inaccessible of all the formal systems in mathematics. It is probably fair to say that most mathematicians do not feel comfortable with their understanding of differential geometry. In addition, there is little agreement about which formalisms to use or how to describe them, with the result that the starting definitions, notations and analytic descriptions vary widely from text to text. What all of these different approaches have in common are underlying geometric intuitions of the basic notions such as straightness (geodesic), smooth, tangent, curvature, and parallel transport.

In this book we will study a foundation for differential geometry based not on analytic formalisms but rather on these underlying geometric intuitions. This foundation should be accessible to anyone with a flexible geometric imagination. It may then be possible that this foundation will serve as a common starting point for the various analytic formalizations. We will explore some of these analytical formalisms. In addition, this geometric foundation relates more directly with our actual experiences of curves and surfaces both in the physical world and in the context of computer graphics.

I invite the reader to explore the basic ideas of differential geometry. I am interested in conveying a different approach to mathematics, stimulating the reader to a broader and deeper experience of mathematics. Active participation with these ideas, including exploring and writing about them, will give the reader a broader context and experience, which is vital for anyone who wishes to understand differential geometry at a deeper level. More and more of the formal analytical aspects of differential geometry have now been mechanized, and this mechanization is widely available on personal computers, but the experience of meaning in differential geometry is still a human enterprise that is necessary for creative work.

I believe that mathematics is a natural and deep part of human experience and that experiences of meanings in mathematics are accessible to everyone. Much of mathematics is not accessible through formal approaches except to those with specialized learning. However, through the use of non-formal experience and geometric imagery, many levels of meaning in mathematics can be opened up in a way that more human beings can experience and find intellectually challenging and stimulating.

This text builds on a foundation of intuitive geometric ideas and then ties them into the formalisms of extrinsic and intrinsic differential geometry. The first chapter is an extensive collection of examples of surfaces which are discussed as much as can be done using elementary techniques and geometric

intuition. Many of the concepts in the text are introduced in Chapter 1. Throughout the text there is an emphasis on looking at curves and surfaces in as many different ways as possible but with a particular emphasis on intrinsic, coordinate-free approaches in order to highlight the geometry.

The book is written for undergraduate mathematics majors and thus assumes of the reader a corresponding level of interest and mathematical sophistication. Previous experience with multivariable calculus and linear algebra are strongly recommended. There is more material in this text than I cover in one semester, so the instructor can choose to leave out certain topics. To assist in this process some problems or parts of problems are preceded by an asterisk (\*), indicating that they are not essential for the rest of the text.

There are some results in this text which (insofar as I know) have never before appeared in print. These include the annular hyperbolic plane (that I learned from William Thurston, see Problems **1.8** and **5.7**); the use of zooming and fields of view in defining smoothness (see the beginnings of Chapters 2 and 3, especially Problems **2.1**, **2.2**, and **3.1**); and the Ribbon Test for a geodesic (Problems **3.4** and **7.5**).

There is a unique problem-centered approach in the presentation of this material. The main geometric notions, both theory and concepts, are introduced through problems which are designed to give students an opportunity to experience their own meanings in the material. This is similar to the approach used in the author's *Experiencing Geometry on Plane and Sphere* (Prentice Hall, 1996), *Experiencing Geometry In Euclidean, Spherical, and Hyperbolic Spaces* (Prentice Hall, 2001), and (with Daina Taimina) *Experiencing Geometry: Euclidean and Non-Euclidean With History* (Pearson Prentice Hall, 2005). Two discussions describing different aspects of this approach can be found in [David Henderson, "I Learn Mathematics From My Students — Multiculturalism in Action," *For the Learning of Mathematics*, v.16, n.2, pp.46-52] and [Jane-Jane Lo, Kelly Gaddis and David Henderson, "Learning Mathematics Through Personal Experiences: A Geometry Course in Action," *For the Learning of Mathematics*, v.16, n.1, pp.34-40].

### ***Useful Supplements***

Those readers who have access to computer systems running Maple©, Mathematica©, Derive©, or similar software can use these systems to facilitate gaining geometric intuition of the concepts of differential geometry. In Appendix C, I have included several computer exercises for Maple, and these and additional scripts are also available for downloading on-line at

<http://www.math.cornell.edu/~henderson/books/dg.html>

I will also include at this site errata and additional information and updates

### ***Acknowledgments for the First Edition***

I acknowledge my debt to all the students who have attended my differential geometry courses. Without them this book would have been an impossibility.

I started writing problems such as those that appear in this book while teaching differential geometry in the spring of 1992. Again in the spring of 1994, I wrote more problems and used them together with a published textbook for the course. In the spring of 1995 I taught the course using only my problems and altered them and extended them as we went along. Finally, the first preliminary version of this text was available in photocopy form in the fall of 1995. It was used by Brian Mortimer in his differential geometry class at Carleton University in Ottawa, Canada, and by James West in his differential geometry course at Cornell University. Both of these mathematicians gave me valuable feedback. In addition, I received many helpful suggestions and comments from the reviewers of the Fall 1995 version and later the Fall 1996 version. These reviewers were Brian Mortimer, Bruce Piper (Rensselaer Polytechnic Institute), Nicola Garofalo (Purdue University), George C. Johnson (University of California at Berkeley), Steven L. Kent (Youngstown State University), Larry Cusick (California State University at Fresno), Bruce Hughes (Vanderbilt University), and several anonymous reviewers.

In addition, I received valuable feedback from Jane-Jane Lo, Dexter Luthulli, Cathy Stenson, and Alex Tsow. In the spring of 1997 I used a near final version of the text in a course at Cornell. From these students, instructors, and others, I received encouragement and much valuable feedback that resulted in what I consider to be a better book. Cathy Stenson also wrote the Maple© computer scripts.

In October 1995, I gave a copy of the preliminary text to Daina Taimina, faculty member of the Faculty of Mathematics and Physics at the University of Latvia in Riga. Much of the final rewriting and extending of this text was completed with her assistance during my two-month visit to Latvia in the summer of 1996 and her visit to Ithaca during the spring of 1997.

The entire production of the manuscript (typing, formatting, drawings, and final layout and typesetting) has been accomplished using AmiPro, an integrated word-processing software, and its successor WordPro.

Finally, I wish to thank George Lobell, former Senior Editor at Prentice Hall, for his contagious enthusiasm and for the vision with which he shepherded the first edition of this book through the publication process.

### ***Acknowledgments for the Second and Third Edition***

This edition has been revised based on my many years of teaching with the first edition and the comments from my students and is also based on the many comments I have received from users of the earlier editions. The people whose feedback I have used include: Robert Lubarsky (Florida National University), Steve Martel (University of Hawaii), Nathaniel Miller (University of Northern Colorado), Colm Mulcahy (Spelman College), Preston Nichols (Gustavus Adolphus College), Patricia Sipe (Smith College), John Snygg, Peter Veerman (Portland State University), Nancy C. Wrinkle (Northeastern Illinois University), and Luis Zambrano (California State University, Los Angeles).

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