Compactness of the automorphism group of a topological parallelism on real projective 3-space: The disconnected case

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Abstract

We prove that the automorphism group of a topological parallelism on real projective 3-space is compact. This settles a conjecture stated in [1], where it was proved that at least the connected component of the identity is compact.

1   Introduction

Clifford parallelism on real projective 3-space $\mathrm{PG}(3, \mathbb{R})$ is the classical example of a topological parallelism. In general, a topological parallelism is an equivalence relation on the set of lines such that every equivalence class $\mathcal{C}$ is a spread (that is, its elements form a simple cover of the point set) and such that some continuity condition is satisfied. Examples of topological parallelisms abound, see, e.g., [2], [3], [6], and further references given there. The group $\Phi = \mathrm{Aut}\,\Pi$ of automorphisms of a topological parallelism is a closed subgroup of $\mathrm{PGL}(4, \mathbb{R})$ and, hence, a Lie group; compare [5] or the introduction of [1]. In the case of Clifford parallelism, Φ is the compact 6-dimensional group $\mathrm{PSO}(4, \mathbb{R})$. According to [9], Clifford parallelism is the only one with $\dim \Phi \geq 4$; compare also [5], where the weaker dimension bound 5 was obtained. Here we prove:

THEOREM 1.1. *Let* Π *be a topological parallelism on* $\mathrm{PG}(3, \mathbb{R})$. *Then the automorphism group* $\Phi = \mathrm{Aut}\,\Pi$ *is compact and hence is conjugate to a closed subgroup of* $\mathrm{PO}(4, \mathbb{R})$.

The theorem will be proved in Sections 3 and 4 without using the result of [1]. In fact, the present result implies that of [1], because the connected component is closed in Φ. Essentially, the proof in [1] consisted in showing that every one-parameter subgroup of Φ has compact closure. The present result is obtained by proving the same property for cyclic subgroups instead of one-parameter subgroups. Some arguments in the proof are close to those used in [1], but other parts of the proof require new ideas.

The result considerably reduces the list of possible groups Φ. The possible connected groups are enumerated[1] in the introduction of [1], and in each case the possibilities for Φ itself are very limited. This will be indispensable for the classification of parallelisms with large groups, see [5], [9] and [10] for some initial steps. As an illustration of the method and an immediate application of Theorem 1.1, we present a quick way to determine the automorphism group of Clifford parallelism. For other proofs of this fact, see [4] and [5].

COROLLARY 1.2. *The automorphism group* Δ *of Clifford parallelism is the compact, connected 6-dimensional group* $\mathrm{PSO}(4, \mathbb{R})$.

Proof. The group $\Sigma = \mathrm{PSO}(4, \mathbb{R})$ is a direct product of two subgroups Λ and Ψ induced by the left and right multiplications by quaternions of norm one, respectively. Clifford parallelism is most easily defined as the set of all Λ-orbits in the line space of $\mathrm{PG}(3, \mathbb{R})$. These orbits are permuted by the other factor Ψ because Ψ centralizes Λ. Thus Δ contains Σ. On the other hand, by Theorem 1.1, Δ is contained in a maximal compact subgroup of $\mathrm{PGL}(4, \mathbb{R})$, so up to conjugacy, Δ is contained in $\mathrm{PO}(4, \mathbb{R})$. Thus if Δ is disconnected, then it contains an involution γ with determinant -1. Now γ can be represented by a diagonal matrix with diagonal entries $-1, 1, 1, 1$, and it follows easily from Lemma 3.5 below that such an involution cannot leave any topological parallelism invariant. Thus we have equality $\Delta = \Sigma$. ∎

I wish to thank the referee for remarks that helped to make the exposition clearer.

2 Preliminaries

We consider real projective 3-space $\mathrm{PG}(3, \mathbb{R})$ with its point space P and line space $\mathcal{L}$. Elements of P and $\mathcal{L}$ are the 1- and 2-dimensional subspaces of the vector space $\mathbb{R}^4$, respectively. The star of all lines passing through a point p is denoted $\mathcal{L}_p$. The topologies of P and $\mathcal{L}$ are the usual topologies of the Grassmann manifolds. This means that a sequence X_n of subspaces converges to a subspace X if and only if the spaces X_n have bases converging to a basis of X. See, e.g., [8] for the continuity properties of geometric operations in this topological projective

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[1]We point out that there are infinitely many conjugacy classes of 1-tori in $\mathrm{PO}(4, \mathbb{R})$.

space. We shall mainly use the facts that P and $\mathcal{L}$ are compact and that the line $L = p \vee q$ joining two distinct points p, q depends continuously on the pair (p, q).

A *spread* on $\mathrm{PG}(3, \mathbb{R})$ is a set of lines $\mathcal{C} \subseteq \mathcal{L}$ such that every point is incident with exactly one of them. A spread defines an affine translation plane whose points are the elements of $\mathbb{R}^4$ and whose lines are the cosets of the vector spaces $L \in \mathcal{C}$. This translation plane is topological (i.e., joining points and intersecting lines are continuous operations) if and only if the spread $\mathcal{C}$ is a compact subset of $\mathcal{L}$ (which then is homeomorphic to the 2-sphere $\mathbb{S}_2$); see [11], 64.4.

A *parallelism* Π on $\mathrm{PG}(3, \mathbb{R})$ is a set of spreads such that every line is contained in exactly one of them. Thus Π is an equivalence relation on $\mathcal{L}$ whose classes are spreads. There are several ways of looking at a parallelism. Accordingly, the symbol Π shall be used with different meanings. If considered as a relation on lines, $\Pi \subseteq \mathcal{L} \times \mathcal{L}$ is a set of pairs of lines. If considered as a set of spreads, Π becomes a set of subsets of $\mathcal{L}$. Finally, Π may be considered as a map $\Pi : P \times \mathcal{L} \to \mathcal{L}$ sending a pair (p, L) to the unique line $\Pi(p, L)$ parallel (i.e., equivalent) to L that contains p, or as a map $\mathcal{L} \to \Pi$ sending a line L to the class $\Pi(L)$ containing L. We trust that this usage will not create any confusion.

A parallelism Π is said to be *topological* if the map $\Pi : P \times \mathcal{L} \to \mathcal{L}$ is continuous. It is not hard to see that this is equivalent to the condition that $\Pi \subseteq \mathcal{L} \times \mathcal{L}$ is a closed and, hence, compact subset, and that all classes of a topological parallelism are compact spreads. If we presuppose that the classes are compact spreads then Π is topological precisely if Π is a compact subset of the hyperspace $H(\mathcal{L})$ of all compact subsets of $\mathcal{L}$, endowed with the Hausdorff topology or, equivalently, if the map $\Pi : \mathcal{L} \to H(\mathcal{L})$ is continuous. Note that the restriction of this map to a line star $\mathcal{L}_p$ is a homeomorphism from $\mathcal{L}_p$ (which is a real projective plane) onto $\Pi \subseteq H(\mathcal{L})$.

The automorphism group $\Phi = \operatorname{Aut}\Pi$ of a parallelism Π is defined as the subgroup of $\mathrm{PGL}(4, \mathbb{R})$ consisting of all collineations of $\mathrm{PG}(3, \mathbb{R})$ that leave Π invariant. Suppose that Π is topological. From the compactness of $\Pi \subseteq H(\mathcal{L})$ one sees immediately that then $\operatorname{Aut}\Pi$ is a closed subgroup of the Lie group $\mathrm{PGL}(4, \mathbb{R})$ and, hence, is again a Lie group.

3   Foundations of the proof

The proof for the connected case given in [1] rests on the theorem of Malcev-Iwasawa, which implies that a non-compact Lie group contains some closed but non-compact one-parameter subgroup. This reduced the proof to the examination of all possible one-parameter groups. Here, we use instead the following theorem of Djocović:

THEOREM 3.1. ([7], Theorem 3) *Let G be a connected, locally compact group and let $H \leq G$ be a closed subgroup. Suppose that every cyclic subgroup generated by an element of H has compact closure (in H or, equivalently, in G). Then H is compact.*

We use this in order to reduce the proof of Theorem 1.1 to the consideration of cyclic groups of automorphisms, as follows. Our group G will be the connected

component $PGL(4, \mathbb{R})^1$. It has index 2 in $PGL(4, \mathbb{R})$, hence it suffices to show that $H = \Phi \cap PGL(4, \mathbb{R})^1$ is compact. This will follow from Theorem 3.1 if we show that every cyclic subgroup of Φ has compact closure in $PGL(4, \mathbb{R})$. In other words, in order to prove Theorem 1.1, all we have to do is to prove

PROPOSITION 3.2. *Every cyclic subgroup of* $\Phi = \operatorname{Aut} \Pi$ *has compact closure in* $PGL(4, \mathbb{R})$.

In proving Proposition 3.2, it suffices to consider one cyclic group from every conjugacy class. In order to obtain these representatives, we use the Jordan normal form of real 4×4 matrices. We also did this in [1], but there we applied this method to generators of one-parameter groups rather than generators of cyclic groups. Consequently, we may now replace a matrix by any nonzero scalar multiple as we are dealing with the projective group, whereas in [1] we were allowed to *add* an arbitrary scalar matrix. This leads to a different list of candidates. In compiling this list, we observe that all eigenvalues will be nonzero. We may replace a matrix by its square without loss, hence we may assume that real eigenvalues are positive. In the list which follows, the cases labelled 'a' are the complex 2×2 matrices, label 'b' indicates the presence of both real and complex eigenvalues, and label 'c' is attached to matrices all of whose eigenvalues are real. The case distinction is not identical to the one used in [1]. We allow complex eigenvalues to be real eigenvalues with a 2-dimensional eigenspace, and in this case we use a complex coordinate on that eigenspace. We shall write γ for a real 4×4 matrix and sometimes also for the projective collineation induced by that matrix. The cyclic group generated by γ is denoted $\langle \gamma \rangle$.

Case (a1): γ *is a complex diagonal* 2×2 *matrix with (possibly real) eigenvalues* $a, c \in \mathbb{C} \setminus \{0\}$.

$$\gamma = \begin{pmatrix} a & \\ & c \end{pmatrix}.$$

Up to multiplication by a real number, we have that $|a| = 1$. If also $|c| = 1$, then $\langle \gamma \rangle$ is contained in the 2-torus of all complex diagonal matrices with diagonal entries of norm one and, hence, has compact closure.

Case (a2): γ *is a complex Jordan block with (possibly real) eigenvalue* $a \in \mathbb{C} \setminus \{0\}$.
Multiplying by $b = |a|^{-1} \in \mathbb{R}$ and renaming, we get $|a| = 1$ and

$$\gamma = \begin{pmatrix} a & b \\ & a \end{pmatrix}, \quad \gamma^n = \begin{pmatrix} a^n & na^{n-1}b \\ & a^n \end{pmatrix}.$$

(To compute the powers of γ, write γ as a sum of a scalar matrix plus a nilpotent matrix and use the binomial theorem. This works in more complicated cases as well, cf. case (c1) below.)

Case (b1): γ *has one eigenvalue* $a \in \mathbb{C} \setminus \{0\}$ *and two real ones* $r, s \in \mathbb{R} \setminus \{0\}$, *the latter with Jordan blocks of size 1.*

If a happens to be real, we assume that it has an eigenspace of real dimension 2. As before, we assume $|a| = 1$ and obtain

$$\gamma = \begin{pmatrix} a & & \\ & r & \\ & & s \end{pmatrix}.$$

Case (b2): γ *has one eigenvalue* $a \in \mathbb{C} \setminus \{0\}$ *and one real eigenvalue* r, *the latter with a Jordan block of size 2.*

Again a is assumed to have a 2-dimensional eigenspace if it is real. Dividing by $|a|$ and renaming, we get

$$\gamma = \begin{pmatrix} a & & \\ & r & s \\ & & r \end{pmatrix}, \quad \text{where } |a| = 1 \quad \text{and} \quad r, s \in \mathbb{R} \setminus \{0\}.$$

Case (c1): γ *has one real eigenvalue* $s \neq 0$ *with a single Jordan block of size 4.*
Multiplying by $r = s^{-1} \in \mathbb{R} \setminus \{0\}$, we get

$$\gamma = \begin{pmatrix} 1 & r & & \\ & 1 & r & \\ & & 1 & r \\ & & & 1 \end{pmatrix}, \quad \gamma^n = \begin{pmatrix} 1 & nr & \binom{n}{2}r^2 & \binom{n}{3}r^3 \\ & 1 & nr & \binom{n}{2}r^2 \\ & & 1 & nr \\ & & & 1 \end{pmatrix}.$$

Case (c2): γ *has two real eigenvalues with Jordan blocks of size 3 and 1, respectively.*
After dividing by one of the eigenvalues, we have

$$\gamma^n = \begin{pmatrix} 1 & nr & \binom{n}{2}r^2 & \\ & 1 & nr & \\ & & 1 & \\ & & & s^n \end{pmatrix}, \quad r, s \in \mathbb{R} \setminus \{0\}.$$

Case (c3): γ *has two distinct real eigenvalues, both with one Jordan block of size 2.*

(The corresponding case with two equal eigenvalues is covered by case (a2), up to a change of coordinates.) By passing to γ^2 we obtain that the eigenvalues are positive. After dividing by one of the eigenvalues, we have

$$\gamma = \begin{pmatrix} r & s & & \\ & r & & \\ & & 1 & s \\ & & & 1 \end{pmatrix}, \quad \gamma^n = \begin{pmatrix} r^n & nr^{n-1}s & & \\ & r^n & & \\ & & 1 & ns \\ & & & 1 \end{pmatrix}, \quad 0 < r, s \in \mathbb{R}, r \neq 1.$$

By inverting γ, if necessary, we can enforce $0 < r < 1$.
Case (c4): γ *has three real eigenvalues with Jordan blocks of size 2, 1, 1, respectively.*
After dividing by one of the eigenvalues and passing to γ^2, if necessary, we have

$$\gamma = \begin{pmatrix} 1 & t & & \\ & 1 & & \\ & & r & \\ & & & s \end{pmatrix}, \quad 0 < r, s, t \in \mathbb{R}.$$

We may assume that $r \neq s$, the case $r = s$ being covered by case (b2).

Case (c5): γ *is diagonal with four distinct eigenvalues* $1, r, s, t$. (Again if two eigenvalues are equal, we have case (b1).)

We complete this section by providing some tools that will be useful in the main part of the proof and possibly elsewhere.

LEMMA 3.3. *Given a topological parallelism* Π *and* $m + 1$ *lines* $M, L_1, ..., L_m$, *there is always a line* N *parallel to* M *and not meeting any of the* L_i.

Proof. Lines L_j that happen to be parallel to M are easily dealt with. Suppose that this is not the case for L_i. Then the set of parallels to M that meet L_i is the image of the point set of L_i under the injective map $p \mapsto \Pi(p, M)$, hence like L_i it is a topological circle. The union of a finite number of circles must be a proper subset of the 2-sphere $\Pi(L)$. All lines N in the complement satisfy our assertion. ∎

LEMMA 3.4. *Let* Π *be a topological parallelism and let* K, L *be two lines with trivial intersection. Then there exist two distinct parallel lines* M, N *meeting both* K *and* L.

Proof. The point sets $[K]$ and $[L]$ are topological circles. If the map $(p, q) \rightarrow \Pi(p \vee q)$ from the torus $[K] \times [L]$ into $\Pi \subseteq H(\mathcal{L})$ is injective, then it is also surjective by domain invariance, and we have a homeomorphism between a torus and a projective plane. This is impossible, hence there are two pairs (p, q) and (p', q') whose joining lines are parallel. ∎

The next lemma is a variation of [1], 3.8.

LEMMA 3.5. *Suppose that* γ *leaves* Π *invariant and fixes two points* $p, q \in P$. *Then the actions induced by* γ *on the factor spaces* $\mathbb{R}^4/p$ *and* $\mathbb{R}^4/q$ *are projectively equivalent.*

Proof. The points of the projective space associated with the vector space $\mathbb{R}^4/p$ are in $1 - 1$ correspondence to the lines of PG$(3, \mathbb{R})$ passing through p. The map $L \rightarrow \Pi(q, L)$ is a γ-equivariant bijection of line stars $\mathcal{L}_p \rightarrow \mathcal{L}_q$. ∎

4   Completion of the proof

We look at the matrices γ listed in the previous section one by one. In each case we shall show that either the cyclic group $\langle \gamma \rangle$ has compact closure in GL$(4, \mathbb{R})$ or γ cannot belong to the automorphism group $\Phi = \text{Aut}\,\Pi$ of a topological parallelism. Types (ai) and (bi) are the ones that require new ideas.

PROPOSITION 4.1. *If Φ contains an element γ of type (a1), then both eigenvalues have norm 1, and $\langle \gamma \rangle$ has compact closure.*

Proof. 1) We write the second eigenvalue as $c = rb$ with $0 < r \in \mathbb{R}$ and $|b| = 1$. If $r = 1$, then γ belongs to the 2-torus T of unitary diagonal matrices. If $r \neq 1$, we may assume that $r > 1$. Consider the diagonal matrix $\tau \in T$ with eigenvalues a, b. If this has infinite order, then its powers must accumulate at some element of T. Hence in any case, there is a sequence $n(k)$ of positive integers tending to infinity and such that the powers $\tau^{n(k)}$ converge to the unit matrix. We set $\delta_k := \gamma^{n(k)}$.

2) We write elements of $\mathbb{C}^2$ as column vectors and consider a line

$$M = \left\langle \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \right\rangle$$

meeting both factors of $\mathbb{C} \times \mathbb{C}$ nontrivially. The line M^{δ_k} has basis vectors

$$\begin{pmatrix} a^{n(k)} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b^{n(k)} y \end{pmatrix}$$

and hence converges to M as $k \to \infty$.

3) On the other hand, a line N disjoint from $\mathbb{C} \times 0$ may be described using a real 2×2 matrix A; complex numbers are thought of as column vectors in $\mathbb{R}^2$ when they are multiplied by the matrix A:

$$N = \left\{ \begin{pmatrix} Ay \\ y \end{pmatrix} ; y \in \mathbb{C} \right\}.$$

The images N^{δ_k} converge to $0 \times \mathbb{C}$ because y is arbitrary and

$$r^{-n(k)} \begin{pmatrix} Ay \\ y \end{pmatrix}^{\delta_k} \to \begin{pmatrix} 0 \\ y \end{pmatrix}.$$

4) Now the line M considered above cannot be parallel to the line $0 \times \mathbb{C}$ since the two lines have a point in common. However by Lemma 3.3, there is a line N parallel to M and disjoint from $\mathbb{C} \times 0$. Then the δ_k-images of M and N are also parallel, and passing to the limit we obtain that M is parallel to $0 \times \mathbb{C}$, a contradiction. ∎

PROPOSITION 4.2. *The group Φ does not contain any nontrivial element γ of type (a2).*

Proof. 1) As in the previous proof, we take a sequence $n(k)$ of positive integers such that $n(k) \to \infty$ and $a^{n(k)} \to 1$, and we set $\delta_k := \gamma^{n(k)}$.

2) For $x, y \in \mathbb{C}, y \neq 0$ we have

$$(n(k)b)^{-1} \begin{pmatrix} x \\ ay \end{pmatrix}^{\delta_k} = \begin{pmatrix} (n(k)b)^{-1} a^{n(k)} x + a^{n(k)} y \\ (n(k)b)^{-1} a^{n(k)+1} y \end{pmatrix} \to \begin{pmatrix} y \\ 0 \end{pmatrix}.$$

3) Now consider the line

$$L = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ a \end{pmatrix} \right\rangle.$$

By step (2), if L^{δ_k} accumulates at any line X then X contains $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Moreover, X contains the limit of the sequence

$$\begin{pmatrix} -n(k)b \\ a \end{pmatrix}^{\delta_k} = \begin{pmatrix} 0 \\ a^{n(k)+1} \end{pmatrix},$$

which is equal to $\begin{pmatrix} 0 \\ a \end{pmatrix}$. Thus we see that $L^{\delta_k} \to L$ as $k \to \infty$.

4) By Lemma 3.3, there is a line M parallel to L that has trivial intersection with $W := \mathbb{C} \times 0$. Then M contains two vectors whose second components are $\mathbb{R}$-linearly independent, hence $M^{\delta_k} \to W$ by step (2). Applying δ_k to the parallel pair (L, M) and passing to the limit, we obtain that L is parallel to W, a contradiction because L and W have a point in common. ∎

PROPOSITION 4.3. *If Φ contains an element γ of type (b1), then the eigenvalues a, r, s have norm 1, and $\langle \gamma \rangle$ has compact closure.*

Proof. By assumption, $|a| = 1$. We consider the case that not both of r, s have modulus 1 and aim for a contradiction.

1) Assume first that $r = s$. Taking the square of γ, we obtain $r > 0$, and passing to γ^{-1}, if necessary, we get $r > 1$. We use a sequence $n(k)$ of integers as in the previous proofs and also retain the definition of δ_k. We denote the lines $\mathbb{C} \times 0$ and $0 \times \mathbb{C}$ by W and S, respectively.

2) Since we assume that $r = s$, a line L that meets both S and W has images L^{δ_k} converging to L itself. On the other hand, a line M that intersects S, W trivially may be described, using a matrix $A \in \mathrm{GL}(2, \mathbb{R})$, as

$$M = \left\{ \begin{pmatrix} x \\ Ax \end{pmatrix}; \quad x \in \mathbb{C} \right\},$$

and for its images we have

$$M^{\delta_k} = \left\{ \begin{pmatrix} a^{n(k)} x \\ r^{n(k)} Ax \end{pmatrix}; \quad x \in \mathbb{C} \right\} \to \left\{ \begin{pmatrix} 0 \\ Ax \end{pmatrix}; \quad x \in \mathbb{C} \right\} = S.$$

By Lemma 3.3, the line M can be chosen to be parallel to L, and passing to the limit we obtain that L is parallel to S, a contradiction.

3) Now suppose that $r \neq s$. We denote by e_i the standard basis vector of $\mathbb{R}^4$ having coordinates 0 in position $j \neq i$ and coordinate 1 in position i. The corresponding points will be denoted $p_i = \langle e_i \rangle$. Applying Lemma 3.5 to $(p, q) = (p_3, p_4)$, we see that $1 \notin \{r, s\}$ and also that 1 does not lie between r and s. Thus we may assume that $1 < r < s$. This leads to a contradiction as in the proof

of [1], 3.13. For the convenience of the reader, we sketch the argument: We let $K = p_2 \vee p_4$, $L = p_3 \vee p_4 = L^{\delta_k}$. We have $p_i^{\delta_k} \to p_i$ for $i \in \{2, 4\}$, hence $K^{\delta_k} \to K$. We consider two hyperplanes $G = e_1^{\perp}$ and $H = e_4^{\perp}$. By Lemma 3.3 there is a line M parallel to K but not equal to K and not intersecting the lines $G \cap H$ or $p_1 \vee p_2$. Thus M may be written as $M = p \vee q$, where p is a point incident with G but not with H, and q is a point incident with H but not with $(p_1 \vee p_2)$. Then $p^{\delta_k} \to p_4$ and $q^{\delta_k} \to p_3$, hence $M^{\delta_k} \to L$. However,

$$M^{\delta_k} = \Pi(p, K)^{\delta_k} = \Pi(p^{\delta_k}, K^{\delta_k}) \to \Pi(p_4, K) = K.$$

This contradicts $M^{\delta_k} \to L$.   ∎

PROPOSITION 4.4. *The group Φ does not contain any nontrivial element Γ of type (b2).*

Proof. As usual, we may assume that $r > 1$, and we have sequences $n(k)$ and δ_k with the usual properties. We consider the lines $W = \mathbb{C} \times 0$ and $S = 0 \times \mathbb{C}$, which have no common point. If a line L meets both W and S, then there are nonzero complex numbers $x, y \in \mathbb{C}$ such that

$$L = L(x, y) = \left\langle \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \right\rangle.$$

It is easily seen that

$$\begin{pmatrix} x \\ 0 \end{pmatrix}^{\delta_k} \to \begin{pmatrix} x \\ 0 \end{pmatrix} \quad \text{and} \quad \left\langle \begin{pmatrix} 0 \\ y \end{pmatrix} \right\rangle^{\delta_k} \to \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle,$$

whence

$$L(x, y)^{\delta_k} \to M(x) = \left\langle \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle.$$

According to Lemma 3.4, there is a pair of distinct parallel lines $L(x, y)$, $L(x', y')$. Then $x \neq x'$, but $M(x)$ is parallel to $M(x')$ by continuity, a contradiction because these lines have a common point.   ∎

In contrast to the cases with complex eigenvalues treated so far, the remaining cases with only real eigenvalues do not require completely new ideas but can be treated by adapting the proofs given in the connected case in [1]. We shall describe the necessary changes and in one case we shall give an alternative proof. Some simplification over the proof in [1] is achieved by shifting subcases from the 'c' set to the 'a' or 'b' sets.

PROPOSITION 4.5. *If the group Φ contains an element γ of type (ci), $1 \leq i \leq 5$, then γ has order at most two.*

Proof. The square of γ has positive eigenvalues, hence it suffices to show that Φ does not contain a nontrivial element of type (ci) with positive eigenvalues.

   Type (c1). This is the most subtle part of the whole proof. Therefore, we briefly give the argument, although it is almost identical to the one used in [1], 3.5. The matrix γ can be written as $1 + R$, where 1 denotes the unit matrix and

$$R = \begin{pmatrix} 0 & r & & \\ & 0 & r & \\ & & 0 & r \\ & & & 0 \end{pmatrix}.$$

Using the binomial theorem for commuting matrices together with the relation $R^4 = 0$, we obtain $\gamma^n = 1 + nR + \frac{1}{2}(n^2 - n)R^2 + \frac{1}{6}(n^3 - 3n^2 + 2n)R^3$ for positive integers n. Moreover, we find $\gamma^{-n} = 1 - nR + \frac{1}{2}(n^2 + n)R^2 - \frac{1}{6}(n^3 - 9n^2 + 2n)R^3$. The matrix entries are polynomials in n whose degree increases as we move away from the main diagonal of the matrix. From this observation it follows that the dynamics of $\langle \gamma \rangle$ acting on P and on $\mathcal{L}$ is the same as for a nontrivial cyclic subgroup of a one-parameter group of type (c1) as considered in [1]. This means that the images $p^{\gamma^{\pm n}}$ of an arbitrary point p converge to $p_1 = \langle e_1 \rangle$ as $n \to \infty$; moreover, if L_n is a sequence of lines that contain p_1 and the sequence converges to a line L not contained in $H = e_4^{\perp}$, then $L_n^{\gamma^{\pm n}}$ converges to L_{12}, where we set $L_{ij} = \langle e_i, e_j \rangle$. We say that $\gamma^{\pm n}$ continuously converges to the constant map with value L_{12}.

   Now we let $s_n = p_3^{\gamma^{-n}}$ and $M_n = s_n \vee p_4$; then $s_n \to p_1$ and, hence, $M_n \to L_{14}$. The images $M_n^{\gamma^n} = p_3 \vee p_4^{\gamma^n}$ converge to L_{13}. By continuity of the parallelism, the parallel lines $\Pi(M_n^{\gamma^n}, p_1)$ converge to $\Pi(L_{13}, p_1) = L_{13}$. But on the other hand, $\Pi(M_n, p_1)$ contains p_1 and converges to $\Pi(L_{14}, p_1) = L_{14}$, a line not contained in H. Thus by continuous convergence, $\Pi(M_n^{\gamma^n}, p_1) = \Pi(M_n, p_1)^{\gamma^n}$ converges to L_{12}, a contradiction. Thus, type (c1) cannot occur.

   Type (c2). The argument from [1], 3.7 goes through, but we give a variation. The line $L = \langle e_1, e_4 \rangle$ is fixed by γ. There is a unique line K parallel to L which is contained in the hyperplane $H = \langle e_1, e_2, e_3 \rangle$ corresponding to the Jordan block of size 3, see [11], 64.10a. Hence K is fixed by γ and must contain an eigenvector for the unique eigenvalue of γ on H. Thus $e_1 \in K$, a contradiction.

   Type (c3). The proof from [1] can be adapted, but we prefer to sketch a more straightforward (though perhaps less beautiful) argument.

   Any line which has trivial intersection with $W = \mathbb{C} \times 0$ can be described as

$$L_A = \left\{ \begin{pmatrix} Ay \\ y \end{pmatrix} ; y \in \mathbb{C} \right\},$$

where A is a real 2×2 matrix. A block diagonal matrix

$$\begin{pmatrix} B & \\ & C \end{pmatrix}$$

transforms L_A into $L_{BAC^{-1}}$, and an easy computation shows in view of $0 < r < 1$ that $L_A^{\gamma^n}$ converges to $L_0 = S = 0 \times \mathbb{C}$. Now by 3.3, the line $K = \langle e_1, e_3 \rangle$, which is fixed by γ, is parallel to some line L_A. This implies that K is parallel to S, a contradiction.

Type (c4). As in [1], this is easily ruled out by applying Lemma 3.5 to the points $\langle e_1 \rangle$ and $\langle e_3 \rangle$.

Type (c5). Both proofs of [1], 3.13 can be used without change. Note that we need only consider the case where the eigenvalues are pairwise different.   ∎

The proof of Proposition 3.2 and, hence, of Theorem 1.1 is now complete.

References

[1] D. Betten and R. Löwen, Compactness of the automorphism group of a topological parallelism on real projective 3-space, *Results in Mathematics* 72, 1021 – 1030, 2017

[2] D. Betten and R. Riesinger, Generalized line stars and topological parallelisms of the real projective 3-space. *J. Geometry* 91, 1-20, 2008.

[3] D. Betten and R. Riesinger, Parallelisms of $PG(3; \mathbb{R})$ composed of non-regular spreads, *Aequationes Math.* 81, 227-250, 2011.

[4] D. Betten and R. Riesinger, Clifford parallelism: old and new definitions, and their use, *J. Geometry* 103, 31 – 73, 2012.

[5] D. Betten and R. Riesinger, Collineation groups of topological parallelisms, *Adv. in Geometry* 14, 175 – 189, 2014.

[6] D. Betten and R. Riesinger, Automorphisms of some topological regular parallelisms of $PG(3; \mathbb{R})$, *Results in Math.* 66, 291-326, 2014.

[7] D. Djocović: The union of compact subgroups of a connected locally compact group, *Math. Zeitschr.* 158, 99 – 105, 1978.

[8] R. Kühne and R. Löwen, Topological projective spaces, *Abh. Math. Sem. Univ. Hamb.* 62, 1 – 9, 1992.

[9] R. Löwen, A characterization of Clifford parallelism by automorphisms, arXiv:1702.03328; *Innovations in Incidence Geometry*, to appear.

[10] R. Löwen, Rotational spreads and rotational parallelisms and oriented parallelisms of PG(3, $\mathbb{R}$), arXiv:1804.07615.

[11] H. Salzmann, D. Betten, T. Grundhöfer, H. Hähl, R. Löwen, M. Stroppel, *Compact projective planes*, Berlin etc.: de Gruyter 1995.

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