Super-biderivations and super-commuting maps on the topological $N = 2$ superconformal algebra

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Abstract

Let $\mathcal{T}$ be the well-known topological $N = 2$ superconformal algebra. In this paper, we prove that every super-skewsymmetric super-biderivation of $\mathcal{T}$ is inner. Based on the result of super-biderivations, we show that all the linear super-commuting maps on $\mathcal{T}$ which have the form $\psi(x) = \lambda x + f(x)c$ are not standard.

1 Introduction

Lie superalgebras as a generalization of Lie algebras came from supersymmetry in mathematical physics. The theory of Lie superalgebras plays an important role in modern mathematics. Derivations, biderivations and super-biderivations are very important topics in the theory of both algebras and their generalizations. They also have attracted many scholars' great interests ([1], [2], [5]-[7]). Let $\mathcal{S}$ be a Lie superalgebra with $\mathbb{Z}_2$-grading $\mathcal{S} = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$ and $\mathcal{S}_{\bar{0}}$ and $\mathcal{S}_{\bar{1}}$ are even and odd parts of $\mathcal{S}$, respectively. Recall that [13] (Section 1.1.4) a super-derivation of degree $|d|$, $d \in \mathbb{Z}_2$, of a Lie superalgebra $\mathcal{S}$ is an endomorphism $D \in End_{|d|}\mathcal{S}$ such that

$$D([x, y]) = [D(x), y] + (-1)^{|d||x|}[x, D(y)].$$

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It is clear that the super-derivation is odd if $|d| = \overline{1}$, otherwise $|d| = \overline{0}$ and the super-derivation is even. Here, and in what follows, we use the notation $|x|$ to denote the $\mathbb{Z}_2$-degree of a homogeneous element $x \in L$, and we always assume that x is homogeneous if x appears in an expression. In addition, for a super-derivation d of $\mathcal{S}$, we can get $D(\mathcal{S}_\alpha) \subseteq \mathcal{S}_{\alpha+|d|}$ for $\alpha \in \mathbb{Z}_2$ from [13] (Example 1). Derivation algebra and automorphism group of generalized topological $N = 2$ superconformal algebra has been determined by Yang, Yu and Yao in [20]. Brešar showed that all biderivations on commutative prime rings are inner biderivations and determined the biderivations of semiprime rings in [6]. The notion of biderivations of Lie algebras was introduced in [18]. Super-biderivations are the extension of biderivations. The definition of super-biderivations was introduced in [10] and [19], respectively.

For super-biderivations, recall a bilinear map $\varphi : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ which satisfies every $x_{\bar{0}} \in \mathcal{S}_{\bar{0}}$ the maps $x \mapsto \varphi(x_{\bar{0}}, x)$ and $x \mapsto \varphi(x, x_{\bar{0}})$ are even super-derivations, and for every $x_{\bar{1}} \in \mathcal{S}_{\bar{1}}$ the maps $x \mapsto \varphi(x_{\bar{1}}, x)$ and $x \mapsto \varphi(x, x_{\bar{1}})^\sigma$ are odd super-derivations, where σ is defined by $(x_{\bar{0}} + x_{\bar{1}})^\sigma = x_{\bar{0}} - x_{\bar{1}}$ for $x_{\bar{0}} \in \mathcal{S}_{\bar{0}}$ and $x_{\bar{1}} \in \mathcal{S}_{\bar{1}}$. Then φ is called a *super-skewsymmetric super-biderivation* of $\mathcal{S}$ and it is equivalent to

$$\varphi([x,y],z) = [x, \varphi(y,z)] + (-1)^{|y||z|}[\varphi(x,z), y], \tag{1.1}$$

$$\varphi(x,[y,z]) = [\varphi(x,y), z] + (-1)^{|x||y|}[y, \varphi(x,z)], \tag{1.2}$$

$$\varphi(x,y) = -(-1)^{|x||y|}\varphi(y,x) \tag{1.3}$$

for all $x, y, z \in \mathcal{S}$. Considering the $\mathbb{Z}_2$-degree of a homogeneous element $x \in \mathcal{S}$, it is easy to obtain that the map φ_λ with $\lambda \in \mathbb{C}$ given by

$$\varphi_\lambda(x,y) = \lambda[x,y] \quad \text{for all} \quad (x,y) \in \mathcal{S} \times \mathcal{S},$$

is a super-biderivation of $\mathcal{S}$. The above form map is called an *inner super-biderivation* of $\mathcal{S}$. Any super-biderivations of other forms are said to be *non-inner*. Finally, we comment that the definition of super-biderivations here is actually the super-biderivations of degree $\overline{0} \in \mathbb{Z}_2$ in [10].

Commuting maps which are involved in the various aspects have a long and rich history [3], [4] and [5] on associative algebras. Recently, the commuting maps on some Lie algebras have been studied. For instance, Wang and Yu proved that all the linear commuting maps on the Schrödinger-Virasoro Lie algebra were standard in [17] and all linear super-commuting maps on the super-Virasoro algebras were also proved standard by Xia, Wang and Han in [19]. The concept of commuting maps has been introduced in [12]. A commuting map ψ on $\mathcal{L}$ is called *standard* if it has the following form

$$\psi(x) = \lambda x + f(x) \quad \text{for all} \quad x \in \mathfrak{L},$$

where λ is a complex number, and f is a map from $\mathfrak{L}$ to its center. All commuting maps of other forms are said to be *non-standard*.

From the above definition, the concept of super-commuting maps on Lie superalgebras was given in [10] and [19], respectively. Let $\mathcal{S}$ be a Lie superalgebra with $\mathbb{Z}_2$-grading $\mathcal{S} = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$ and $\mathcal{S}_{\bar{0}}$ and $\mathcal{S}_{\bar{1}}$ are even and odd parts of

$\mathcal{S}$, respectively. A map $\psi : \mathcal{S} \to \mathcal{S}$ is called *super-commuting* if it satisfies the $\mathbb{Z}_2$-grading of $\mathcal{S}$ and

$$[\psi(x), x] = 0 \quad \text{for all} \quad x \in \mathcal{S}.$$

A super-commuting map ψ on $\mathcal{S}$ is called *standard* if it maps the even part $\mathcal{S}_{\bar{0}}$ of $\mathcal{S}$ to the center of $\mathcal{S}$, and maps the odd part $\mathcal{S}_{\bar{1}}$ of $\mathcal{S}$ to zero. All super-commuting maps of other forms are said to be *non-standard*. Furthermore, compared with that in [10], the definition of super-commuting maps here has an additional condition of preserving gradation.

In this paper, we mainly study the topological $N = 2$ superconformal algebra which is one of the $N = 2$ superconformal algebras. There are four classes of $N = 2$ superconformal algebras in [11]. In [16], the author studied topological field theories. The topological $N = 2$ superconformal algebra was presented in [8], which is the symmetry algebra of topological conformal field theory in two dimensions. This algebra can be obtained from the Neveu-Schwarz $N = 2$ superconformal algebra by "twisting" the stress-energy tensor by adding the derivative of the $U(1)$ current, procedure known as "topological twist" [9], [14] and [15]. And it is obvious to find that this Lie superalgebra $\mathcal{T} = \mathcal{T}_{\bar{0}} \oplus \mathcal{T}_{\bar{1}}$ is given by

$$\mathcal{T}_{\bar{0}} = \{L_m,\, H_n,\, c \,|\, m, n \in \mathbb{Z}\},$$
$$\mathcal{T}_{\bar{1}} = \{G_m,\, Q_n \,|\, m, n \in \mathbb{Z}\}.$$

The topological $N = 2$ superconformal algebra reads

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n}, \\
[L_m, G_n] &= (m-n)G_{m+n}, \\
[L_m, Q_n] &= -nQ_{m+n}, \\
[L_m, H_n] &= -nH_{m+n} + \frac{c}{6}(m^2+m)\delta_{m+n,0}, \\
[G_m, Q_n] &= 2L_{m+n} - 2nH_{m+n} + \frac{c}{3}(m^2+m)\delta_{m+n,0}, \\
[H_m, H_n] &= \frac{c}{3}\delta_{m+n,0}, \\
[H_m, G_n] &= G_{m+n}, \\
[H_m, Q_n] &= -Q_{m+n},
\end{aligned}$$

where $m, n \in \mathbb{Z}$. It is easy to see that the center of this algebra is $Z(\mathcal{T}) = \mathbb{C}c$.

The structure of the paper is as follows. In Section 2, we recall some basic results on super-biderivations of Lie superalgebras in [19]. In Section 3, we study the super-skewsymmetric super-biderivations of the topological $N = 2$ superconformal algebra $\mathcal{T}$. Finally, in Section 4, we show that all the linear super-commuting maps on $\mathcal{T}$ are not standard which is based on the result of super-skewsymmetric super-biderivations. In addition, comparing our main results with both Virasoro Lie algebra *Vir* in [17] and super-Virasoro algebra *SVir* in [10] and [19], we find some differences (see Table 1 and Table 2).

Table 1: Are all the (super-)skewsymmetric (super-)biderivations of $\mathcal{T}$ inner ?

	Answer for the question	Reference
Vir	Yes	[17, Theorem 3.1]
$SVir$	Yes	[19, Theorem 3.1]
$\mathcal{T}$	Yes	Theorem 3.1

Table 2: Are all the linear (super-)commuting maps on $\mathcal{T}$ standard ?

	Answer for the question	Reference
Vir	Yes	[17, Theorem 4.1]
$SVir$	Yes	[19, Theorem 4.1]
$\mathcal{T}$	No	Theorem 4.1

2 Basic results on super-biderivations of Lie superalgebras

In this section, we mainly give two useful results that are quoted from [19]. Firstly, Let $\mathcal{S}$ be a Lie superalgebra with the center $Z(\mathcal{S})$ and $\varphi : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ be a bilinear map. In addition,

$$F(x,y,u,v) = (-1)^{|u||y|}([\varphi(x,y),[u,v]] - [[x,y],\varphi(u,v)]) \quad \text{for} \quad x,y,u,v \in \mathcal{S}.$$

Lemma 2.1. *Let φ be a super-biderivation on $\mathcal{S}$. Then*

$$F(x,y,u,v) = (-1)^{|y||v|}F(x,v,u,y) \quad \textit{for} \quad x,y,u,v \in \mathcal{S}.$$

Lemma 2.2. *Let φ be a super-skewsymmetric super-biderivation on $\mathcal{S}$.*
(1) *$F(x,y,u,v) = 0$ for $x,y,u,v \in \mathcal{S}$.*
(2) *For $x,y \in \mathcal{S}$, if $|x| + |y| = 0$, then $[\varphi(x,y),[x,y]] = 0$.*
(3) *Suppose $\mathcal{S}$ is perfect. For $x,y \in \mathcal{S}$, if $[x,y] = 0$, then $\varphi(x,y) \in Z(\mathcal{S})$.*

3 Super-skewsymmetric super-biderivations of the topological $N = 2$ superconformal algebra

In this section, we shall give a description of the super-skewsymmetric super-biderivations of the topological $N = 2$ superconformal algebra $\mathcal{T}$. We know that all the super-skewsymmetric super-biderivations of the super-Virasoro algebra are inner in [10] and [19]. Clearly, the inner super-biderivation $(x,y) \mapsto \lambda[x,y]$ with $\lambda \in \mathbb{C}$ of any Lie superalgebra is super-skewsymmetric.

Theorem 3.1. *Let φ be a super-skewsymmetric super-biderivation of the topological $N = 2$ superconformal algebra $\mathcal{T}$. We have*

$$\varphi(x,y) = \lambda[x,y] \quad \textit{for} \quad x,y \in \mathcal{T},$$

where $\lambda \in \mathbb{C}$.

Proof. Obviously, we notice that $|\varphi(x,y)| = |x| + |y|$ for any homogeneous elements $x, y \in \mathcal{T}$ from the definition of super-skewsymmetric super-biderivations.

We will give the proof of the theorem by the following several claims.

Claim 1. There exists $\lambda \in \mathbb{C}$ such that

$$\varphi(L_m, L_n) \equiv \lambda[L_m, L_n] \pmod{\mathbb{C}c} \quad \text{for} \quad m, n \in \mathbb{Z}.$$

Notice that $|\varphi(L_m, L_n)| = |L_m| + |L_n| = \bar{0}$. So we can suppose that

$$\varphi(L_m, L_n) = \sum_{i\in\mathbb{Z}} a_i^{(1)} L_i + \sum_{j\in\mathbb{Z}} b_j^{(1)} H_j + \theta^{(1)} c,$$

$a_i^{(1)}, b_j^{(1)}, \theta^{(1)} \in \mathbb{C}$, for any fixed $m, n \in \mathbb{Z}$.

If $m = n$, then $[L_m, L_n] = 0$. By Lemma 2.2 (3), we have $\varphi(L_m, L_n) \in Z(\mathcal{T})$. Since the center of $\mathcal{T}$ is $Z(\mathcal{T}) = \mathbb{C}c$, this claim holds.

Next, we assume that $m \neq n$. By Lemma 2.2 (2), we have

$$\frac{1}{m-n}[\varphi(L_m, L_n), [L_m, L_n]] = 0,$$

that is,

$$\sum_{i\in\mathbb{Z}} a_i^{(1)}(i-m-n)L_{m+n+i} + \sum_{j\in\mathbb{Z}} b_j^{(1)} j M_{m+n+j} = 0,$$

from which it follows that

$$a_i^{(1)}(i-m-n) = 0; \quad b_j^{(1)} j = 0. \tag{3.1}$$

So we have $a_i^{(1)} = 0$ if $i \neq m+n$ and $b_j^{(1)} j = 0$ if $j \neq 0$. Then we get

$$\varphi(L_m, L_n) \equiv a_{m+n}^{(1)} L_{m+n} + b_0^{(1)} H_0 \pmod{\mathbb{C}c}.$$

Furthermore, we use Lemma 2.2 (1) to get

$$[\varphi(L_m, L_n), [L_k, L_0]] = [[L_m, L_n], \varphi(L_k, L_0)],$$

and then we have

$$a_{m+n}^{(1)}(m+n-k)L_{m+n+k} = (m-n)a_1^{(1)}(m+n-k)L_{m+n+k}.$$

By the arbitrariness of k, m and n, we must have

$$a_{m+n}^{(1)} = (m-n)a_1^{(1)}.$$

Taking $\lambda = a_1^{(1)}$, we have

$$\varphi(L_m, L_n) \equiv \lambda[L_m, L_n] + b_0^{(1)} H_0 \pmod{\mathbb{C}c}.$$

We will prove $b_0^{(1)} = 0$ in the proof of the Claim 2.

Claim 2. $\varphi(L_m, G_n) \equiv \lambda[L_m, G_n] \pmod{\mathbb{C}c}$ for $m, n \in \mathbb{Z}$.

Note that $|\varphi(L_m, G_n)| = |L_m| + |G_n| = \bar{1}$. For any fixed $m, n \in \mathbb{Z}$, we may suppose that

$$\varphi(L_m, G_n) = \sum_{i\in\mathbb{Z}} a_i^{(2)} G_i + \sum_{j\in\mathbb{Z}} b_j^{(2)} Q_j,$$

where $a_i^{(2)}, b_j^{(2)} \in \mathbb{C}$.

If $m = n$, then $[L_m, G_n] = 0$. By Lemma 2.2 (3), we have $\varphi(L_m, G_n) \in Z(\mathcal{T})$. This claim holds.

Next, we assume that $m \neq n$. By Lemma 2.2 (1), we have

$$[\varphi(L_m, G_n), [L_1, L_0]] = [[L_m, G_n], \varphi(L_1, L_0)],$$

from which it follows that

$$\sum_{i\in\mathbb{Z}} a_i^{(2)}(i-1)G_{i+1} + \sum_{j\in\mathbb{Z}} b_j^{(2)} j Q_{j+1} = \lambda(m-n)(m+n-1)G_{m+n+1} - (m-n)b_0^{(1)} G_{m+n}.$$

So we have $a_i^{(2)}(m+n-2) = -(m-n)b_0^{(1)}$ if $i = m+n-1$; $(m-n)b_0^{(1)} = 0$ if $i \neq m+n-1$ and $b_j^{(2)} = 0$ if $j \neq 0$. By the arbitrariness of m and n, we must have $b_0^{(1)} = 0$.

Then we get

$$\varphi(L_m, L_n) \equiv \lambda[L_m, L_n] \pmod{\mathbb{C}c} \quad \text{for} \quad m, n \in \mathbb{Z}.$$

The Claim 1 holds.

Since $b_j^{(2)} = 0$ if $j \neq 0$ we can get

$$\varphi(L_m, G_n) \equiv \sum_{i\in\mathbb{Z}} a_i^{(2)} G_i + b_0^{(2)} Q_0 \pmod{\mathbb{C}c}.$$

Furthermore, by Lemma 2.2 (1), we have

$$[\varphi(L_m, G_n), [L_k, L_0]] = [[L_m, G_n], \varphi(L_k, L_0)],$$

that is,

$$\sum_{i\in\mathbb{Z}} k a_i^{(2)}(i-k)G_{i+k} = k\lambda(m-n)(m+n-k)G_{m+n+k}.$$

We must have $a_{m+n}^{(2)} = \lambda(m-n)$ if $i = m+n \neq k$; $a_i^{(2)} = 0$ if $i \neq m+n$. By the arbitrariness of k, we know $a_{m+n}^{(2)} = \lambda(m-n)$. Then we obtain

$$\varphi(L_m, G_n) \equiv \lambda[L_m, G_n] + b_0^{(2)} Q_0 \pmod{\mathbb{C}c}.$$

We will prove $b_0^{(2)} = 0$ in the proof of the Claim 4.

Claim 3. $\varphi(L_m, Q_n) \equiv \lambda[L_m, Q_n] \pmod{\mathbb{C}c}$ for $m, n \in \mathbb{Z}$.

Note that $|\varphi(L_m, Q_n)| = |L_m| + |Q_n| = \bar{1}$. For any fixed $m, n \in \mathbb{Z}$, we may suppose that

$$\varphi(L_m, Q_n) = \sum_{i\in\mathbb{Z}} a_i^{(3)} G_i + \sum_{j\in\mathbb{Z}} b_j^{(3)} Q_j,$$

where $a_i^{(3)}, b_j^{(3)} \in \mathbb{C}$.

By Lemma 2.2 (1), we have

$$[\varphi(L_m, Q_n), [L_k, L_0]] = [[L_m, Q_n], \varphi(L_k, L_0)],$$

that is,

$$\sum_{i\in\mathbb{Z}} k a_i^{(3)} (i-k) G_{i+k} + \sum_{j\in\mathbb{Z}} k b_j^{(3)} j Q_{j+k} = -\lambda k n (m+n) Q_{m+n+k}.$$

Thus we have $a_i^{(3)} = 0$ if $i \neq k$; $b_j^{(3)} = -\lambda n$ if $j = m+n \neq 0$ and $b_j^{(3)} j = 0$ if $j \neq m+n$. By the arbitrariness of k, m and n, we must have $a_i^{(3)} = 0$. Then

$$\varphi(L_m, Q_n) \equiv \lambda[L_m, Q_n] + b_0^{(3)} Q_0 \pmod{\mathbb{C}}. \tag{3.2}$$

Furthermore, using Lemma 2.2 (1), we have

$$[\varphi(L_m, Q_n), [L_1, G_0]] = [[L_m, Q_n], \varphi(L_1, G_0)],$$

then we get

$$\begin{aligned} -\lambda n(2(m+n)H_{m+n+1} - 2L_{m+n+1}) + 2b_0^{(3)} L_1 = \\ -\lambda n(2(m+n)H_{m+n+1} - 2L_{m+n+1}). \end{aligned}$$

So we must have $b_0^{(3)} = 0$. Hence, we know this claim holds from (3.2).

Claim 4. $\varphi(L_m, H_n) \equiv \lambda[L_m, H_n] \pmod{\mathbb{C}}$ for $m, n \in \mathbb{Z}$.

Note that $|\varphi(L_m, H_n)| = |L_m| + |H_n| = \bar{0}$. For any fixed $m, n \in \mathbb{Z}$, we may suppose that

$$\varphi(L_m, H_n) = \sum_{i\in\mathbb{Z}} a_i^{(4)} L_i + \sum_{j\in\mathbb{Z}} b_j^{(4)} H_j + \theta^{(4)} c,$$

where $a_i^{(4)}, b_j^{(4)}, \theta^{(4)} \in \mathbb{C}$.

If $n = 0$, then $[L_m, H_n] = 0$. By Lemma 2.2 (3), we have $\varphi(L_m, H_n) \in Z(\mathcal{T})$. This claim holds.

Next, we assume that $n \neq 0$, by Lemma 2.2 (2), we have

$$\frac{1}{n}[\varphi(L_m, H_n), [L_m, H_n]] = 0,$$

that is,

$$\sum_{i\in\mathbb{Z}} a_i^{(4)} (m+n) H_{m+n+i} = 0.$$

By the arbitrariness of m and n, we must have $a_i^{(4)} = 0$. Then

$$\varphi(L_m, H_n) \equiv \sum_{j\in\mathbb{Z}} b_j^{(4)} H_j \pmod{\mathbb{C}c}.$$

Furthermore, by Lemma 2.2 (1), we have

$$[\varphi(L_m, H_n), [L_k, L_0]] = [[L_m, H_n], \varphi(L_k, L_0)],$$

and that means

$$\sum_{j\in\mathbb{Z}} k b_j^{(4)} j H_{j+k} = -k\lambda n(m+n) H_{m+n+k}.$$

Thus we obtain $b_j^{(4)} = -\lambda n$ if $j = m+n \neq 0$; $b_j^{(4)} j = 0$ if $j \neq m+n$. By the arbitrariness of m and n, we know

$$\varphi(L_m, H_n) \equiv \lambda[L_m, H_n] + b_0^{(4)} H_0 \pmod{\mathbb{C}c}.$$

Due to

$$[\varphi(L_m, H_n), [L_1, G_0]] = [[L_m, H_n], \varphi(L_1, G_0)],$$

we get

$$-\lambda n G_{m+n+1} + b_0^{(4)} G_1 = -\lambda n G_{m+n+1} + n b_0^{(3)} Q_{m+n},$$

that implies $b_0^{(3)} = 0$ and $b_0^{(4)} = 0$. Hence we have

$$\varphi(L_m, G_n) \equiv \lambda[L_m, G_n] \pmod{\mathbb{C}c} \quad \text{for} \quad m, n \in \mathbb{Z}.$$

The Claim 2 holds. And

$$\varphi(L_m, H_n) \equiv \lambda[L_m, H_n] \pmod{\mathbb{C}c} \quad \text{for} \quad m, n \in \mathbb{Z}.$$

This Claim holds.

Claim 5. $\varphi(G_m, Q_n) \equiv \lambda[G_m, Q_n] \pmod{\mathbb{C}c} \quad \text{for} \quad m, n \in \mathbb{Z}.$

Note that $|\varphi(G_m, Q_n)| = |G_m| + |Q_n| = \bar{0}$. For any fixed $m, n \in \mathbb{Z}$, we may suppose that

$$\varphi(G_m, Q_n) = \sum_{i\in\mathbb{Z}} a_i^{(5)} L_i + \sum_{j\in\mathbb{Z}} b_j^{(5)} H_j + \theta^{(5)} c,$$

where $a_i^{(5)}, b_j^{(5)}, \theta^{(5)} \in \mathbb{C}$.

From Lemma 2.2 (2), we have

$$[\varphi(G_m, Q_n), [G_m, Q_n]] = 0,$$

from which it follows that

$$\sum_{i\in\mathbb{Z}} 2a_i^{(5)}(i-m-n) L_{m+n+i} + \sum_{j\in\mathbb{Z}} 2b_j^{(5)} j H_{m+n+j} + \sum_{i\in\mathbb{Z}} 2n a_i^{(5)}(m+n) H_{m+n+i} = 0.$$

Then we have that $a_i^{(5)} = 0$ if $i \neq m+n$; $b_j^{(5)} = -na_{m+n}^{(5)}$ if $j = m+n \neq 0$ and $b_j^{(5)} j = 0$ if $j \neq m+n$. Thus we obtain

$$\varphi(G_m, Q_n) \equiv a_{m+n}^{(5)} L_{m+n} - na_{m+n}^{(5)} H_{m+n} + b_0^{(5)} H_0 \pmod{\mathbb{C}c}.$$

Furthermore , using Lemma 2.2 (1), we have

$$[\varphi(G_m, Q_n), [L_k, L_0]] = [[G_m, Q_n], \varphi(L_k, L_0)],$$

and then we have

$$\begin{aligned} ka_{m+n}^{(5)}(m+n-k)L_{m+n+k} - nka_{m+n}^{(5)}(m+n)H_{m+n+k} = \\ 2k\lambda((m+n-k)L_{m+n+k} - 2k\lambda n(m+n)H_{m+n+k}. \end{aligned}$$

By comparing both sides of the equation, by the arbitrariness of k, m and n, we get $a_{m+n}^{(5)} = 2\lambda$. Furthermore, we have

$$\varphi(G_m, Q_n) = \lambda[G_m, Q_n] + b_0^{(5)} H_0 \pmod{\mathbb{C}c}.$$

And by

$$[\varphi(G_m, Q_n), [L_1, G_0]] = [[G_m, Q_n], \varphi(L_1, G_0)],$$

we have

$$\begin{aligned} 2\lambda(m+n-1)G_{m+n+1} - 2\lambda nG_{m+n+1} + b_0^{(5)} G_1 = \\ 2\lambda(m+n-1)G_{m+n+1} - 2\lambda nG_{m+n+1}, \end{aligned}$$

that means $b_0^{(5)} = 0$. Hence, we get

$$\varphi(G_m, Q_n) \equiv \lambda[G_m, Q_n] \pmod{\mathbb{C}c} \quad \text{for} \quad m, n \in \mathbb{Z}.$$

Claim 6. $\varphi(H_m, G_n) \equiv \lambda[H_m, G_n] \pmod{\mathbb{C}c}$ for $m, n \in \mathbb{Z}$.

Note that $|\varphi(H_m, G_n)| = |H_m| + |G_n| = \bar{1}$. For any fixed $m, n \in \mathbb{Z}$, we can suppose that

$$\varphi(H_m, G_n) = \sum_{i \in \mathbb{Z}} a_i^{(6)} G_i + \sum_{j \in \mathbb{Z}} b_j^{(6)} Q_j,$$

where $a_i^{(6)}, b_j^{(6)} \in \mathbb{C}$.

By Lemma 2.2 (1), we have

$$[\varphi(H_m, G_n), [L_k, L_0]] = [[H_m, G_n], \varphi(L_k, L_0)],$$

we have

$$\sum_{i \in \mathbb{Z}} ka_i^{(6)}(i-k)G_{i+k} + \sum_{j \in \mathbb{Z}} kb_j^{(6)} jQ_{j+k} = k\lambda(m+n-k)G_{m+n+k}.$$

Then we have that $a_i^{(6)} = \lambda$ if $i = m + n \neq k$; $a_i^{(6)} = 0$ if $i \neq m + n$, and $b_j^{(6)} = 0$ if $j \neq 0$, and by the arbitrariness of k, $a_i^{(6)} = \lambda$. Therefore,

$$\varphi(H_m, G_n) \equiv \lambda[H_m, G_n] + b_0^{(6)} Q_0 \pmod{\mathbb{C}c}. \tag{3.3}$$

Since Lemma 2.2 (1), we have

$$[\varphi(H_m, G_n), [L_1, G_0]] = [[H_m, G_n], \varphi(L_1, G_0)],$$

that is

$$2b_0^{(6)} L_1 = 0.$$

So we get $b_0^{(6)} = 0$. Hence, considering (3.3), this claim holds.

Claim 7. $\varphi(H_m, Q_n) \equiv \lambda[H_m, Q_n] \pmod{\mathbb{C}c}$ for $m, n \in \mathbb{Z}$.

Notice that $|\varphi(H_m, Q_n)| = |H_m| + |Q_n| = \bar{1}$. For any fixed $m, n \in \mathbb{Z}$, we can suppose that

$$\varphi(H_m, Q_n) = \sum_{i \in \mathbb{Z}} a_i^{(7)} G_i + \sum_{j \in \mathbb{Z}} b_j^{(7)} Q_j,$$

where $a_i^{(7)}, b_j^{(7)} \in \mathbb{C}$.

By Lemma 2.2 (1), we have

$$[\varphi(H_m, Q_n), [L_k, L_0]] = [[H_m, Q_n], \varphi(L_k, L_0)],$$

we have

$$\sum_{i \in \mathbb{Z}} k a_i^{(7)} (i - k) G_{i+k} + \sum_{j \in \mathbb{Z}} k b_j^{(7)} j Q_{j+k} = -k\lambda(m + n) Q_{m+n+k}.$$

Thus we obtain that $a_i^{(7)} = 0$ if $i \neq k$; $b_j^{(7)} j = 0$ if $j \neq m + n$ and $b_j^{(7)} j = -\lambda$ if $j = m + n \neq 0$ and by the arbitrariness of m,n and k, we get

$$\varphi(H_m, Q_n) \equiv -\lambda[H_m, Q_n] + b_0^{(7)} Q_0 \pmod{\mathbb{C}c}. \tag{3.4}$$

Using Lemma 2.2 (1), we have

$$[\varphi(H_m, Q_n), [L_1, G_0]] = [[H_m, Q_n], \varphi(L_1, G_0)].$$

That is,

$$-\lambda(2(m + n)H_{m+n+1} - 2L_{m+n+1}) - 2b_0^{(7)} L_1 = -\lambda(2(m + n)H_{m+n+1} - 2L_{m+n+1}).$$

So we have $b_0^{(7)} = 0$. Hence, combining (3.4), this claim holds.

Claim 8. $\varphi(H_m, H_n) \equiv \lambda[H_m, H_n] \pmod{\mathbb{C}c}$ for $m, n \in \mathbb{Z}$.

Since $[H_m, H_n] \equiv 0 \pmod{\mathbb{C}c}$. By Lemma 2.2 (3), we must have $\varphi(H_m, H_n) \in Z(\mathcal{T})$. And the center of $\mathcal{T}$ is $Z(\mathcal{T}) = \mathbb{C}c$, so

$$\varphi(H_m, H_n) \equiv 0 \pmod{\mathbb{C}}.$$

This claim holds.

Claim 9. $\varphi(x,y) \equiv \lambda[x,y] \pmod{\mathbb{C}c}$ for $x,y \in \mathcal{T}$.

If $[x,y]=0$, then this claim clearly holds.
If $[x,y] \neq 0$, then this claim follows from Claim 1–Claim 8.

Now, by Claim 9, we may assume that

$$\varphi(x,y) = \lambda[x,y] + f(x,y)c \quad \text{for} \quad x,y \in \mathcal{T},$$

where f is a bilinear function from $\mathcal{T} \times \mathcal{T}$ to $\mathbb{C}$. Then, for $x,y,z \in \mathcal{T}$, due to

$$\varphi([x,y],z) = [x,\varphi(y,z)] + (-1)^{|y||z|}[\varphi(x,z),y],$$

we have

$$f([x,y],z)c = 0.$$

Since $[\mathcal{T},\mathcal{T}] = \mathcal{T}$, that is, $\mathcal{T}$ is perfect, one sees that f must be the zero functions. Hence, $\varphi(x,y) = \lambda[x,y]$. It is desired. ∎

4 Super-commuting maps on the the topological $N=2$ super-conformal algebra

In this section, we shall give the form of the linear super-commuting maps on the the topological $N=2$ superconformal algebra $\mathcal{T}$ based on Theorem 3.1. We have the following result, which generalizes the result for the super-Virasoro algebra *SVir* given in [10] and [19], respectively.

Theorem 4.1. *Each linear super-commuting map ψ on $\mathfrak{L}$ has the following form*

$$\psi(x) = \lambda x + f(x)c \quad \textit{for all} \quad x \in \mathcal{T},$$

where f is a linear function from $\mathcal{T}$ to $\mathbb{C}$ mapping the odd part $\mathcal{T}_{\bar{1}}$ of $\mathcal{T}$ to zero. That implies all linear super-commuting maps on the topological $N=2$ superconformal algebra $\mathcal{T}$ are not standard.

Proof. Let ψ be a linear super-commuting map on the topological $N=2$ superconformal algebra $\mathcal{T}$. Define

$$\begin{aligned} \varphi: \quad & \mathcal{T} \times \mathcal{T} \to \mathcal{T} \\ & (x,y) \mapsto [\psi(x),y] \end{aligned}$$

for $x,y \in \mathcal{T}$. Notice that ψ maintains the $\mathbb{Z}_2$-grading of $\mathcal{T}$. By the definition of φ, one can easily versify that

$$\varphi(x,[y,z]) = [\varphi(x,y),z] + (-1)^{|x||y|}[y,\varphi(x,z)] \quad \text{for} \quad x,y,z \in \mathcal{T}.$$

Namely, φ satisfies the equation (1.2). Recalling $[\psi(x),y] = (-1)^{|x||y|}[x,\psi(y)]$ (ψ is a linear super-commuting map), one can easily check the other equation (1.1). In addition, φ is super-skewsymmetric by its definition. Thus, φ is a super-skewsymmetric super-biderivation of $\mathcal{T}$. By Theorem 3.1, there exists $\lambda \in \mathbb{C}$ such that

$$\varphi(x,y) = \lambda[x,y] \quad \text{for} \quad x,y \in \mathcal{T}.$$

Considering the definition of φ, we have

$$[\psi(x) - \lambda x, y] = 0. \tag{4.1}$$

From the above Theorem 3.1, we also have

$$\varphi(x,y) \equiv [\psi(x), y] \pmod{\mathbb{C}c} \quad \text{for} \quad x,y \in \mathcal{T}.$$

Then, by (4.1), we see that

$$[\psi(x) - \lambda x, y] \equiv 0 \pmod{\mathbb{C}c}.$$

This means that

$$\psi(x) - \lambda x \in \mathbb{C}c \quad \text{for} \quad x \in \mathcal{T}.$$

Thus, we may assume that

$$\psi(x) - \lambda x = f(x)c,$$

where f is a linear functions from $\mathcal{T}$ to $\mathbb{C}$. Hence, $\psi(x) = \lambda x + f(x)c$. This completes the proof. ■

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