# Coarse Lipschitz embeddings of James spaces

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**Abstract**

We prove that, for $1 < p \neq q < \infty$, there does not exist any coarse Lipschitz embedding between the two James spaces $J_p$ and $J_q$, and that, for $1 < p < q < \infty$ and $1 < r < \infty$ such that $r \notin \{p, q\}$, $J_r$ does not coarse Lipschitz embed into $J_p \oplus J_q$.

## 1 Introduction

Let $(M, d)$ and $(N, \delta)$ be two metric spaces and $f : M \to N$.
The map $f$ is said to be a *coarse Lipschitz embedding* if there exist $\theta$, $A$, $B > 0$ such that

$$\forall\, x,\ y \in M \ \ d(x,y) \geq \theta \Rightarrow Ad(x,y) \leq \delta(f(x), f(y)) \leq Bd(x,y).$$

Then we say that $M$ *coarse Lipschitz embeds* into $N$.

R.C. James introduced in [7] a non-reflexive space defined by :

$$J = \Bigg\{ x : \mathbb{N} \to \mathbb{R} \text{ s.t. } x(n) \to 0 \text{ and}$$
$$\|x\|_J = \sup_{p_1 < \ldots < p_n} \left( \sum_{i=1}^{n-1} |x(p_{i+1}) - x(p_i)|^2 \right)^{\frac{1}{2}} < \infty \Bigg\}$$

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We will use the following spaces (where $1 < p < \infty$), which are variants of $J$ :

$$J_p = \Bigg\{ x : \mathbb{N} \to \mathbb{R} \text{ s.t. } x(n) \to 0 \text{ and } \|x\|_{J_p} = \sup_{p_1 < \ldots < p_n} \left( \sum_{i=1}^{n-1} |x(p_{i+1}) - x(p_i)|^p \right)^{\frac{1}{p}} < \infty \Bigg\}$$

Like in the case of $J$, the codimension of $J_p$ in $J_p^{**}$ is 1.
In this respect, we precise that $J_p^{**}$ can be seen as :

$$J_p^{**} = \left\{ x : \mathbb{N} \to \mathbb{R} \text{ s.t. } \sup_{p_1 < \ldots < p_n} \left( \sum_{i=1}^{n-1} |x(p_{i+1}) - x(p_i)|^p \right)^{\frac{1}{p}} < \infty \right\}$$

All those spaces are studied in [13].

In 2008, N.J. Kalton and N.L. Randrianarivony [10] proved that, if $r \notin \{p_1, \ldots, p_n\}$ where $1 \le p_1 < p_2 < \ldots < p_n < \infty$, then $\ell_r$ does not coarse Lipschitz embed into $\ell_{p_1} \oplus \ldots \oplus \ell_{p_n}$.
The aim of this article is to prove similar results for the $J_p$ spaces. One of the main obstacles is the lack of reflexivity, which was crucial in Kalton-Randrianarivony's work. However, the James spaces have nice properties of asymptotic uniform smoothness and weak$^*$ asymptotic uniform convexity that we shall use (see [10] or [11] for the definitions). We shall not refer to these notions in our paper, but we will build concrete equivalent norms on $J_p$ that will serve our purpose. Some compactness arguments will also be used to deal with the extra dimension in $J_p^{**}$.

This paper is organized as follows. In Section 2 we summarize the notation and terminology and we give the basic results. Section 3 contains the proof of the nonexistence of a coarse Lipschitz embedding between two James spaces $J_p$ and $J_q$ for $1 < p \neq q < \infty$. At the end of this last section, we show that, for $1 < p < q < \infty$ and $1 < r < \infty$ such that $r \notin \{p, q\}$, $J_r$ does not coarse Lipschitz embed into $J_p \oplus J_q$.

## 2 Preliminaries

**Notation 2.1.** Let $e_n$ defined by $e_n(k) = \delta_{n,k}$ for $k \in \mathbb{N}$. The sequence $(e_n)_{n=1}^{\infty}$ is a Schauder basis of $J_p$ (where $p > 1$).
Moreover, the sequence $(e_n^*)_{n=1}^{\infty}$ of the coordinate functionals associated with $(e_n)_{n=1}^{\infty}$ is a Schauder basis of $J_p^*$.
For $x \in J_p$, we denote $\text{supp}(x) = \{n \in \mathbb{N},\ e_n^*(x) \neq 0\}$ (support of $x$).
When $u$ and $v$ in $J_p$ have consecutive and disjoint finite supports with respect to $(e_n)_{n=1}^{\infty}$, we will denote $u \prec v$.
Likewise, when $u^*$ and $v^*$ in $J_p^*$ have a consecutive and disjoint finite supports with respect to $(e_n^*)_{n=1}^{\infty}$, we will denote $u^* \prec v^*$.

We start with the construction of an ad'hoc equivalent norm on $J_p$. We follow the construction given in [12] for $J_2$.

**Lemma 2.2.** *Let $x_1, \dots, x_n$ in $J_p$ such that their supports are consecutive and finite with respect to the basis $(e_n)_{n=1}^{\infty}$. Then*

$$\|\sum_{i=1}^{n} x_i\|_{J_p}^p \leq (2^p+1)\sum_{i=1}^{n}\|x_i\|_{J_p}^p.$$

*Proof.* We can find disjoint intervals in $\mathbb{N}$, $[s_i, s'_i]$, with $1 \leq i \leq n$ and $s'_i < s_{i+1}$, such that :
$\forall\ 1 \leq i \leq n$, $\mathrm{supp}(x_i) \subset [s_i, s'_i]$ (for convenience, we fix $s_1 = 0$ et we denote $s_{n+1} = \infty$).
Let now $q_1 < \ldots < q_k$ be an arbitrary sequence in $\mathbb{N}$. We must show that

$$\sum_{j=1}^{k-1} \left|y(q_j) - y(q_{j+1})\right|^p \leq (2^p+1)\sum_{i=1}^{n}\|x_i\|_{J_p}^p,$$

where $y = \sum\limits_{i=1}^{n} x_i$.
There exist an increasing sequence $(i_m)_{m=1}^{l}$ in $\{1, \ldots, n\}$ and an increasing sequence $(j_m)_{m=1}^{l}$ in $\{1, \ldots, k\}$ with $j_1 = 1$ such that,
for any $1 \leq m \leq l-1$, $\{q_{j_m}, \ldots, q_{j_{m+1}-1}\} \subset [s_{i_m}, s_{i_m+1})$. Therefore

$$\sum_{j=1}^{k-1} \left|y(q_j) - y(q_{j+1})\right|^p = \sum_{j=1}^{j_2-2} \left|y(q_j) - y(q_{j+1})\right|^p + \left|y(q_{j_2-1}) - y(q_{j_2})\right|^p +$$
$$\sum_{j=j_2}^{j_3-2} \left|y(q_j) - y(q_{j+1})\right|^p + \ldots + \left|y(q_{j_{l-1}-1}) - y(q_{j_{l-1}})\right|^p + \sum_{j=j_{l-1}}^{j_l-2} \left|y(q_j) - y(q_{j+1})\right|^p.$$

We have that

$$\forall\ 1 \leq m \leq l-1, \sum_{j=j_m}^{j_{m+1}-2} \left|y(q_j) - y(q_{j+1})\right|^p \leq \|x_{i_m}\|_{J_p}^p.$$

And for all $2 \leq m \leq l-1$,

$$\left|y(q_{j_m-1}) - y(q_{j_m})\right|^p \leq 2^{p-1}\left|y(q_{j_m-1})\right|^p + 2^{p-1}\left|y(q_{j_m})\right|^p \leq$$
$$2^{p-1}\|x_{i_{m-1}}\|_{J_p}^p + 2^{p-1}\|x_{i_m}\|_{J_p}^p.$$

Then

$$\sum_{j=1}^{k-1} \left|y(q_j) - y(q_{j+1})\right|^p \leq (2^{p-1}+1)\|x_{i_1}\|_{J_p}^p + (2^p+1)\|x_{i_2}\|_{J_p}^p + \ldots +$$
$$(2^p+1)\|x_{i_{l-1}}\|_{J_p}^p + (2^{p-1}+1)\|x_{i_l}\|_{J_p}^p.$$

This concludes the proof of our lemma. ∎

We now define a new norm on $J_p^*$ as follows. Let $q$ be the conjugate exponent of $p$, in other words $\dfrac{1}{p}+\dfrac{1}{q}=1$, where $p\in(1,\infty)$. For $x^*\in J_p^*$, we set

$$|x^*|_{J_p^*}=\sup\Big\{(\sum_{i=1}^n\|x_i^*\|_{J_p^*}^q)^{\frac{1}{q}}\ :x^*=x_1^*+\ldots+x_n^*\text{ and }x_1^*\prec\ldots\prec x_n^*\Big\},$$

where $\|\cdot\|_{J_p^*}$ denotes the dual norm of $\|\cdot\|_{J_p}$. Note that $x_n^*$ is not supposed to be finitely supported.
We can now state the following proposition.

**Proposition 2.3.** *The norm $|\cdot|_{J_p^*}$ is the dual norm of an equivalent norm on $J_p$ (that we shall denote $|\cdot|_{J_p}$).*
*Moreover, $|\cdot|_{J_p^*}$ satisfies the following property: for any $x^*,y^*$ in $J_p^*$ such that $x^*\prec y^*$, we have that*

$$|x^*+y^*|_{J_p^*}^q\geq|x^*|_{J_p^*}^q+|y^*|_{J_p^*}^q.$$

*Proof.* To show that $|\cdot|_{J_p^*}$ is a norm, we only detail the proof of the triangle inequality : let $(x^*,y^*)\in\left(J_p^*\right)^2$ that we may assume with finite supports. Let now $u_1^*\prec u_2^*\prec\ldots\prec u_n^*$ in $J_p^*$ such that

$$x^*+y^*=u_1^*+\ldots+u_n^*$$

.
We write, for $i\in[1,n]\cap\mathbb{N}$, $u_i^*=x_i^*+y_i^*$, where $x^*=\sum\limits_{i=1}^n x_i^*$ and $y^*=\sum\limits_{i=1}^n y_i^*$.
Thank to the triangle inequality for $\|\cdot\|_{J_p^*}$, we get:

$$\left(\sum_{i=1}^n\|u_i^*\|_{J_p^*}^q\right)^{\frac{1}{q}}\leq\left(\sum_{i=1}^n\left(\|x_i^*\|_{J_p^*}+\|y_i^*\|_{J_p^*}\right)^q\right)^{\frac{1}{q}}.$$

It then follows from Minkowski's inequality that

$$\left(\sum_{i=1}^n\|u_i^*\|_{J_p^*}^q\right)^{\frac{1}{q}}\leq\left(\sum_{i=1}^n\|x_i^*\|_{J_p^*}^q\right)^{\frac{1}{q}}+\left(\sum_{i=1}^n\|y_i^*\|_{J_p^*}^q\right)^{\frac{1}{q}}\leq|x^*|_{J_p^*}+|y^*|_{J_p^*}.$$

We have shown that the triangle inequality is valid for $|.|_{J_p^*}$.

Next we show that for $x_1^*,\ldots,x_n^*$ in $J_p^*$ satisfying $x_1^*\prec\ldots\prec x_n^*$ with respect to the basis $(e_n^*)_{n=1}^\infty$, we have:

$$\|\sum_{i=1}^n x_i^*\|_{J_p^*}^q\geq\frac{1}{2^q(2^p+1)^{q-1}}\sum_{i=1}^n\|x_i^*\|_{J_p^*}^q. \tag{2.1}$$

So, let $x_1^*\prec\ldots\prec x_n^*$, with, for $i\in[1,n-1]\cap\mathbb{N}$, $\operatorname{supp}(x_i^*)\subseteq[r_i,s_i]$, where $s_i<r_{i+1}$ for $i\in[1,n-2]$, and $\operatorname{supp}(x_n^*)\subseteq[r_n,\infty)$. Fix now $\varepsilon>0$.

$$\exists\, y_i\in J_p,\ \begin{cases} x_i^*(y_i) & \geq\ \|x_i^*\|_{J_p^*}^q-\varepsilon\\ \|y_i\|_{J_p} & \leq\ \|x_i^*\|_{J_p^*}^{q-1}\end{cases}$$

For $r \in \mathbb{N}$, denote $P_r$ the projection onto the linear span of $\{e_i,\ 1 \le i \le r\}$ with kernel $\overline{\text{span}}\{e_i,\ i > r\}$.
Since $(e_i)_{i=1}^{\infty}$ is a monotone basis, $\|P_{s_i} - P_{r_i-1}\| \le 2$ and, for $x_i = (P_{s_i} - P_{r_i-1})(y_i)$, we have $x_i^*(x_i) = x_i^*(y_i)$. So:

$$\exists\, x_i \in J_p, \quad \left\{ \begin{array}{rcl} x_i^*(x_i) & \ge & \|x_i^*\|_{J_p^*}^q - \varepsilon \\ \|x_i\|_{J_p} & \le & 2\|x_i^*\|_{J_p^*}^{q-1} \\ \text{supp}(x_i) & \subseteq & [r_i, s_i] \end{array} \right.$$

Thank to Lemma 2.2: $\|\sum\limits_{i=1}^n x_i\|_{J_p}^p \le (2^p+1)\sum\limits_{i=1}^n \|x_i\|_{J_p}^p$.
Since $\|\cdot\|_{J_p^*}$ is the dual norm of $\|\cdot\|_{J_p}$, we have that

$$\|\sum_{i=1}^n x_i^*\|_{J_p^*} \ge \left(\sum_{i=1}^n x_i^*\right)\left(\sum_{i=1}^n x_i\right)\left(\|\sum_{i=1}^n x_i\|_{J_p}\right)^{-1}$$

and

$$\|\sum_{i=1}^n x_i^*\|_{J_p^*} \ge \left(\sum_{i=1}^n x_i^*(x_i)\right)\left(\|\sum_{i=1}^n x_i\|_{J_p}\right)^{-1} \ge$$

$$\left(\sum_{i=1}^n \|x_i^*\|_{J_p^*}^q - n\varepsilon\right)\left((2^p+1)^{\frac{1}{p}}\left(\sum_{i=1}^n \|x_i\|_{J_p}^p\right)^{\frac{1}{p}}\right)^{-1}.$$

Moreover, $\left(\sum\limits_{i=1}^n \|x_i\|_{J_p}^p\right)^{\frac{1}{p}} \le 2\left(\sum\limits_{i=1}^n \|x_i^*\|_{J_p^*}^{p(q-1)}\right)^{\frac{1}{p}} = 2\left(\sum\limits_{i=1}^n \|x_i^*\|_{J_p^*}^q\right)^{\frac{1}{p}}$.
Letting $\varepsilon$ tend to 0, we obtain :

$$\|\sum_{i=1}^n x_i^*\|_{J_p^*} \ge \frac{1}{2(2^p+1)^{\frac{1}{p}}}\left(\sum_{i=1}^n \|x_i^*\|_{J_p^*}^q\right)^{1-\frac{1}{p}} = \frac{1}{2(2^p+1)^{\frac{1}{p}}}\left(\sum_{i=1}^n \|x_i^*\|_{J_p^*}^q\right)^{\frac{1}{q}}.$$

So, we have established inequality (2.1).
It follows easily that

$$\|x^*\|_{J_p^*} \le |x^*|_{J_p^*} \le 2(2^p+1)^{1-\frac{1}{q}}\|x^*\|_{J_p^*} = 2(2^p+1)^{\frac{1}{p}}\|x^*\|_{J_p^*}.$$

Moreover, $|\cdot|_{J_p^*}$ is the dual norm of an equivalent norm on $J_p$. Indeed, it is clear that $|\cdot|_{J_p^*}$ is $\sigma(J_p^*, J_p)$ lower semi-continuous.
Finally, it follows clearly from the definition of $|\cdot|_{J_p^*}$ that for all $x^*, y^*$ in $J_p^*$ such that $x^* \prec y^*$, we have that

$$|x^* + y^*|_{J_p^*}^q \ge |x^*|_{J_p^*}^q + |y^*|_{J_p^*}^q. \qquad \blacksquare$$

**Corollary 2.4.** *The dual norm $|\cdot|_{J_p^{**}}$ of $|\cdot|_{J_p^*}$ satisfies the following property.*
*For $x \in J_p$ with a finite support and $y \in J_p^{**}$ (not necessarily with finite support) such that $x \prec y$, we have*

$$|x+y|_{J_p^{**}}^p \le |x|_{J_p^{**}}^p + |y|_{J_p^{**}}^p.$$

*Proof.* Let $x \in J_p$ which has a finite support and $y \in J_p^{**}$ such that $x \prec y$, with $\mathrm{supp}(x) \subset [m,n]$, $\mathrm{supp}(y) \subset [m',\infty)$ and $n < m'$. Fix $\varepsilon > 0$.
There exists $z^* \in J_p^*$ such that

$$|z^*|_{J_p^*} = |x+y|_{J_p^{**}}^{p-1} \quad \text{and} \quad z^*(x+y) \geq |x+y|_{J_p^{**}}^{p} - \varepsilon.$$

Moreover, we can write $z^* = x^* + y^*$, with $x^* \prec y^*$, $z^*(x) = x^*(x)$ and $z^*(y) = y^*(y)$.
We deduce that $|x+y|_{J_p^{**}}^{p} \leq x^*(x) + y^*(y) + \varepsilon$. Then Hölder's inequality and Proposition 2.3 yield

$$|x+y|_{J_p^{**}}^{p} \leq (|x^*|_{J_p^*}^{q} + |y^*|_{J_p^*}^{q})^{\frac{1}{q}}(|x|_{J_p^{**}}^{p} + |y|_{J_p^{**}}^{p})^{\frac{1}{p}} + \varepsilon \leq (|x^*+y^*|_{J_p^*})(|x|_{J_p^{**}}^{p} + |y|_{J_p^{**}}^{p})^{\frac{1}{p}} + \varepsilon.$$

Since $|z^*|_{J_p^*} = |x+y|_{J_p^{**}}^{p-1}$, we get

$$|x+y|_{J_p^{**}}^{p} \leq (|x+y|_{J_p^{**}}^{p-1})(|x|_{J_p^{**}}^{p} + |y|_{J_p^{**}}^{p})^{\frac{1}{p}} + \varepsilon.$$

We conclude our proof by letting $\varepsilon$ tend to 0. ∎

We now turn to the study of the coarse Lipschitz embeddings between James spaces. Let us first recall some notation.

**Definition 2.5.** Let $(M,d)$ and $(N,\delta)$ be two metric spaces and $f : M \to N$ be a mapping. If $(M,d)$ is unbounded, we define

$$\forall s > 0,\ Lip_s(f) = \sup\left\{\frac{\delta((f(x),f(y))}{d(x,y)},\ d(x,y) \geq s\right\} \text{ and } Lip_\infty(f) = \inf_{s>0} Lip_s(f).$$

Note that $f$ is coarse Lipschitz if and only if $Lip_\infty(f) < \infty$.

We also recall a classical definition.

**Definition 2.6.** Given a metric space $X$, two points $x,y \in X$, and $\delta > 0$, the approximate metric midpoint set between $x$ and $y$ with error $\delta$ is the set :

$$Mid(x,y,\delta) = \left\{z \in X \ :\ \max\{d(x,z),d(y,z)\} \leq (1+\delta)\frac{d(x,y)}{2}\right\}$$

The use of approximate metric midpoints in the study of nonlinear geometry is due to Enflo in an unpublished paper and has been used elsewhere, e.g. [2], [4] and [8]. The next proposition and its proof can be found for instance in [10] and [11].

**Proposition 2.7.** *Let $X$ be a normed space and suppose $M$ is a metric space. Let $f : X \to M$ be a coarse Lipschitz map. If $Lip_\infty(f) > 0$, then for any $t, \varepsilon > 0$ and any $0 < \delta < 1$, there exist $x,y \in X$ with $\|x-y\| > t$ and*

$$f\left(Mid(x,y,\delta)\right) \subset Mid\left(f(x),f(y),(1+\varepsilon)\delta\right).$$

Let us now recall the definition of the metric graphs introduced in [10] that will be crucial in our proofs.

**Notation 2.8.** Let $\mathbb{M}$ be an infinite subset of $\mathbb{N}$ and $k \in \mathbb{N}$. We denote

$$G_k(\mathbb{M}) = \{\overline{n} = (n_1, \dots, n_k),\ n_i \in \mathbb{M} \quad n_1 < \dots < n_k\}.$$

Then we equip $G_k(\mathbb{M})$ with the Hamming distance $d_H(\overline{n}, \overline{m}) = |\{j,\ n_j \neq m_j\}|$.

We end these preliminaries by recalling Ramsey's theorem and one of its immediate corollaries (see [5] for instance).

**Theorem 2.9.** *Let $k, r \in \mathbb{N}$ and $f : G_k(\mathbb{N}) \to \{1, \dots, r\}$ be any map. Then there exists an infinite subset $\mathbb{M}$ of $\mathbb{N}$ and $i \in \{1, \dots, r\}$ such that, for every $\overline{n} \in G_k(\mathbb{M})$, $f(\overline{n}) = i$.*

**Corollary 2.10.** *Let $(K, d)$ be a compact metric space, $k \in \mathbb{N}$ and $f : G_k(\mathbb{N}) \to K$. Then for every $\epsilon > 0$, there exist an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that for every $\overline{n}, \overline{m} \in G_k(\mathbb{M})$, $d(f(\overline{n}), f(\overline{m})) < \epsilon$.*

## 3 The main results

Our first lemma gives a description of approximate metric midpoints in $J_p$ that is analogous to the situation in $\ell_p$ (see [10] or [11]). However, we need to use both the original and our new norm on $J_p$.

**Lemma 3.1.** *Let $1 < p < \infty$. We denote $E_N$ the closed linear span of $\{e_i,\ i > N\}$. Let now $x, y \in J_p$, $\delta \in (0,1)$, $u = \dfrac{x+y}{2}$ and $v = \dfrac{x-y}{2}$. Then*
*(i) There exists $N \in \mathbb{N}$ such that:*

$$u + \delta^{\frac{1}{p}} |v|_{J_p} B_{(E_N, |\cdot|_{J_p})} \subset Mid_{|\cdot|_{J_p}}(x, y, \delta).$$

*(ii) There is a compact subset $K$ of $J_p$ such that:*

$$Mid_{\|\cdot\|_{J_p}}(x, y, \delta) \subset K + 2\delta^{\frac{1}{p}} \|v\|_{J_p} B_{(J_p, \|\cdot\|_{J_p})}.$$

*Proof.* Fix $\lambda > 0$.
Let $N \in \mathbb{N}$ such that $|v - v_N|_{J_p} \leq \lambda |v|_{J_p}$, where $v_N = \sum_{i=1}^N v(i) e_i$.

(i) Let now $z \in E_N$ so that $|z|_{J_p}^p \leq \delta |v|_{J_p}^p$. Then

$$|x - (u+z)|_{J_p}^p = |v - z|_{J_p}^p = |v - v_N + v_N - z|_{J_p}^p$$

It follows from the Corollary 2.4 that:

$$|x - (u+z)|_{J_p}^p \leq |v - v_N - z|_{J_p}^p + |v_N|_{J_p}^p \leq (|v - v_N|_{J_p} + |z|_{J_p})^p + |v_N|_{J_p}^p$$

Therefore, thanks to the last inequality and the triangle inequality for $|\cdot|_{J_p}$, we obtain :

$$|x - (u+z)|_{J_p}^p \leq \big((\lambda + \delta^{1/p})^p + (\lambda + 1)^p\big) |v|_{J_p}^p \leq (1+\delta)^p |v|_{J_p}^p,$$

if $\lambda$ was chosen initially small enough.
We argue similarly to show that $|y-(u+z)|_{J_p} = |v+z|_{J_p} \leq (1+\delta)|v|_{J_p}$ and deduce that $u+z \in Mid(x,y,\delta)$.

(ii) Fix $\nu > 0$ and choose $N \in \mathbb{N}$ such that $\|v_N\|_{J_p}^p \geq (1-\nu^p)\|v\|_{J_p}^p$. We assume now that $u+z \in Mid_{\|\cdot\|_{J_p}}(x,y,\delta)$ and write $z = z' + z''$ with $z' \in F_N = \text{span}\{e_i, i \leq N\}$ and $z'' \in E_N$.
Since $\|v-z\|_{J_p}$, $\|v+z\|_{J_p} \leq (1+\delta)\|v\|_{J_p}$, we get, by convexity, that

$$\|z'\|_{J_p} \leq \|z\|_{J_p} \leq (1+\delta)\|v\|_{J_p}.$$

Therefore, $u+z'$ belongs to the compact set $K = u + (1+\delta)\|v\|_{J_p} B_{(F_N,\|\cdot\|_{J_p})}$.
Moreover, for any $(m,n) \in (\mathbb{N}^*)^2$, with $m > n$:

$$\max\{|v(n)-v(m)|^p,\ |z(n)-z(m)|^p\} \leq \frac{1}{2}(|(v(n)-z(n))-(v(m)-z(m))|^p + |(v(n)+z(n))-(v(m)+z(m))|^p).$$

Therefore

$$(1-\nu^p)\|v\|_{J_p}^p + \|z''\|_{J_p}^p \leq \|v_N\|_{J_p}^p + \|z''\|_{J_p}^p \leq \frac{1}{2}(\|v-z\|_{J_p}^p + \|v+z\|_{J_p}^p) \leq (1+\delta)^p\|v\|_{J_p}^p.$$

Then, if $\nu$ was chosen small enough, we get

$$\|z''\|_{J_p}^p \leq [(1+\delta)^p - (1-\nu^p)]\|v\|_{J_p}^p \leq 2^p\delta\|v\|_{J_p}^p. \qquad \blacksquare$$

**Proposition 3.2.** *Let $1 < p < q < \infty$ and $f : (J_q, |\cdot|_{J_q}) \to (J_p, \|\cdot\|_{J_p})$ be a coarse Lipschitz embedding. Then, for any $\tau > 0$ and for any $\varepsilon > 0$, there exist $u \in J_q$, $\theta > \tau$, $N \in \mathbb{N}$ and $K$ a compact subset of $J_p$ such that*

$$f(u + \theta B_{(E_N,|\cdot|_{J_q})}) \subset K + \varepsilon\theta B_{(J_p,\|\cdot\|_{J_p})}.$$

*Proof.* If $Lip_\infty(f) = 0$, the conclusion is clear. So we assume that $Lip_\infty(f) > 0$.
We choose a small $\delta > 0$ (to be detailed later). Then we choose $s$ large enough so that $Lip_s(f) \leq 2Lip_\infty(f)$.
Then, by Proposition 2.7,

$$\exists\, x,y \in J_q,\ |x-y|_{J_q} \geq s \ \text{ and } \ f(Mid_{|\cdot|_{J_q}}(x,y,\delta)) \subset Mid_{\|\cdot\|_{J_p}}(f(x),f(y),2\delta).$$

Denote $u = \dfrac{x+y}{2}$, $v = \dfrac{x-y}{2}$ and $\theta = \delta^{\frac{1}{q}}|v|_{J_q}$. By Lemma 3.1, there exists $N \in \mathbb{N}$ such that $u + \theta B_{(E_N,|\cdot|_{J_q})} \subset Mid_{|\cdot|_{J_q}}(x,y,\delta)$ and there exists a compact subset $K$ of $J_p$ so that $Mid_{\|\cdot\|_{J_p}}(f(x),f(y),2\delta) \subset K + (2\delta)^{\frac{1}{p}}\|f(x)-f(y)\|_{J_p} B_{(J_p,\|\cdot\|_{J_p})}$. But :

$$(2\delta)^{\frac{1}{p}}\|f(x)-f(y)\|_{J_p} \leq 2Lip_\infty(f)(2\delta)^{\frac{1}{p}}|x-y|_{J_q} \leq 4Lip_\infty(f)2^{\frac{1}{p}}\delta^{\frac{1}{p}-\frac{1}{q}}\theta \leq \varepsilon\theta,$$

if $\delta$ was chosen initially small enough.
Then an appropriate choice of a large $s$ will ensure that $\theta \geq \frac{1}{2}\delta^{\frac{1}{q}}s > \tau$. This finishes the proof. $\blacksquare$

**Corollary 3.3.** *Let $1 < p < q < \infty$.*
*Then $J_q$ does not coarse Lipschitz embed into $J_p$.*

*Proof.* We proceed by contradiction and suppose that there exists a coarse Lipschitz embedding $f : (J_q, |\cdot|_{J_q}) \to (J_p, \|\cdot\|_{J_p})$.
With the notation of the previous proposition, we can find a sequence $(u_n)_{n=1}^{\infty}$ in $u + \theta B_{(E_N, |\cdot|_{J_q})}$, such that $|u_n - u_m|_{J_q} \geq \theta$ for $n \neq m$. Then $f(u_n) = k_n + \varepsilon\theta v_n$, with $k_n \in K$ et $v_n \in B_{(J_p, \|\cdot\|_{J_p})}$. Since $K$ is compact, by extracting a subsequence, we may assume that $\|f(u_n) - f(u_m)\|_{J_p} \leq 3\varepsilon\theta$.
Since $\varepsilon$ can be chosen arbitrarily small and $\theta$ arbitrarily large, this yields a contradiction. ∎

In order to treat the coarse Lipschitz embeddability in the other direction, we shall use the Kalton-Randrianarivony graphs and some special sets of pairs of elements of these graphs that we introduce now.

**Definition 3.4.** Let $\overline{n}, \overline{m} \in G_k(\mathbb{M})$ (where $\mathbb{M}$ is an infinite subset of $\mathbb{N}$).
We say that $(\overline{n}, \overline{m}) \in I_k(\mathbb{M})$ if $n_1 < m_1 < n_2 < m_2 < \ldots < n_k < m_k$.

**Proposition 3.5.** *Let $\varepsilon > 0$ and $f : G_k(\mathbb{N}) \to (J_p^{**}, |\cdot|_{J_p^{**}})$ be a Lipschitz map.*
*Then, for any infinite subset $\mathbb{M}$ of $\mathbb{N}$, there exists $(\overline{n}, \overline{m}) \in I_k(\mathbb{M})$ such that*

$$|f(\overline{n}) - f(\overline{m})|_{J_p^{**}} \leq 2Lip(f)k^{\frac{1}{p}} + \varepsilon$$

*Proof.* We shall prove this statement by induction on $k \in \mathbb{N}$.
The proposition is clearly true for $k = 1$.
Assume now that it is true for $k \geq 1$.
Let $f : G_k(\mathbb{M}) \to J_p^{**}$ be a Lipschitz map and $\varepsilon > 0$.
By a diagonal extraction process and thank to weak$^*$-compactness, we can find an infinite subset $\mathbb{M}_1$ of $\mathbb{M}$ such that

$$\forall\, \overline{n} \in G_{k-1}(\mathbb{M}_1),\ w^* - \lim_{n_k \in \mathbb{M}_1} f(\overline{n}, n_k) = g(\overline{n}) \in J_p^{**} \tag{3.2}$$

Then $Lip(g) \leq Lip(f)$, by weak$^*$-lower semicontinuity of $|\cdot|_{J_p^{**}}$.
We recall that the codimension of $J_p$ in $J_p^{**}$ is 1.
So, we can denote $g(\overline{n}) = v(\overline{n}) + c_{\overline{n}}\mathbb{1}$ (where $v(\overline{n}) \in J_p$, $c_{\overline{n}} \in \mathbb{R}$ and $\mathbb{1}$ is the constant sequence $(1, 1, 1, \ldots)$).
Let $\eta > 0$ (small enough : to be detailed later).
By Ramsey's theorem, there exists an infinite subset $\mathbb{M}_2$ of $\mathbb{M}_1$ such that

$$\forall\, \overline{n}, \overline{m} \in G_{k-1}(\mathbb{M}_2), \quad |c_{\overline{n}} - c_{\overline{m}}| = |(v(\overline{n}) - v(\overline{m})) - (g(\overline{n}) - g(\overline{m}))|_{J_p^{**}} < \eta. \tag{3.3}$$

For $\overline{n}, \overline{m} \in G_{k-1}(\mathbb{M}_2)$ and $t, l \in \mathbb{M}_2$, set

$$u_{\overline{n},\overline{m},t,l} = f(\overline{n}, t) - g(\overline{n}) + g(\overline{m}) - f(\overline{m}, l).$$

Using (3.2), we have

$$\forall\, \overline{n} \in G_{k-1}(\mathbb{M}_2),\ w^* - \lim_{n_k \in \mathbb{M}_2} (f(\overline{n}, n_k) - g(\overline{n})) = 0.$$

Then, with Corollary 2.4, we deduce that there exists $l_0 \in \mathbb{N}$ such that for all $t, l \in \mathbb{M}_2 \cap [l_0, +\infty)$:

$$|g(\overline{n}) - g(\overline{m}) + u_{\overline{n},\overline{m},t,l}|^p_{J_p^{**}} \leq |v(\overline{n}) + c_{\overline{n}}\mathbb{1} - v(\overline{m}) - c_{\overline{m}}\mathbb{1}|^p_{J_p^{**}} + |u_{\overline{n},\overline{m},t,l}|^p_{J_p^{**}} + \eta.$$

Note that $f(\overline{n}, t) - f(\overline{m}, l) = g(\overline{n}) - g(\overline{m}) + u_{\overline{n},\overline{m},t,l}$.
Then it follows from (3.3) and the triangle inequality, that for all $t, l \in \mathbb{M}_2 \cap [l_0, +\infty)$:

$$|f(\overline{n}, t) - f(\overline{m}, l)|^p_{J_p^{**}} \leq |u_{\overline{n},\overline{m},t,l}|^p_{J_p^{**}} + (|v(\overline{n}) - v(\overline{m})|_{J_p} + \eta)^p + \eta.$$

Moreover : $f(\overline{n}, t) - g(\overline{n}) = w^* - \lim_i (f(\overline{n}, t) - f(\overline{n}, i))$.
Therefore, by weak$^*$-lower semicontinuity of $|\cdot|_{J_p^{**}}$ : $|f(\overline{n}, t) - g(\overline{n})|_{J_p^{**}} \leq Lip(f)$.
Likewise : $|f(\overline{m}, l) - g(\overline{m})|_{J_p^{**}} \leq Lip(f)$.
Then, we deduce the following inequality : $|u_{\overline{n},\overline{m},t,l}|^p_{J_p^{**}} \leq 2^p Lip(f)^p$.
On the other hand, it follows from our induction hypothesis that:

$$\exists\ (\overline{n},\ \overline{m}) \in\ I_{k-1}(\mathbb{M}_2),\ |g(\overline{n}) - g(\overline{m})|_{J_p^{**}} \leq 2Lip(f)(k-1)^{\frac{1}{p}} + \eta.$$

Then, for $t,\ l \in \mathbb{M}_2 \cap [l_0, +\infty)$ such that $m_{k-1} < t < l$, we have $((\overline{n}, t), (\overline{m}, l)) \in I_k(\mathbb{M}_2)$, and

$$|f(\overline{n}, t) - f(\overline{m}, l)|^p_{J_p^{**}} \leq 2^p Lip(f)^p + (2Lip(f)(k-1)^{\frac{1}{p}} + 2\eta)^p + \eta.$$

So :

$$|f(\overline{n}, t) - f(\overline{m}, l)|^p_{J_p^{**}} \leq 2^p Lip(f)^p k + \varphi(\eta),\ \text{with}\ \varphi(\eta) \underset{\eta \to 0}{\longrightarrow} 0.$$

Thus, if $\eta$ was chosen small enough :

$$|f(\overline{n}, t) - f(\overline{m}, l)|_{J_p^{**}} \leq 2Lip(f)k^{\frac{1}{p}} + \varepsilon.$$

This finishes our inductive proof. ■

**Corollary 3.6.** *Let* $1 < q < p < \infty$.
*Then* $J_q$ *does not coarse Lipschitz embed into* $J_p$.

*Proof.* Suppose that $g\ : J_q \to J_p$ is a map such that there exist $\theta$, $A$ and $B$ real positive numbers such that :

$$\forall x, y \in J_q, \|x - y\|_{J_q} \geq \theta \Rightarrow A\|x - y\|_{J_q} \leq |g(x) - g(y)|_{J_p} \leq B\|x - y\|_{J_q}.$$

Let us rescale by defining $f(v) = (A\theta)^{-1} g(\theta v)$, for $v \in J_q$. We have that there exists $C \geq 1$ such that

$$\forall x, y \in J_q, \|x - y\|_{J_q} \geq 1 \Rightarrow \|x - y\|_{J_q} \leq |f(x) - f(y)|_{J_p} \leq C\|x - y\|_{J_q}. \quad (3.4)$$

We still denote $(e_n)_{n=1}^{\infty}$ the canonical basis of $J_q$.

Consider the map $\varphi : (G_k(\mathbb{N}), d_H) \to (J_q, \|\cdot\|_{J_q})$ defined by $\varphi(\overline{n}) = e_{n_1} + \ldots + e_{n_k}$.
Note that for all $\overline{n}, \overline{m} \in G_k(\mathbb{N})$,

$$\|(e_{n_1} + \ldots + e_{n_k}) - (e_{m_1} + \ldots + e_{m_k})\|_{J_q} \leq \sum_{n_i \neq m_i} \|e_{n_i} - e_{m_i}\|_{J_q} \leq 2d_H(\overline{n}, \overline{m}).$$

Thus, $Lip(\varphi) \leq 2$. Since moreover $\|\varphi(\overline{n}) - \varphi(\overline{m})\|_{J_q} \geq 1$ whenever $\overline{n} \neq \overline{m}$, we have that $Lip(f \circ \varphi) \leq 2C$. It then follows from Proposition 3.5 that there exists $(\overline{n}, \overline{m}) \in I_k(\mathbb{N})$ such that:

$$|(f \circ \varphi)(\overline{n}) - (f \circ \varphi)(\overline{m})|_{J_p} \leq 5Ck^{\frac{1}{p}}.$$

On the other hand, since $(\overline{n}, \overline{m}) \in I_k(\mathbb{N})$, we have that

$$\|\varphi(\overline{n}) - \varphi(\overline{m})\|_{J_q} \geq 2k^{\frac{1}{q}}, \text{ for } k \geq 2.$$

This is in contradiction with (3.4), for $k$ large enough.
Therefore, there is no coarse Lipschitz embedding from $J_q$ into $J_p$. ■

We now explain how to get a quantitative version of the above result. A similar study was done by F. Baudier [1] for $\ell_p$-spaces and by B.M. Braga [3] for the p-convexified Tsirelson spaces. First we introduce a definition due to E. Guentner and J. Kaminker [6].

**Definition 3.7.** Let $X$ and $Y$ be two Banach spaces. The *compression exponent* of $X$ in $Y$, denoted $\alpha_Y(X)$ is the supremum of all $\alpha \in (0,1]$ such that there exist a constant $C > 0$ and a map $f : X \to Y$ so that

$$\forall x, x' \in X \;\; C^{-1}\|x - x'\|^{\alpha} - C \leq \|f(x) - f(x')\| \leq C\|x - x'\| + C.$$

The next result follows from a straightforward adaptation of the previous proof.

**Theorem 3.8.** *Let $1 < q < p < \infty$. Then $\alpha_{J_p}(J_q) \leq \frac{q}{p}$.*

We conclude with a result combining the approximate midpoint principle and the use of Kalton-Randrianarivony's graphs.

**Corollary 3.9.** *Let $1 < p < q < \infty$, and $r > 1$ such that $r \notin \{p, q\}$.*
*Then $J_r$ does not coarse Lipschitz embed into $J_p \oplus J_q$.*

*Proof.* When $r > q$, the argument is based on a midpoint technique like in the proof of Corollary 3.3.

If $r < p$, we mimic the proof of Corollary 3.6.

So we assume, as we may, that $1 < p < r < q < \infty$ and $f : J_r \to J_p \oplus_\infty J_q$ is a map such that there exists $C \geq 1$ such that

$$\forall\, x,\, y \in J_r \;\; |x - y|_{J_r} \geq 1 \Rightarrow |x - y|_{J_r} \leq \|f(x) - f(y)\| \leq C|x - y|_{J_r}. \tag{3.5}$$

We follow the proof in [10] and write $f = (g,h)$. We still denote $(e_n)_{n=1}^{\infty}$ the canonical basis of $J_r$. We fix $k \in \mathbb{N}$ and $\varepsilon > 0$. We recall that

$$\exists\, \gamma > 0,\ \forall\, x \in J_r,\ \gamma \|x\|_{J_r} \leq |x|_{J_r} \leq \|x\|_{J_r}.$$

We start by applying the midpoint technique to the coarse Lipschitz map $g$ and deduce from Proposition 3.2 that there exist $\theta > \gamma^{-1}(2k)^{\frac{1}{r}}$, $u \in J_r$, $N \in \mathbb{N}$ and $K$ a compact subset of $J_p$ such that :

$$g(u + \theta B_{(E_N, |\cdot|_{J_r})}) \subset K + \varepsilon\theta B_{(J_p, \|\cdot\|_{J_p})}. \tag{3.6}$$

Let $\mathbb{M} = \{n \in \mathbb{N},\ n > N\}$ and $\varphi\ : G_k(\mathbb{M}) \mapsto J_r$ be defined as follows

$$\forall\, \overline{n} = (n_1, \ldots, n_k) \in G_k(\mathbb{M}),\ \varphi(\overline{n}) = u + \theta(2k)^{-\frac{1}{r}}(e_{n_1} + \ldots + e_{n_k}).$$

Then $\varphi(\overline{n}) \in u + \theta B_{(E_N, |\cdot|_{J_r})}$ for all $\overline{n} \in G_k(\mathbb{M})$.
And, from (3.6) we deduce that $(g \circ \varphi)(G_k(\mathbb{M})) \subset K + \varepsilon\theta B_{(J_p, \|\cdot\|_{J_p})}$. Thus, by Ramsey's theorem, there is an infinite subset $\mathbb{M}'$ of $\mathbb{M}$ such that

$$\mathrm{diam}_{\|\cdot\|_{J_p}}(g \circ \varphi)(G_k(\mathbb{M}')) \leq 3\varepsilon\theta. \tag{3.7}$$

Since for $\overline{n} \neq \overline{m}$, we have

$$|\varphi(\overline{n}) - \varphi(\overline{m})|_{J_r} \geq \gamma \|\varphi(\overline{n}) - \varphi(\overline{m})\|_{J_r} \geq \gamma\theta(2k)^{-\frac{1}{r}} > 1,$$

it follows from (3.5) that

$$\forall \overline{n}, \overline{m} \in G_k(\mathbb{M})\ \ \|h \circ \varphi(\overline{n}) - h \circ \varphi(\overline{m})\|_{J_q} \leq C|\varphi(\overline{n}) - \varphi(\overline{m})|_{J_r} \leq C||\varphi(\overline{n}) - \varphi(\overline{m})||_{J_r}.$$

We recall that for all $\overline{n}, \overline{m} \in G_k(\mathbb{N})$,

$$\|(e_{n_1} + \ldots + e_{n_k}) - (e_{m_1} + \ldots + e_{m_k})\|_{J_q} \leq \sum_{n_i \neq m_i} \|e_{n_i} - e_{m_i}\|_{J_q} \leq 2d_H(\overline{n}, \overline{m}).$$

Since moreover $|\cdot|_{J_q} \leq \|\cdot\|_{J_q}$, $Lip(h \circ \varphi) \leq 2C\theta(2k)^{-\frac{1}{r}}$, when $h \circ \varphi$ is considered as a map from $G_k(\mathbb{M}')$ to $(J_q, |\cdot|_{J_q})$. Thus, we can apply Proposition 3.5 to obtain:

$$\exists\, (\overline{n}, \overline{m}) \in I_k(\mathbb{M}'),\ |h \circ \varphi(\overline{n}) - h \circ \varphi(\overline{m})|_{J_q} \leq 5C\theta(2k)^{-\frac{1}{r}} k^{\frac{1}{q}}.$$

Then, if $k$ was chosen large enough, we have:

$$\exists\, (\overline{n}, \overline{m}) \in I_k(\mathbb{M}'),\ |h \circ \varphi(\overline{n}) - h \circ \varphi(\overline{m})|_{J_q} \leq \varepsilon\theta.$$

This, combined with (3.7) implies that

$$\exists\, (\overline{n}, \overline{m}) \in I_k(\mathbb{M}')\ \ \|f \circ \varphi(\overline{n}) - f \circ \varphi(\overline{m})\| \leq 3\varepsilon\theta.$$

But,

$$\forall (\overline{n}, \overline{m}) \in I_k(\mathbb{M}')\ \ |\varphi(\overline{n}) - \varphi(\overline{m})|_{J_r} \geq \gamma \|\varphi(\overline{n}) - \varphi(\overline{m})\|_{J_r} \geq \gamma\theta.$$

If $\varepsilon$ was initially chosen such that $\varepsilon < \frac{\gamma}{3}$, this yields a contradiction with (3.5), which concludes our proof. ∎

**Remark.** This result can be easily extended as follows. Assume $r \in (1,\infty) \setminus \{p_1,\ldots,p_n\}$ where $1 < p_1 < p_2 < \ldots < p_n < \infty$, then $J_r$ does not coarse Lipschitz embed into $J_{p_1} \oplus \ldots \oplus J_{p_n}$.

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## References

[1] F. Baudier, Quantitative nonlinear embeddings into Lebesgue sequence spaces, *J. Topol. Anal.*, **08**, (2016), 117–150.

[2] J. Bourgain, Remarks on the extension of Lipschitz maps defined on discrete sets and uniform homeomorphisms, Geometrical aspects of functional analysis (1985/86), *Lecture Notes in Math., 1267*, Springer Berlin, (1987), 157–167.

[3] B.M. Braga, Asymptotic structure and coarse Lipschitz geometry of Banach spaces, *Studia Mathematica*, **237**, (2017), 71–97.

[4] E. Gorelik, The uniform nonequivalence of $L_p$ and $l_p$, *Israel J. Math*, **87**, (1994), 1–8.

[5] W.T. Gowers, Ramsey methods in Banach spaces, *Handbook of the geometry of Banach spaces*, **2**, North-Holland, Amsterdam, (2003), 1071–1097.

[6] E. Guentner and J. Kaminker, Exactness and uniform embeddability of discrete groups, *J. London Math. Soc. (2)*, **70**, (2004), no. 3, 703-718.

[7] R.C. James, A separable somewhat reflexive Banach space with nonseparable dual, *Bull. A.M.S.*, **80**, (1974), no. 4, 738–743.

[8] W.B. Johnson, J. Lindenstrauss and G. Schechtman, Banach spaces determined by their uniform structures, *Geom. Funct. Anal.*, **6**, (1996), 430–470.

[9] N.J. Kalton, Coarse and uniform embeddings into reflexive spaces, *Quart. J. Math. (Oxford)*, **58**, (2007), no. 3, 393–414.

[10] N.J. Kalton, N.L. Randrianarivony, The coarse Lipschitz geometry of $\ell_p \oplus \ell_q$, *Math. Ann.*, **341**, (2008), 223–237.

[11] G. Lancien, A short course on non linear geometry of Banach spaces. *Topics in Functional and Harmonic Analysis, Theta Series in Advanced Mathematics*, (2012), 77–102.

[12] G. Lancien, Réflexivité et normes duales possédant la propriété uniforme de Kadec-Klee, *Publ. Math. de la Fac. des sciences de Besançon, Fascicule 14*, (1993-94), 69–74.

[13] G. Pisier, The dual $J^*$ of the James Space has cotype 2 and the Gordon-Lewis property, *Proc. Cambridge Phil.*, **103**, (1988), 323–331.

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