Attached primes and Artinian modules

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Abstract

In this paper, we calculate the annihilator of an Artinian module over a complete local ring; moreover, we present a certain generalization of Grothendieck's Non-Vanishing Theorem for local cohomology modules over a local ring which is the homomorphic image of a Gorenstein local ring. Finally, as application of our results, we study the finiteness of local cohomology modules.

1 Introduction

Throughout this paper, let R denote a commutative Noetherian ring (with identity) and I be an ideal of R. For an R-module M, the i^{th} local cohomology module of M with respect to I is defined as

$$H_I^i(M) = \varinjlim_{n \geq 1} \operatorname{Ext}_R^i(R/I^n, M).$$

We refer the reader to [1] for more details about local cohomology. I.G. Macdonald in [5] developed the theory of attached prime ideals and secondary representations of module, which is in a certain sense dual to the theory of associated prime ideals and primary decompositions. This theory was successfully applied to the theory of local cohomology by him and R. Y. Sharp [7]. An R-module M is said to be secondary if $M \neq 0$ and , for each $a \in R$, the endomorphism $\varphi_a : M \to M$ defined by $\varphi_a(m) = am$ (for $m \in M$) is either surjective

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or nilpotent. If M is secondary, then $\mathfrak{p} = \mathrm{Rad}(0 : M)$ is a prime ideal, and M is said to be $\mathfrak{p}$-secondary. Any non-zero quotient of a $\mathfrak{p}$-secondary module is a $\mathfrak{p}$-secondary.

Before going on, let us fix the following notation: first of all, given any ideal $\mathfrak{b}$ of R, the *radical* of $\mathfrak{b}$, denoted by $\mathrm{Rad}(\mathfrak{b})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b}$ for some $n \in \mathbb{N}\}$; moreover, we denote by $V(\mathfrak{b})$ the set $\{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{b}\}$. Finally, given an R-module L, we denote by $\mathrm{Att}_R L$ its set of attached prime ideals and also we denote by $\mathrm{Assh}_R L$ the set $\{\mathfrak{p} \in \mathrm{Ass}_R L : \dim R/\mathfrak{p} = \dim L\}$. Given an Artinian R-module A, it is known (see [1, 7.2.11]) that its set of attached prime ideals is finite, and $\mathrm{Rad}(0 :_R A) = \cap_{\mathfrak{p} \in \mathrm{Att}_R A} \mathfrak{p}$.

In Section 2, building upon a result due to L. Melkersson (see [8, Theorem 1.6]), we provide an alternative expression for $\mathrm{Rad}(0 :_R A)$ in case R is a complete local ring (see Theorem 2.4).

One important result in the study of local cohomology modules is the so-called *Non-Vanishing Theorem* (see [1, 7.3.2]); it says that, for a local ring $(R, \mathfrak{m})$ and a finitely generated R-module M of (Krull) dimension d, one has that $H^d_{\mathfrak{m}}(M) \neq 0$, and

$$\mathrm{Att}_R(H^d_{\mathfrak{m}}(M) = \{\mathfrak{p} \in \mathrm{Ass}_R(M) \mid \dim R/\mathfrak{p} = d\}.$$

In Section 3, we show (see Theorem 3.2) that if, in addition, R is a homomorphic image of a Gorenstein local ring, then for any $0 \leq t \leq d$,

$$\{\mathfrak{p} \in \mathrm{Att}_R(H^t_{\mathfrak{m}}(M) \ : \ \dim R/\mathfrak{p} = t\} = \{\mathfrak{p} \in \mathrm{Ass}_R(M) \ : \ \dim R/\mathfrak{p} = t\}.$$

In particular, we will see (see Remark 3.3 and Theorem 3.2) that [2, Corollary 1] and [3, Theorem 2.2], can be deduced from our Theorem 3.2. For any unexplained notation and terminology we refer the reader to [1] and [6].

2 Artinian modules

Remember that one says that a module is representable provided it can be expressed as finite direct sum of secondary modules.

Lemma 2.1. *Let R be a commutative Noetherian ring, let A be a representable R-module, and let $x \in R$ which is not a unit. Then, for any $n \gg 0$, $\mathrm{Att}_R(A) \cap V(x) \subseteq \mathrm{Att}_R((0 :_A x^n))$.*

Proof. Let $\mathrm{Att}_R(A) \cap V(x) = \{\mathfrak{p}_i\}_{i=1}^t$, and $\mathrm{Att}_R(A) - V(x) = \{\mathfrak{p}_i\}_{i=t+1}^s$. Now let $A = T_1 + ... + T_s$, be a minimal secondary representation of A, where T_i is $\mathfrak{p}_i$-secondary for every $i = 1, ..., s$. By definition for all $n \gg 0$, we have $x^n T_i = 0$, for all $0 \leq i \leq t$. Since $x \notin \mathfrak{p}_i$, for all $t+1 \leq i \leq s$, it follows that $x^n T_i = T_i$ for all $t+1 \leq i \leq s$, and hence $x^n A = T_{t+1} + \cdots + T_s$ is a minimal secondary representation for $x^n A$ where T_i is $\mathfrak{p}_i$-secondary for every $i = t+1, ..., s$. Therefore by definition of attached prime ideals, $\mathrm{Att}_R(x^n A) = \{\mathfrak{p}_i\}_{i=t+1}^s$=$\mathrm{Att}_R(A) - V(x)$ for all $n \gg 0$. Now for all $n \gg 0$ there is an exact sequence:

$$0 \to (0 :_A x^n) \to A \to x^n A \to 0$$

That implies $\mathrm{Att}_R(A) \subseteq \mathrm{Att}_R(x^n A) \cup \mathrm{Att}_R((0 :_A x^n))$, now by $\mathrm{Att}_R(x^n A) = \mathrm{Att}_R(A) - V(x)$, it is easy to see that $\mathrm{Att}_R(A) \cap V(x) \subseteq \mathrm{Att}_R((0 :_A x^n))$. ∎

Definition 2.2. (See [4]) Let M be an R-module and I be an ideal of R. The R-module M is said to be I-cofinite if $\operatorname{Supp} M \subseteqq V(I)$ and $Ext^j_R(R/I, M)$ is finitely generated for all j.

The main reason for introducing cofinite modules is the below statement, due to Melkersson, because it plays a key role in the proof of the main result of this section (namely, Theorem 2.4); the interested reader may like to consult [8, Theorem 1.6].

Theorem 2.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, let I be an ideal of R, and let A be an Artinian R-module. Then:*
A is $I-$cofinite if and only if $\operatorname{Hom}_R(R/I, A)$ *is a finitely generated R-module.*
If, in addition, R is complete, then A is I-cofinite if and only if $\operatorname{Rad}(I + \mathfrak{p}) = \mathfrak{m}$ *for any* $\mathfrak{p} \in \operatorname{Att} A$.

Theorem 2.4. *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring. Let M be a non-zero Artinian R-module. Set $J := \operatorname{Rad}(0 :_R M)$ and $S = \{\mathfrak{p} \in \operatorname{Spec} R : M$ is not $\mathfrak{p} - cofinite$ and $\dim R/\mathfrak{p} = 1\}$. If M is not finitely generated, then $J = \bigcap\limits_{\mathfrak{p}\in S} \mathfrak{p}$.*

Proof. (i). First we show that S is non-empty. By our assumption and by [1, Corollary 7.2.12], $\operatorname{Att} M \nsubseteq \{\mathfrak{m}\}$. Therefore, we can deduce that there exists $\mathfrak{q} \in \operatorname{Att} M$ such that $\mathfrak{q} \neq \mathfrak{m}$. Hence there exists $\mathfrak{p} \in V(\mathfrak{q})$ such that $\dim R/\mathfrak{p} = 1$. Now in view of Theorem 2.3, M is not $\mathfrak{p}$-cofinite and hence $\mathfrak{p} \in S$, that means $S \neq \emptyset$. Now assume that $\mathfrak{p} \in S$. Hence by [8, Theorem 1.6], there exists $\mathfrak{q} \in \operatorname{Att} M$ such that $\operatorname{Rad}(\mathfrak{p} + \mathfrak{q}) \neq \mathfrak{m}$. Consequently, $\dim R/(\mathfrak{p} + \mathfrak{q}) = \dim R/\mathfrak{p} = 1$ that implies $\mathfrak{q} \subseteq \mathfrak{p}$. Since $0 :_R M \subseteq \mathfrak{q} \subseteq \mathfrak{p}$, it follows that $J \subseteq \mathfrak{p}$ and so $J \subseteq \bigcap\limits_{\mathfrak{p}\in S} \mathfrak{p}$. In order to prove $\bigcap\limits_{\mathfrak{p}\in S} \mathfrak{p} \subseteq J$, suppose on the contrary that $\bigcap\limits_{\mathfrak{p}\in S} \mathfrak{p} \nsubseteq J$. Since $J = \bigcap\limits_{\mathfrak{q}\in \operatorname{Att} M} \mathfrak{q}$, it follows that there is $x_1 \in \bigcap\limits_{\mathfrak{p}\in S} \mathfrak{p}$ such that $x_1 \notin \mathfrak{q}$ for some $\mathfrak{q} \in \operatorname{Att}(M)$. Suppose that $\dim R/\mathfrak{q} = t$. Then $t \geq 1$ and there are elements $x_2, ..., x_t \in \mathfrak{m}$ such that $x_1, x_2, ..., x_t$ forms a system of parameters for R-module $R/\mathfrak{q}$. Hence $\dim R/\mathfrak{q} + (x_2, ..., x_t) = 1$ and so there exists a prime ideal $\mathfrak{p}$ of R such that $\mathfrak{p} \in \operatorname{Ass} R/\mathfrak{q} + (x_2, ..., x_t)$ and $\dim R/\mathfrak{p} = 1$. Therefore, as $\mathfrak{q} \subseteq \mathfrak{p}$, according to Melkersson's theorem [8, Theorem 1.6], it follows that M is not $\mathfrak{p}$-cofinite and so $\mathfrak{p} \in S$ and hence $x_1 \in \mathfrak{p}$. But $0 = \dim R/\mathfrak{q} + (x_1, ..., x_t) \geq \dim R/\mathfrak{p} = 1$, which is a contradiction. ∎

3 Attached primes

In [2, Corollary 1], the set $\operatorname{Att} H^{\dim M}_{\mathfrak{m}}(M)$ is calculated and Theorem 3.2, for $t = \dim M$ easily follows from this. Therefore, the results of this paper at point $i = \dim M$ are very interesting and general.
The aim of this section is to show a certain variant of the *Non-Vanishing Theorem* for local cohomology modules; before doing so, we want to review the below statement ([1, 11.2.12]), because it plays some role in the proof of the main result of this section.

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let M be a non-zero finitely generated R-module. Let $\mathfrak{p} \in \operatorname{Ass} M$ and $\dim R/\mathfrak{p} = n$. Then $H^n_{\mathfrak{m}}(M) \neq 0$ and $\mathfrak{p} \in \operatorname{Att} H^n_{\mathfrak{m}}(M)$.*

Now, building upon our previous results, we are ready to establish the main result of this paper, namely:

Theorem 3.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, which is a homomorphic image of a Gorenstein local ring, and let M be a finitely generated R-module of dimension d, then for all $0 \leq t \leq d$, we have:*

$$\{\mathfrak{p} \in \operatorname{Att}_R(H^t_{\mathfrak{m}}(M) \mid \dim R/\mathfrak{p} = t\} = \{\mathfrak{p} \in \operatorname{Ass}_R(M) \mid \dim R/\mathfrak{p} = t\}.$$

Proof. Suppose that $0 \leq t \leq d$. By Theorem 3.1, we have $\{\mathfrak{p} \in \operatorname{Ass}_R(M) \mid \dim R/\mathfrak{p} = t\} \subseteq \{\mathfrak{p} \in \operatorname{Att}_R(H^t_{\mathfrak{m}}(M) \mid \dim R/\mathfrak{p} = t\}$. Now set,

$$S := \{N' \mid N' \text{ is a submodule of } M \text{ and } \dim N' < t\}.$$

It is easy to see that as in [1, Lemma, 7.3.1], S has a largest element with respect to inclusion, as, N. Set $G := M/N$, we have $H^t_{\mathfrak{m}}(G) \cong H^t_{\mathfrak{m}}(M)$, and G has no non-zero submodule of dimension less than t. In fact, if $N \neq 0$ then we set, $\{\mathfrak{p} \in \operatorname{Ass}_R(M) \mid \dim R/\mathfrak{p} < t\} := \{\mathfrak{p}_i\}_{i=1}^{s}$, and $I := \bigcap_{i=1}^{s} \mathfrak{p}_i$. It is easy to see that $N = \Gamma_I(M)$, and hence $\operatorname{Ass}_R(G) = \{\mathfrak{p} \in \operatorname{Ass}_R(M) \mid \dim R/\mathfrak{p} \geq t\}$. Hence we may assume, without loss of generality, that M is a non-zero finitely generated R-module such that $\{\mathfrak{p} \in \operatorname{Ass}_R(M) \mid \dim R/\mathfrak{p} < t\} = \emptyset$. Now, if

$$\{\mathfrak{p} \in \operatorname{Att}_R(H^t_{\mathfrak{m}}(M) \mid \dim R/\mathfrak{p} = t\} \neq \{\mathfrak{p} \in \operatorname{Ass}_R(M) \mid \dim R/\mathfrak{p} = t\},$$

then there is $\mathfrak{q} \in \operatorname{Att}_R(H^t_{\mathfrak{m}}(M))$, such that $\dim R/\mathfrak{q} = t$, and $\mathfrak{q} \notin \operatorname{Ass}_R(M)$. Now if $\mathfrak{q} \subseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(M)} \mathfrak{p}$, then by*Prime Avoidance Theorem* we have $\mathfrak{q} \subseteq \mathfrak{p}$, for some $\mathfrak{p} \in \operatorname{Ass}_R(M)$, but since, $t = \dim R/\mathfrak{q} \geq \dim R/\mathfrak{p} \geq t$, it follows that $\mathfrak{q} = \mathfrak{p} \in \operatorname{Ass}_R(M)$, which is a contradiction. So we have $\mathfrak{q} \nsubseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R M} \mathfrak{p}$. Hence there is an element $x \in \mathfrak{q}$, such that $x \notin \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R M} \mathfrak{p}$. Now, for all $n \gg 0$ the exact sequence

$$0 \longrightarrow M \xrightarrow{x^n} M \longrightarrow M/x^n M \longrightarrow 0$$

induces a long exact sequence

$$\cdots \longrightarrow H^j_{\mathfrak{m}}(M) \xrightarrow{x^n} H^j_{\mathfrak{m}}(M) \longrightarrow H^j_{\mathfrak{m}}(M/x^n M) \longrightarrow$$
$$H^{j+1}_{\mathfrak{m}}(M) \xrightarrow{x^n} H^{j+1}_{\mathfrak{m}}(M) \longrightarrow \cdots$$

Consequently, it yields the following short exact sequence,

$$0 \longrightarrow H^{t-1}_{\mathfrak{m}}(M)/x^n H^{t-1}_{\mathfrak{m}}(M) \longrightarrow H^{t-1}_{\mathfrak{m}}(M/x^n M) \longrightarrow (0 :_{H^t_{\mathfrak{m}}(M)} x^n) \longrightarrow 0.$$

But by Lemma 2.1 for all $n \gg 0$, we have $\mathfrak{q} \in \operatorname{Att}_R(H^t_{\mathfrak{m}}(M)) \cap V(x) \subseteq \operatorname{Att}_R((0 :_{H^t_{\mathfrak{m}}(M)} x^n))$, and hence by the above exact sequence we have $\mathfrak{q} \in \operatorname{Att}_R(H^{t-1}_{\mathfrak{m}}(M/x^n M))$, and so by [1, Corollary, 11.3.5], it follows that

$\dim R/\mathfrak{q} \leq t-1$, which is a contradiction, so we obtained the desired conclusion; therefore,

$$\{\mathfrak{p} \in \operatorname{Att}_R(H^t_{\mathfrak{m}}(M) \mid \dim R/\mathfrak{p} = t\} = \{\mathfrak{p} \in \operatorname{Ass}_R(M) \mid \dim R/\mathfrak{p} = t\}. \qquad \blacksquare$$

In what follows, we would like to pinpoint the connections of our Theorem 3.2 with earlier results that have already appeared in the literature.

Remark 3.3. In [9], the authors obtained (see [9, Theorem 1.4 and Theorem 2.9 (ii)]) an expression for the attached prime ideals of local cohomology modules supported at the maximal ideal, whereas in Theorem 3.2 we only have calculated a subset of the attached prime ideals of $H^i_{\mathfrak{m}}(M)$; however, the reader will easily note that the description we provide of the elements of the set of attached primes of $H^i_{\mathfrak{m}}(M)$ is quite different from the one given in [9], so it is not clear for us the connection between both descriptions. On the other hand, in [2, Corollary 1], the set of attached primes of $H^{\dim M}_{\mathfrak{m}}(M)$ is calculated; one can easily see that this result immediately follows from our Theorem 3.2 (just taking in our statement t = dimM).
Our next goal is to provide an alternative proof of [3, Theorem 2.2], using Theorem 3.2; from this point of view [3, Theorem 2.2], can be regarded as a consequence of our Theorem 3.2.

Theorem 3.4. *Let $(R,\mathfrak{m})$ be a complete local ring of dimension d. Let I be an ideal of R. Assume that $H^{d-1}_I(R)$ is representable and $H^d_I(R) = 0$. Then*
1) $\operatorname{Att}_R H^{d-1}_I(R) \subseteq \{P \in \operatorname{Spec}(R) \mid \dim R/P = d-1, \operatorname{Rad}(I+P) = \mathfrak{m}\} \cup \operatorname{Assh} R$.
2) $\{P \in \operatorname{Spec}(R) \mid \dim R/P = d-1, \operatorname{Rad}(I+P) = \mathfrak{m}\} \subseteq \operatorname{Att}_R H^{d-1}_I(R)$.

Proof. Let $P \in \operatorname{Spec}(R)$ be such that $\dim R/P = d-1$ and $\operatorname{Rad}(I+P) = \mathfrak{m}$. Then $H^{d-1}_I(R/P) = H^{d-1}_{I+P}(R/P)) = H^{d-1}_{\mathfrak{m}}(R/P)$. Since R is complete, it follows that R is a homomorphic image of a Gorenstein local ring. Therefore, by our Theorem 3.2, we conclude that $P \in \operatorname{Att} H^{d-1}_{\mathfrak{m}}(R/P) = \operatorname{Att} H^{d-1}_I(R/P)$. On the other hand, $H^d_I(R) = 0$ and so $H^d_I(P) = 0$. The following exact sequence,

$$0 \longrightarrow P \longrightarrow R \longrightarrow R/P \longrightarrow 0,$$

induce the following exact sequence

$$H^{d-1}_I(R) \longrightarrow H^{d-1}_I(R/P) \longrightarrow 0,$$

Since $H^{d-1}_I(R)$ has a P-secondary representation, it follows that $P \in \operatorname{Att}_R H^{d-1}_I(R)$ and so

$$\{P \in \operatorname{Spec}(R) \mid \dim R/P = d-1, \operatorname{Rad}(I+P) = \mathfrak{m}\} \subseteq \operatorname{Att}_R H^{d-1}_I(R).$$

Now let $q \in \operatorname{Att}_R H^{d-1}_I(R)$. Then by definition, there is a homomorphic image of $H^{d-1}_I(R)$ which has annihilator equal to q. This implies that $qH^{d-1}_I(R) \neq H^{d-1}_I(R)$ (indeed, otherwise, one can easily check through snake's Lemma that $H^{d-1}_I(R) = 0$, a contradiction), and therefore $H^{d-1}_I(R) \otimes R/q \neq 0$. Clearly

$\mathrm{Cd}(I, R) = d-1$ and so by [1, Exercise, 6.1.10], $H_I^{d-1}(R/q) \cong H_I^{d-1}(R) \otimes R/q \neq 0$. Therefore by Grothendick's vanishing Theorem, $\dim R/q \geq d-1$. Now, if $\dim R/q = d-1$, then by Lichtenbaum Hartshorne vanishing Theorem, $\mathrm{Rad}(I+q) = \mathfrak{m}$ and consequently

$$q \in \{P \in \mathrm{Spec}(R) \mid \dim R/P = d-1, \ \ \mathrm{Rad}(I+P) = \mathfrak{m}\}.$$

If $\dim R/q = d$, then by definition $q \in \mathrm{Assh}\, R$.

Definition 3.5. (See [1, definition, 9.1.3]) Let M be a finitely generated R-module and I be an ideal of R. We define the finiteness dimension $f_I(M)$ of M relative to I by

$$f_I(M) = \inf\{i \in \mathbb{N} \mid H_I^i(M) \text{ is not finitely generated }\}$$

Corollary 3.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, which is a homomorphic image of a Gorenstein local ring, and let M be a finitely generated R-module of dimension $d \geq 1$, then the followings hold:*
i) $\mathrm{Att}_R(H_\mathfrak{m}^1(M)) - \{\mathfrak{m}\} = \{\mathfrak{p} \in \mathrm{Ass}_R(M) \mid \dim R/\mathfrak{p} = 1\}$.
ii) We have $f_\mathfrak{m}(M) \geq 2$, if and only if $\{\mathfrak{p} \in \mathrm{Ass}_R(M) \mid \dim R/\mathfrak{p} = 1\} = \emptyset$.

Proof. i) By [1, Corollary, 11.3.5], it follows that for all $\mathfrak{p} \in \mathrm{Att}_R(H_\mathfrak{m}^1(M)) - \{\mathfrak{m}\}$, we have $\dim R/\mathfrak{p} = 1$. Now the assertion follows from Theorem 3.2.
ii) The assertion follows from the part i) and [1, Corollary, 7.2.12]. ■

Theorem 3.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, which is a homomorphic image of a Gorenstein local ring, and let M be a finitely generated R-module of dimension d, then for all $\mathfrak{p} \in \mathrm{Att}_R(H_\mathfrak{m}^t(M)) - V(\mathfrak{m})$, we have*

$$\mathrm{depth}(M_\mathfrak{p}) + \dim R/\mathfrak{p} \leq t$$

.

Proof. We use induction on t. When $t = 1$, the assertion holds by Corollary 3.4(i). So suppose that $t \geq 2$ and that the result has been proved for smaller values of t. Suppose that $\mathfrak{p} \in \mathrm{Att}_R(H_\mathfrak{m}^t(M)) - V(\mathfrak{m})$. Now if $\dim R/\mathfrak{p} = t$, then the assertion holds by Theorem 3.2. So by [1, Corollary, 11.3.5], we can suppose that $\dim R/\mathfrak{p} = j \leq t-1$. Now if $\mathfrak{p} \in \mathrm{Ass}_R(M)$, then we have $\mathrm{depth}\, M_\mathfrak{p} = 0$, and hence the assertion holds. Therefore we can assume that $\mathfrak{p} \notin \mathrm{Ass}_R(M)$. Now we set,

$$S := \{N' \mid N' \text{is a submodule of } M \text{ and } \dim N' < j\},$$

it is easy to see that as in [1, Lemma, 7.3.1], S has a largest element with respect to inclusion, as, N. Set, $G := M/N$. We have $H_\mathfrak{m}^t(G) \cong H_\mathfrak{m}^t(M)$, and G has no non-zero submodule of dimension less than t. In fact if $N \neq 0$, set, $\{\mathfrak{p} \in \mathrm{Ass}_R(M) \mid \dim R/\mathfrak{p} < j\} = \{\mathfrak{p}_i\}_{i=1}^s$, and $I := \bigcap_{i=1}^s \mathfrak{p}_i$. It is easy to see that $N = \Gamma_I(M)$, and hence $\mathrm{Ass}_R(G) = \{\mathfrak{p} \in \mathrm{Ass}_R(M) \mid \dim R/\mathfrak{p} \geq j\}$. Also by $N_\mathfrak{p} = 0$, we have $G_\mathfrak{p} \cong M_\mathfrak{p}$, hence by replacing M by G, we may assume without loss of generality that M is a non-zero finitely generated R-module such that $\{\mathfrak{p} \in \mathrm{Ass}_R(M) \mid \dim R/\mathfrak{p} < j\} = \emptyset$. Now if $\mathfrak{p} \subseteq \bigcup_{\mathfrak{q} \in \mathrm{Ass}_R M} \mathfrak{q}$, then by *Prime Avoidance Theorem* we have $\mathfrak{p} \subseteq \mathfrak{q}$, for some $\mathfrak{q} \in \mathrm{Ass}_R(M)$. But since,

$j \leq \dim R/\mathfrak{q} \leq \dim R/\mathfrak{p} = j$, it follows that $\mathfrak{p} = \mathfrak{q} \in \mathrm{Ass}_R(M)$, which is a contradiction. So we have $\mathfrak{p} \nsubseteq \bigcup_{\mathfrak{q}\in \mathrm{Ass}_R(M)} \mathfrak{q}$, hence there is an element $x \in \mathfrak{p}$, such that $x \notin \bigcup_{\mathfrak{q}\in \mathrm{Ass}_R M} \mathfrak{q}$. Arguing like in the proof of Theorem 3.2 (switching the roles of $\mathfrak{q}$ and $\mathfrak{p}$, one can show that $\mathfrak{p} \in \mathrm{Att}_R(H^{t-1}_{\mathfrak{m}}(M/x^nM))$, for all $n \gg 0$. So by the inductive hypothesis we have

$$\mathrm{depth}((M/x^nM)_{\mathfrak{p}}) + \dim R/\mathfrak{p} \leq t-1,$$

But this implies

$$\mathrm{depth}(M_{\mathfrak{p}}) + \dim R/\mathfrak{p} = \mathrm{depth}((M/x^nM)_{\mathfrak{p}}) + 1 + \dim R/\mathfrak{p} \leq t.$$

This completes the inductive step. ∎

As an immediate cosequence of Theorem 3.7 and the so-called Grothendieck's Finiteness Theorem (see [1, Theorem, 9.5.2]), we obtain the following:

Corollary 3.8. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, which is a homomorphic image of a regular local ring, and let M be a finitely generated R-module of dimension d, and let $t = f_{\mathfrak{m}}(M)$. Then for all $\mathfrak{p} \in \mathrm{Att}_R(H^t_{\mathfrak{m}}(M)) - V(\mathfrak{m})$, we have*

$$\mathrm{depth}(M_{\mathfrak{p}}) + \dim R/\mathfrak{p} = t.$$ ∎

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