Observations on spaces with property $(DC(\omega_1))^*$

Wei-Feng Xuan Wei-Xue Shi

Abstract

A topological space X has property $(DC(\omega_1))$ if it has a dense subspace every uncountable subset of which has a limit point in X. In this paper, we make some observations on spaces with property $(DC(\omega_1))$. In particular, we prove that the cardinality of a space X with property $(DC(\omega_1))$ does not exceed $\mathfrak{c}$ if X satisfies one of the following conditions: (1) X is normal and has a rank 2-diagonal; (2) X is perfect and has a rank 2-diagonal; (3) X has a rank 3-diagonal; (4) X is perfect and has countable tightness. We also prove that if X is a regular space with a G_δ-diagonal and property $(DC(\omega_1))$ then the cardinality of X is at most $2^{\mathfrak{c}}$.

1 Introduction

All topological spaces in this paper are assumed to be Hausdorff unless otherwise stated.

The property $(DC(\omega_1))$ was first introduced and studied by Ikenaga in [10]. We say that a topological space X has *property* $(DC(\omega_1))$ ([10]) if it has a dense subspace every uncountable subset of which has a limit point in X. Obviously, every separable space or every space with countable extent has property $(DC(\omega_1))$.

The properties of the diagonal often imply restrictions on the cardinality. For example, Ginsburg and Woods in [8] proved that the cardinality of a space with countable extent and a G_δ-diagonal is at most $\mathfrak{c}$. Buzyakova in [3] proved that if

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a space X with the countable Souslin number has a regular G_δ-diagonal then the cardinality of X does not exceed $\mathfrak{c}$. Arhangel'skii and Bella in [2] proved that if X is a space with a rank 4-diagonal and cellularity at most $\mathfrak{c}$ then the cardinality of X does not exceed $\mathfrak{c}$. In [14], we prove that the cardinality of a star Lindelöf space X does not exceed $\mathfrak{c}$ if X satisfies one of the following conditions: (1) X has a rank 3-diagonal; (2) X is normal and has a rank 2-diagonal; (3) X is first countable, normal and has a G_δ-diagonal. For more results one can refer to [4, 6, 7].

In this paper, by developing the idea of [8], we prove that the cardinality of a space X with property $(DC(\omega_1))$ does not exceed $\mathfrak{c}$ if X satisfies one of the following conditions: (1) X is normal and has a rank 2-diagonal; (2) X is perfect and has a rank 2-diagonal; (3) X has a rank 3-diagonal; (4) X is perfect and has countable tightness. We also prove that if X is a regular space with a G_δ-diagonal and property $(DC(\omega_1))$ then the cardinality of X is at most $2^{\mathfrak{c}}$.

2 Notation and terminology

The cardinality of a set X is denoted by $|X|$, and $[X]^2$ will denote the set of two-element subsets of X. As usual, $w(X), \chi(X), d(X), nw(X)$ and $\psi(X)$ denote respectively the *weight*, *character*, *density*, *network weight* and *pseudocharacter* of X. We write ω for the first infinite cardinal and $\mathfrak{c}$ for the cardinality of the continuum.

If A is a subset of a space X and $\mathcal{U}$ is a family of subsets of X, then $\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. We also put $\mathrm{St}^0(A,\mathcal{U}) = A$ and for a natural number n, $\mathrm{St}^{n+1}(A,\mathcal{U}) = \mathrm{St}(\mathrm{St}^{n}(A,\mathcal{U}),\mathcal{U})$. For simplicity, we write $\mathrm{St}^{n}(x,\mathcal{U})$ instead of $\mathrm{St}^{n}(\{x\},\mathcal{U})$.

Definition 2.1. ([1]) A *diagonal sequence of rank k* on a space X, where $k \in \omega$, is a countable family $\{\mathcal{U}_n : n \in \omega\}$ of open covering of X such that $\{x\} = \bigcap\{\mathrm{St}^{k}(x,\mathcal{U}_n) : n \in \omega\}$ for each $x \in X$.

Definition 2.2. ([1]) A space X has a *rank k-diagonal*, where $k \in \omega$, if there is a diagonal sequence $\{\mathcal{U}_n : n \in \omega\}$ on X of rank k. The rank of the diagonal of X is defined as the greatest natural number k such that X has a rank k-diagonal, if such a number k exists.

Definition 2.3. ([1]) Recall that a space X has a *strong rank* 1*-diagonal* if there exists a sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X such that for each $x \in X$, we have the equality $\{x\} = \bigcap\{\overline{\mathrm{St}(x,\mathcal{U}_n)} : n \in \omega\}$.

Definition 2.4. ([15]) We say that a space X has a *G_δ-diagonal* if there is a countable family $\{U_n : n \in \omega\}$ of open neighbourhoods of the diagonal Δ_X in the square $X \times X$ such that $\Delta_X = \bigcap\{U_n : n \in \omega\}$.

Definition 2.5. ([15]) We say that a space X has a *regular G_δ-diagonal* if there is a countable family $\{U_n : n \in \omega\}$ of open neighbourhoods of the diagonal Δ_X in the square $X \times X$ such that $\Delta_X = \bigcap\{\overline{U_n} : n \in \omega\}$.

Zenor in [15] pointed out that a space X has a G_δ-diagonal if and only if X has a rank 1-diagonal. If the rank of the diagonal of a space X is at least 3 then X has a regular G_δ-diagonal. It is evident that every rank 2-diagonal is a strong rank 1-diagonal and every strong rank 1-diagonal is a G_δ-diagonal (see [1]).

Definition 2.6. A topological space X is called *perfect* if every closed subset of X is a G_δ-set.

Therefore, every perfect T_1-space has countable pseudocharacter.

Definition 2.7. We say that X has *countable tightness* if for any $A \subset X$, if $x \in \overline{A}$, then there exists a countable set $A_0 \subset A$ such that $x \in \overline{A_0}$.

Definition 2.8. A topological space X is called a *sequential* space if a set $A \subset X$ is closed if and only if together with any sequence it contains all its limits.

Definition 2.9. If X is a topological space and $A \subset X$, say that a family $\mathcal{U}$ is an *open expansion* of A if $\mathcal{U} = \{U_a : a \in A\}$ and $U_a \in \tau(a, X)$ for any $a \in A$.

All notations and terminology not explained in the paper are given in [5].

3 Results

We will use a following set-theoretic theorem due to Erdös and Radó.

Lemma 3.1. ([9, p.8]) *Let X be a set with $|X| > \mathfrak{c}$ and suppose $[X]^2 = \bigcup\{P_n : n \in \omega\}$. Then there exists $n_0 < \omega$ and a subset S of X with $|S| > \omega$ such that $[S]^2 \subset P_{n_0}$.*

Proposition 3.2. *If a space X has property $(DC(\omega_1))$, then any discrete family of non-empty open subsets of X is countable.*

Proof. Assume the contrary. Then there exists a discrete family $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\}$ of non-empty open subsets in X. Let Y be a dense subspace of X such that every uncountable subset of Y has a limit point in X. For each $\alpha < \omega_1$ take $d_\alpha \in U_\alpha \cap Y$. Then $D = \{d_\alpha : \alpha < \omega_1\}$ is an uncountable closed and discrete subset of X, which leads a contradiction. ∎

Proposition 3.3. *If D is a closed and discrete subset of a normal space X and $\mathcal{U} = \{U(d) : d \in D\}$ is a pairwise disjoint open expansion of D, then there is a discrete disjoint open expansion $\mathcal{V} = \{V(d) : d \in D\}$ of D such that $d \in V(d) \subset U(d)$ for each $d \in D$.*

Proof. Since X is normal, there exists an open set $W \subset X$ such that $D \subset W \subset \overline{W} \subset \bigcup \mathcal{U}$. For each $d \in D$, let $V(d) = U(d) \cap W$. It is not difficult to show that $\mathcal{V} = \{V(d) : d \in D\}$ is a discrete disjoint open expansion of D. This completes the proof. ∎

Proposition 3.4. *If X is a perfect space and D is an uncountable discrete subset of X, then there exists an uncountable subset $E \subset D$ which is closed and discrete in X.*

Proof. Let $\mathcal{U} = \{U(d) : d \in D\}$ be a family of open subsets of X such that $U(d) \cap D = \{d\}$ for each $d \in D$. Since X is perfect, there are closed subsets F_n for $n \in \omega$ such that $\bigcup_{d \in D} U_d = \bigcup_{n \in \omega} F_n$. Clearly, there is an uncountable subset $E = D \cap F_{n_0} \subset X$ for some $n_0 \in \omega$. Now we show that E is closed and discrete in X. Suppose not, then there is a limit point ξ for E. Since F_{n_0} is closed, we have

$$\xi \in F_{n_0} \subset \bigcup_{n \in \omega} F_n = \bigcup_{d \in D} U_d.$$

Therefore, there exists $d' \in D$ such that $\xi \in U(d')$, and hence $U(d')$ contains infinite points of E, which contradicts the choice of $\mathcal{U}$. This completes the proof. ■

Proposition 3.5. *If a regular space X has countable pseudocharacter and countable tightness, then $|\overline{Y}| \leq \mathfrak{c}$ for any subset $Y \subset X$ with $|Y| \leq \mathfrak{c}$.*

Proof. Let $\mathcal{U}(x) = \{U_n(x) : n \in \omega\}$ be a family of open subsets of X such that $\{x\} = \bigcap_n \overline{U_n(x)}$ for each $x \in \overline{Y}$, since X is regular and has countable pseudocharacter. Since X has countable tightness, for each $x \in \overline{Y}$ there is a countable set $A_x \subset Y$ such that $x \in \overline{A_x}$. Now define a map $f : \overline{Y} \to (Y^\omega)^\omega$ by

$$f(x) = \{U_n(x) \cap A_x : n \in \omega\}.$$

Since $|Y| \leq \mathfrak{c}$, it follows that $|(Y^\omega)^\omega| \leq \mathfrak{c}$.

To complete the proof, we will show that such a mapping is injective. Fix any two distinct points $a, b \in \overline{Y}$. Then there exists $n_0 \in \omega$ such that $b \notin \overline{U_{n_0}(a)}$. It is obvious that $b \notin \overline{U_{n_0}(a) \cap A_a}$ and $b \in \overline{U_{n_0}(b) \cap A_b}$, which implies that $U_{n_0}(a) \cap A_a \neq U_{n_0}(b) \cap A_b$ for some $n_0 \in \omega$. Thus $f(a) \neq f(b)$. So the mapping f is injective and this completes the proof. ■

Note that the regularity is necessary in Proposition 3.5, which can be seen in the following example.

Example 3.6. ([11, p.64]) Let kN denote the Katetov's extension of the natural numbers with the discrete topology. The space kN has the following properties: (a) kN is a Hausdorff non-regular space; (b) kN is separable; (c) kN has countable tightness; (d) kN has countable pseudocharacter; (e) $|kN| = 2^{\mathfrak{c}}$.

Proposition 3.7. *If a space X has a rank 2-diagonal and $|X| > \mathfrak{c}$, then there exists an uncountable closed and discrete subset of X which has a disjoint open expansion.*

Proof. Assume the contrary. Since X has a rank 2-diagonal, there exists a sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X such that $\{x\} = \bigcap\{\mathrm{St}^2(x, \mathcal{U}_n) : n \in \omega\}$ for every $x \in X$. Note that $x \in \mathrm{St}^2(y, \mathcal{U}_n)$ if and only if $y \in \mathrm{St}^2(x, \mathcal{U}_n)$ for any distinct points $x, y \in X$ by symmetry. For each $n \in \omega$, let

$$P_n = \Big\{\{x, y\} \in [X]^2 : x \notin \mathrm{St}^2(y, \mathcal{U}_n)\Big\}.$$

Thus, $[X]^2 = \bigcup\{P_n : n \in \omega\}$ and hence there exists a subset $D \subset X$ with $|D| > \omega$ and $[D]^2 \subset P_{n_0}$ for some $n_0 \in \omega$ by Lemma 3.1. It is evident that D is a closed and discrete set and $\{\mathrm{St}(x, \mathcal{U}_{n_0}) : x \in D\}$ is an uncountable pairwise disjoint family of non-empty open sets of X by symmetry. This completes the proof. ■

Corollary 3.8. If X is a normal space with a rank 2-diagonal and property $(DC(\omega_1))$, then the cardinality of X is at most $\mathfrak{c}$.

Proof. Assume the contrary. Then there exists an uncountable closed and discrete subset $D \subset X$ which has a disjoint open expansion by Proposition 3.7, since X has a rank 2-diagonal. Therefore, D shall have a discrete disjoint open expansion by Proposition 3.3 and normality of X. But every discrete family of non-empty open subsets of X is countable by Proposition 3.2, since X has property $(DC(\omega_1))$. This contradiction completes the proof. ■

The following corollary shows that the condition "normal" in Corollary 3.8 can be replaced by "perfect".

Corollary 3.9. If X is a perfect space with a rank 2-diagonal and property $(DC(\omega_1))$, then the cardinality of X is at most $\mathfrak{c}$.

Proof. Assume the contrary. Then there exists an uncountable closed and discrete subset $S \subset X$ which has a disjoint open expansion $\{U(x) : x \in S\}$ by Proposition 3.7, since X has a rank 2-diagonal. Let Y be a dense subspace of X such that every uncountable subset of Y has a limit point in X. For each $x \in S$ take $d_x \in U(x) \cap Y$. Then $D = \{d_x : x \in S\}$ is an uncountable discrete subset of Y. It follows from Proposition 3.4 that there exists an uncountable subset $E \subset D$ which is closed and discrete in X, since X is perfect. This contradicts the choice of Y and completes the proof. ∎

Corollary 3.10. If X is a Moore space with property $(DC(\omega_1))$, then the cardinality of X is at most $\mathfrak{c}$.

Proof. Since every Moore space is perfect and has a rank 2-diagonal ([1]), we could conclude that $|X| \leq \mathfrak{c}$ by Corollary 3.9. ∎

The following questions look interesting.

Question 3.11. Let X be a Hausdorff (regular, Tychonoff) space with a rank 2-diagonal and property $(DC(\omega_1))$. Must the cardinality of X be at most $\mathfrak{c}$?

Question 3.12. ([7]) Let X be a weakly Lindelöf space with a rank 2-diagonal. Must the cardinality of X be at most $\mathfrak{c}$?

Question 3.13. ([7]) Let X be a weakly Lindelöf Moore space. Must the cardinality of X be at most $\mathfrak{c}$?

Theorem 3.14. *If X is a regular space with a G_δ-diagonal and property $(DC(\omega_1))$, then the cardinality of X is at most $2^{\mathfrak{c}}$.*

Proof. Since X has a G_δ-diagonal, there exists a sequence $\{G_k : k \in \omega\}$ of open sets of X^2 such that $\Delta_X = \bigcap\{G_k : k \in \omega\}$. For each $k \in \omega$ and $x \in X$, there exists an open subset $V_k(x)$ of X such that $(x, x) \in V_k(x) \times V_k(x) \subset G_k$. Thus without loss of generality, we assume that $G_k = \bigcup\{V_k(x) \times V_k(x) : x \in X\}$ and $G_{k+1} \subset G_k$.

Assume that Y is the dense subspace of X which witnesses that X has property $(DC(\omega_1))$. We shall show that $|Y| \leq \mathfrak{c}$. Suppose not. For each $n \in \omega$, let

$$P_n = \Big\{\{x, y\} \in [Y]^2 : (x, y) \notin G_n\Big\}.$$

Clearly, for any $\{x, y\} \in [Y]^2$, there exists $n \in \omega$ such that $\{x, y\} \in P_n$. Thus, $[Y]^2 = \bigcup\{P_n : n \in \omega\}$. Then by Lemma 3.1 there exists a subset $S \subset Y$ with $|S| > \omega$ and $[S]^2 \subset P_{n_0}$ for some $n_0 \in \omega$. It follows that S has a limit point $x \in X$ by the choice of Y. Since X is T_1, each neighborhood of x meets infinitely many members of S. In particular, there exist distinct points y and z in $S \cap V_{n_0}(x)$. Thus

$(y,z) \in V_{n_0}(x) \times V_{n_0}(x) \subset G_{n_0}$. However, since $\{y,z\} \in P_{n_0}$, $(y,z) \notin G_{n_0}$, which is a contradiction. This shows that $|Y| \le \mathfrak{c}$.

Since $w(X) \le 2^{d(X)}$ holds for any regular space X and $d(X) \le |Y| \le \mathfrak{c}$, we have $w(X) \le 2^{\mathfrak{c}}$. Therefore, $|X| \le nw(X)^{\psi(X)} \le w(X)^{\psi(X)} \le (2^{\mathfrak{c}})^{\omega} = 2^{\mathfrak{c}}$. ■

The conclusion in Theorem 3.14 is also true for Hausdorff spaces if we replace "G_δ-diagonal" with "strong rank 1-diagonal".

Proposition 3.15. *If X is a Hausdorff space with a strong rank 1-diagonal and property $(DC(\omega_1))$, then the cardinality of X is at most $2^{\mathfrak{c}}$.*

Proof. Since every strong rank 1-diagonal is a G_δ-diagonal, by using the proof of Theorem 3.14, we could conclude that there exists a dense set $Y \subset X$ of cardinality at most $\mathfrak{c}$, thus $d(X) \le \mathfrak{c}$. Since X has a strong rank 1-diagonal, it follows that $s\Delta(X) = \omega$ (see [4]). It has been established in [4] that $|X| \le 2^{d(X)s\Delta(X)}$ for any Hausdorff space X so we have $|X| \le 2^{\mathfrak{c}\cdot\omega} = 2^{\mathfrak{c}}$. This completes the proof. ■

Theorem 3.16. *If X is a space with a rank 3-diagonal and property $(DC(\omega_1))$, then the cardinality of X is at most $\mathfrak{c}$.*

Proof. Assume the contrary. Since X has a rank 3-diagonal, there exists a sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X such that $\{x\} = \bigcap\{\mathrm{St}^3(x,\mathcal{U}_n) : n \in \omega\}$ for every $x \in X$. Note that $x \in \mathrm{St}^3(y,\mathcal{U}_n)$ if and only if $y \in \mathrm{St}^3(x,\mathcal{U}_n)$ for any distinct points $x, y \in X$ by symmetry. For each $n \in \omega$, let

$$P_n = \Big\{\{x,y\} \in [X]^2 : x \notin \mathrm{St}^3(y,\mathcal{U}_n)\Big\}.$$

Thus, $[X]^2 = \bigcup\{P_n : n \in \omega\}$. Then by Lemma 3.1 there exists a subset S of X with $|S| > \omega$ and $[S]^2 \subset P_{n_0}$ for some $n_0 \in \omega$. It is evident that $\{\mathrm{St}(x,\mathcal{U}_{n_0}) : x \in S\}$ is an uncountable discrete family of non-empty open subsets of X. But every discrete family of non-empty open subsets of X is countable by Proposition 3.2, since X has property $(DC(\omega_1))$. This contradiction completes the proof. ■

Note that every rank 3-diagonal is a regular G_δ-diagonal, however the converse doesn't hold in general. Thus the following question arises naturally.

Question 3.17. Let X be a space with a regular G_δ-diagonal and property $(DC(\omega_1))$. Is the cardinality of X at most $\mathfrak{c}$? What if X is additionally first countable?

Theorem 3.18. *If X is a regular perfect space of countable tightness with property $(DC(\omega_1))$, then the cardinality of X is at most $\mathfrak{c}$.*

Proof. Let Y be a dense subspace of X which witnesses that X has property $(DC(\omega_1))$. We shall show that $|Y| \le \mathfrak{c}$. Suppose not. Since X is a perfect space, X has countable pseudocharacter. For each $x \in Y$, let $\mathcal{B}(x) = \{B_n(x) : n \in \omega\}$ be a family of open sets of X such that $\bigcap \mathcal{B}(x) = \{x\}$ and $B_{n+1} \subset B_n$ for each $n \in \omega$. For each $n \in \omega$, let

$$P_n = \Big\{\{x,y\} \in [Y]^2 : y \notin B_n(x); x \notin B_n(y)\Big\}.$$

It is easy to check that $[Y]^2 = \bigcup\{P_n : n \in \omega\}$. We can apply Lemma 3.1 to conclude that there exists an uncountable subset S of Y and $[S]^2 \subset P_{n_0}$ for some $n_0 \in \omega$. Note that for each $x \in S$, $B_{n_0}(x) \cap S = \{x\}$. It follows from Proposition 3.4 that there exists an uncountable subset $E \subset S$ which is closed and discrete in X. This contradiction the choice of Y shows that $|Y| \leq \mathfrak{c}$. Now we could conclude that $|X| = |\overline{Y}| \leq \mathfrak{c}$ by Proposition 3.5, since X has countable pseudocharacter and countable tightness and Y is dense in X. This completes the proof. ∎

Since every first countable (Fréchet, sequential) space has countable tightness, we have the following corollaries by Theorem 3.18.

Corollary 3.19. If X is a regular, perfect and sequential space with property $(DC(\omega_1))$, then the cardinality of X is at most $\mathfrak{c}$.

Corollary 3.20. If X is a regular, perfect and Fréchet space with property $(DC(\omega_1))$, then the cardinality of X is at most $\mathfrak{c}$.

Corollary 3.21. If X is a regular, perfect and first countable space with property $(DC(\omega_1))$, then the cardinality of X is at most $\mathfrak{c}$.

If we drop the condition "countable tightness" in Theorem 3.18, then $2^{\mathfrak{c}}$ would be the least upper bound of X.

Proposition 3.22. *If X is a regular perfect space with property $(DC(\omega_1))$, then the cardinality of X is at most $2^{\mathfrak{c}}$.*

Proof. By using the proof of Theorem 3.18, we could conclude that there exists a dense set $Y \subset X$ of cardinality at most $\mathfrak{c}$, thus $d(X) \leq \mathfrak{c}$. Since $|X| \leq 2^{d(X)\psi(X)}$ holds for any regular space X, we conclude that $|X| \leq 2^{\mathfrak{c}\cdot\omega} = 2^{\mathfrak{c}}$ which completes the proof. ∎

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College of Science,
Nanjing Audit University,
Nanjing, China, 210093
email: wfxuan@nau.edu.cn

Department of Mathematics,
Nanjing University,
Nanjing, China, 210093
email: wxshi@nju.edu.cn