Generalizing nil clean rings

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Abstract

We introduce the class of *unipotently nil clean* rings as these rings R in which for every $a \in R$ there exist an idempotent e and a nilpotent q such that $a - e - 1 - q \in (1-e)Ra$. Each unipotently nil clean ring is weakly nil clean as well as each nil clean ring is unipotently nil clean. Our results obtained here considerably extend those from [8] and [7], respectively.

1 Introduction and Backgrounds

Let R be an associative ring with identity. Following [17], we say that R is π-regular if, for every $a \in R$, there exists a positive integer n such that $a^n \in a^n R a^n$. Valuable examples are the classical von Neumann regular rings, Artinian rings and perfect rings. In a way of similarity a ring R is said to be strongly π-regular if, for every $a \in R$, there exists a positive integer n such that $a^n \in a^{n+1}R \cap Ra^{n+1}$. It is well known that each strongly π-regular ring is π-regular and the two notions coincide in the case when the ring is abelian, that is, when all idempotents are central. In this aspect, in [1] was showed that each π-regular ring with bounded index of nilpotence is strongly π-regular. Major examples of strongly π-regular rings include Artinian rings and perfect rings as well as algebraic algebras over a field, whereas von Neumann regular rings are in general not strongly π-regular. For a more detailed information about von Neumann regular rings, π-regular and strongly π-regular rings, we refer the interested reader to [9] and [17], respectively.

Mimicking [13], a ring R is called exchange if, for every $a \in R$, there exists an idempotent $e \in Ra$ such that $1 - e \in R(1-a)$, and R is called clean if every

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element in R can be written as a sum of an idempotent and a unit. Clean rings are exchange, but the converse does not hold. However, in the abelian case these two classes do coincide. It is principally known that all π-regular rings are exchange, while strongly π-regular rings are clean, but none of these two implications is reversible.

Imitating [8], a ring R is called nil clean if every its element can be represented as the sum of an idempotent and a nilpotent. If they commute, R is said to be strongly nil clean. Nil clean rings are clean and strongly nil clean rings are strongly π-regular. Examples of nil clean rings are the classical Boolean rings as well as the rings $\mathbb{Z}_{2^n}$ for any $n \in \mathbb{N}$.

The motivation of the present paper is to generalize the notion of nil cleanness in such a way that is appropriate for a further development and which has a close connection with other well-known related concepts. It is organized as follows: After the short historical facts presented above, we put in the next section the main definitions and give some examples in order to clarify the situation. After that, we show in a subsequent section how the new concept is situated between the classical notions of nil cleanness and π-regularity. We also examine in the sequel the uniqueness conditions set on some special ring elements, and finally we close with a few unresolved problems of certain interest and importance.

Throughout the text, all rings are assumed to be associative and unital, mainly non-commutative. The standard notations $\mathrm{U}(R)$, $\mathrm{J}(R)$, $\mathrm{Id}(R)$, $\mathrm{Nil}(R)$ and $\mathrm{Z}(R)$ will stand for the set of units, the Jacobson radical, the set of idempotents, the set of nilpotents and the center of the ring R, respectively. We also denote by $\mathbb{M}_n(R)$ the ring of all $n \times n$ matrices over R. As usual, the symbol $\mathbb{N}$ stands for the set of all positive integers (= naturals).

2 Definitions and Examples

The following concept arisen in both [15] and [16].

Definition 2.1. A ring R is said to be *weakly clean* if, for each $r \in R$, there exist $e \in \mathrm{Id}(R)$ and $u \in \mathrm{U}(R)$ such that $r - e - u \in (1-e)Rr$.

It was proved there that weakly clean rings are exchange and clean rings are weakly clean. Thus all abelian weakly clean rings are precisely the clean rings.

The next concept appeared in [7].

Definition 2.2. A ring R is said to be *weakly nil clean* if, for each $r \in R$, there exist $e \in \mathrm{Id}(R)$ and $q \in \mathrm{Nil}(R)$ such that $r - e - q \in eRr$.

It was shown in [7] that weakly nil clean rings are weakly clean. Moreover, all nil clean as well as all π-regular rings are both weakly nil clean. In addition, abelian weakly nil clean rings are exactly the π-regular rings. Thus, opposite to above, abelian weakly nil clean rings need not be nil clean. Other examples of such rings include also semi-perfect rings with nil Jacobson radical (which, in general, are not necessarily π-regular), finite direct products of π-regular rings, and upper triangular matrix rings over π-regular rings. In [7], we also have generalized the aforementioned result from [1] by showing that each weakly nil clean ring of bounded index of nilpotence is strongly π-regular; in particular, weakly nil clean PI rings are strongly π-regular. We also proved there that the center of a weakly nil clean ring is strongly π-regular, thus extending the analogous result in [12] related for π-regular rings. This also implies that each weakly nil clean ring is a corner of a clean ring.

Our central tool, however, is the following new notion:

Definition 2.3. A ring R is called *unipotently weakly clean* if, for every $x \in R$, there are $e \in \mathrm{Id}(R)$ and $q \in \mathrm{Nil}(R)$ with the property $x - e - 1 - q \in (1-e)Rx$.

In other words, we replace in Definition 2.1 the existing unit u by the unipotent $1+q$. Thereby the substitution $u = 1+q$ makes sense to truly choose the name "unipotently weakly clean". Therefore, weakly clean UU rings are obviously unipotently weakly clean. Even much more, according to [6], weakly clean UU rings R are strongly nil clean; especially $R/\,\mathrm{J}(R)$ is Boolean and $\mathrm{J}(R)$ is nil.

Clearly, if $2 = 0$, Definitions 2.2 and 2.3 are equivalent. We will establish in the sequel even more that if $2 \in \mathrm{Nil}(R)$ such an equivalence remains still true.

Before we establish certain critical results on unipotently weakly clean rings, we need to recollect some further facts about weakly nil clean rings. Specifically, the next two criteria are fulfilled:

Criterion 2.4. *A ring R is weakly nil clean if, and only if, one of the following equivalencies holds:*

- $\forall r \in R, \exists e \in \mathrm{Id}(R), \exists q \in \mathrm{Nil}(R) \ : \ r + e + q \in eRr.$
- $\forall r \in R, \exists e \in \mathrm{Id}(R), \exists q \in \mathrm{Nil}(R) \ : \ r - e + 1 + q \in (1-e)Rr.$
- $\forall r \in R, \exists e \in \mathrm{Id}(R), \exists q \in \mathrm{Nil}(R) \ : \ r + e - 1 - q \in (1-e)Rr.$
- $\forall r \in R, \exists f \in \mathrm{Id}(Rr), \exists t \in \mathrm{Nil}(R) \ : \ 1 - f = (1-f)(1+t)(1-r).$

Proof. The first point is precisely Definition 2.2, where r is replaced by $-r$. The second point follows from the first one via the substitution $e \to 1-e$. Furthermore, the third point follows either from Definition 2.2 again using the same substitution $e \to 1-e$, or by replacing r with $-r$ in point two. And finally, the fourth point is just [7, Proposition 2.5]. ∎

Criterion 2.5. *A ring R is unipotently nil clean if, and only if, one of the following equivalencies holds:*

- $\forall x \in R, \exists e \in \mathrm{Id}(R), \exists q \in \mathrm{Nil}(R) \ : \ x + e + 1 + q \in (1-e)Rx.$
- $\forall x \in R, \exists e \in \mathrm{Id}(R), \exists q \in \mathrm{Nil}(R) \ : \ x - e + 2 + q \in eRx.$
- $\forall x \in R, \exists e \in \mathrm{Id}(R), \exists q \in \mathrm{Nil}(R) \ : \ x + e - 2 - q \in eRx.$
- $\forall x \in R, \exists f \in \mathrm{Id}(Rx), \exists t \in \mathrm{Nil}(R) \ : \ 1 - f = (1-f)(-1-t)(1-x).$

Proof. The first point is exactly Definition 2.3, replacing x via $-x$. For the next second point we just substitute $e \to 1-e$. Further, to obtain the third point, we again replace x by $-x$ in point two, or use the same substitution $e \to 1-e$ in Definition 2.3.

Finally, point four follows like this: Let us prove only the forward direction, the converse can be proved similarly. To that aim, given $a \in R$ and, by assumption, write with the aid of point three that $-a = e - 2 - q + eba$, where $e \in \mathrm{Id}(R)$, $q \in \mathrm{Nil}(R)$ and $b \in R$. Denoting $h = 1 - e \in \mathrm{Id}(R)$ and $u = -1 - q \in \mathrm{U}(R)$, we have $e + u - 1 = -a - eba = -(1 + eb)a \in Ra$, whence $u^{-1}eu = u^{-1}e(e + u - 1) = f \in \mathrm{Id}(Ra)$. Furthermore, we deduce that $ha = h(-e + 2 + q - eba) = h(2 + q)$ and hence $hu = -h(1 + q) = h - h(2 + q) = h - ha = h(1 - a)$. So, $u^{-1}hu = u^{-1}hu.u^{-1}(1 - a)$ and consequently $u^{-1}hu = u^{-1}(1 - e)u = 1 - u^{-1}eu = 1 - f = (1 - f)u^{-1}(1 - a)$. Since it is elementary checked by the classical binomial formula of Newton that $u^{-1} = (-1 - q)^{-1} = -1 - t$ for some $t \in \mathrm{Nil}(R)$, we are done. ∎

Again, if $\mathrm{char}(R) = 2$, then Criterion 2.4 is tantamount to Criterion 2.5. The last characterization also gives a direct proof that unipotently weakly clean rings have to be exchange.

In what follows we provide some crucial examples. We shall give more concrete examples later when we have more information at our disposal. For instance, we will show below that unipotently weakly clean rings include all π-regular rings having 2 as a nilpotent. Specifically, summarizing the necessary and sufficient conditions stated above, we are now ready to proceed by proving with the following structural characterization.

Theorem 2.6. *A ring R is unipotently weakly clean if, and only if, R is weakly nil clean and* $2 \in \mathrm{Nil}(R)$.

Proof. Taking $x = 0$ in some of the first three relationships from Criterion 2.5, we deduce that $2 = e - q$, i.e., $1 - e = -1 - q \in \mathrm{Id}(R) \cap \mathrm{U}(R) = \{1\}$. Hence $e = 0$ and $2 = q$ has to be a nilpotent. Same will be valid if we take $x = 0$ in the last equality of Criterion 2.5.

Since $-1 - t = 1 + (-2 - t)$ and $1 + t = -1 - (-2 - t)$, accomplished with $-2 - t \in \mathrm{Nil}(R)$ whenever $2, t \in \mathrm{Nil}(R)$, we just apply Criteria 2.4 and 2.5 to conclude the claim. ∎

Remark 2.7. Actually, the third point from Criterion 2.5 can be written as $x + e + z \in eRx$, where $z = -2 - q \in \mathrm{Nil}(R)$. Thus we obtain the first point of Criterion 2.4, so this directly means that unipotently weakly clean rings are weakly nil clean. The converse also holds by the same token using this trick, provided 2 is a nilpotent.

As a nontrivial consequence to this criterion, we derive the following chief statement which motivated the writing of this article.

Corollary 2.8. *Nil clean rings are unipotently weakly clean.*

Proof. It was shown in [7] that nil clean rings are themselves weakly nil clean. Since in view of [8] we have that $2 \in \mathrm{Nil}(R)$, Theorem 2.6 is applicable to get the wanted assertion. ∎

Recall that an element v of a ring R is an involution, provided $v^2 = 1$. The next somewhat surprising assertion describes all involutions in commutative unipotently weakly clean rings.

Proposition 2.9. *Suppose R is a commutative unipotently weakly clean ring. Then every involution in R is a unipotent.*

Proof. First, let $\mathrm{char}(R) = 2$. Given $v \in \mathrm{U}(R)$ with $v^2 = 1$, we infer by virtue of Criterion 2.5 that $v + e + 1 + q \in (1-e)Rv$ for some $e \in \mathrm{Id}(R)$ and $q \in \mathrm{Nil}(R)$. If we assume in a way of contradiction that $e \neq 0$, then multiplying both sides of the above relation by e from the left, we deduce that $e(v+q) = 0$. But $v + q \in \mathrm{U}(R)$ and hence $e = 0$ which contradicts our assumption. Finally, one can take $e = 0$ and so $v + 1 + q \in Rv$ leads to $(1-b)v = -1-q$ for some $b \in R$. Furthermore, lifting both sides of this equality by degree 2, we obtain that $b^2 = q^2$ whence $b \in \mathrm{Nil}(R)$. Consequently, one easily checks that $v = -(1+q)(1-b)^{-1} \in -1 + \mathrm{Nil}(R)$, as required.

Now, let R be of arbitrary characteristic. Again take $v \in \mathrm{U}(R)$ with $v^2 = 1$. Hence $v + 2R \in R/2R$ is also an involution. By what we have shown so far, $v + 2R = (1 + 2R) + (q + 2R)$, where the latter element is a nilpotent in $R/2R$. Since $2R$ is a nil ideal of R owing to the fact deduced from Criterion 2.5 that $2 \in \mathrm{Nil}(R)$, it must be that $q \in \mathrm{Nil}(R)$. Therefore, $v = 1 + q + 2c \in 1 + \mathrm{Nil}(R)$ for some $c \in R$, as expected. ■

As a direct valuable consequence, we yield the following strengthening from [4] (notice that, however, the result remains valid for non necessarily commutative rings).

Corollary 2.10. *Each involution in a commutative nil clean ring is a unipotent.*

It was established in [4] that in nil clean rings every involution is a unipotent. This was strengthened in [5, Proposition 2.20], where it was proved that any involution is a unipotent $\iff$ 2 is a nilpotent. By what we have shown above, the element 2 is a nilpotent in unipotently weakly clean rings. Thus Proposition 2.9 is true even without the "commutativity" restriction.

We are now in a position to provide the reader with more element-wise examples and crucial properties in unipotently weakly clean rings.

Example 2.11. (1) Every idempotent in an arbitrary ring and every nilpotent in a ring in which 2 is a nilpotent are unipotently weakly clean elements. In particular, Boolean rings are unipotently weakly clean (compare with Corollary 2.8 above).

In fact, for any $x \in \mathrm{Id}(R)$, one can write that

$$x - x + 2 + (-2) = x.0.x$$

and moreover, for any $x \in \mathrm{Nil}(R)$, one may write that

$$x - 0 + 2 + (-2 - x) = 0.1.x,$$

where by assumption $2 \in \mathrm{Nil}(R)$, as required in order to apply Criterion 2.5.

(2) Every invertible element in a ring is unipotently weakly clean.

Indeed, for any $a \in \mathrm{U}(R)$, we can write that $u + 0 + 1 + 0 = 1.(1 + u^{-1}).u$, where $1 \in \mathrm{Id}(R)$ and $0 \in \mathrm{Nil}(R)$, as needed to employ Criterion 2.5. In particular, this shows that each division ring is unipotently weakly clean. Note however that division rings are never nil clean, except the case when $R = \mathbb{Z}_2$.

Proposition 2.12. *Unipotently weakly clean rings are closed under the formation of homomorphic images, direct limits and finite direct products.*

Proof. All the checks are straightforward, so we omit the details. ■

The following is an analogue of [8, Proposition 3.15].

Proposition 2.13. *Let R be a ring with a nil ideal I. Then R is unipotently weakly clean if and only if R/I is unipotently weakly clean.*

Proof. According to Proposition 2.12, we only need to prove the backward direction. So, assume that R/I is unipotently weakly clean. We put $\overline{R} = R/I$ and $\overline{x} = x + I \in \overline{R}$ for any $x \in R$. Now, taking $a \in R$, we write by assumption that $\overline{a} = 1 + \epsilon + \eta + (1 - \epsilon)\chi\overline{a}$ for some $\epsilon \in \mathrm{Id}(\overline{R})$, $\eta \in \mathrm{Nil}(\overline{R})$ and $\chi \in \overline{R}$. Since idempotents lift modulo every nil ideal, there exists $e \in \mathrm{Id}(R)$ with $\overline{e} = \epsilon$. Taking $x \in R$ with $\overline{x} = \chi$, we have $\overline{a - 1 - e - (1 - e)xa} = \eta \in \mathrm{Nil}(\overline{R})$, so that $\overline{(a - 1 - e - (1 - e)xa)^n} = 0$ for some $n \geq 1$, or equivalently, $(a - 1 - e - (1 - e)xa)^n \in I$. Since I is nil, it finally follows that $a - 1 - e - (1 - e)xa$ is a nilpotent, as desired. ■

Remark 2.14. This claim also follows from [7, Proposition 2.7] accomplished with Theorem 2.6, bearing in mind that $2 \in \mathrm{Nil}(R)$ only when $2 + I \in \mathrm{Nil}(R/I)$. However, the current proof is more conceptual and self-contained.

Proposition 2.15. *A ring R is unipotently weakly clean if and only if $R/\,\mathrm{J}(R)$ is unipotently weakly clean and $\mathrm{J}(R)$ is nil.*

Proof. Referring to Propositions 2.12 and 2.13, we only need to show that $\mathrm{J}(R)$ is nil whenever R is unipotently weakly clean. To that goal, take $a \in \mathrm{J}(R)$ and, by assumption, write $a = 1 + e + q + (1 - e)xa$ with $e \in \mathrm{Id}(R)$, $q \in \mathrm{Nil}(R)$ and $x \in R$. Since $1 + e + q = a - (1 - e)xa \in \mathrm{J}(R)$, we have $e + \mathrm{J}(R) = -1 - q + \mathrm{J}(R)$, so that this element is simultaneously an idempotent and a unit in $R/\,\mathrm{J}(R)$. This forces $e \in 1 + \mathrm{J}(R) \subseteq \mathrm{U}(R)$ and hence $e = 1$, thus giving that $a = 2 + q$ is a nilpotent, because with Theorem 2.6 in hand we know that $2 \in \mathrm{Nil}(R)$. ■

The above results allow us to give some more pivotal examples.

Example 2.16. (1) Every ideal of a unipotently weakly clean ring is a unipotently weakly clean (non-unital) ring. More generally, an ideal I of a ring R is unipotently weakly clean if and only if every $a \in I$ is a unipotently weakly clean element in R. This can be proved in a similar way as Proposition 2.15.

(2) Proposition 2.13 leads us to the construction of unipotently weakly clean rings from known ones. For instance, if R is a unipotently weakly clean ring, then the upper triangular matrix ring $\mathbb{T}_n(R)$ is unipotently weakly clean by combining Propositions 2.13 and 2.12. In fact, $\mathbb{T}_n(R)/I \cong R \times \cdots \times R$ (n-times), where $I = \{(a_{ij}) \in \mathbb{T}_n(R) : \text{all } a_{ii} = 0\}$ is a nil ideal in $\mathbb{T}_n(R)$.

(3) As another example, if R is a unipotently weakly clean ring and M is an R-bimodule, then the *trivial extension* of R by M (i.e., the direct sum $R \oplus M$, equipped with multiplication $(a, x)(b, y) = (ab, ay + xb)$) is a unipotently weakly clean ring by Proposition 2.13.

(4) Proposition 2.15 provides a rich source of exchange (or clean) rings that are not unipotently weakly clean. For instance, by this proposition, a local ring is unipotently weakly clean if and only if its Jacobson radical is nil and $2 \in \mathrm{J}(R)$. Note that local rings are always clean.

(5) The ring $R(Q, L)$ in [13, Example 1.7] is an example of an exchange ring with zero Jacobson radical which is not unipotently weakly clean.

(6) If R is unipotently weakly clean, then $R[X]/(X^n)$ is unipotently weakly clean for every n, which is another easy consequence of Proposition 2.13.

3 Relations to Nil Clean and π-Regular Rings

In conjunction with Definition 2.3, we shall say that the element r of a ring R is *unipotently weakly clean* in R, provided that $r \in e + 1 + q + (1-e)Rr$ for some $e \in \mathrm{Id}(R)$ and $q \in \mathrm{Nil}(R)$.

The following technicality is useful.

Lemma 3.1. *Let R be a ring with $2 \in \mathrm{Nil}(R)$, and let $a \in R$. If there exists $e = e^2 = 1 - f \in Ra$ such that faf is unipotently weakly clean in fRf, then a is unipotently weakly clean in R.*

Proof. Supposing faf is unipotently weakly clean in fRf, we write that

$$faf = g + f + t + (f-g)dfaf \tag{1}$$

for some $g \in \mathrm{Id}(fRf)$, $t \in \mathrm{Nil}(fRf)$ and $d \in fRf$. Clearly, $fg = g = gf$ as well as $ft = tf = t$. Thus $f - g = g' \in \mathrm{Id}(fRf)$ and hence (1) takes the form

$$faf = -g' + 2f + t + g'dfaf. \tag{2}$$

Since $(f-g')g' = g(f-g) = 0$, multiplying (2) by $f - g'$ one sees that $(f-g')faf = (f-g')(2f+t)$. But $2f + t \in \mathrm{Nil}(R)$, denoting it by t'. Now, defining $\mu = t' + fae$, we can directly verify that $\mu^n = (t')^n + (t')^{n-1}fae$ for all $n \geq 1$, so that μ is a nilpotent in R. Letting $\pi = 1 - g = 1 - (f - g') = e + g'$, one easily checks that $\pi \in \mathrm{Id}(R)$. Since $(f-g')faf = (f-g')t'$, $(f-g')f = gf = g = f - g'$ and $e + f = 1$, we deduce that $(1-\pi)a = (f-g')a = (f-g')fae + (f-g')faf = (f-g')fae + (f-g')t' = (f-g')\mu = (1-\pi)\mu$, i.e., $a - \pi a = \mu - \pi\mu$. Thus $a + \pi - \mu = \pi + \pi a - \pi\mu \in \pi R$. Furthermore, since $e \in Ra$ and $af = a - ae \in Ra$, we obtain $-\pi + \mu = -e - g' + t' + fae = -e + fae + faf - g'afaf \in Ra$ because $af = a(1-e) = a - ae \in Ra$. Therefore $a + \pi - \mu \in \pi R \cap Ra = \pi Ra$ whence $a \in -\pi + \mu + \pi Ra$. Setting $\pi = 1 - \varepsilon$, we finally infer that $\varepsilon \in \mathrm{Id}(R)$ and $a \in \varepsilon + 1 + (\mu - 2) + (1-\varepsilon)Ra$ with $\mu - 2 \in \mathrm{Nil}(R)$, as desired. ∎

It was proved in [16] that every π-regular ring is weakly clean and in [7] that it is even weakly nil clean. The following statement somewhat strengthens this result.

Proposition 3.2. *Every π-regular ring for which 2 is a nilpotent is unipotently weakly clean.*

Proof. Let $a \in R$ be a π-regular element. Choose $n \geq 1$ and $r \in R$ such that $a^n = a^n r a^n$. Then $e = ra^n$ is an idempotent in Ra. Setting $f = 1 - e$, we see as in [16, Example 2.7 (1)] that faf is a nilpotent in fRf. Since in virtue of Example 2.11(1) every nilpotent element is unipotently weakly clean, the conclusion follows at once by Lemma 3.1. ■

Remark 3.3. This assertion can also be deduced from [7, Proposition 3.2] in combination with Theorem 2.6. However, the present proof is more direct and uses the major element-wise Lemma 3.1 which is not deducible from [7] nor the method used in [16, Lemma 2.6] can be successfully applied.

The last result provides many other examples of unipotently weakly clean rings.

Example 3.4. (1) Every von Neumann regular ring is π-regular and hence it must be unipotently weakly clean, provided its characteristic is 2. Moreover, Artinian rings, (right or left) perfect rings, and algebraic algebras over a field, all of characteristic 2, are unipotently weakly clean, because they are π-regular.

(2) A finite direct product of π-regular rings for which 2 is a nilpotent is unipotently weakly clean. It is not known to the author if it is actually always π-regular, but it seems that in general this is not the case (see [17] for more details). However, it holds true for abelian rings (see Proposition 3.9 below).

(3) An infinite direct product of unipotently weakly clean rings need not be unipotently weakly clean. For example, Proposition 2.15 works to conclude that $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times \ldots$ is not unipotently weakly clean since it has non-nil Jacobson radical although 2 is a nilpotent in the direct factors but not in the whole direct product.

(4) If R is a π-regular ring with $2 \in \mathrm{Nil}(R)$, then the upper triangular matrix ring $\mathbb{T}_n(R)$ is unipotently weakly clean invoking Proposition 2.13 (see Example 2.16(2) too). It seems to be very unlikely (though it is not known to the author) that this ring has to be π-regular in general.

(5) A semiperfect ring is unipotently weakly clean if and only if the Jacobson radical of the ring is nil. This follows from a simple combination of Proposition 2.13 and Proposition 3.2.

Based on the example of Rowen (see, for instance, [7, Example 3.4]), we can now give an example of a unipotently weakly clean ring which is not π-regular.

Example 3.5. Let F be a field, and $F(X)$ the field of fractions of the polynomial ring $F[X]$. Extend $\{X^n | \, n \in \mathbb{Z}\}$ to a basis $\mathcal{B}$ of $F(X)$ over F. Let T be the free product of $F(X)$ with the (unital) free algebra on two elements $F\langle A, B\rangle$. Let $V = \{BwA | \, w \in \mathcal{B} \setminus \{X^n | \, n < 0\}\} \subseteq T$, and P be the ideal of T generated by A^2, B^2, AwA, AwB, BwB, for all $w \in \mathcal{B} \setminus \{1\}$, V, and $\bigcup_{k=1}^{\infty}\{(BX^{-1}A)^{n_k}, (BX^{-2}A)^{n_k}, \ldots, (BX^{-k}A)^{n_k}\}$, where $n_k > 2^k + 1$. Set $R = T/P$. As in [7], R is a local ring whose Jacobson radical $\mathrm{J}(R)$ is locally nilpotent. In fact, $\mathrm{J}(R) = R\overline{A}R + R\overline{B}R$, where $\overline{A}, \overline{B}$ are homomorphic images of A, B in R, and $R/\mathrm{J}(R) \cong F(X)$. Setting

$S = \mathbb{M}_2(R)$, we then infer that S is semiperfect and the Jacobson radical $\mathrm{J}(S) = \mathbb{M}_2(\mathrm{J}(R))$ is nil because $\mathrm{J}(R)$ is locally nilpotent; even $S/\mathrm{J}(S) \cong \mathbb{M}_2(R/\mathrm{J}(R))$, where $R/\mathrm{J}(R)$ is a division ring, being even a field. Hence, an appeal to [7, Example 3.4] guarantees that S is weakly nil clean and, therefore, $S/2S$ is also weakly nil clean. A utilization of Theorem 2.6 ensures that $S/2S$ is unipotently weakly clean. But this ring is not π-regular as indicated in Example 3.4 from [7].

We can also find a unipotently weakly clean ring with trivial Jacobson radical which is not π-regular, namely the following construction holds:

Example 3.6. Let S be as in Example 3.5. We may additionally choose $\mathrm{char}(S) = 2$; in fact, in the preceding example we just can take $\mathrm{char}(F) = 2$. We also embed S in a π-regular ring S_1 of $\mathrm{char}(S_1) = 2$ with $J(S_1) = 0$. For example, considering S as an algebra over a field F, we can embed S in the endomorphism ring $S_1 = End_F(S)$ (which is von Neumann regular). Now, the ring

$$T = \{(a_1, \ldots, a_n, a, a, \ldots) \mid n \in \mathbb{N},\ a_i \in S_1 (1 \leq i \leq n),\ a \in S\},$$

equipped with componentwise operations and $2a_i = 2a = 0$, is unipotently weakly clean of characteristic 2 since S and S_1 are weakly nil clean, but T is not π-regular since it has a homomorphic image S. Note also that $J(T) = 0$ by direct computation.

In [16, Proposition 4.9] it was shown that if R is a ring with an ideal I such that both I and R/I are π-regular then R is weakly clean, and in [7] that it is even weakly nil clean. So, one can state the following:

Proposition 3.7. *Let R be a ring with an ideal I such that I and R/I are both π-regular. Then R is unipotently weakly clean, provided $2 \in \mathrm{Nil}(R)$.*

Proof. As remarked above, it follows from [7, Proposition 3.6] that R is weakly nil clean. Henceforth, we apply Theorem 2.6 to get the wanted claim. ∎

Remark 3.8. Note that, in general, the ring R in Proposition 3.7 need not be π-regular. Consider, for instance, Example 3.5.

In the next few statements we show that, for a wide range of examples, Proposition 3.2 is also true in the converse direction.

Proposition 3.9. *An abelian ring in which 2 is a nilpotent is unipotently weakly clean if and only if it is strongly π-regular.*

Proof. Employing [7, Proposition 3.8] and Theorem 2.6, we derive the desired assertion. ∎

Remark 3.10. In view of this proposition, it follows that there exists a π-regular ring that is not unipotently weakly clean, namely there is an abundance of such π-regular rings in which 2 is not a nilpotent.

A ring R is said to have *bounded index of nilpotence* if there exists $n \in \mathbb{N}$ such that $x^n = 0$ for every nilpotent $x \in R$. In [1] was proved that every π-regular ring with bounded index of nilpotence is strongly π-regular. We will extend this result to the wider class of unipotently weakly clean rings.

The next technicality also appeared in [7].

Lemma 3.11. *Suppose that R is an exchange ring with bounded index of nilpotence, and suppose that every homomorphic image of R has nil Jacobson radical. Then R is strongly π-regular.*

Proof. By virtue of [17, Theorem 23.2], it suffices to prove that every prime factor ring of R is strongly π-regular. Thus, let P be a prime ideal of R. By Zorn's lemma, we can find a minimal prime ideal P_0 of R with the property that $P_0 \subseteq P$. By [17, Remark 14.4 (1)], the ring R/P_0 is with bounded index of nilpotence, hence owing to [17, Proposition 14.5 (1)] and [2, Corollary 2] the quotient R/P_0 is semiperfect noticing that R/P_0 is exchange. Since R/P_0 is also semiprime and $J(R/P_0)$ is nil by assumption, [17, Corollary 14.3 (2)] guarantees that $J(R/P_0) = 0$. Thus R/P_0 is semisimple Artinian, and hence strongly π-regular. The ring R/P, being a homomorphic image of R/P_0, is therefore also strongly π-regular, as desired. ∎

The following extends the aforementioned result [1, Theorem 5].

Proposition 3.12. *Every unipotently weakly clean ring of bounded index of nilpotence is strongly π-regular.*

Proof. Utilizing Propositions 2.12 and 2.15, every homomorphic image of a unipotently weakly clean ring has nil Jacobson radical, whence the result follows immediately from Lemma 3.11. ∎

Recall that a ring R is said to be a PI-ring if R satisfies a polynomial identity with coefficients in $\mathbb{Z}$ and at least one of them is invertible. In particular, each commutative ring is a PI-ring.

Using Proposition 3.12 and the classical result that semiprime PI-rings have bounded index of nilpotence, we can state the following strengthening.

Corollary 3.13. *Every unipotently weakly clean PI-ring is strongly π-regular.*

Proof. Let R be a unipotently weakly clean PI-ring, and let $P(R)$ denote the prime radical of R. Then it is readily verified the factor ring $R/P(R)$ has bounded index of nilpotence. Thus Proposition 3.12 shows that $R/P(R)$ is strongly π-regular, which in turn forces that R is strongly π-regular, as promised. ∎

If e is an idempotent of a ring R, then eRe is a subring of R with identity e, called the *corner ring* of R. It is easy to see that every corner of a (strongly) π-regular ring is again (strongly) π-regular. So, it is natural to ask:

Question 3.14. Is every corner of a unipotently weakly clean ring also unipotently weakly clean? In particular, if the full $n \times n$ matrix ring $\mathbb{M}_n(R)$ over a ring R is unipotently weakly clean, is then R unipotently weakly clean, too?

For clean rings the answer in general is negative (cf. [15] and [16]). As our definition seems to be closer to that of nil clean rings, there is some hope for the above question to have a positive answer. Unfortunately, a general result cannot be deduced at this stage, but however we are able to give a partial settling.

Proposition 3.15. *Let R be an abelian ring. If $\mathbb{M}_n(R)$ is unipotently weakly clean for some $n \in \mathbb{N}$, then R is unipotently weakly clean.*

Proof. If is fairly clear that $2 \in \mathrm{Nil}(\mathbb{M}_n(R))$ implies that $2 \in \mathrm{Nil}(R)$. In fact, one sees by induction that

$$\begin{pmatrix} 2 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2 \end{pmatrix}^n = \begin{pmatrix} 2^n & 0 & \cdots & 0 \\ 0 & 2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2^n \end{pmatrix}$$

for any $n \in \mathbb{N}$, which equality assures at once the wanted implication. Hereafter, we just combine [7, Proposition 3.15] and Theorem 2.6. ∎

Remark 3.16. The matrix ring $\mathbb{M}_n(R)$ in Proposition 3.15 need not be in general (strongly) π-regular, as Example 3.5 demonstrates. In fact, to be more precise, $\mathbb{M}_n(R)$ need not be abelian although R is.

The reverse form of Question 3.14 asks the following:

Question 3.17. If e is an idempotent of a ring R such that both eRe and $(1-e)R(1-e)$ are unipotently weakly clean rings, is then R also unipotently weakly clean? In particular, if R is a unipotently weakly clean ring, is then $M_n(R)$ unipotently weakly clean for every $n \in \mathbb{N}$?

These two questions have a positive resolution for weakly clean rings (cf. [15, Proposition 3.3]) as well as for clean rings (cf. [10, Lemma on p. 2590]). For the class of nil clean rings, the question is still open. However, it looks unlikely for the above question to have an answer in the affirmative, because for strongly π-regular and π-regular rings the answer is negative as the above Example 3.5 illustrates.

Notice that, in some cases, a positive answer to Question 3.17 can be given by making use of known results. For example, if R is either of bounded index of nilpotence or is a PI-ring, then it is known that the strongly π-regular property of R and of $\mathbb{M}_n(R)$ are principally known to be equivalent. In particular, for PI-rings we have the following extension:

Corollary 3.18. *For any PI-ring R with $2 \in \mathrm{Nil}(R)$ and any $n \in \mathbb{N}$, the following conditions are equivalent:*

(i) *R is unipotently weakly clean.*

(ii) *R is π-regular.*

(iii) *R is strongly π-regular.*

(iv) $\mathbb{M}_n(R)$ *is unipotently weakly clean.*

(v) $\mathbb{M}_n(R)$ *is* π*-regular.*

(vi) $\mathbb{M}_n(R)$ *is strongly* π*-regular.*

Proof. We already know that points (i)–(iii) are equivalent, and since matrices over a PI-ring are again a PI-ring, points (iv)–(vi) are also equivalent via the same reason. Moreover, implication (vi)⇒(iii) is valid for every ring, while implication (iii)⇒(vi) is well-known. ■

We close this section with one more property of unipotently weakly clean rings. It is principally known that the center of every π-regular ring is (strongly) π-regular (see [12, Theorem 1]). On the other hand, it is well known that the center of an exchange ring, as well as the center of a clean ring, need not be exchange. Hence, the center of a weakly clean ring is not necessarily clean, too. But now we however have the following affirmative result:

Proposition 3.19. *The center of a unipotently weakly clean ring is again unipotently weakly clean.*

Proof. Letting R be a unipotently weakly clean ring, it follows from [7, Proposition 3.19] and Theorem 2.6 that $\mathrm{Z}(R)$ is weakly nil clean. Since $2 \in \mathrm{Nil}(R)$ and $2 \in \mathrm{Z}(R)$, we deduce that $2 \in \mathrm{Nil}(\mathrm{Z}(R))$. Thus Theorem 2.6 applies once again to infer that $\mathrm{Z}(R)$ is unipotently weakly clean, as formulated. ■

Remark 3.20. A parallel verification can also be made directly with Definition 2.3 at hand.

4 Uniqueness Conditions

In [14] was introduced a class of clean rings with the property that the clean decomposition of every element is *unique* in the sense that every element can be written as the sum of an idempotent and a unit in a unique way. These rings are called *uniquely clean*. By virtue of [3] this is equivalent to the existence of a unique idempotent in that record.

On the other vein, in [8] were analogously defined *uniquely nil clean* rings.

Thus, one might ask what happens if one adds an additional uniqueness condition to our initial definition of unipotently weakly clean rings. Certainly, the uniqueness condition might be required either on the idempotent or on the nilpotent, so that we could get two different classes of rings. However, the next two propositions show that in both cases we get nothing new.

Proposition 4.1. *For a ring R, the following are equivalent:*

(i) *For every* $a \in R$ *there exist a unique idempotent* $e \in R$ *and a nilpotent* $q \in R$ *such that* $a - e - 1 - q \in (1 - e)Ra$.

(ii) *R is abelian strongly* π*-regular with* $2 \in \mathrm{J}(R)$.

Proof. Firstly, we prove the implication (i)⇒(ii). Applying Proposition 3.9, we only need to show that all idempotents in R are central. In fact, for every $e \in \mathrm{Id}(R)$ and every $x \in R$, we can write two unipotently weakly clean decompositions, namely

$$-e-(1-e)-1-(-2)=0,$$

observing that $1-e \in \mathrm{Id}(R)$,
and

$$-e-(1-e+ex(1-e))-1-(-2-ex(1-e))=0,$$

seeing that $1-e+ex(1-e) \in \mathrm{Id}(R)$ and that $ex(1-e) \in \mathrm{Nil}(R)$, whence $-2-ex(1-e) \in \mathrm{Nil}(R)$ as well, because in view of Theorem 2.6 we have that $2 \in \mathrm{Nil}(R)$.

Now, the uniqueness of the idempotent will force $ex(1-e) = 0$. Similarly, we see that $(1-e)xe = 0$, and thus $ex = exe = xe$ ensures that e is central, as desired. Furthermore, since 2 is a central nilpotent by Theorem 2.6, it must be that $2 \in \mathrm{J}(R)$, and thereby we are finished.

To prove the converse implication (ii)⇒(i), assume that R is strongly π-regular with central idempotents, and let

$$a = e+1+q+(1-e)ba \tag{3}$$

be a unipotently weakly clean decomposition of an arbitrary element $a \in R$ with $e \in \mathrm{Id}(R)$, $q \in \mathrm{Nil}(R)$ and $b \in R$. Since R is by assumption abelian strongly π-regular, we can find $h \in \mathrm{Id}(R)$ such that $a(1-h) \in \mathrm{U}(R(1-h))$ and $ah \in \mathrm{Nil}(Rh)$. We shall now prove that $e = h$, which will lead to the uniqueness of e, as required.

To that purpose, we first of all multiply the above equation (3) by $(1-h)e$ to get that

$$a(1-h)e = (1-h)e(2+q) \in \mathrm{Nil}(R)$$

because we know that for strongly π-regular rings R the ideal $\mathrm{J}(R)$ has to be nil and thus $2 \in \mathrm{Nil}(R)$. That is why $(a(1-h))^n e = 0$ for some $n \in \mathbb{N}$. Since $a(1-h)$ is invertible in $R(1-h)$, this leads to $(1-h)e = 0$, that is, $e = he$.

On the other side, multiplying (3) by $h(1-e)$, we obtain

$$ah(1-e) = (1+q)h(1-e) + bah(1-e)$$

which is tantamount to

$$(1+q)^{-1}(1-b)ah(1-e) = h(1-e).$$

Therefore, $ah(1-e)$ is left invertible in $Rh(1-e)$, where $h(1-e) \in \mathrm{Id}(R)$, whereas $ah(1-e) \in \mathrm{Nil}(R)$ because $ah \in \mathrm{Nil}(R)$. This immediately yields that $h(1-e) = 0$, i.e., $h = he$. Comparing these two equalities, we conclude that $e = h$, as pursued. ∎

Recall that a ring R is *strongly regular* if, for every $a \in R$, there exists $r \in R$ such that $a = a^2r$. It is well known that a ring is strongly regular if, and only if, it is abelian regular [9, Theorem 3.5] or, equivalently, it is reduced regular. It is also pretty easy to prove that a ring is strongly regular exactly when it is strongly π-regular and has no nonzero nilpotents. Note that rings with no nonzero nilpotent elements (called reduced rings) are themselves rings with central idempotents (called abelian rings).

Proposition 4.2. *For a ring R, the following are equivalent:*

(i) *For every $a \in R$ there exist an idempotent $e \in R$ and a unique nilpotent $q \in R$ such that $a - e - 1 - q \in (1 - e)Ra$.*

(ii) *R is strongly regular of characteristic* 2.

Proof. To establish the forward direction, we first observe analogously as in the proof of Proposition 4.1 that R must be abelian, and thus it has to be strongly π-regular by Proposition 3.9. Now, taking any $q \in \mathrm{Nil}(R)$, we have two unipotently weakly clean decompositions as follows $1 - 0 - 1 - q = 1 \cdot (-q) \cdot 1$ and $1 - 0 - 1 - 0 = 1 \cdot (-0) \cdot 1$. We further deduce that $q = 0$ and hence $\mathrm{Nil}(R) = \{0\}$. Thus R is reduced of characteristic 2 since $2 \in \mathrm{Nil}(R)$ in virtue of Theorem 2.6. Hence R is a reduced strongly π-regular ring and so it is strongly regular, as claimed.

The converse direction is obvious by employing once again Proposition 3.9, because every strongly regular ring is reduced strongly π-regular. ∎

Remark 4.3. It will be interesting to explore certain uniqueness conditions stated on weakly clean rings from Definition 2.1.

5 Open Problems

In closing, we state a few still unsettled questions.

Problem 5.1. Does it follows that weakly nil clean (in particular, π-regular) rings are clean if, and only if, they are strongly π-regular?

As a starting point of view, it could be considered the endomorphism ring $End(V)$, where V is an infinite dimensional vector space.

In the sense of the usual left-right symmetric question, we can ask the following:

Problem 5.2. Does it follow that

(i) a ring R is weakly nil clean $\iff$ $\forall a \in R, \exists f \in \mathrm{Id}(aR), \exists t \in \mathrm{Nil}(R)$: $1 - f = (1 - a)(1 + t)(1 - f)$;

(ii) a ring R is unipotently weakly clean $\iff$ $\forall a \in R, \exists f \in \mathrm{Id}(aR)$, $\exists t \in \mathrm{Nil}(R)$: $1 - f = (1 - a)(-1 - t)(1 - f)$?

Problem 5.3. Characterize weakly nil clean rings and unipotently weakly clean rings both having the "strong property" in the sense that the existing idempotent and nilpotent commute.

In addition, is it true that a weakly nil clean ring having the "strong property" is (strongly) π-regular? Same for a unipotently weakly clean ring.

If yes, this will enlarge the fact from [8] that strongly nil clean rings are strongly π-regular.

We also pose the following two related problems.

Problem 5.4. Describe those rings R in which for every $a \in R$ there are an idempotent e and an involution v such that $a - e - v \in (1-e)Ra$. We call them invo weakly clean.

Problem 5.5. Characterize those elements a of a ring R for which the matrix $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ is

(i) strongly clean;
(ii) nil clean;
(iii) unipotently clean (i.e., the sum of a unipotent and an idempotent)
in $\mathbb{M}_2(R)$?

In that aspect it follows easily that $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ is strongly nil clean in $\mathbb{M}_2(R)$ if and only if a is strongly nil clean in R utilizing the fact that an element b is strongly nil clean precisely when $b^2 - b$ is a nilpotent. Likewise, in [15] was established a criterion when $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ is clean for any $a \in R$, thus defining the class of weakly clean rings mentioned above (see also [16]). Generally, the wanted element-wise characterization will help us to introduce some new ring classes of certain interest and importance.

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