Fixed point sets of equivariant fiber-preserving maps

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Abstract

Given a selfmap $f : X \to X$ on a compact connected polyhedron X, H. Schirmer gave necessary and sufficient conditions for a nonempty closed subset A to be the fixed point set of a map in the homotopy class of f. R. Brown and C. Soderlund extended Schirmer's result to the category of fiber bundles and fiber-preserving maps. The objective of this paper is to prove an equivariant analogue of Brown-Soderlund theorem result in the category of G-spaces and G-maps where G is a finite group.

1 Introduction and statement of results

A well-known and important question in classical topology is the *fixed point property*. Recall that a topological space X is said to have the fixed point property if every (continuous) map $f : X \to X$ must have a fixed point $x_0 \in X$ such that $f(x_0) = x_0$. A related question is the so-called *complete invariance property for deformation* (CIPD). We say that X has the CIPD if for any nonempty closed subset $A \subset X$, there exists a selfmap $f : X \to X$ homotopic to the identity 1_X such that $A = \text{Fix}(f) = \{x \in X \mid f(x) = x\}$. In [9], H. Schirmer generalized the concept of CIPD and gave necessary and sufficient conditions for a nonempty closed subset A to be the fixed point set of a map g in the homotopy class of a given selfmap f. That is, given a map $f : X \to X$, Schirmer determined when a closed nonempty subset A can be realized as $A = \text{Fix}(g)$ for some g homotopic to f. Upon relaxing the conditions given by Schirmer, C. Soderlund together with R. Brown [3] generalized Schirmer's result to fiber-preserving maps of fiber bundles.

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Suppose that X is a compact connected polyhedron without local cutpoints and A is a closed subset imbedded inside a subpolyhedron K that can be *by-passed in* X, that is, every path C in X with $C(0), C(1) \in X - K$, is homotopic to a path C' in $X - K$ relative to the endpoints. H. Schirmer [9] introduced the following two conditions to realize A as the fixed point set of a selfmap in the homotopy class $[f]$.

(C1) if there exists a homotopy $H_A : A \times [0,1] \to X$ from $f|_A$ to the inclusion $i : A \hookrightarrow X$;

(C2) if for every essential fixed point class $\mathbb{F}$ of f, there exists a path $\alpha : [0,1] \to X$ with $\alpha(0) \in \mathbb{F}, \alpha(1) \in A$ and $\{\alpha(t)\} \sim \{f \circ \alpha(t)\} * \{H_A(\alpha(1), t)\}$ relative to the endpoints.

Soderlund [10, Theorem 3.5] showed, by relaxing the assumption on A given by Schirmer, the following result.

Theorem 1.1. Let X be a compact, connected polyhedron with no local cut points and A be a closed locally contractible subspace of X such that $X - A$ is not a 2-manifold and A can be by-passed in X. Then $A = \mathrm{Fix}(g)$ for some $g \sim f$ if and only if (C1) and (C2) are satisfied.

Subsequently, R. Brown and C. Soderlund [3] introduced analogous conditions in the fiber-preserving setting. Let $\mathfrak{F} = (E, p, B; Y)$ be a (locally trivial) fiber bundle and $f : E \to E$ a fiber preserving map.

($C1_{\mathfrak{F}}$) if there exists a fiber preserving homotopy $H_A : A \times [0,1] \to E$ from $f|_A$ to the inclusion $i : A \hookrightarrow E$;

($C2_{\mathfrak{F}}$) if for every essential fixed point class $\mathbb{F}$ of f, there exists a path $\alpha : [0,1] \to E$ with $\alpha(0) \in \mathbb{F}, \alpha(1) \in A$ and $\{\alpha(t)\} \sim \{f \circ \alpha(t)\} * \{H_A(\alpha(1), t)\}$ relative to the endpoints.

Following the terminology of [3], we call (X, A) a *suitable pair* if X is a finite polyhedron with no local cut points and A is a closed locally contractible subspace of X such that $X - A$ is not a 2-manifold and A can be by-passed in X.

In [3], it was shown that conditions ($C1_{\mathfrak{F}}$) and ($C2_{\mathfrak{F}}$) are also sufficient. The following is their main result.

Theorem 1.2. Let $\mathfrak{F} = (E, p, B; Y)$ be a fiber bundle where E, B and Y are connected finite polyhedra, $f : E \to E$ a fiber preserving map and A a closed locally contractible sub-bundle of E such that each component $p(A)_j$ of $p(A)$ is contractible and $(B, p(A)), (Y, Y_j)$ for all sub-bundle fibers Y_j of A, are suitable pairs. Suppose ($C1_{\mathfrak{F}}$) and ($C2_{\mathfrak{F}}$) are satisfied and A intersects every essential fixed point class of $f_{b_j} : p^{-1}(b_j) \to p^{-1}(b_j)$ for at least one b_j in each component $p(A)_j$. If Z is a closed bundle subset of A that intersects every component of A, then there exists a map $g : E \to E$ that is fiber preserving and fiberwise homotopic to f ($g \sim_{\mathfrak{F}} f$) such that $\mathrm{Fix}(g) = Z$.

In particular, when $Z = A$, this theorem shows that $(C1_{\mathfrak{F}})$ and $(C2_{\mathfrak{F}})$ are necessary and sufficient for $A = \text{Fix}(g)$ for some $g \sim_{\mathfrak{F}} f$.

Many applications involve symmetries in the presence of a group action. As a result, equivariant topology has been proven to be useful in the study of nonlinear problems. In the equivariant setting, we are concerned with a group G acting on a space X together with a G-map $f : X \to X$ which respects the group action, that is, for all $\alpha \in G$, $f(\alpha x) = \alpha f(x)$ for all $x \in X$. In this case, the fixed point set $\text{Fix}(f)$ is *a priori* a G-invariant subset of X.

In [9], Schirmer observed that for a given selfmap $f : \mathbb{S}^n \to \mathbb{S}^n$ of an n-sphere, $n \geq 2$, *any* closed nonempty proper subset A of $\mathbb{S}^n$ can be realized as the fixed point set of a map $g \in [f]$ with $\text{Fix}(g) = A$. However, such phenomenon does not hold if we impose a group action as we show in the following example, which gives the underlying motivation for this paper.

Example 1.3. Let $G = \mathbb{Z}_2$, $X = \mathbb{S}^2$ and the action is given by $\xi(x, y, z) \mapsto (-x, -y, z)$. If $A = \{(x, y, 0) \in \mathbb{S}^2\}$ then A is $\mathbb{Z}_2$-invariant, but there is no $\mathbb{Z}_2$-map $h : \mathbb{S}^2 \to \mathbb{S}^2$ that is $\mathbb{Z}_2$-homotopic to the identity map $Id : \mathbb{S}^2 \to \mathbb{S}^2$ such that $\text{Fix}(h) = A$.

In fact, suppose there is a $\mathbb{Z}_2$-homotopy H from Id to $h : \mathbb{S}^2 \to \mathbb{S}^2$ such that $\text{Fix}(h) = A$. Then, h preserves $X^G = \{N, S\}$, where $N = (0, 0, 1)$ and $S = (0, 0, -1)$. Hence, $h(N) = S$ and $h(S) = N$ and the path $p : I \to X^G$ defined by $p(t) = H^G(N, t)$ is such that $p(0) = Id(N) = N$ and $p(1) = h(N) = S$. But, this is impossible.

In this situation, the location of A in X is more important than its topology, because if we replace A by $A' = \{(x, 0, z) \in \mathbb{S}^2\}$ then:

$$H(t, \cos\theta \sin\psi, \sin\theta \sin\psi, \cos\psi) = \\ (\cos(\theta + t\epsilon \sin\theta) \sin\psi, \sin(\theta + t\epsilon \sin\theta) \sin\psi, \cos\psi),$$

is a $\mathbb{Z}_2$-homotopy (with polar coordinates) between the identity and the $\mathbb{Z}_2$-map h such that $\text{Fix}(h) = A'$.

Example 1.4. It is easy to see, by modifying the last example, that the equivariant analogue of Schirmer's result does not hold in general. Let $G = \mathbb{Z}_2$, $X = \mathbb{S}^2 \times \mathbb{S}^2$ and the action is given by $\xi((x, y, z), (x', y', z')) \mapsto ((-x, -y, z), (1, 0, 0))$. The set $A = \{((x, y, 0), (1, 0, 0)) \in X\}$ is $\mathbb{Z}_2$-invariant and $X^G = \{N, S\} \times \{(1, 0, 0)\}$ consists of two points. The same argument as in Example 1.3 shows that A cannot be the fixed point set of any map $\mathbb{Z}_2$-homotopic to the identity map while (X, A) satisfies the conditions of Schirmer's result for A can be by-passed since A has codimension 3 in X.

The main objective of this paper is to give an equivariant analogue of Schirmer's result and of Brown-Soderlund's result. This paper is organized as follows. In the first section, we briefly recall the non-equivariant results of [9] and [3] and review some basic background on G-maps and G-spaces where G denotes a compact Lie group. Then we review the necessary equivariant Nielsen fixed point theory from [13]. In section 2, we prove our first main result, an equivariant analogue of [9]:

Theorem 1.5. Let G be a compact Lie group, X be a compact and smooth G-manifold and A be a nonempty, closed, locally contractible G-subset of X such that for each finite WK we assume that $\dim(X^K) \geq 3$, $\dim(X^K) - \dim(X^K - X_K) \geq 2$ and A^K is by-passed in X^K, for all $(K) \in \mathrm{Iso}(X)$. Suppose that the following conditions holds for a G-map $f : X \to X$:

(C_G1) there exists a G-homotopy $H_A : A \times I \to X$ from $f|_A$ to the inclusion $i : A \hookrightarrow X$;

(C_G2) for each finite WK, for every WK-essential fixed point class F of $f^K : X^K \to X^K$ there exists a path $\alpha : I \to X^K$ with $\alpha(0) \in F$, $\alpha(1) \in A^K$, and $\{\alpha(t)\} \sim \{f^K \circ \alpha(t)\} * \{H_A^K(\alpha(1), t)\}$.

Then for every closed G-subset Φ of A that has nonempty intersection with every component of A there exists a G-map $h : X \to X$, G-homotopic to f with $\mathrm{Fix}(h) = \Phi$.

In the last section, we apply Theorem 1.5 to prove an equivariant analogue of [3] when G is finite:

Theorem 1.6. Let G be a finite group, $\mathfrak{F} = (X, p, B, Y)$ be a G-fiber bundle where X, B and Y are compact and smooth G-manifolds, $\dim(B^K) \geq 3$, $\dim(B^K) - \dim(B^K - B_K) \geq 2$, for all $(K) \in \mathrm{Iso}(B)$, $\dim(Y^K) \geq 3$, $\dim(Y^K) - \dim(Y^K - Y_K) \geq 2$, for all $(K) \in \mathrm{Iso}(Y)$.

Let A be a nonempty, closed, locally contractible G-subset of X such that (X, A) is G-fiber bundle pair with respect to the fiber bundle $\mathfrak{F}$, $p(A)$ be a closed G-subset of B such that each component $p(A)_j$ of $p(A)$ is equivariantly contractible and $p^K(A^K)$ is by-passed in B^K, for all $(K) \in \mathrm{Iso}(B)$. Let Y_j be a sub-bundle fiber of A such that Y_j is a closed and locally contractible G-subset of Y and Y_j^K is by-passed in Y^K, for all $(K) \in \mathrm{Iso}(Y)$, and $f : X \to X$ be a G-fiber-preserving map such that A^K intersects every essential WK-fixed point class of $f_{b_j}^K : WK(p^K)^{-1}(\{b_j\}) \to WK(p^K)^{-1}(\{b_j\})$ for at least one b_j in each component $p^K(A^K)_j$, for all $(K) \in \mathrm{Iso}(X)$. Suppose that the following conditions hold for f and A:

$(C_G1)_{\mathfrak{F}}$ there exists a G-fiberwise-homotopy $H_A : A \times I \to X$ from $f|_A$ to the inclusion $i : A \hookrightarrow X$;

$(C_G2)_{\mathfrak{F}}$ for every WK-essential fixed point class F of $f^K : X^K \to X^K$ there exists a path $\alpha : I \to X^K$ with $\alpha(0) \in F$, $\alpha(1) \in A^K$, and $\{\alpha(t)\} \sim \{f^K \circ \alpha(t)\} * \{H_A^K(\alpha(1), t)\}$.

Then for every nonempty closed G-bundle subset Φ of A that intersects every component of A there exists a G-fiber-preserving map h, G-fiberwise homotopic to f with $\mathrm{Fix}(h) = \Phi$.

In order to establish the notations, let G be a topological group and X be a (left) G-space. Given a subgroup K of G we denote by NK the normalizer of K in G,

$WK = \frac{NK}{K}$ is the Weyl group of K in G. The orbit type of K is the conjugacy class of K in G denoted by (K). If (K_1) is subconjugate to (K_2), we write $(K_1) \leq (K_2)$.

If $x \in X$, then $G_x = \{g \in G;\ gx = x\}$ denotes the isotropy subgroup of $x \in X$, and (G_x) is called an isotropy type of X. We denote by $\mathrm{Iso}(X)$ the set of isotropy types of X. Moreover, $X^K = \{x \in X; K \leq G_x\}$, $X^{(K)} = \{x \in X; (K) \leq (G_x)\}$, $X_K = \{x \in X; G_x = K\}$ and $X_{(K)} = \{x \in X; G_x \subset (K)\}$.

If $\mathrm{Iso}(X)$ is finite (in particular when G is finite), we can choose an admissible ordering on $\mathrm{Iso}(X)$ such that $(K_i) \leq (K_j)$ implies $i \leq j$. Then we have a filtration of G-subspaces $X_1 \subset \cdots \subset X_n = X$ where $X_i = \{x \in X;\ (G_x) = (H_j)$ for some $j \leq i\}$

If $f : X \to X$ is a G-map, then $f^K = f|_{X^K} : X^K \to X^K$ is a WK-map. Let $\mathcal{F} = \{(K) \in \mathrm{Iso}(X) \mid |WK| < \infty\}$ and $(K) \in \mathcal{F}$. If $x, y \in \mathrm{Fix}(f^K)$ then $x \sim_K y$ if either $y = \alpha x$ for some $\alpha \in WK$ or $\exists\ \sigma : [0,1] \to X^K$ such that $\sigma \sim f^K \circ \sigma$ relative to endpoints. Then $\sim_K$ is an equivalence relation on $\mathrm{Fix}(f^K)$ and the equivalence classes are called the *$WK-$fixed point classes* of f^K. Evidently, a WK-fixed point class $\mathcal{W}$ is a disjoint union of a finite number of ordinary fixed point classes $W_1, ..., W_r$ of f^K and thus the fixed point index $ind(\mathcal{W})$ is defined as $ind(\mathcal{W}) = \sum_i ind(W_i)$. A WK-fpc (fixed point class) $\mathcal{W}$ is *essential* if $ind(\mathcal{W}) \neq 0$. For further information on equivariant Nielsen fixed point theory, see [13]. Throughout, by a smooth G-manifold X, we assume that the fixed point set X^H is a smooth connected submanifold for each isotropy subgroup $H \leq G$.

2 Proof of Theorem 1.5 - An equivariant analogue of a result of Soderlund-Schirmer

If X is a smooth G-manifold and A is a closed smooth G-submanifold of X, G being a finite group, then there exists a smooth equivariant triangulation $f_1 : (K, K_0) \to (X, A)$ as proved in [8]. If B is another closed smooth G-submanifold of X then there is a smooth equivariant triangulation $f_2 : (L, L_0) \to (X, B)$ and G-subdivisions K' of K and L' of L such that $f_1'^{-1} \circ f_2' : |L'| \to |K'|$ is a simplicial G-homeomorphism, where f_1' and f_2' are smooth G-triangulations (see [8]).

By Corollary 3.3.5 of [11] and G being finite, we can find unique G-subcomplexes L_0' of L and K_0' of K such that L_0' is a refinement of L_0 and K_0' is a refinement of K_0. Then, $f_1'^{-1} \circ f_2'(L_0') = K_1$ is a G-subcomplex of K' and a G-triangulation of B. In fact, $f_1'(K_1) = f_1' \circ f_1'^{-1} \circ f_2'(L_0') = f_2'(L_0') = B$. Hence, by induction if $\{A_i\}_{i=1}^n$ is a finite collection of closed smooth G-submanifolds of X then there exists a smooth equivariant triangulation $f : K \to X$ and a finite collection of G-subcomplexes $\{L_i\}_{i=1}^n$ of K such that L_i is a G-triangulation of A_i, for $i = 1, \ldots, n$.

To realize A as the fixed point set of some $h : X \to X$, it is necessary to remove every fixed point $x \in X$ of $f : X \to X$ outside of A. Hence, we need to extend the notion of *neighborhood by-passed* for a closed subset A as in [10, Definition 2.1] in order to handle these undesired fixed points.. Thus, a G-invariant subset A is said to be *G-neighborhood by-passed* if there exists an invariant open subset $U \subset X$ such that $A \subset U$ and U can be by-passed in X.

We observe that if (X, A) is a G-ENR pair then A is an invariant neighborhood retract in X and if $\{Y_i\}_{i=1}^n$ is a finite collection of closed smooth G-submanifolds such that $\dim(Y_i) + 1 < \dim(X)$, then $A \cup Y$ ($Y = \bigcup_{i=1}^n Y_i$) remains a by-passed G-subset of X provided A is by-passed in X. Furthermore, a close inspection of the proof of Theorem 2.2 of [10] indicates that the same argument works for the same result in the equivariant setting. That is, if A is a by-passed locally contractible G-subset of X then A is G-neighborhood by-passed, for X a compact smooth G-manifold with $\dim(X) \geq 3$. To see that, we note that if K is the G-triangulation of X then there is a by-passed neighborhood (may not be equivariant) U of A in $|K|$. We obtain the open G-subpolyhedron:

$$St(A, K) = \bigcup_{\substack{\overline{|t|} \cap A \neq \emptyset \\ t \in K}} |t|,$$

such that $\overline{St(A, K)}$ is a subset of U by taking a G-refinement K' of K if necessary, where t is a simplex of K. Therefore, if $p : I \to X$ is a path with endpoints in $U - \overline{St(A, X)}$ and outside $\overline{St(A, X')}$ then using Corollary 3.3.11 of [11] we deform p out of $\overline{St(A, X')}$.

Thus, if $\{Y_i\}_{i=1}^n$ is a finite collection of closed smooth G-submanifolds such that $\dim(Y_i) + 1 < \dim(X)$ (thus each Y_i has codimension at least 2 in X so that Y_i can be by-passed in X), then $A \cup Y$ ($Y = \bigcup_{i=1}^n Y_i$) remains a by-passed G-subset of X using a finite collection of G-subcomplex $\{L_i\}_{i=1}^n$ of K such that L_i is a G-triangulation of Y_i, for $i = 1, \ldots, n$.

The next lemma shows how the fixed points outside A may be removed (see also [7]).

Lemma 2.1. Let $\{Y_i\}_{i=1}^n$ be a finite collection of closed G-submanifolds of the G-manifold X such that $\dim(Y_i) + 1 < \dim(X)$ and the action of G outside $Y = \bigcup_{i=1}^n Y_i$ is free, where G is a finite group. Let $f : X \to X$ be a G-selfmap, A be a non-empty closed locally contractible and by-passed G-subset of X such that $A \subset \mathrm{Fix}(f)$, there are no fixed points of f in $Y - A$, and f has a finite number of fixed points in $X - (A \cup Y)$. Let x_0 and x_1 be two fixed points of f that are G-Nielsen equivalent from different orbits such that $x_0 \in X - (A \cup Y)$ and $x_1 \in X - (A \cup Y)$ or $x_1 \in \partial(A)$, where $\partial(A)$ is the boundary of A in X and $q : I \to X$ a path with end points $q(0) = x_0$ and $q(1) = x_1$ such that $f \circ q$ is homotopic to q relative to the endpoints.

Then, f is G-homotopic, relative to $(A \cup Y)$, to a G-selfmap $h : X \to X$ such that $\mathrm{Fix}(h) = \mathrm{Fix}(f) - G\{x_0\}$.

Proof of Lemma 2.1: Since A is locally contractible and can be by-passed in X, the discussion above shows that A is G-neighborhood by-passed in X. Furthermore, $A \cup Y$ can be by-passed in X. Thus, the path q is homotopic, relative to endpoints, to a path $q'(t)$ such that for $0 \leq t < 1$, $q'(t) \in X - (A \cup Y)$ with $q'(0) = x_0, q'(1) = x_1$. Since G acts freely on $X - Y$ and hence on $X - (A \cup Y)$, taking the G-translates of q' yields $|G|$ paths from the orbit $G\{x_0\}$ to the orbit $G\{x_1\}$. Note that the segments $G\{q'([0, 1))\}$ are disjoint while $\{G\{q'(1)\}\}$ consists of $[G : G_{x_1}]$ distinct endpoints. Here, the isotropy subgroup G_{x_1} at x_1 is trivial if $x_1 \in X - (A \cup Y)$. Now we coalesce these two fixed orbits in the same fashion as in [14, Lemma 3.1]. (For slightly more general spaces in which normal arcs are used, see [7, Theorem 2].) ∎

We will prove Theorem 2.2 before Theorem 1.5 and for the same reason we prove Theorem 2.2 by first establishing Lemma 2.3 and Lemma 2.5.

Theorem 2.2. Let G be a compact Lie group, X be a compact smooth G-manifold and A be a nonempty, closed, locally contractible G-subset of X such that for each finite WK we assume that $\dim(X^K) \geq 3$, $\dim(X^K) - \dim(X^K - X_K) \geq 2$ and A^K is by-passed in X^K, for all $(K) \in \mathrm{Iso}(X)$. Then, given a G-map $f : X \to X$ there exists a G-map $h : X \to X$ G-homotopic to f with $\mathrm{Fix}(h) = A$ if, and only if, the conditions (C_G1) and (C_G2), given in Theorem 1.5, hold for f relative to A.

Lemma 2.3. Let G be a compact Lie group, X be a G-space G-ANR and A be a nonempty closed G-subset of X. If $f : X \to X$ is a G-map G-homotopic to $h : X \to X$ such that $\mathrm{Fix}(h) = A$ then the conditions (C_G1) and (C_G2) given by Theorem 1.5 hold for f relative to A.

Proof of Lemma 2.3: Let $H : X \times I \to X$ be a G-homotopy which starts at f and ends at h. Then $\overline{H} = H|_{(X\times\{0\})\cup(A\times I)} : (X \times \{0\}) \cup (A \times I) \to X$ satisfies (C_G1). If F is a WK-essential fixed point class of f^K, then, there exists a path $p : I \to X^K$ such that $p(0) \in F$ and $p(1) \in J$, where $J \subset A^K$ is a WK-essential fixed point class of h^K, H^K-related to F and $\{p(t)\} \sim \{\overline{H}^K(p(t), t)\}$. In fact,

$$\{\overline{H}^K(p(t), t)\} \sim \underbrace{\{\overline{H}^K(p(t), 0)\}}_{=\{f\circ p(t)\}} * \{\overline{H}^K(p(1), t)\}.$$

So, (C_G2) is satisfied. ∎

Lemma 2.3 shows that the conditions (C_G1) and (C_G2) are necessary for $A = \mathrm{Fix}(h)$. The example below shows that these two conditions are independent of each other.

Example 2.4. Let $G = \mathbb{Z}_2$, $X = \mathbb{S}^2$ and the action given by $\xi(x, y, z) \mapsto (-x, -y, z)$. Then, there is no $\mathbb{Z}_2$-homotopy H from the identity Id to h such that $\mathrm{Fix}(h) = \{(x, y, 0) \in \mathbb{S}^2\}$. Note that (C_G1) occurs, because the map is the identity, but (C_G2) does not. On the other hand, let $G = \mathbb{Z}_2$, $X = \mathbb{S}^3$ and the action given by $\xi(x, y, z, w) \mapsto (x, y, z, -w)$. Then, there is no $\mathbb{Z}_2$-homotopy H from the antipodal map $-Id$ to h such that $\mathrm{Fix}(h) = \{(x, y, z, 0) \in \mathbb{S}^3\}$. This time (C_G2) holds because the map is fixed point free but (C_G1) does not hold.

Lemma 2.5. Let G be a compact Lie group, X be a compact smooth G-manifold and A be a nonempty, closed, locally contractible G-subset of X such that for each finite WK we assume that $\dim(X^K) \geq 3$, $\dim(X^K) - \dim(X^K - X_K) \geq 2$ and A^K is by-passed in X^K, for all $(K) \in \mathrm{Iso}(X)$. If the conditions (C_G1) and (C_G2), given in Theorem 1.5, hold for a G-map $f : X \to X$ relative to A, then there exists a G-map $h : X \to X$, G-homotopic to f with $\mathrm{Fix}(h) = A$.

Proof of Lemma 2.5: This proof follows the steps of the proof of Theorem 3.2 of [9]. Consider a G-map $\overline{H} : (X \times \{0\}) \cup (A \times I) \to X$ given by $(C_G 1)$. It is possible to extend $\overline{H}$ to a G-homotopy ${}_1\overline{H}_1 : \Big(X \times \{0\}\Big) \cup \Big((A \cup X_1) \times I\Big) \to X$. As commented above, there is a closed G-invariant neighborhood V of A_1 inside X_1 and V retracts onto A_1 equivariantly. Note that WK_1 acts freely on $X_1^{K_1} = X_{K_1}$ and ${}_1h_1^{K_1}$ is a WK_1-map. Hence, if WK_1 has positive dimension we apply Lemma 3.3 of [12] and Lemma 2.1 of [6] to extend ${}_1\overline{H}_1$ to a G-homotopy $\overline{H}_1 : (X \times \{0\}) \cup ((A \cup X_1) \times I) \to X$, relative to V. Moreover, h_1 has no fixed points in $X_1 - A_1$ and $\mathrm{Fix}(h_1) = A$, where $h_1 = \overline{H}_1(\bullet, 1) : A \cup X_1 \to X$.

On the other hand, if WK_1 is a finite group then X^{K_1} is a WK_1-polyhedron such that $A_1^{K_1}$ is a WK_1-subpolyhedron and $St(A_1, X^{K_1})$ is neighborhood by-passed in X^{K_1}. We apply Lemma 3.1 of [12] and Lemma 2.1 to obtain a WK_1-homotopy $H : (A_1 \cup X_1)^{K_1} \times I \to X^{K_1}$ which can be extended by Lemma 2.1 of [6] to a G-homotopy $\overline{H}_1 : (X \times \{0\}) \cup ((A \cup X_1) \times I) \to X$, relative to V, such that h_1 has no fixed points in $X_1 - A_1$ and $\mathrm{Fix}(h_1) = A$, where $h_1 = \overline{H}_1(\bullet, 1) : A \cup X_1 \to X$.

By induction, we may assume that we have a G-map $\overline{H}_{i-1} : (X \times \{0\}) \cup ((A \cup X_{i-1}) \times I) \to X$ such that $\mathrm{Fix}(h_{i-1}) = A$, where $h_{i-1} = \overline{H}_{i-1}(\bullet, 1) : A \cup X_{i-1} \to X$ and the proof follows the steps we did for WK_1. ■

Now Theorem 2.2 follows easily from Lemma 2.3 and Lemma 2.5.

Proof of Theorem 1.5: First of all, by Theorem 2.2, there is a G-map $h_1 : X \to X$ G- homotopic to f such that $\mathrm{Fix}(h_1) = A$. We may apply Proposition 2.5 of [12] and Theorem 4.3 of [13] to conclude that h_1 is G-homotopic to h_2 such that $h_2|_{X^K}$ has a finite number of fixed points, all of which inside $St(A^K)$ and lying in the interior of a maximal simplex of X^K and h_2 is a G-proximity map in $St(A)$ (for some G-triangulation of X).

Since Φ has nonempty intersection with every component of A we can pull the fixed points of h_2 to Φ. Let α be the G-map of Lemma VIII.C.1 of [2] and $\overline{d}$ the equivariant bounded distance in X then we define

$$\overline{H}_3 : (X \times \{0\}) \cup (St(A) \times I) \to X$$

given by:

$$(x,t) \mapsto \begin{cases} \alpha(x, h_2(x), 1 - (1 - \overline{d}(x, \Phi))t) & if\ (x,t) \in St(A) \times I; \\ h_2(x) & if\ t = 0. \end{cases}$$

Then, we extend $\overline{H}_3$, relative to $\partial(St(A))$, to a G-map $\overline{H}_4 : X \times I \to X$. By Lemma 3.1 of [12], we eliminate the fixed points of $\overline{H}_4(\bullet, 1)$ inside $X - St(A) \times \{1\}$. This finite set of fixed points can be removed because these fixed points lie in some non essential fixed point classes of $\overline{H}_4(\bullet, 1)$ since $h_2|_{X - Int(A)}$ is fixed point free. Thus, the resulting G-map is a G-homotopy $H : X \times I \to X$ connecting f to a G-map h such that $\mathrm{Fix}(h) = \Phi$.

3 Proof of Theorem 1.6 - An equivariant analogue of a theorem of Brown-Soderlund

Throughout this last section, G will denote a finite group. Given a G-fiber-preserving map $f : X \to X$ of the total space X of a G-fiber bundle $\mathfrak{F} = (X, p, B, Y)$, it is known that the fixed point set of f is related with the fixed point set of the induced map $\overline{f} : B \to B$. However, there are equivariant homotopies that are not fiber-preserving as in the example below:

Example 3.1. Let $G = \mathbb{Z}_2$ and $X = \mathbb{S}^2 \times \mathbb{S}^1$ and the action is given by $\xi((a,b,c), \cos x + i \sin x) \mapsto ((a,b,c), \cos x - i \sin x)$. The G-map f, defined on X by setting $f((a,b,c), \cos x + i \sin x) = ((-a,-b,-c), \cos x + i \sin x)$, is the start of the following equivariant homotopy:

$$H\Big(\big(\cos\theta\sin\psi, \sin\theta\sin\psi, \cos\psi\big), \cos x + i\sin x, t\Big) =$$
$$\Big(\big(-\cos(\theta + t|\sin x|\pi)\sin\psi, -\sin(\theta + t|\sin x|\pi)\sin\psi, -\cos\psi\big), \cos x + i\sin x\Big).$$

Then, $A = \{(a,b,0) \in \mathbb{S}^2\} \times \{-i, i\} = \mathbb{S}^1 \times \{i, -i\}$ is the fixed point set of $h \in [f]_G$ where $h = H(\bullet, 1)$. Let $p = \pi_1 : \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2$ be the projection, then $(\mathbb{S}^2 \times \mathbb{S}^1, \pi_1, \mathbb{S}^2)$ is a $\mathbb{Z}_2$-fiber bundle, f is a fiber-preserving map and the induced map $\overline{f} = a : \mathbb{S}^2 \to \mathbb{S}^2$ is the antipodal map. However, $p((x,y,z),1) = (x,y,z) = p((x,y,z),i)$ and $p \circ h((x,y,z),1) = (-x,-y,-z)$ is different from $p \circ h((x,y,z),i) = (x,y,-z)$. So, h is not a fiber preserving map and H is not a fiber-preserving homotopy. In fact, A *cannot* be realized as the fixed point set of *any* map equivariantly fiberwise homotopic to f. To see that, we note that $X^G = \mathbb{S}^2 \times \{\pm 1\} = \mathbb{S}^2_1 \sqcup \mathbb{S}^2_{-1}$, where $(w, \pm 1) \in \mathbb{S}^2_{\pm 1}$, consists of two disjoint 2-spheres $\mathbb{S}^2$. If F_t is a $\mathbb{Z}_2$ fiber-preserving homotopy such that $F_0 = f$ and $\mathrm{Fix}(F_1) = A$, then F_t^G is a homotopy on X^G. Now, $f^G = F_0^G$ maps $\mathbb{S}^2_1$ to $\mathbb{S}^2_1$ and $\mathbb{S}^2_{-1}$ to $\mathbb{S}^2_{-1}$. On the other hand, F_1 is fiber-preserving and A is the fixed point of F_1, it follows that the induced map $\overline{F_1}$ fixes the circle $\{(a,b,0) \in \mathbb{S}^2\}$ pointwise. This implies that F_1 maps the (non-fixed) point $((a,b,0),1)$ to the point $((a,b,0),-1)$ so that F_1 maps the equator of $\mathbb{S}^2_1$ to that of $\mathbb{S}^2_{-1}$, and vice versa. Thus F_1^G maps X^G to itself by interchanging the two disjoint spheres $\mathbb{S}^2_{\pm 1}$. The images of X^G under F_0^G and F_1^G contradict the continuity of F_t^G. Hence such an equivariant fiber-preserving homotopy F_t cannot exist.

The example above indicates the importance of modifying the conditions (C_G1) and (C_G2) and replacing them by $(C_G1)_{\mathfrak{F}}$ and $(C_G2)_{\mathfrak{F}}$ for the fiber-preserving map setting.

Lemma 3.2. Let $f : X \to X$ be a G-fiber preserving map in the total space of the G-fiber bundle $\mathfrak{F} = (X, p, B, Y)$, where X, B and Y are G-ANR spaces. Suppose that there is a G-fiber preserving homotopy connecting a G-fiber preserving map $h : X \to X$ to f such that $\mathrm{Fix}(h) = A$ for a nonempty and closed G-subset A of X. Then the conditions $(C_G1)_{\mathfrak{F}}$ and $(C_G2)_{\mathfrak{F}}$ given in Theorem 1.6 hold for f and A.

The proof of Lemma 3.2 follows the steps of Lemma 2.3. Since $\mathfrak{F} = (X, p, B, Y)$ is a G-fiber bundle where X, B and Y are compact smooth G-manifolds, we observe that (X, p, B) is a G-fibration and there is a G-lift map $\Lambda : \Omega_p \to E^I$ such that $\Lambda(e, \alpha)(0) = e$, $p \circ \Lambda(e, \alpha)(t) = \alpha(t)$ and $\Lambda(e, p(e))(t) = e$, for all $t \in I$, where $E^I = \{\alpha : I \to E;\ \alpha \text{ is a path}\}$ and $\Omega_p = \{(e, \alpha) \in X \times B^I;\ p(e) = \alpha(0)\}$.

Remark 3.3. We should point out that Lemma 3.2 holds for any compact Lie group G if we modify condition $(C_G2)_{\mathfrak{F}}$ by only considering those (K)'s with $|WK| < \infty$.

The next proposition is an equivariant analogue of Theorem 2.1 of [1].

Proposition 3.4. Let $\overline{H} : (X \times \{0\}) \cup (A \times I) \to E$ be a G-map in the G-fibration $\mathfrak{F} = (E, p, B)$, where E is a G-ANR, A is a closed G-subset of X, (X, A) is a G-metric pair and $p \circ \overline{H}(x, 0) = p \circ \overline{H}(x, t)$ for all $(x, t) \in A \times I$. Then $\overline{H}$ can be extended to a G-homotopy $H : X \times I \to E$ such that $p \circ H(x, 0) = p \circ H(x, t)$ for all $(x, t) \in X \times I$.

Proof of Proposition 3.4: Let $H' : X \times I \to E$ a G-extension of $\overline{H}$. Then H' is given by:

$$\begin{array}{rcccccc} H': & X & \to & E^I & & & \\ & x & \mapsto & H'(x, \bullet): & I & \to & E \\ & & & & t & \mapsto & H'(x,t). \end{array}$$

Then define $H(x, t) = \Lambda(H'(x, t), p(H'(x, \bullet))_t)(1)$, where $p(H'(x, \bullet))_t(s) = p(H'(x, (1 - s)t))$ and Λ is a G-lift map. ∎

Lemma 3.5. Let $\mathfrak{F} = (X, p, B, Y)$ be a G-fiber bundle where X, B and Y are compact and smooth G-manifolds, $\dim(B^K) \geq 3$, $\dim(B^K) - \dim(B^K - B_K) \geq 2$, for all $(K) \in \mathrm{Iso}(B)$, A be a nonempty, closed, locally contractible G-subset of X such that $p(A)$ be a closed G-subset of B and $p^K(A^K)$ is by-passed in B^K, for all $(K) \in \mathrm{Iso}(B)$, and $f : X \to X$ a G- fiber preserving map such that conditions $(C_G1)_{\mathfrak{F}}$ and $(C_G2)_{\mathfrak{F}}$ given in Theorem 1.6 hold for f and A.

Then there exists a G-fiber-preserving map h, G-fiberwise homotopic to f with $A \subset \mathrm{Fix}(h) \subset p^{-1}(p(A))$ and $\mathrm{Fix}(\overline{h}) \cap (B - p(A))$ is a finite set.

Proof of Lemma 3.5: $p(A)$ is a closed G-subset of B then the G-fiber-preserving map $H_A : A \times I \to X$ given by $(C_G1)_{\mathfrak{F}}$ induces a G-map $\overline{H}_A : p(A) \times I \to B$ such that $\overline{H}_A(\bullet, 0) = \overline{f}$ and $\overline{H}_A(\bullet, 1) = i_{p(A)} : p(A) \hookrightarrow B$ the inclusion map.

Observe that we have almost the same conditions that we had in Theorem 1.5 except for (C_G2). In this situation, suppose we have a G-map $\overline{H}_{i-1,A} : (p(A) \cup B_{i-1}) \times I \to B$. As commented in Lemma 2.5, it is possible to extend $\overline{H}_{i-1,A}$ to a G-map $\overline{H}_{i,1} : (B_i \cup p(A)) \times I \to B$ relative to $p(A) \cup B_{i-1}$.

Since WK_i is a finite group, $B_i^{K_i}$ is a WK_i- polyhedron such that $B_{i-1}^{K_i}$ is a WK_i-subpolyhedron of $B_i^{K_i}$ and $St(p(A_i^{K_i}))$ is neighborhood by-passed in $B_i^{K_i}$. Let V be a G-invariant neighborhood retract of $St(p(A_i)) \cup B_{i-1}$. It follows from Lemma 3.1 of [12] and Lemma 2.1 that there exists of a WK_i-homotopy $\overline{H}_i : B_i^{K_i} \times I \to B_i^{K_i}$ from $\overline{H}_{i,1}^{K_i}(\bullet, 1)$ to $\overline{h} = \overline{H}_i(\bullet, 1)$ such that:

1. $p(A)_i^{K_i} \subset \mathrm{Fix}(\overline{h})$;

2. $\overline{h}$ has a finite number of fixed points in $B_i^{K_i} - V^{K_i}$;

3. given a WK_i-fixed point class F of $\overline{h}$ such that $F \cap p(A)_i^{K_i} = \emptyset$ then $F = WK_i\{x\}$, where $x \in B_i^{K_i} - V^{K_i}$ and F is an essential WK_i-fixed point class of $\overline{h}$.

Then, the G-map given by:

$$\overline{h}_t(x) \quad = \quad \begin{cases} g\overline{H}_i(g^{-1}x, t), & for\ x \in X - A,\ where\ G_x = gWK_ig^{-1}; \\ \overline{h}(x), & for\ x \in V. \end{cases}$$

extends a WK_i-homotopy to a G-homotopy $\overline{H}_i : (B_i \cup p(A)) \times I \to B$ relative to V and such that

$$\mathrm{Fix}(\overline{H}_i(\bullet, 1)) = p(A) \cup \Big(\bigcup_{j \in T,\, j \leq i} (G\{b_{j,1}\} \cup \cdots \cup G\{b_{j,m_j}\}) \Big)$$

and $WK_i\{b_{i,l}\}$ is a essential WK_i-fixed point class of $\overline{H}_i^{K_i}(\bullet, 1)$, for $1 \leq l \leq m_i$.

Observe that if $p^K(F) = WK\{b_{i,l}\}$ for an essential WK-fixed point class F of f^K where $(K) \in \mathrm{Iso}(X)$, then we have a path $\overline{\alpha}$ such that:

$$\{\overline{\alpha}\} \sim \{\overline{f}^K \circ \overline{\alpha}\} * \{\overline{H}_A^K(\overline{\alpha}(1), t)\} \sim \{\overline{H}^K(\overline{\alpha}(t), t)\}.$$

Hence, $\overline{\alpha}(1) = gb_{i,l}$, for some $g \in WK$ and $\overline{\alpha}(1) \in p^K(A)$. However, this cannot occur because $b_{i,l} \notin p^K(A)$ and $p^K(A)$ is WK-invariant. By induction we extend the G-map $\overline{H}_A : p(A) \times I \to B$ to a G-homotopy $\overline{H} : B \times I \to B$ with the properties above.

Note that $H' : X \times I \to B$ defined by $H'(x, t) = \overline{H}(p(x), t)$ is such that

$$H'(x, 0) = \overline{H}(p(x), 0) = \overline{f} \circ p(x) = p \circ f(x).$$

Therefore, the lift of H' is a fiber-preserving G-homotopy $H_1 : X \times I \to X$ such that $f(x) = H_1(x, 0)$ and $h_1(x) = H_1(x, 1)$. Thus,

$$\mathrm{Fix}(h_1) \subset p^{-1}(\mathrm{Fix}(\overline{h})) = p^{-1}(p(A) \cup G\{b_1\} \cup \cdots \cup G\{b_l\}).$$

For each G-orbit $G\{b_j\}$ take the restriction h_{1,b_j} of h_1 for $Gp^{-1}(b_j) = p^{-1}(G\{b_j\})$, so $h_{1,b_j} : Gp^{-1}(b_j) \to Gp^{-1}(b_j)$ has no essential fixed point classes. In fact, suppose that h_{1,b_j}^K has an essential WK-fixed point class F. Then, given $x \in F$ we have $WK\{x\}$ lying inside an essential WK-fixed point class of h_1^K. Thus, there exists a WK-fixed point class Q of h_1^K which contains $WK\{x\}$. But, h_1^K is fiber-preserving WK-homotopic to f^K, so, there exists an essential WK-fixed point class D of f^K H_1^K-related to Q. Note that $p^K(D)$ cannot be $\overline{H}^K$-related to $WK\{b_j\}$. Consequently, h_{1,b_j} is fiber-preserving G-homotopic to $h_{2,b_j} : Gp^{-1}(b_j) \to Gp^{-1}(b_j)$ fixed point free.

Consider the G-map

$$\widetilde{H}_2 : (X \times \{0\}) \cup (p^{-1}(\mathrm{Fix}(\overline{h})) \times I) \to X$$

defined by:

$$\widetilde{H}_2(x,t) \quad = \quad \begin{cases} h_1(x) & if\ t = 0\ or\ if\ x \in p^{-1}(p(A)); \\ H_{2,b_j}(x,t) & if\ x \in p^{-1}(G\{b_j\}). \end{cases}$$

With Proposition 3.4 we extend $\widetilde{H}_2$ to a fiber-preserving G-homotopy $H_2 : X \times I \to X$ and $h_2 = H_2(\bullet,1)$ is such that $\overline{H}_2(\bullet,1) = \overline{h}$. By $(C_G1)_{\mathfrak{F}}$, $h|_2$ is fiber-preserving G-homotopic to i_A. Let $\widetilde{H}_A$ such that $h_2|_A = \widetilde{H}_A(\bullet,0)$ and $i_A = \widetilde{H}_A(\bullet,1)$. Define $\widetilde{H} : (X \times \{0\}) \cup ((A \cup p^{-1}(G\{b_1,\dots,b_r\})) \times I) \to X$ given by:

$$\widetilde{H}(x,t) \quad = \quad \begin{cases} h_2(x), & if\ t = 0; \\ h_{2,b_j}(x), & if\ x \in Gp^{-1}(b_j); \\ \widetilde{H}_A(x,t), & if\ x \in A. \end{cases}$$

Applying Proposition 3.4 again we extend $\widetilde{H}$ to a fiber-preserving G- homotopy $H : X \times I \to X$ such that $A \subset \mathrm{Fix}(h) \subset p^{-1}(p(A))$ and $\mathrm{Fix}(\overline{h}) \cap (B - p(A))$ is a finite set. ■

Lemma 3.6. Let $(\mathfrak{F},\mathfrak{F}_0) = ((X,A),p,B,(Y,Y_0))$ be a G-fiber bundle pair, where X, B and Y are compact and smooth G-manifolds, B retracts equivariantly to a point $b_0 \in B$ and $\dim(Y^K) \geq 3$ and $\dim(Y^K) - \dim(Y^K - Y_K) \geq 2$, for all $(K) \in \mathrm{Iso}(Y)$. Let Y_0 be a closed and locally contractible G-subset of Y such that Y_0^K is by-passed in Y^K, for all $(K) \in \mathrm{Iso}(Y)$, A be a nonempty, closed, locally contractible G-subset of X and $f : X \to X$ be a G-map such that $p \circ f = p$, $A \subset \mathrm{Fix}(f)$, A^K intersects every essential WK-fixed point class of $f_{b_0}^K : WK(p^K)^{-1}(\{b_0\}) \to WK(p^K)^{-1}(\{b_0\})$, for all $(K) \in \mathrm{Iso}(X)$.

Then for every closed G-invariant subset Z of A that intersects every component of A and (A,Z) is G-fiber bundle pair of $\mathfrak{F}_0$ there exists a fiber-preserving G-map h, G-fiberwise homotopic to f with $\mathrm{Fix}(h) = Z$.

Proof of Lemma 3.6: (X,p,B) is G-equivalent to a trivial G-fibration $(B \times Y,\pi,B)$, where π is a projection in B. So, there exists a G- homeomorphism $\Phi : B \times Y \to X$ such that $\Phi(B \times Y_0) = A$ and $p \circ \Phi = \pi$. Define $f^* = \Phi^{-1} \circ f \circ \Phi : B \times Y \to B \times Y$ and note that:

$$\pi \circ f^* = \underbrace{(p \circ \Phi) \circ (\Phi^{-1}}_{=p} \circ f \circ \Phi) = \underbrace{p \circ f}_{=p} \circ \Phi = p \circ \Phi = \pi.$$

Therefore, $f^*(b,y) = (b, f_b^*(y))$ and $gf^*(b,y) = (gb, f_{gb}^*(gy))$, for all $g \in G$.

B retracts equivariantly to b_0, so, there exists a G-homotopy $D : B \times I \to B$ such that for each $b \in B$ we have $D(b,0) = b$ and $D(b,1) = b_0$. Then, define $U^* : B \times Y \times I \to B \times Y$ given by:

$$U^*(b,y,t) = \begin{cases} (b, f^*_{D(b,2t)}(y)), & if\ 0 \le t \le \frac{1}{2} \\ (b, f^*_{D(b_0,2-2t)}(y)), & if\ \frac{1}{2} \le t \le 1. \end{cases}$$

Note that $U^*(b,y,0) = (b, f^*_b(y)) = f^*(b,y)$ and $U^*(b,y,1) = (b, f^*_{b_0}(y))$. Then, f^* is G-homotopic to $id \times f^*_{b_0}$. Since $\Phi(B \times Y_0) = A \subset \mathrm{Fix}(f)$ we have, for each $(b,y) \in B \times Y_0$:

$$f^*(b,y) = \Phi^{-1} \circ \underbrace{f \circ \underbrace{\Phi(b,y)}_{\in A}}_{=\Phi(b,y)} = \Phi^{-1} \circ \Phi(b,y) = (b,y).$$

Then, $B \times Y_0 \subset \mathrm{Fix}(f^*)$ and $Y_0 \subset \mathrm{Fix}(f^*_b)$ because $(b, f^*_b(y)) = f^*(b,y) = (b,y)$.

By hypothesis, A intersects each essential WK-fixed point class of $f^K_{b_0} : WK(p^K)^{-1}(b_0) \to WK(p^K)^{-1}(b_0)$. So, $A \cap WK(p^K)^{-1}(b_0)$ intersects each essential WK-fixed point class of $f^K_{b_0}$. So, Y_0 intersects each essential WK-fixed point class of $(f^*_{b_0})^K$ because:

$$(\Phi^{-1})^K(A^K \cap WK(p^K)^{-1}(b_0)) = WK\{b_0\} \times Y_0^K.$$

The G-fiber bundle pair $\Big((A,Z), p, B, (Y_0, \Omega)\Big)$ is such that Ω intersects every component of Y_0 because Z intersects every component of A. Therefore, $(C_G1)_{\mathfrak{F}}$ and $(C_G2)_{\mathfrak{F}}$ hold for Y_0 and $f^*_{b_0}$. By Theorem 2.2 there exists a homotopy $V^* : Y \times I \to Y$ such that $f^*_{b_0} = V^*(\bullet, 0)$, $V^*(\bullet, 1) = g^*_{b_0} : Y \to Y$ and $\mathrm{Fix}(g^*_{b_0}) = \Omega$. Define a fiber-preserving G-homotopy $H^* : B \times Y \times I \to B \times Y$ given by:

$$H^*(b,y,t) = \begin{cases} U^*(b,y,2t), & if\ 0 \le t \le \frac{1}{2} \\ (b, V^*(y, 2t-1)), & if\ \frac{1}{2} \le t \le 1. \end{cases}$$

Therefore, f^* is G-homotopic to $id \times g^*_{b_0}$ and $\mathrm{Fix}(id \times g^*_{b_0}) = B \times \Omega$. Then, $H : X \times I \to X$ given by $H(e,t) = \Phi \circ H^*(\Phi^{-1}(e), t)$ is a G-homotopy such that $H(x,0) = f(x)$ and $\mathrm{Fix}(H(\bullet,1)) = \Phi(B \times \Omega) = Z$. ∎

Proof of Theorem 1.6: With Lemma 3.5 we assume that:

1. $A \subset \mathrm{Fix}(f) \subset p^{-1}(p(A))$;

2. $F = \mathrm{Fix}(\overline{f}) \cap (B - p(A))$ is a finite set.

Let $f_j = f|_{p^{-1}(p(A)_j)} : p^{-1}(p(A)_j) \to p^{-1}(p(A)_j)$ a restriction of f, so $p \circ f_j = p$. Using $X = p^{-1}(p(A)_j)$, $A = A \cap p^{-1}(p(A)_j)$, $B = p(A)_j$, $b_0 = b_j$, $Y_0 = Y_j$ and $f = f_j$ the hypotheses of Lemma 3.6 are satisfied and there exists a fiber-preserving G-homotopy $H_j : p^{-1}(p(A)_j) \times I \to p^{-1}(p(A)_j)$ from f_j to $h_j = H_j(\bullet, 1)$ such that $\mathrm{Fix}(h_j) = Z_j$.

Define $\widetilde{H}_2 : (X \times \{0\}) \cup (p^{-1}(F \cup p(A)) \times I) \to X$ by:

$$\widetilde{H}_2(x,t) = \begin{cases} f(x), & if\ t = 0\ or\ p(x) \in F \\ H_j(x,t), & if\ p(x) \in p(A)_j. \end{cases}$$

With Proposition 3.4 there is a fiber-preserving G-homotopy $H : X \times I \to X$ such that $p(H(x,t)) = \overline{f} \circ p(x)$. Therefore, $h = H(\bullet, 1) : X \to X$ is such that $\mathrm{Fix}(h) = Z$ and h is fiber-preserving G-homotopic to f. ∎

Corollary 3.7. Let $\mathfrak{F} = (X, p, B, Y)$ be a G-fiber bundle where X, B and Y are compact and smooth G-manifolds, $\dim(B^K) \geq 3$, $\dim(B^K) - \dim(B^K - B_K) \geq 2$, for all $(K) \in \mathrm{Iso}(B)$, $\dim(Y^K) \geq 3$, $\dim(Y^K) - \dim(Y^K - Y_K) \geq 2$, for all $(K) \in \mathrm{Iso}(Y)$.

Let A be a nonempty, closed, locally contractible G-subset of X such that (X, A) is a G-fiber bundle pair with respect to the fiber bundle $\mathfrak{F}$, $p(A)$ be a closed G-subset of B such that each component $p(A)_j$ of $p(A)$ is equivariantly contractible and $p^K(A^K)$ is by-passed in B^K, for all $(K) \in \mathrm{Iso}(B)$. Let Y_j be a sub-bundle fiber of A such that Y_j is a closed and locally contractible G-subset of Y and Y_j^K is by-passed in Y^K, for all $(K) \in \mathrm{Iso}(Y)$, and $f : X \to X$ be a G-fiber-preserving map such that A^K intersects every essential WK-fixed point class of $f_{b_j}^K : WK(p^K)^{-1}(\{b_j\}) \to WK(p^K)^{-1}(\{b_j\})$ for at least one b_j in each component $p^K(A^K)_j$, for all $(K) \in \mathrm{Iso}(X)$.

Then there exists a G-fiber-preserving map h, G-fiberwise homotopic to f with $\mathrm{Fix}(h) = A$ if, and only if, the following conditions holds for f and A:

$(C_G1)_{\mathfrak{F}}$ there exists a G-fiber-homotopy $H_A : A \times I \to X$ from $f|_A$ to the inclusion $i : A \hookrightarrow X$;

$(C_G2)_{\mathfrak{F}}$ for every WK-essential fixed point class F of $f^K : X^K \to X^K$ there exists a path $\alpha : I \to X^K$ with $\alpha(0) \in F$, $\alpha(1) \in A^K$, and $\{\alpha(t)\} \sim \{f^K \circ \alpha(t)\} * \{H_A^K(\alpha(1), t)\}$.

Proof of Corollary 3.7: If the conditions hold then we apply Theorem 1.6 for $Z = A$. If there exists h then by Lemma 3.2 $(C_G1)_{\mathfrak{F}}$ and $(C_G2)_{\mathfrak{F}}$ hold. ∎

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