The Nielsen Borsuk-Ulam number

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Abstract

A Nielsen-Borsuk-Ulam number ($NBU(f, \tau)$) is defined for continuous maps $f : X \to Y$ where X and Y are closed orientable triangulable n-manifolds and X has a free involution τ. This number is a lower bound, in the homotopy class of f, for the number of pairs of points in X satisfying $f(x) = f \circ \tau(x)$. It is proved that $NBU(f, \tau)$ can be realized (Wecken type theorem) when $n \geq 3$.

1 Introduction

The classical Borsuk-Ulam Theorem of maps from the sphere S^n in the Euclidean space $\mathbb{R}^n$ has been discussed and generalized in many different directions (see [1, 2, 4, 5, 6]).

Given a triple $(X, \tau; Y)$, where X and Y are finite n-dimensional complexes and τ is a free simplicial involution, one possible approach is to study the question - in the homotopy classes of maps - of the existence of points $x \in X$ such that $f(x) = f \circ \tau(x)$.

In a previous work ([1]) some notions, which can be seen as a Nielsen theory approach for Borsuk-Ulam type problems, were defined. In the context of maps between finite n-dimensional complexes, Nielsen Borsuk-Ulam coincidence classes (named BU-coincidence classes) were defined and a mild version of an index is proposed with the property that when such index is non-zero the class is geometrically essential.

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This work went further in the same direction. In the context of closed orientable triangulable manifolds, we define a *"pseudo-index"* for BU-coincidence classes, then a Nielsen-Borsuk-Ulam number in such situation, demonstrating that said number is a lower bound for the number of pairs of coincidences between f and $f \circ \tau$ in the homotopy class of f and that it can be realized (Wecken type theorem) when the dimension of the manifolds are greater than 2 (as usual in Nielsen theory).

In the last section an example where said number is greater than 1 is presented, showing that this approach can contribute for the description of the set of Borsuk-Ulam coincidences.

2 Nielsen Borsuk-Ulam theory

In [1] some ideas about a Nielsen Borsuk-Ulam theory were presented. In fact, the theory was constructed using an index with image in $\mathbb{Z}_2$ for the Nielsen Borsuk-Ulam classes. Following [1] we have:

Definition 2.1. *Let $(X, \tau; Y)$ be a triple where X and Y are finite n-dimensional complexes, τ is a free simplicial involution on X for any map $f : X \to Y$ with $Coin(f, f \circ \tau) = \{x_1, \tau(x_1), \cdots, x_m, \tau(x_m)\}$ we define the Borsuk-Ulam coincidence set for the pair (f, τ), as the set of pairs:*

$$BUCoin(f;\tau) = \{(x_1, \tau(x_1)); \cdots ; (x_m, \tau(x_m))\}$$

and we say that two pairs $(x_i, \tau(x_i)), (x_j, \tau(x_j))$ are in the same BU-coincidence class if there exists a path γ from a point in $\{x_i, \tau(x_i)\}$ to a point in $\{x_j, \tau(x_j)\}$ such that $f \circ \gamma$ is homotopic to $f \circ \tau \circ \gamma$ with fixed endpoints.

Definition 2.2. *A BU-coincidence class C is called single if for one (or any) pair $(x, \tau(x)) \in C$ there exists a path γ from x to $\tau(x)$ such that $f \circ \gamma$ is homotopic to $f \circ \tau \circ \gamma$ with fixed endpoints.*

If we consider:

Remark 2.3. *[1, Proposition 4.3] If C' is a usual Nielsen coincidence class for the pair $(f, f \circ \tau)$ then there exists a BU-coincidence class C of the pair (f, τ) such that $C' \subseteq C$.*

We obtain:

Proposition 2.4. *A BU-coincidence class C is single if, and only if, it is composed of just one usual coincidence class of the pair $(f, f \circ \tau)$. Moreover, if C is a finite BU-coincidence class of the pair (f, τ) that is not single (called double) then we can change the labels of the elements of C in a way that:*

- $C = \{(x_1, \tau(x_1)), \ldots, (x_k, \tau(x_k))\}$;
- $C = C_1 \cup C_2$ *where C_1 and C_2 are usual coincidence classes of the pair $(f, f \circ \tau)$;*
- $C_1 = \{x_1, \ldots, x_k\}$ *and* $C_2 = \{\tau(x_1), \ldots, \tau(x_k)\}$. ■

Furthermore, for an isolated coincidence c of the pair $(f, f\circ\tau)$ between closed orientable n-manifolds, we have:

$$ind(f, f\circ\tau; c) = \begin{cases} ind(f, f\circ\tau; \tau(c)) & \text{if } \tau \text{ preserves orientation,} \\ -ind(f, f\circ\tau; \tau(c)) & \text{if } \tau \text{ reverses orientation.} \end{cases}$$

where $ind(f, f\circ\tau; c)$ is the usual local index for coincidence.

Now it is possible to define a pseudo-index[1] for BU-coincidence classes:

Definition 2.5. *Let X and Y be closed orientable triangulable n-manifolds, τ a free involution on X and $f : X \to Y$ a continuous map such that $BUCoin(f,\tau)$ is finite. If $C = \{(x_1, \tau(x_1)), \ldots, (x_k, \tau(x_k))\}$ is a BU-coincidence class of the pair (f,τ) we define the pseudo-index of C by*

$$|ind|(f,\tau;C) = \begin{cases} \sum ind(f, f\circ\tau; x_i) \bmod 2 & \text{if } C \text{ is single and } \tau \text{ reverses orientation;} \\ \dfrac{ind(f, f\circ\tau; C)}{2} & \text{if } C \text{ is single and } \tau \text{ preserves orientation;} \\ |ind(f, f\circ\tau; C_1)| & \text{if } C \text{ is double, } C = C_1\cup C_2 \text{ and } \tau \text{ reverses orientation;} \\ ind(f, f\circ\tau; C_1) & \text{if } C \text{ is double, } C = C_1\cup C_2 \text{ and } \tau \text{ preserves orientation.} \end{cases}$$

where C_1 and C_2 are disjoint usual coincidence classes of the pair $(f, f\circ\tau)$.

We note that when τ reverses orientation, a single BU-coincidence class has similar properties to the defective classes defined for coincidences of maps between non-orientable manifolds (see [3, 7, 8]).

Definition 2.6. *As usual, we call a BU-coincidence class C essential if $|ind|(f,\tau;C) \neq 0$ and we define $NBU(f,\tau)$, the Nielsen Borsuk-Ulam number of the pair (f,τ), as the number of essential BU-coincidences classes.*

Proposition 2.7. *If f' is homotopic to f then f' has at least $NBU(f,\tau)$ pairs of BU-coincidence points.*

Proof: Given an essential BU-coincidence class C of the pair (f,τ) then we can have

1. C is double; so $C = C_1 \cup C_2$, two disjoint usual coincidence classes of the pair $(f, f\circ\tau)$ both with non-zero index;
2. C is single and τ preserves orientation; so $|ind|(f,\tau;C) \neq 0$ implies $ind(f, f\circ\tau; C) \neq 0$ as a usual coincidence class;
3. C is single and τ reverses orientation; in this case the geometric essentiality of C is a result of [1, Lemma 5.1];

in all cases C is geometrically essential and the result follows. ∎

[1]See [3, 7, 8] for a definition of a semi-index on coincidence classes for non-orientable closed manifolds

3 Realization

From classic coincidence theory it is easy to prove the following lemma:

Lemma 3.1. *Let X and Y be closed triangulable n-manifolds, τ a free involution on X and $f : X \to Y$ a continuous map, suppose $c \in X$ an isolated point such that the pair $(c, \tau(c))$ is a BU-coincidence pair of points (i.e. $f(c) = f(\tau(c))$) with $ind(f, f \circ \tau; c) = 0$, then, by a deformation of f in a small neighborhood of c we can obtain a map f', homotopic to f, such that $BUCoin(f', \tau) = BUCoin(f, \tau) \setminus \{c, \tau(c)\}$.* ■

The following lemma corresponds to the geometric realizations of the *join* procedure defined in [1, page 3744]:

Lemma 3.2. *Let X and Y be closed orientable triangulable n-manifolds, $n \geq 3$, τ a free involution on X and $f : X \to Y$ a continuous map. Suppose that*

- *$BUCoin(f, \tau) = \{(x_1, \tau(x_1)); \cdots ; (x_m, \tau(x_m))\}$;*
- *x_1 and x_2 are in the same usual coincidence class of the pair $(f, f \circ \tau)$ (so the pairs $(x_1, \tau(x_1))$, $(x_2, \tau(x_2))$ are in the same BU-coincidence class);*

then there exists a map $f' \sim f$ such that:

- *$BUCoin(f', \tau) = \{(x'_1, \tau(x'_1)); (x_3, \tau(x_3)); \cdots ; (x_m, \tau(x_m))\}$;*
- *$ind(f, f \circ \tau; x'_1) = ind(f, f \circ \tau; x_1) + ind(f, f \circ \tau; x_2)$;*

Proof: There exists a path γ, from x_1 to x_2, realizing the Nielsen relation (i.e. $f(\gamma)$ is homotopic relative to the endpoints to $f\tau(\gamma)$), and a closed neighborhood $\overline{U}$ of γ in X, such that $\overline{U} \cap \tau(\overline{U}) = \varnothing$, and $\overline{U} \cap BUCoin(f, f \circ \tau) = \{x_1, x_2\}$.

We can suppose that there exists a homeomorphism φ from $\overline{U}$ to a δ-neighborhood $U(I, \delta)$ of the interval I (the line segment from the origin to $(1, 0, \cdots, 0)$) in $\mathbb{R}^n$, with $\varphi(\gamma) = I$.

The idea is to follow the steps used to define f' in the proof of Theorem 2.1 in [1] until the $(n-2)$-skeleton of $\overline{U}$ without changing the map on the boundary of $\overline{U}$. Such construction consists in changing the definition of f in the simplexes in the interior of $\overline{U}$, using a triangularization of Y with small diameter, in a way that the image of any point by f' is so close to the image by f that the two maps are homotopic.

Now, for the maximal simplexes of $\overline{U}$ and its faces (all the n and $(n-1)$-simplexes) we proceed in the following way: First we note that all n-simplexes of $\overline{U}$ can be ordered by $\sigma_1^n, \sigma_2^n, \cdots, \sigma_r^n$ in a way that all σ_i^n with $i < r$, contains one face (named σ_i^{n-1}) which is a face of one σ_j^n with $r \geq j > i$.

We will define f' without coincidences with $f' \circ \tau$ in $\sigma_1^n, \sigma_2^n, \cdots, \sigma_{r-1}^n$.

In σ_1^n, using a geometric construction similar to that one in the non maximal simplexes we can extend f' over all $(n-1)$-simplexes of $\partial\sigma_1^n - \sigma_1^{n-1}$ where f' is not defined yet.

We can choose $p \notin \overline{\sigma}_1^n$ in a way that σ_1^n can be bijected over $\partial\sigma_1^n - \overline{\sigma}_1^{n-1}$ by a linear projection from p, (imagine p inside the other n-simplex that has σ_1^{n-1} as a face σ_j^n).

For each $\alpha_0 \in \partial\sigma_1^n - \overline{\sigma}_1^{n-1}$ let α_1 be the intersection of $\overrightarrow{p\alpha_0}$ with σ_1^{n-1} and we define $\alpha_t = (1-t)\alpha_0 + t\alpha_1$, for $0 \leq t \leq 1$.

We can suppose $\left[f \circ \tau(\overline{\sigma}_1^n) \cup f(\partial\sigma_1^n - \sigma_1^{n-1})\right] \subset V_1$ where $V_1 \subset Y$ is homeomorphic, by φ_1, to the unitary ball $B_1^n(0)$ in $\mathbb{R}^n$.

So, for all $\alpha_1 \in \sigma_1^{n-1}$ we can associate, in a continuous way, a positive number $\lambda(\alpha_1) = |\overrightarrow{\varphi_1(f \circ \tau(\alpha_1))\varphi_i(f(\alpha_1))}|$. In the same way we define $\lambda(\alpha_0) = |\overrightarrow{\varphi_1(f \circ \tau(\alpha_0))\varphi_1(f(\alpha_0))}|$, for all $\alpha_0 \in \partial\sigma_1^n - \overline{\sigma}_1^{n-1}$. Then, for each $t \in [0,1]$, we define $f(\alpha_t)$ satisfying:

$$\overrightarrow{0\varphi_1(f'(\alpha_t))} = \overrightarrow{0\varphi_1(f \circ \tau(\alpha_t))} + \left[1 + t\left(\frac{\lambda(\alpha_1)}{\lambda(\alpha_0)} - 1\right)\right] \overrightarrow{\varphi_1(f \circ \tau(\alpha_0))\varphi_1(f(\alpha_0))}.$$

We can see that for all $t \in [0,1]$, the vector $\overrightarrow{0\varphi_1(f(\alpha_t))}$ is entirely contained in $B_1^n(0)$, then the map is well defined. So, f' is extended in a continuous way in σ_1^n.

Correspondingly, following the sequence, the map f' can be defined in σ_2^n, $\sigma_3^n, \cdots, \sigma_{r-1}^n$.

The map f' is already defined in $\partial\sigma_r^n$ close enough to $f' \circ \tau$, then we can use the same geometric constructions as in the proof of Theorem 2.1 in [1] to define f' in σ_r^n in a way that it produces at most one coincidence with $f' \circ \tau$ in σ_r^n.

We finish with a map f', homotopic to f relatively to the set $X \setminus \overline{U}(\gamma)$, such that f' and $f' \circ \tau$ have, at most, one coincidence in $\overline{U}(\gamma)$. The conclusion about the index of said coincidence follows from properties of the index. ■

Remark 3.3. *The geometrical equivalent to the procedure named blend, defined in [1, page 3744], is exactly an interchange of the names in one pair $(x_j, \tau(x_j)) \in BUCoin(f, \tau)$ and the geometric version of the split can be stated as the Lemma 3.4 below and it can be seen as the reverse of Lemma 3.2.*

Lemma 3.4. *Let X and Y be compact connected orientable triangulable n-manifolds, $n \geq 3$, τ a free involution on X and $f : X \to Y$ a continuous map. Suppose that*

$$BUCoin(f, \tau) = \{(x_1, \tau(x_1)); \cdots ; (x_m, \tau(x_m))\}$$

then there exists a map $f' \sim f$ such that:

- $BUCoin(f', \tau) = \{(x_1', \tau(x_1')); (x_1'', \tau(x_1'')); (x_2, \tau(x_2)); \cdots$
  $\cdots ; (x_m, \tau(x_m))\}$;
- $ind(f, f \circ \tau; x_1) = ind(f, f \circ \tau; x_1') + ind(f, f \circ \tau; x_1'')$; ■

Now the tools are complete to prove a Wecken type theorem:

Theorem 3.5. *Let X and Y be closed orientable triangulable n-manifolds, τ a free involution on X and $f : X \to Y$ a continuous map, if $n \geq 3$ then there exists a map f' homotopic to f such that f' has exact $NBUCoin(f, \tau)$ pairs of BU-coincidence points.*

Proof: Using [1, Theorem 2.1] we can suppose $BUCoin(f, \tau)$ finite, moreover, Theorem 3.5, Corollaries 3.8 and 3.9 in [1] prove that the pseudo-index is invariant by homotopies, in the sense that BU-coincidence classes related by one homotopy have the same pseudo-index.

Using Lemma 3.2 we can produce a map f' with exactly one BU-coincidence pair in each BU-coincidence essential class, additionally, it can be done in a way that the local index of one point of the pair is equal to the pseudo-index of its class, so the non essential ones can be removed (Lemma 3.1). ■

4 Examples

Consider the torus $T = \dfrac{\mathbb{R}\times\mathbb{R}}{\mathbb{Z}\times\mathbb{Z}}$ that we will denote by:

$$T = [0,1]\times[0,1] \quad \mathrm{mod}\ 1$$

Let $\tau : T \to T$ be the free involution given by

$$\tau(x,y) = (x+\frac{1}{2}, -y).$$

Define $f : T \to T$ by $f(x,y) = (2x+y, y)$. The set $BUCoin(f,\tau)$ corresponds to the solutions of

$$(2x+y,y) = (2(x+\frac{1}{2}) - y, -y) \quad \mathrm{mod}\ 1,$$

so all points with $y = 0$ or $y = \frac{1}{2}$ are in $BUCoin(f,\tau)$.

Taking $\epsilon(x) : [0,1] \to [0,1]$ such that

- $\epsilon(x) = 0$ if $x = 0$ or $x \in [\frac{1}{2}, 1]$;
- $0 < \epsilon(x) < \frac{1}{10}$ if $x \in]0, \frac{1}{2}[$

It is not difficult to see that it is possible to deform f (by an ϵ-homotopy) to a map:

$$f'(x,y) = f(x,y) + (\epsilon(x), 0),$$

such that the solutions to $f'(x,y) = f'\circ\tau(x,y)$ satisfy:

$$(2x+y+\epsilon(x), y) = (2(x+\frac{1}{2}) + y + \epsilon(x+\frac{1}{2}), -y) \quad \mathrm{mod}\ 1.$$

Which corresponds to

$$f(x,y) + (\epsilon(x), 0) = f\circ\tau(x,y) + \epsilon(x+\frac{1}{2}).$$

So there exist 4 exact points:

$$\{(0,0), (\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\}$$

such that $f'(x,y) = f'\circ\tau(x,y)$.

We have two usual coincidence classes:

$$C_1 = \{(0,0), (\frac{1}{2}, 0)\} \qquad C_2 = \{(0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\}$$

each of them is equal to one (single) BU-class, and both of them with pseudo-index equals to 1.

So these two BU-classes are essential, and $NBU(f, \tau) = 2$.

In the examples in Theorem 5.2 in [1] (self-maps of the sphere S^n) there exists only one BU-coincidence class, which is single, and its pseudo-index depends on whether the involution (in that case the antipodal map) reverses or preserves orientation.

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