Linear systems over $\mathbb{Z}[Q_{16}]$ and roots of maps of some 3-complexes into $M_{Q_{16}}$

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Abstract

Let $\mathbb{Z}[Q_{16}]$ be the group ring where $Q_{16} = \langle x, y | x^4 = y^2, xyx = y \rangle$ is the quaternion group of order 16 and ε the augmentation map. We show that, if $PX = K(x-1)$ and $PX = K(-xy+1)$ has solution over $\mathbb{Z}[Q_{16}]$ and all $m \times m$ minors of $\varepsilon(P)$ are relatively prime, then the linear system $PX = K$ has a solution over $\mathbb{Z}[Q_{16}]$, where $P = [p_{ij}]$ is an $m \times n$ matrix with $m \leq n$. As a consequence of such results, we show that there is no map $f : W \to M_{Q_{16}}$ that is strongly surjective, i.e., such that $MR[f, a] = \min\{\#(g^{-1}(a)) | g \in [f]\} \neq 0$. Here, $M_{Q_{16}}$ is the orbit space of the 3-sphere $\mathbb{S}^3$ with respect to the action of Q_{16} determined by the inclusion $Q_{16} \subseteq \mathbb{S}^3$ and W is a CW-complex of dimension 3 with $H^3(W; \mathbb{Z}) = 0$.

1 Introduction

Given a map $f : W \to M$ between topological spaces, and an arbitrary point $a \in M$, recall that $MR[f, a] = \min\{\#(g^{-1}(a)) | g \in [f]\}$, where $[-]$ means a homotopy class. We say that a map $f : W \to M$ is *strongly surjective*, if any map homotopic to it is surjective or, equivalently, if $MR[f, a] \neq 0$ for some $a \in M$. The problem of the existence of a map $f : W \to M$ which is strongly surjective has been studied in the paper [1] when W is a CW-complex and M is a closed manifold, both of dimension 3. The main results are:

Received by the editors in August 2016 - In revised for in December 2016.
Communicated by K. Dekimpe, D.L. Gonçalves and P. Wong.
2010 *Mathematics Subject Classification* : 55M20, 55S35, 55N25.
Key words and phrases : quaternion group; quaternionic determinant; strongly surjective map.

i) There is no map $f : W \to M$ which is a strongly surjective map if W is a CW-complex with $H^3(W;\mathbb{Z}) = 0$ and M is either $\mathbb{S}^1 \times \mathbb{S}^2$, $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, or a lens space;

ii) There is no map $f : W \to M$ which is a strongly surjective map if W is a CW-complex with $H^3(W;\widetilde{\mathbb{Z}}) = 0$ and M is either $\mathbb{S}^1 \times \mathbb{P}^2$, where $\mathbb{P}^2$ denotes the 2-dimensional real projective space, or $M_A = \mathbb{S}^2 \times [0,1]/(x,0) \sim (-x,1)$. Here, $H^3(W;\widetilde{\mathbb{Z}})$ is the cohomology with an arbitrary local coefficient system $\widetilde{\mathbb{Z}}$.

iii) There exists a strongly surjective map $f : W \to M_A$, where W is a certain 3-complex with $H^3(W;\mathbb{Z}) = 0$.

The above results were obtained making use of the obstruction theory. The existence or non-existence of a strongly surjective map $f : W \to M$ may be determined by an obstruction class $\omega^3(f) \in H^3(W;\mathbb{Z}[\pi])$, where $H^3(W;\mathbb{Z}[\pi])$ is the cohomology group of W with a local coefficient system and $\pi = \pi_1(M)$. In [1], the vanishing of the obstruction to deform a map $f : W \to M$ to a root free map is described in terms of solutions of a liner system $PX = K$ over the group ring $\mathbb{Z}[\pi]$. Let $Q_8 = \langle x, y | x^2 = y^2,\ xyx = y\rangle$ be the quaternion group of order 8 and M_{Q_8} the orbit space of the 3-sphere $\mathbb{S}^3$ with respect to the action determined by the inclusion $Q_8 \subseteq \mathbb{S}^3$. In [2] we studied the problem of the existence of a map $f : W \to M_{Q_8}$ which is strongly surjective, we obtained:

iv) If W is a three dimensional CW-complex with $H^3(W;\mathbb{Z}) = 0$ then there is no strongly surjective map $f : W \to M_{Q_8}$.

This case differs from the previous one because $\pi_1(M_{Q_8}) = Q_8$ is nonabelian. To prove the statement iv) is equivalent to demonstrate that: If $PX = K(x-1)$ and $PX = K(-xy+1)$ have solutions over $\mathbb{Z}[Q_8]$ where $P = [p_{ij}]$ is an $m \times n$ matrix with $m \leq n$, and all $m \times m$ minors of $\varepsilon(P) = [\varepsilon(p_{ij})]$ are relatively prime, then the system $PX = K$ has a solution over $\mathbb{Z}[Q_8]$. The condition about $\varepsilon(P)$ is obtained from the hypothesis that $H^3(W;\mathbb{Z}) = 0$.

In this work we consider linear systems over $\mathbb{Z}[Q_{16}]$ and maps $f : W \to M_{Q_{16}}$. The main results are:

Theorem 1.1. *Let $PX = K$ be a linear system over $\mathbb{Z}[Q_{16}]$, where $P = [p_{ij}]$ is an $m \times n$ with $m \leq n$. If $PX = K(x-1)$ and $PX = K(-xy+1)$ have solutions over $\mathbb{Z}[Q_{16}]$ and all $m \times m$ minors of $\varepsilon(P) = [\varepsilon(p_{ij})]$ are relatively prime, then the system $PX = K$ has a solution over $\mathbb{Z}[Q_{16}]$.*

Theorem 1.2. *If W is a three dimensional CW-complex with $H^3(W;\mathbb{Z}) = 0$, then there is no strongly surjective map $f : W \to M_{Q_{16}}$.*

The techniques used in [2] unfortunately do not apply to this case. Therefore we introduce new techniques to study the case Q_{16}. Let $M_2(\mathbb{Z})$ be the ring of 2×2 matrices with entries in $\mathbb{Z}$ and $H(\mathbb{Q}[\sqrt{2}])$ the quaternion field. Throughout this paper, the problem of solving the linear system $PX = K$ will be converted to

solving linear systems over $\mathbb{Z}$, $M_2(\mathbb{Z})$ and $H(\mathbb{Q}[\sqrt{2}])$. The problem of solving linear equations over the quaternion field $H(\mathbb{R})$, has been studied in [3] and [8] making use of quaternionic determinant and inverse square matrix, subjects that will be used in this work.
The paper is organized as follows. Sections 2, 3, 4, 5, 6, 7 are dedicated to prove theorem 1.1. Section 8 contains the proof of theorem 1.2 and discusses the case Q_{32}.

2 Linear systems over $\mathbb{Z}[Q_{16}]$

Consider the quaternion group $Q_{16} = \langle x, y | x^4 = y^2, \; xyx = y \rangle$ of order 16. Any element $w \in Q_{16}$ has a unique canonical form $w = x^{\mu} y^{\delta}$, with $0 \leq \mu < 8$ and $\delta = 0, 1$. The function $\varepsilon : \mathbb{Z}[Q_{16}] \to \mathbb{Z}$ given by $\sum_{i=1}^{p} r_i w_i \mapsto \sum_{i=1}^{p} r_i$ is a ring homomorphism. Let $\mathbb{Q}$ be the field of rational numbers, $M_2(\mathbb{Q})$ the ring of 2×2 matrices with entries in $\mathbb{Q}$, $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} | \; a, b \in \mathbb{Q}\}$, $H(\mathbb{Q}[\sqrt{2}])$ the quaternion field and $\mathbb{Q}^4 \oplus M_2(\mathbb{Q}) \oplus H(\mathbb{Q}[\sqrt{2}])$ the ring with component-wise addition and multiplication. The quaternion field $H(\mathbb{Q}[\sqrt{2}])$ has dimension 8 over $\mathbb{Q}$ with basis $\beta = \{1, \sqrt{2}, i, \sqrt{2}i, j, \sqrt{2}j, k, \sqrt{2}k\}$ where $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$ and $ki = j = -ik$. Consider the isomorphism of rings $T : \mathbb{Q}(Q_{16}) \to \mathbb{Q}^4 \oplus M_2(\mathbb{Q}) \oplus H(\mathbb{Q}[\sqrt{2}])$ given by:

$$
\begin{array}{rlrl}
1 \longmapsto & (1,1,1,1, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, 1) & x \longmapsto & (1,1,-1,-1, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) \\
x^2 \longmapsto & (1,1,1,1, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, i) & x^3 \longmapsto & (1,1,-1,-1, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) \\
x^4 \longmapsto & (1,1,1,1, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, -1) & x^5 \longmapsto & (1,1,-1,-1, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}) \\
x^6 \longmapsto & (1,1,1,1, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, -i) & x^7 \longmapsto & (1,1,-1,-1, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}) \\
y \longmapsto & (1,-1,1,-1, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, j) & xy \longmapsto & (1,-1,-1,1, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \frac{\sqrt{2}}{2}j + \frac{\sqrt{2}}{2}k) \\
x^2y \longmapsto & (1,-1,1,-1, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, k) & x^3y \longmapsto & (1,-1,-1,1, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, -\frac{\sqrt{2}}{2}j + \frac{\sqrt{2}}{2}k) \\
x^4y \longmapsto & (1,-1,1,-1, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, -j) & x^5y \longmapsto & (1,-1,-1,1, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, -\frac{\sqrt{2}}{2}j - \frac{\sqrt{2}}{2}k) \\
x^6y \longmapsto & (1,-1,1,-1, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, -k) & x^7y \longmapsto & (1,-1,-1,1, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \frac{\sqrt{2}}{2}j - \frac{\sqrt{2}}{2}k)
\end{array}
$$

Given an element $q = q_0 + q_1 x + \cdots + q_7 x^7 + q_8 y + q_9 xy + \cdots + q_{15} x^7 y \in \mathbb{Z}[Q_{16}]$, denote by $T(q) = (\varepsilon(q), q^1, q^2, q^3, q^4, q^5)$, where

$$
\begin{array}{rcl}
\varepsilon(q) & = & q_0 + q_1 + \cdots + q_6 + q_7 + q_8 + q_9 + \cdots + q_{14} + q_{15} \\
q^1 & = & q_0 + q_1 + \cdots + q_6 + q_7 - q_8 - q_9 - \cdots - q_{14} - q_{15} \\
q^2 & = & q_0 - q_1 + \cdots + q_6 - q_7 + q_8 - q_9 + \cdots + q_{14} - q_{15} \\
q^3 & = & q_0 - q_1 + \cdots + q_6 - q_7 - q_8 + q_9 - \cdots - q_{14} + q_{15}. \\
q^4 & = & \begin{bmatrix} n_1 & n_2 \\ n_3 & n_4 \end{bmatrix} \\
q^5 & = & a + b\sqrt{2} + ci + d\sqrt{2}i + ej + f\sqrt{2}j + gk + h\sqrt{2}k
\end{array}
$$

and

$$\begin{aligned}
n_1 &= q_0 - q_2 + q_4 - q_6 - q_9 + q_{11} - q_{13} + q_{15}\\
n_2 &= -q_1 + q_3 - q_5 + q_7 + q_8 - q_{10} + q_{12} - q_{14}\\
n_3 &= q_1 - q_3 + q_5 - q_7 + q_8 - q_{10} + q_{12} - q_{14}\\
n_4 &= q_0 - q_2 + q_4 - q_6 + q_9 - q_{11} + q_{13} - q_{15}
\end{aligned}$$

$$\begin{array}{ll}
a = q_0 - q_4 & b = \dfrac{q_1 - q_3 - q_5 + q_7}{2}\\
c = q_2 - q_6 & d = \dfrac{q_1 + q_3 - q_5 - q_7}{2}\\
e = q_8 - q_{12} & f = \dfrac{q_9 - q_{11} - q_{13} + q_{15}}{2}\\
g = q_{10} - q_{14} & h = \dfrac{q_9 + q_{11} - q_{13} - q_{15}}{2}
\end{array}$$

Let $\phi : \mathbb{Z} \to \mathbb{Z}_2$ be the quotient map, $\phi(n)$, $n \in \mathbb{Z}$ will be abbreviated $\overline{n}$ and should not be confused with $\overline{p}$, $p \in H(\mathbb{R})$ the conjugate of the quaternions. Notice that ϕ will be used also as the obvious extension map from the matrix sets $M_n(\mathbb{Z})$ to $M_n(\mathbb{Z}_2)$.

Lemma 2.1. *The set of all the elements*

$$a + b\sqrt{2} + ci + d\sqrt{2}i + ej + f\sqrt{2}j + gk + h\sqrt{2}k,$$

where a, c, e, g, $2b$, $2d$, $2f$, $2h \in \mathbb{Z}$ *with* $\overline{2b} = \overline{2d}$ *and* $\overline{2f} = \overline{2h}$, *denoted by* $\overline{H}(\mathbb{Q}[\sqrt{2}])$, *is a subring of* $H(\mathbb{Q}[\sqrt{2}])$.

Proof: Let $q_1 = a_1 + b_1\sqrt{2} + c_1 i + d_1\sqrt{2}i + e_1 i + f_1\sqrt{2}j + g_1 k + h_1\sqrt{2}k$ and $q_2 = a_2 + b_2\sqrt{2} + c_2 i + d_2\sqrt{2}i + e_2 j + f_2\sqrt{2}j + g_2 k + h_2\sqrt{2}k$ be elements of $\overline{H}(\mathbb{Q}[\sqrt{2}])$. The element $q_1 - q_2$ lies in $\overline{H}(\mathbb{Q}[\sqrt{2}])$. Consider the product

$$q_1 \cdot q_2 = a + b\sqrt{2} + ci + d\sqrt{2}i + ej + f\sqrt{2}j + gk + h\sqrt{2}k,$$

where

i) $a = a_1a_2 - c_1c_2 - e_1e_2 - g_1g_2 + 2(b_1b_2 - d_1d_2 - f_1f_2 - h_1h_2)$

ii) $b = a_1b_2 + b_1a_2 - c_1d_2 - c_2d_1 - e_1f_2 - f_1e_2 - g_1h_2 - g_2h_1$

iii) $c = a_1c_2 + c_1a_2 + e_1g_2 - g_1e_2 + 2(b_1d_2 + d_1b_2 + f_2h_1 - f_1h_2)$

iv) $d = a_1d_2 + b_1c_2 - c_1b_2 - a_2d_1 - e_1h_2 + f_1g_2 - g_1f_2 - e_2h_1$

v) $e = a_1e_2 - c_1g_2 + e_1a_2 + g_1c_2 + 2(b_1f_2 - d_1h_2 + f_1b_2 + h_1d_2)$

vi) $f = a_1f_2 + b_1e_2 - c_1h_2 - g_2d_1 + e_1b_2 + f_1a_2 + g_1d_2 + c_2h_1$

vii) $g = a_1g_2 + c_1e_2 - e_1c_2 + g_1a_2 + 2(b_1h_2 + d_1f_2 + f_1d_2 + h_1b_2)$

viii) $h = a_1h_2 + b_1g_2 + c_1f_2 + e_2d_1 + e_1d_2 + f_1c_2 + g_1b_2 + a_2h_1$.

Note that $2b_12b_2 - 2d_12d_2 - 2f_12f_2 - 2h_12h_2 = 2k$ with $k \in \mathbb{Z}$, since $\overline{2b_1} = \overline{2d_1}$, $\overline{2f_1} = \overline{2h_1}$, $\overline{2b_2} = \overline{2d_2}$ and $\overline{2f_2} = \overline{2h_2}$. Hence, $2(b_1b_2 - d_1d_2 - f_1f_2 - h_1h_2) = k \in \mathbb{Z}$

and $a \in \mathbb{Z}$. By a similar argument, $c, e, g \in \mathbb{Z}$.
Now,

$$2b = a_1 2b_2 + 2b_1 a_2 - c_1 2d_2 - c_2 2d_1 - e_1 2f_2 - 2f_1 e_2 - g_1 2h_2 - g_2 2h_1$$

and

$$2d = a_1 2d_2 + 2b_1 c_2 - c_1 2b_2 - a_2 2d_1 - e_1 2h_2 + 2f_1 g_2 - g_1 2f_2 - e_2 2h_1.$$

Because $\overline{a_1 2b_2} = \overline{a_1 2d_2}$, $\overline{2b_1 a_2} = \overline{2d_1 a_2}$, $\overline{c_1 2d_2} = \overline{c_1 2b_2}$, $\overline{c_2 2d_1} = \overline{c_2 2b_1}$, $\overline{e_1 2f_2} = \overline{e_1 2h_2}$, $\overline{e_2 2f_1} = \overline{e_2 2h_1}$, $\overline{g_1 2h_2} = \overline{g_1 2f_2}$ and $\overline{2h_1 g_2} = \overline{2f_1 g_2}$, then $\overline{2b} = \overline{2d}$. Analogously, $\overline{2f} = \overline{2h}$. ∎

The restriction $T : \mathbb{Z}[Q_{16}] \to \mathbb{Z}^4 \oplus M_2(\mathbb{Z}) \oplus \overline{H}(\mathbb{Q}[\sqrt{2}])$ gives an embedding of $\mathbb{Z}[Q_{16}]$ in $\mathbb{Z}^4 \oplus M_2(\mathbb{Z}) \oplus \overline{H}(\mathbb{Q}[\sqrt{2}])$. To study the solutions of a linear system $PX = K$ over $\mathbb{Z}[Q_{16}]$, where $P = [p_{ij}]$ is an $m \times n$ matrix with entries in $\mathbb{Z}[Q_{16}]$, and $X = [x_1 \cdots x_n]^t$ and $K = [k_1 \cdots k_m]^t$ are column vectors with coordinates in $\mathbb{Z}[Q_{16}]$, we will consider first the problem in $\mathbb{Z}^4 \oplus M_2(\mathbb{Z}) \oplus \overline{H}(\mathbb{Q}[\sqrt{2}])$.

Theorem 2.2. *If the system $\overline{P}X = \overline{K}$ has a solution over $\mathbb{Z}^4 \oplus M_2(\mathbb{Z}) \oplus \overline{H}(\mathbb{Q}[\sqrt{2}])$, where $\overline{P} = [T(p_{ij})]$ and $\overline{K} = [T(k_1) \cdots T(k_m)]^t$, then the system $PX = 16K$ has a solution over $\mathbb{Z}[Q_{16}]$.*

Proof: Observe that the isomorphism T has the following property:

$$\frac{1 + x + \cdots + x^7 + y + xy + \cdots + x^7 y}{16} \longmapsto (1, 0, 0, 0, \mathbf{0}, 0)$$
$$\frac{1 + x + \cdots + x^7 - y - xy - \cdots - x^7 y}{16} \longmapsto (0, 1, 0, 0, \mathbf{0}, 0)$$
$$\frac{1 - x + \cdots + x^6 - x^7 + y - xy + \cdots + x^6 y - x^7 y}{16} \longmapsto (0, 0, 1, 0, \mathbf{0}, 0)$$
$$\frac{1 - x + \cdots + x^6 - x^7 - y + xy - \cdots - x^6 y + x^7 y}{16} \longmapsto (0, 0, 0, 1, \mathbf{0}, 0)$$
$$\frac{1 - x^4}{2} \longmapsto (0, 0, 0, 0, \mathbf{0}, 1)$$
$$\frac{x^2 - x^6}{2} \longmapsto (0, 0, 0, 0, \mathbf{0}, i)$$
$$\frac{y - x^4 y}{2} \longmapsto (0, 0, 0, 0, \mathbf{0}, j)$$
$$\frac{x^2 y - x^6 y}{2} \longmapsto (0, 0, 0, 0, \mathbf{0}, k)$$
$$\frac{x - x^3 - x^5 + x^7}{2} \longmapsto (0, 0, 0, 0, \mathbf{0}, \sqrt{2})$$
$$\frac{x + x^3 - x^5 - x^7}{2} \longmapsto (0, 0, 0, 0, \mathbf{0}, \sqrt{2}i)$$
$$\frac{xy + x^3 y - x^5 y - x^7 y}{2} \longmapsto (0, 0, 0, 0, \mathbf{0}, \sqrt{2}k)$$
$$\frac{xy - x^3 y - x^5 y + x^7 y}{2} \longmapsto (0, 0, 0, 0, \mathbf{0}, \sqrt{2}j)$$
$$\frac{1 - x^2 + x^4 - x^6 - xy + x^3 y - x^5 y + x^7 y}{8} \longmapsto \left(0, 0, 0, 0, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, 0\right)$$

$$\frac{-x+x^3-x^5+x^7+y-x^2y+x^4y-x^6y}{8} \longmapsto (0,0,0,0,\begin{bmatrix}0&1\\0&0\end{bmatrix},0)$$
$$\frac{x-x^3+x^5-x^7+y-x^2y+x^4y-x^6y}{8} \longmapsto (0,0,0,0,\begin{bmatrix}0&0\\1&0\end{bmatrix},0)$$
$$\frac{1-x^2+x^4-x^6+xy-x^3y+x^5y-x^7y}{8} \longmapsto (0,0,0,0,\begin{bmatrix}0&0\\0&1\end{bmatrix},0)$$

Thus, every element of $\mathbb{Z}^4 \oplus M_2(\mathbb{Z}) \oplus \overline{H}(\mathbb{Q}([\sqrt{2}])$ multiplied by

$$T(16) = (16,16,16,16,\begin{bmatrix}16&0\\0&16\end{bmatrix},16)$$

is an element of $T(\mathbb{Z}[Q_{16}])$. By hypothesis $\overline{P}X = \overline{K}$ has a solution over $\mathbb{Z}^4 \oplus M_2(\mathbb{Z}) \oplus \overline{H}(\mathbb{Q}([\sqrt{2}])$, that is, there exists a column vector X_0 with coordinates in $\mathbb{Z}^4 \oplus M_2(\mathbb{Z}) \oplus \overline{H}(\mathbb{Q}([\sqrt{2}])$ such that $\overline{P}X_0 = \overline{K}$. As $T(16)$ belongs to the center of $\mathbb{Z}^4 \oplus M_2(\mathbb{Z}) \oplus \overline{H}(\mathbb{Q}([\sqrt{2}])$, then $\overline{P}(T(16)X_0) = T(16)\overline{K}$. Let X_1 be a column vector with coordinates in $\mathbb{Z}[Q_{16}]$ such that $T(X_1) = T(16)X_0$, then $PX_1 = 16K$. ∎

Let $M(q)$ be the 16×16 matrix given by multiplication in $\mathbb{Z}[Q_{16}]$, that is,

$$M(q) = \begin{bmatrix} J & L & R & S \\ L & J & S & R \\ S & R & J & L \\ R & S & L & J \end{bmatrix},$$

where

$$J = \begin{bmatrix} q_0 & q_7 & q_6 & q_5 \\ q_1 & q_0 & q_7 & q_6 \\ q_2 & q_1 & q_0 & q_7 \\ q_3 & q_2 & q_1 & q_0 \end{bmatrix} \qquad L = \begin{bmatrix} q_4 & q_3 & q_2 & q_1 \\ q_5 & q_4 & q_3 & q_2 \\ q_6 & q_5 & q_4 & q_3 \\ q_7 & q_6 & q_5 & q_4 \end{bmatrix}$$

and

$$R = \begin{bmatrix} q_{12} & q_{13} & q_{14} & q_{15} \\ q_{13} & q_{14} & q_{15} & q_8 \\ q_{14} & q_{15} & q_8 & q_9 \\ q_{15} & q_8 & q_9 & q_{10} \end{bmatrix} \qquad S = \begin{bmatrix} q_8 & q_9 & q_{10} & q_{11} \\ q_9 & q_{10} & q_{11} & q_{12} \\ q_{10} & q_{11} & q_{12} & q_{13} \\ q_{11} & q_{12} & q_{13} & q_{14} \end{bmatrix}$$

The linear system $PX = K$ over $\mathbb{Z}[Q_{16}]$ is equivalent to a linear system $M(P)X = V(K)$ over $\mathbb{Z}$, where $M(P) = [M(p_{ij})]$ is a block matrix and $V(K) = [v(k_1)\cdots v(k_m)]^t$ with $v(q) = \begin{bmatrix} q_0 & \cdots & q_{15}\end{bmatrix}^t$ is a column vector with coordinates in $\mathbb{Z}$.

Lemma 2.3. *Let $AX = K$ be a linear system over $\mathbb{Z}$, where A is an $m \times n$ matrix with $m \le n$. If the system $AX = 2^rK$ has a solution over $\mathbb{Z}$, where $r \ge 1$ is an integer and the matrix A has at least one odd $m \times m$-minor, then $AX = K$ has solution over $\mathbb{Z}$.*

Proof: By hypothesis, there exists an $m \times m$-submatrix $B = [b_{ij}]$ of A with an odd $\det B$. Therefore, the system $AX = (\det B)K$ has a solution over $\mathbb{Z}$. Suppose that $AX_0 = 2^rK$, $AX_1 = (\det B)K$ and $s \cdot 2^r + p \cdot (\det B) = 1$ with $s, p \in \mathbb{Z}$. Then, $A(sX_0 + pX_1) = s \cdot 2^rK + p \cdot (\det B)K = (s \cdot 2^r + p \cdot \det B)K = K$. ∎

3 Parity between $\det M(q)$ and $\varepsilon(q)$

Let $[T]$ be the matrix of T, then

$$[T]M(q)[T]^{-1} = \left[\begin{array}{cccccccccccccccc}
\varepsilon(q) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & q^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & n_1 & 0 & n_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & n_1 & 0 & n_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & n_3 & 0 & n_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & n_3 & 0 & n_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & Q(q) & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & &
\end{array}\right]$$

where

$$Q(q) = \begin{bmatrix}
a & 2b & -c & -2d & -e & -2f & -g & -2h \\
b & a & -d & -c & -f & -e & -h & -g \\
c & 2d & a & 2b & -g & -2h & e & 2f \\
d & c & b & a & -h & -g & f & e \\
e & 2f & g & 2h & a & 2b & -c & -2d \\
f & e & h & g & b & a & -d & -c \\
g & 2h & -e & -2f & c & 2d & a & 2b \\
h & g & -f & -e & d & c & b & a
\end{bmatrix}$$

Theorem 3.1. *If $A = [A_{ij}]$ is a square block matrix, where the A_{ij} are mutually commutative $m \times m$ matrices, and B is the $m \times m$ matrix obtained by taking the determinant of A with the A_{ij} as elements, then* $\det A = \det B$.

Proof: See theorem 1 of [7]. ■

Let $q = a + b\sqrt{2} + ci + d\sqrt{2}i + ej + f\sqrt{2}j + gk + h\sqrt{2}k$ be an element of $\overline{H}(\mathbb{Q}[\sqrt{2}])$ and $s(q) = a^2 + c^2 + e^2 + g^2 + 2b^2 + 2d^2 + 2f^2 + 2h^2$.

Lemma 3.2. $\sqrt{\det Q(q)} = -8(ba + dc + fe + hg)^2 + s(q)^2$

Proof: Consider the mutually commutative matrices,

$$A = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}, \quad B = \begin{bmatrix} c & 2d \\ d & c \end{bmatrix}, \quad C = \begin{bmatrix} e & 2f \\ f & e \end{bmatrix}, \quad D = \begin{bmatrix} g & 2h \\ h & g \end{bmatrix}.$$

Therefore

$$Q(q) = \begin{bmatrix} A & -B & -C & -D \\ B & A & -D & C \\ C & D & A & -B \\ D & -C & B & A \end{bmatrix}$$

and applying the theorem 3.1, we have

$$\det Q(q) = \det(A^2 + B^2 + C^2 + D^2)^2 = (\det(A^2 + B^2 + C^2 + D^2))^2.$$

If $S = A^2 + B^2 + C^2 + D^2$, then

$$S = \begin{bmatrix} s(q) & 4(ba + dc + fe + hg) \\ 2(ba + dc + fe + hg) & s(q) \end{bmatrix}$$

Thus, $\sqrt{\det Q(q)} = \sqrt{\det S} = -8(ba + dc + fe + hg)^2 + s(q)^2$. ■

Theorem 3.3. *The numbers* $\det Q(q)$ *and* $\varepsilon(q)$ *have the same parity.*

Proof: The term $-8(ba + dc + fe + hg)^2$ is always even, then the parity $\det Q(q)$ is equal to parity $s(q) = a^2 + c^2 + e^2 + g^2 + 2b^2 + 2d^2 + 2f^2 + 2h^2$. The numbers $\varepsilon(q)$ and $a + c + e + g + 2b + 2f$ have the same parity. We have the following possibilities:

i) $\varepsilon(q)$ odd, $2b$ and $2f$ even.

In this case, $a + c + e + g$ is odd, $2b = 2n_1$, $2f = 2n_2$, $2d = 2n_3$ and $2h = 2n_4$ with $n_1, n_2, n_3, n_4 \in \mathbb{Z}$. Therefore, $4b^2 = 4k_1$, $4f^2 = 4k_2$, $4d^2 = 4k_3$, $4h^2 = 4k_4$ with $k_1, k_2, k_3, k_4 \in \mathbb{Z}$ and $2b^2 + 2d^2 + 2f^2 + 2h^2 = 2(k_1 + k_2 + k_3 + k_4)$ is even. Because $a^2 + c^2 + e^2 + g^2$ is odd, then $s(q)$ is odd.

ii) $\varepsilon(q)$ odd, $2b$ and $2f$ odd.

In this case, $a + c + e + g$ is odd, $2b = 2n_1 + 1$, $2f = 2n_2 + 1$, $2d = 2n_3 + 1$ and $2h = 2n_4 + 1$ with $n_1, n_2, n_3, n_4 \in \mathbb{Z}$. Thus, $4b^2 = 4k_1 + 1$, $4f^2 = 4k_2 + 1$, $4d^2 = 4k_3 + 1$, $4h^2 = 4k_4 + 1$ with $k_1, k_2, k_3, k_4 \in \mathbb{Z}$ and

$$2b^2 + 2d^2 + 2f^2 + 2h^2 = 2(k_1 + k_2 + k_3 + k_4) + 2$$

As $a^2 + c^2 + e^2 + g^2$ is odd, then $s(q)$ is odd.

iii) $\varepsilon(q)$ odd, $2b$ odd and $2f$ even.

In this case, $a + c + e + g$ is even, $2b = 2n_1 + 1$, $2f = 2n_2$, $2d = 2n_3 + 1$ and $2h = 2n_4$ with $n_1, n_2, n_3, n_4 \in \mathbb{Z}$. So, $4b^2 = 4k_1 + 1$, $4f^2 = 4k_2$, $4d^2 = 4k_3 + 1$ and $4h^2 = 4k_4$, com $k_1, k_2, k_3, k_4 \in \mathbb{Z}$ and

$$2b^2 + 2d^2 + 2f^2 + 2h^2 = 2(k_1 + k_2 + k_3 + k_4) + 1$$

As $a^2 + c^2 + e^2 + g^2$ is even, then $s(q)$ is odd.

If $\varepsilon(q)$ is even, the argument is analogous. ■

In particular, it was demonstrated that the numbers $\varepsilon(q)$ and $s(q)$ have the same parity.

Theorem 3.4. *The numbers* $\varepsilon(q)$ *and* $\det N(q)$ *have the same parity, where*

$$N(q) = \begin{bmatrix} n_1 & 0 & n_2 & 0 \\ 0 & n_1 & 0 & n_2 \\ n_3 & 0 & n_4 & 0 \\ 0 & n_3 & 0 & n_4 \end{bmatrix}$$

Proof: Because the matrix $N(q)$ is formed by commutative blocks, by theorem 3.1

$$\det N(q) = (n_1 n_4 - n_3 n_2)^2.$$

Notice that $\overline{n}_1 = \overline{n}_4$ and $\overline{n}_3 = \overline{n}_2$. We have two cases to consider:

i) $\varepsilon(q)$ odd.
   In this case, we have $\overline{n}_1 \neq \overline{n}_3$, then $\det N(q)$ is odd.

ii) $\varepsilon(q)$ even.
   In this case, we have $\overline{n}_1 = \overline{n}_3$, then $\det N(q)$ is even. ■

Theorem 3.5. *The numbers* $\det M(q)$ *and* $\varepsilon(q)$ *have the same parity.*

Proof: The determinant of the matrix $M(q)$ is equal to the determinant of the matrix $[T]M(q)[T]^{-1}$. Therefore,

$$\det M(q) = \varepsilon(q) \cdot q^1 \cdot q^2 \cdot q^3 \cdot \det N(q) \cdot \det Q(q).$$

Now apply theorem 3.3 and theorem 3.4. ■

4 Linear systems over $M_2(\mathbb{Z})$

Let $\mathbb{Z}$ be the ring of integer numbers and $M_2(\mathbb{Z})$ the ring of 2×2 matrices with entries in $\mathbb{Z}$. Consider the matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, K = \begin{bmatrix} x_0 & y_0 \\ z_0 & t_0 \end{bmatrix}$$

in $M_2(\mathbb{Z})$. The system $AX = K$ over $M_2(\mathbb{Z})$ is equivalent to the system $N(A)X = v(K)^t$ over $\mathbb{Z}$, where

$$N(A) = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix} \text{ and } v(K) = \begin{bmatrix} x_0 & y_0 & z_0 & t_0 \end{bmatrix}.$$

Let $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ be the ring of integers modulo 2 and $E_4(\mathbb{Z}_2)$ be the set of all 4×4 matrices over $\mathbb{Z}_2$ of the form:

$$E = \begin{bmatrix} a & \bar{0} & b & \bar{0} \\ \bar{0} & a & \bar{0} & b \\ b & \bar{0} & a & \bar{0} \\ \bar{0} & b & \bar{0} & a \end{bmatrix}$$

The set $E_4(\mathbb{Z}_2)$ is a commutative subring with identity of the matrix ring $M_4(\mathbb{Z}_2)$. Since $\det(E) = a + b$ for any $E \in E_4(\mathbb{Z}_2)$, we have

$$\det(E_1 + E_2) = \det(E_1) + \det(E_2)$$

for all $E_1, E_2 \in E_4(\mathbb{Z}_2)$.

Consider $N_2(\mathbb{Z})$ the set of matrices $A = [a_{ij}] \in M_2(\mathbb{Z})$ such that $\overline{a_{11}} = \overline{a_{22}}$ and $\overline{a_{21}} = \overline{a_{12}}$. Observe that, if $A \in N_2(\mathbb{Z})$, then $\phi(N(A)) \in E_4(\mathbb{Z}_2)$.

Lemma 4.1. *Let $A = [A_{ij}]$ be an $n \times n$ matrix over $E_4(\mathbb{Z}_2)$, i.e, a block matrix. If A is regarded as a matrix over $\mathbb{Z}_2$, then its determinant will be*

$$\det(A) = \sum_{\sigma \in S_n} \det(A_{1\sigma(1)}) \cdots \det(A_{n\sigma(n)}),$$

where S_n is the symmetric group on n symbols.

Proof: As blocks A_{ij} of A are commutative pairwise, we can apply theorem 3.1. Thus,

$$\det(A) = \det\left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) A_{1\sigma(1)} \cdots A_{n\sigma(n)}\right).$$

Since the determinant on $E_4(\mathbb{Z}_2)$ is additive, the result follows. ■

Lemma 4.2. *Let $B = [B_{ij}]$ be an $m \times m$ matrix with entries in $N_2(\mathbb{Z})$. If $D(B) = [\det(B_{ij})]$ and $N(B) = [N(B_{ij})]$, then $\overline{\det D(B)} = \overline{\det N(B)}$.*

Proof: As $\phi(N(B_{ij})) \in E_4(\mathbb{Z}_2)$, from lemma 4.1

$$\begin{aligned} \det(\phi N(B)) &= \sum_{\sigma \in S_m} \det(\phi(N(B_{1\sigma(1)}))) \cdots \det(\phi(N(B_{m\sigma(m)}))) \\ &= \sum_{\sigma \in S_m} \phi(\det(N(B_{1\sigma(1)}))) \cdots \phi(\det(N(B_{m\sigma(m)}))) \end{aligned}$$

Since $\det N(B_{ij}) = (\det B_{ij})^2$, then $\overline{\det(N(B_{ij}))} = \overline{\det B_{ij}}$ and

$$\begin{aligned} \phi(\det N(B)) &= \sum_{\sigma \in S_m} \phi(\det(B_{1\sigma(1)})) \cdots \phi(\det(B_{m\sigma(m)})) \\ &= \phi\left(\sum_{\sigma \in S_m} \det(B_{1\sigma(1)}) \cdots \det(B_{m\sigma(m)})\right) \end{aligned}$$

Hence, $\overline{\det N(B)} = \overline{\det D(B)}$. ■

Theorem 4.3. *Consider the linear system $AX = K$ over $M_2(\mathbb{Z})$, where $A = [A_{ij}]$ is an $m \times n$ matrix with $m \leq n$ and $A_{ij} \in N_2(\mathbb{Z})$. If the system $AX = K \cdot \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ has solution over $M_2(\mathbb{Z})$ and the matrix $D(A) = [\det(A_{ij})]$ has at least one odd $m \times m$-minor, then the system $AX = K$ has a solution over $M_2(\mathbb{Z})$.*

Proof: Certainly, the system $AX = K$ over $M_2(\mathbb{Z})$ is equivalent to the system $N(A)X = V(K)$ over $\mathbb{Z}$, where $N(A) = [N(A_{ij})]$ is a block matrix and $V(K) = [v(k_1) \cdots v(k_m)]^t$ is a column vector. Multiplying the equation $AX = K \cdot \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ by $\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$ from the right, we have

$$AX \cdot \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} = K \cdot \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Hence, a solution of $AX = K \cdot \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ over $M_2(\mathbb{Z})$ yields a solution of the system $N(A)X = 2V(K)$ over $\mathbb{Z}$. If we find a $2m \times 2m$ odd minor of $N(A)$, we can apply lemma 2.3 and conclude that $N(A)X = V(K)$ has solution over $\mathbb{Z}$. By hypothesis, there exists an $m \times m$-submatrix $B = [B_{ij}]$ of A with an odd $\det D(B)$. From lemma 4.2, $\det N(B)$ is odd. ■

5 Quaternionic determinants

In the case studied in [2] the 4×4 matrix equivalent to $Q(q)$ had entries in $\mathbb{Z}$ and the techniques of section 4 were sufficient to obtain the expected results. Here $Q(q)$ has rational entries, being necessary the introduction of new concepts such as quaternionic determinants. Let $\mathbb{R}$ be the field of real numbers, we can write $A \in M_n(H(\mathbb{R}))$ uniquely as $A = A_0 + A_1 i + A_2 j + A_3 k$ where A_0, A_1, A_2 and A_3 are real $n \times n$ matrices. Consider the homomorphism $\mu : M_n(H(\mathbb{R})) \to M_{4n}(\mathbb{R})$ given by

$$\mu(A) = \begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix}$$

According to [5], the Study's determinant is $\mathrm{Sdet}(A) = \sqrt{\det_{\mathbb{R}} \mu(A)}$.
For any $a = a_0 + a_1 i + a_2 j + a_3 k \in H(\mathbb{R})$ define

$$\psi(a) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}.$$

Given an matrix $A = [a_{ij}] \in M_n(H(\mathbb{R}))$, the matrix $\psi(A) = [\psi(a_{ij})] \in M_{4n}(\mathbb{R})$ can be transformed by means of exchanging lines, columns and signs in the matrix $\mu(A)$. Therefore, $\mathrm{Sdet}(A) = \sqrt{\det_{\mathbb{R}} \psi(A)}$.

The conjugate of a quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$ is defined to be $\overline{q} = q_0 - q_1 i - q_2 j - q_3 k$ so that $\overline{p+q} = \overline{p} + \overline{q}$, $\overline{p \cdot q} = \overline{q} \cdot \overline{p}$ for all $p, q \in H(\mathbb{R})$. The norm of q is defined by $\eta(q) = q \cdot \overline{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2$ such that $\eta(p \cdot q) = \eta(p) \cdot \eta(q)$ and the trace of q is $t(q) = q + \overline{q}$.
Given a matrix $A \in M_n(H(\mathbb{R}))$ define $A^* = \overline{A}^t$, if $A = A^*$, we say that A is Hermitian. For a Hermitian matrix, the Moore's determinant is defined by specifying a certain ordering of the factors in the $n!$ terms in the sum. Let σ be a permutation of n. Write it as a product of disjoint cycles. Permute each cycle cyclically until the smallest number in the cycle is in front. Then sort the cycles in decreasing order according to the first number of each cycle. In other words, write

$$\sigma = (n_{11} \cdots n_{1l_1})(n_{21} \cdots n_{2l_2}) \cdots (n_{r1} \cdots n_{rl_r}),$$

where for each i, we have $n_{i1} < n_{ij}$ for all $j > 1$, and $n_{11} > n_{21} > \cdots > n_{r1}$. Then we define

$$\mathrm{Mdet}(A) = \sum_{\sigma \in S_n} |\sigma| a_{n_{11} n_{12}} \cdots a_{n_{1l_1} n_{11}} a_{n_{21} n_{22}} \cdots a_{n_r l_r n_{r1}}$$

Observe that, for any matrix $A \in M_n(H(\mathbb{R}))$, the matrix AA^* is Hermitian.

Theorem 5.1. *For any quaternionic matrix A, we have*

$$\mathrm{Sdet}A = \mathrm{Mdet}(AA^*).$$

Proof: See theorem 10 of [5]. ∎

If A is Hermitian, then $\mathrm{Mdet}(A)$ is a real number. In particular, if A is Hermitian and $A \in M_n(\overline{H}(\mathbb{Q}[\sqrt{2}]))$, then $\mathrm{Mdet}(A) \in \overline{H}(\mathbb{Q}[\sqrt{2}])$ and $\mathrm{Mdet}(A)$ is a real number. Therefore, $\mathrm{Mdet}(A) = a + b\sqrt{2}$, with $a, b \in \mathbb{Z}$, because $b \in \mathbb{Q}$ with $2b$ even. Notice that, if $q \in \overline{H}(\mathbb{Q}[\sqrt{2}])$, then

$$\begin{aligned} \eta(q) &= (a+b\sqrt{2})^2 + (c+d\sqrt{2})^2 + (e+f\sqrt{2})^2 + (g+h\sqrt{2})^2 \\ &= (a^2+c^2+e^2+g^2+2b^2+2d^2+2f^2+2h^2) + 2\sqrt{2}(ab+cd+ef+gh) \\ &= s(q) + 2(ab+cd+ef+gh)\sqrt{2} \end{aligned}$$

Therefore, $\eta(q) = s(q) + r\sqrt{2}$ with $s(q), r \in \mathbb{Z}$.

Lemma 5.2. *For any $q_1, q_2 \in \overline{H}(\mathbb{Q}[\sqrt{2}])$ we have $\overline{s(q_1+q_2)} = \overline{s(q_1)+s(q_2)}$ and $\overline{s(q_1 \cdot q_2)} = \overline{s(q_1) \cdot s(q_2)}$.*

Proof: Notice that,

$$\begin{aligned} \eta(q_1+q_2) &= (q_1+q_2) \cdot \overline{(q_1+q_2)} = q_1\overline{q_1} + q_1\overline{q_2} + q_2\overline{q_1} + q_2\overline{q_2} \\ &= \eta(q_1) + \eta(q_2) + q_1\overline{q_2} + \overline{q_1\overline{q_2}} = \eta(q_1) + \eta(q_2) + t(q_1\overline{q_2}) \\ &= s(q_1) + r_1\sqrt{2} + s(q_2) + r_2\sqrt{2} + (2m + n\sqrt{2}) \\ &= s(q_1) + s(q_2) + 2m + (r_1 + r_2 + n)\sqrt{2} \end{aligned}$$

Hence, $s(q_1+q_2) = s(q_1) + s(q_2) + 2m$, that is, $\overline{s(q_1+q_2)} = \overline{s(q_1)+s(q_2)}$.

Moreover,

$$\begin{aligned} \eta(q_1 \cdot q_2) &= \eta(q_1) \cdot \eta(q_2) \\ &= (s(q_1) + r_1\sqrt{2}) \cdot (s(q_2) + r_2\sqrt{2}) \\ &= s(q_1) \cdot s(q_2) + 2r_1r_2 + (s(q_1) + s(q_2))\sqrt{2} \end{aligned}$$

Thus, $s(q_1 \cdot q_2) = s(q_1) \cdot s(q_2) + 2r_1r_2$, that is, $\overline{s(q_1 \cdot q_2)} = \overline{s(q_1) \cdot s(q_2)}$. ∎

Lemma 5.3. *If* $A = \begin{bmatrix} q_1 & q_2 \\ q_3 & q_4 \end{bmatrix} \in M_2(\overline{H}(\mathbb{Q}[\sqrt{2}]))$ *and* $s(A) = \begin{bmatrix} s(q_1) & s(q_2) \\ s(q_3) & s(q_4) \end{bmatrix}$, *then* $\mathrm{Mdet}(AA^*) = S + R\sqrt{2}$ *with* $R, S \in \mathbb{Z}$ *and* $\overline{S} = \overline{\det s(A)}$.

Proof: Consider the matrix

$$A \cdot A^* = \begin{bmatrix} q_1 & q_2 \\ q_3 & q_4 \end{bmatrix} \cdot \begin{bmatrix} \overline{q}_1 & \overline{q}_3 \\ \overline{q}_2 & \overline{q}_4 \end{bmatrix} = \begin{bmatrix} q_1\overline{q}_1 + q_2\overline{q}_2 & q_1\overline{q}_3 + q_2\overline{q}_4 \\ q_3\overline{q}_1 + q_4\overline{q}_2 & q_3\overline{q}_3 + q_4\overline{q}_4 \end{bmatrix}.$$

The Moore's determinant is

$$\begin{aligned} \mathrm{Mdet}(AA^*) &= (q_1\overline{q}_1 + q_2\overline{q}_2)(q_3\overline{q}_3 + q_4\overline{q}_4) - (q_3\overline{q}_1 + q_4\overline{q}_2)(q_1\overline{q}_3 + q_2\overline{q}_4) \\ &= q_1\overline{q}_1q_3\overline{q}_3 + q_1\overline{q}_1q_4\overline{q}_4 + q_2\overline{q}_2q_3\overline{q}_3 + q_2\overline{q}_2q_4\overline{q}_4 \\ &- q_3\overline{q}_1q_1\overline{q}_3 - q_3\overline{q}_1q_2\overline{q}_4 - q_4\overline{q}_2q_1\overline{q}_3 - q_4\overline{q}_2q_2\overline{q}_4 \end{aligned}$$

Because $q_1\overline{q}_1q_3\overline{q}_3 = q_3\overline{q}_1q_1\overline{q}_3$, $q_2\overline{q}_2q_4\overline{q}_4 = q_4\overline{q}_2q_2\overline{q}_4$, $q_3\overline{q}_1q_2\overline{q}_4 = \overline{q_4\overline{q}_2q_1\overline{q}_3}$, then

$$\begin{aligned} \mathrm{Mdet}(AA^*) &= q_1\bar{q}_1q_4\bar{q}_4 + q_2\bar{q}_2q_3\bar{q}_3 - t(q_3\bar{q}_1q_2\bar{q}_4) \\ &= (s(q_1) + r_1\sqrt{2}) \cdot (s(q_4) + r_2\sqrt{2}) + (s(q_3) + r_3\sqrt{2}) \cdot (s(q_2) + r_4\sqrt{2}) \\ &+ (2k_1 + r_5\sqrt{2}) \\ &= s(q_1) \cdot s(q_4) + s(q_3) \cdot s(q_2) + 2k_2 + R\sqrt{2} \end{aligned}$$

If $S = s(q_1) \cdot s(q_4) + s(q_3) \cdot s(q_2) + 2k_2$, then $\mathrm{Mdet}(AA^*) = S + R\sqrt{2}$ with $\overline{\det s(A)} = \overline{S}$. ∎

The technique used to prove the lemma 5.3 requires the analysis of an excessively large number of combinations to n greater than two, so we will use Ivan Kyrchei's ideas in [8] to extend it. In [8], for an $A \in M_n(H(\mathbb{R}))$ is defined the *ith* row determinant, denoted by $\mathrm{rdet_i A}$ and *jth* column determinant, denoted by $\mathrm{cdet_i A}$. For the purpose of this work it is only necessary to define the *ith* row determinant. In [8], definition 2.4, the *ith* row determinant of $A \in M_n(H(\mathbb{R}))$ is defined as the alternating sum of $n!$ products of entries of A, during which the index permutation of every product is written by the direct product of disjoint cycles. That is

$$\mathrm{rdet_i A} = \sum_{\sigma \in S_n} (-1)^{n-r} a_{i\, i_{k_1}} a_{i_{k_1} i_{k_1+1}} \cdots a_{i_{k_1+l_1} i} \cdots a_{i_{k_r} i_{k_r+1}} \cdots a_{i_{k_r+l_r} i_{k_r}}$$

The cycle notation of the permutation σ is written as follows

$$\sigma = (i\, i_{k_1} i_{k_1+1} \cdots i_{k_1+l_1})(i_{k_2} i_{k_2+1} \cdots i_{k_2+l_2}) \cdots (i_{k_r} i_{k_r+1} \cdots i_{k_r+l_r})$$

Here the index i opens the first cycle from the left and the other cycles satisfy the following conditions

$$i_{k_2} < i_{k_3} < \cdots < i_{k_r} \text{ and } i_{k_t} < i_{k_{t+s}}, \text{ for } s = 1, \ldots, l_t \text{ and } t = 2, \ldots, r$$

Theorem 5.4. *If $A \in M_n(H(\mathbb{R}))$ is a Hermitian matrix, then*

$$\mathrm{rdet}_1 A = \cdots = \mathrm{rdet}_n A = \mathrm{cdet}_1 A = \cdots = \mathrm{cdet}_n A \in \mathbb{R}$$

Proof: See theorem 3.1 of [8]. ■

Since all column and row determinants of a Hermitian matrix over $H(\mathbb{R})$ are equal, in [8] remark 3.1, put

$$\det(A) = \mathrm{rdet}_i A = \mathrm{cdet}_i A \text{ for } i = 1, \ldots, n.$$

Given a matrix $A = [a_{ij}] \in M_n(\overline{H}(\mathbb{Q}[\sqrt{2}]))$, we define $A_2 = [a_{ij}^2]$. The next result was inspired by theorem 3.1 of [8].

Theorem 5.5. *For any matrix $A = [a_{ij}] \in M_n(\overline{H}(\mathbb{Q}[\sqrt{2}]))$, if*

$$\det(AA^*) = S_1 + R_1\sqrt{2} \text{ and } \det(A_2 A_2^*) = S_2 + R_2\sqrt{2},$$

with S_1, R_1, S_2, $R_2 \in \mathbb{Z}$, then $\overline{S}_1 = \overline{S}_2$.

Proof: Initially suppose that the matrix $A = [a_{ij}]$ is Hermitian, then $a_{ii} \in \mathbb{R}$ and $a_{ij} = \overline{a_{ij}}$. We divide the monomials of some $\mathrm{rdet}_i A$ into two subsets. If the indices of the coefficients of a monomial form a permutation as product of disjoint cycles of lengths 1 and 2 then we include this monomial in the first subset, the other monomials belong to the second subset. If the indices of the coefficients form a disjoint cycle of length 1, then these coefficients are entries of the principal diagonal of the Hermitian matrix A, hence they are real number of the form $s_1 + r_1\sqrt{2}$ with s_1, $r_1 \in \mathbb{Z}$. If the indices of the coefficients form a disjoint cycle of length 2, then these elements are conjugated, $a_{i_k i_{k+1}} = \overline{a_{i_{k+1} i_k}}$, and their product takes on a real value number as well,

$$a_{i_k i_{k+1}} \cdot a_{i_{k+1} i_k} = a_{i_{k+1} i_k} \overline{a_{i_{k+1} i_k}} = \eta(a_{i_{k+1} i_k}) = s(a_{i_{k+1} i_k}) + r\sqrt{2}$$

Let d be a monomial from the second subset and assume that the indices of its coefficients form a permutation as a product of r disjoint cycles, by the proof of the theorem 3.1 of [8], there exist another $2^p - 1$ monomials, where $p = r - \rho$ and ρ is the number of disjoint cycles of length 1 and 2, such that the sum of these $2^p - 1$ monomials and d is given by

$$(-1)^{n-r} \alpha t(h_{v_1}) \cdots t(h_{v_p})$$

Here α is the product of the coefficients whose indices form disjoint cycles of lengths 1 and 2, $v_k \in \{1, \dots, r\}$ and $k = 1, \dots p$. As $t(q) = 2m + n\sqrt{2}$ with $m, n \in \mathbb{Z}$, if $\det A = s_2 + r_2\sqrt{2}$, then the parity of s_1 is determined by the monomials of the first subset.
Now we consider the Hermitian matrix $C = AA^*$ and the Hermitian matrix $D = A_2A_2^*$. Note that,

$$c_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj}^* \text{ and } d_{ij} = \sum_{k=1}^{n} a_{ik}^2(a_{kj}^*)^2.$$

From lemma 5.2, $\overline{s(c_{ij})} = \overline{s(d_{ij})}$. Let us look at the elements of the principal diagonal,

$$c_{ii} = \sum_{k=1}^{n} a_{ik}a_{ki}^* = \sum_{k=1}^{n} a_{ik}\overline{a_{ik}} = \sum_{k=1}^{n} \eta(a_{ik}) = \sum_{k=1}^{n} s(a_{ik}) + r_i\sqrt{2}$$

and

$$d_{ii} = \sum_{k=1}^{n} a_{ik}^2(a_{ki}^*)^2 = \sum_{k=1}^{n} a_{ik}^2\overline{a_{ik}^2} = \sum_{k=1}^{n} \eta(a_{ik}^2) = \sum_{k=1}^{n} s(a_{ik}^2) + r_i'\sqrt{2},$$

since $\overline{s(c_{ii})} = \overline{s(d_{ii})}$, if $c_{11}\cdots c_{nn} = s_3 + r_3\sqrt{2}$ and $d_{11}\cdots d_{nn} = s_4 + r_4\sqrt{2}$, then $\overline{s_3} = \overline{s_4}$. Now, let's consider the monomials whose indices of coefficients form a disjoint cycle of length 2, in this case these elements are conjugated, $c_{i_ki_{k+1}} = \overline{c_{i_{k+1}i_k}}$ and $d_{i_ki_{k+1}} = \overline{d_{i_{k+1}i_k}}$, then

$$c_{i_ki_{k+1}} \cdot c_{i_{k+1}i_k} = \overline{c_{i_{k+1}i_k}} \cdot c_{i_{k+1}i_k} = \eta(c_{i_{k+1}i_k}) = s(c_{i_{k+1}i_k}) + r_5\sqrt{2}$$

and

$$d_{i_ki_{k+1}} \cdot d_{i_{k+1}i_k} = \overline{d_{i_{k+1}i_k}} \cdot d_{i_{k+1}i_k} = \eta(d_{i_{k+1}i_k}) = s(d_{i_{k+1}i_k}) + r_6\sqrt{2},$$

with $\overline{s(c_{i_{k+1}i_k})} = \overline{s(d_{i_{k+1}i_k})}$.
Therefore, if $\det(AA^*) = S_1 + R_1\sqrt{2}$ and $\det(A_2A_2^*) = S_2 + R_2\sqrt{2}$, then $\overline{S_1} = \overline{S_2}$. ∎

In [8], definition 7.2, for any $A \in M_n(H(\mathbb{R}))$, the determinant of its corresponding Hermitian matrix is called its double determinant, that is,

$$\text{ddet}A = \det(A^*A) = \det(AA^*).$$

Corollary 5.6. *For any matrix $A = [a_{ij}] \in M_n(\overline{H}(\mathbb{Q}[\sqrt{2}]))$, if*

$$\text{Mdet}(AA^*) = S_1 + R_1\sqrt{2} \textit{ and } \text{Mdet}(A_2A_2^*) = S_2 + R_2\sqrt{2},$$

with $S_1, R_1, S_2, R_2 \in \mathbb{Z}$, then $\overline{S}_1 = \overline{S}_2$.

Proof: By remark 7.2 of [8], we have $\text{ddet}A = \text{Mdet}(A^*A)$. ∎

6 Linear systems over $\overline{H}(\mathbb{Q}[\sqrt{2}])$

Lemma 6.1. *If $q \in \overline{H}(\mathbb{Q}[\sqrt{2}])$, then $q^2 \in H(\mathbb{Z}[\sqrt{2}])$.*

Proof: Let $q = (a + b\sqrt{2}) + (c + d\sqrt{2})i + (e + f\sqrt{2})j + (g + h\sqrt{2})k$ be an element of $q \in \overline{H}(\mathbb{Q}[\sqrt{2}])$, we have

$$q^2 = (A + B\sqrt{2}) + (C + D\sqrt{2})i + (E + F\sqrt{2})j + (G + H\sqrt{2})k,$$

where

i) $(a^2 - c^2 - e^2 - g^2 + 2(b^2 - d^2 - f^2 - h^2)) + 2(ab - cd - ef - gh)\sqrt{2} = A + B\sqrt{2}$, with $A \in \mathbb{Z}$ and $B \in \mathbb{Z}$

ii) $(2ac + 2(2bd)) + 2(-eh)\sqrt{2} = C + D\sqrt{2}$, with $C \in \mathbb{Z}$ and $D \in \mathbb{Z}$

iii) $(2ae + 2(2bf)) + 2(af + be)\sqrt{2} = E + F\sqrt{2}$, with $E \in \mathbb{Z}$ and $F \in \mathbb{Z}$

iv) $2ag + 2(2(bh + df))) + 2(ah + bg + cf + ed)\sqrt{2} = G + H\sqrt{2}$, with $G \in \mathbb{Z}$ and $H \in \mathbb{Z}$. ■

Consider the ring homomorphism $\varphi : \mathbb{Q}[\sqrt{2}] \to M_2(\mathbb{Q})$ given by

$$\varphi(a + b\sqrt{2}) = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}.$$

Lemma 6.2. *If $A = [a_{ij}]$ is an element of $M_n(\overline{H}(\mathbb{Q}[\sqrt{2}]))$, then*

$$\det Q(A) = (\det(\varphi(\mathrm{Mdet}(AA^*))))^2,$$

where $Q(A) = [Q(a_{ij})]$.

Proof. The matrix $\psi(A)$ is an element of $M_{4n}(\mathbb{Q}[\sqrt{2}])$. Notice that $\varphi(\psi(A)) = Q(A) \in M_{8n}(\mathbb{Q})$, then $\det Q(A) = \det(\varphi(\det_{\mathbb{R}} \psi(A)))$. Because $\det_{\mathbb{R}} \psi(A) = \mathrm{Sdet}^2(A)$ and $\mathrm{Mdet}(AA^*) = \mathrm{Sdet}(A)$, we have

$$\det Q(A) = (\det(\varphi(\mathrm{Mdet}(AA^*))))^2.$$ ■

Lemma 6.3. *If $q_1, q_2 \in H(\mathbb{Z}[\sqrt{2}])$, then the matrices $\phi(Q(q_1))$ and $\phi(Q(q_2))$ are mutually commutative and* $\det(\phi(Q(q_1)) + \phi(Q(q_2))) = \det\phi(Q(q_1)) + \det\phi(Q(q_2))$.

Proof: The matrices

$$Q(q_i) = \begin{bmatrix} A_i & -B_i & -C_i & -D_i \\ B_i & A_i & -D_i & C_i \\ C_i & D_i & A_i & -B_i \\ D_i & -C_i & B_i & A_i \end{bmatrix},$$

where A_i, B_i, C_i, $D_i \in M_2(\mathbb{Z})$ and are mutually commutative for $i = 1, 2$. Hence

$$\phi(Q(q_i)) = \begin{bmatrix} \phi(A_i) & \phi(B_i) & \phi(C_i) & \phi(D_i) \\ \phi(B_i) & \phi(A_i) & \phi(D_i) & \phi(C_i) \\ \phi(C_i) & \phi(D_i) & \phi(A_i) & \phi(B_i) \\ \phi(D_i) & \phi(C_i) & \phi(B_i) & \phi(A_i) \end{bmatrix},$$

and $\phi(Q(q_1)) \cdot \phi(Q(q_2)) = \phi(Q(q_2)) \cdot \phi(Q(q_1))$. Moreover

$$\det(\phi(Q(q_1)) + \phi(Q(q_2))) = \det(\phi(Q(q_1) + Q(q_2))) = \phi(\det(Q(q_1) + Q(q_2))).$$

Now, $Q(q_1) + Q(q_2) = Q(q_1 + q_2)$ and from lemma 6.2,

$$\det Q(q_1 + q_2) = (\det(\varphi(\eta(q_1 + q_2))))^2 .$$

From lemma 5.2, $\eta(q_1 + q_2) = s(q_1) + s(q_2) + 2m + (r_1 + r_2 + n)\sqrt{2}$. Hence,

$$(\det(\varphi(\eta(q_1 + q_2))))^2 = \Big((s(q_1) + s(q_2) + 2m)^2 - 2(r_1 + r_2 + n)\Big)^2 .$$

Thus,

Proof.

$$\phi(\det Q(q_1 + q_2)) = \phi(s(q_1)) + \phi(s(q_2)) = \phi(\det Q(q_1)) + \phi(\det Q(q_2)). \qquad \blacksquare$$

Theorem 6.4. *If $A = [a_{ij}]$ is an element of $M_n(\overline{H}(\mathbb{Q}[\sqrt{2}]))$, then $\det s(A)$ and $\det Q(A)$ have the same parity.*

Proof. From lemma 6.1, the matrix $Q(A_2) = [Q(a_{ij}^2)]$ has integer entries. Using the techniques of section 4 and the lemma 6.3, we conclude that $\det Q(A_2)$ and $\det s(A_2)$ have the same parity. As $\overline{s(q)} = \overline{s(q^2)}$, then $\overline{\det Q(A_2)} = \overline{\det s(A)}$. From lemma 6.2

$$\det Q(A_2) = (\det(\varphi(\mathrm{Mdet}(A_2 A_2^*))))^2 = (S_1^2 - 2R_1^2)^2$$

and

$$\det Q(A) = (\det(\varphi(\mathrm{Mdet}(AA^*))))^2 = (S_2^2 - 2R_2^2)^2$$

From theorem 5.4, $\overline{S_1} = \overline{S_2}$. Therefore

$$\overline{\det s(A)} = \overline{\det Q(A_2)} = \overline{S_1} = \overline{S_2} = \overline{\det Q(A)}. \qquad \blacksquare$$

Corollary 6.5. *If $A = [a_{ij}]$ is an element of $M_n(\overline{H}(\mathbb{Q}[\sqrt{2}]))$, then*

$$\mathrm{Mdet}(AA^*) = S + R\sqrt{2},$$

with $\overline{S} = \overline{\det s(A)}$.

Lemma 6.6. *If $A \in M_n(\overline{H}(\mathbb{Q}[\sqrt{2}]))$ and K is column vector with entries in $\overline{H}(\mathbb{Q}[\sqrt{2}])$, then the system $AX = \mathrm{Mdet}(AA^*) \cdot K$ has a solution in $\overline{H}(\mathbb{Q}[\sqrt{2}])$.*

Proof: If $\mathrm{Mdet}(AA^*) = 0$, then $X = 0$. On the other hand, if $\mathrm{Mdet}(AA^*) \neq 0$, by theorem 8.1 and remark 8.1 of [8], there exists $\mathrm{Adj}[[A]] \in M_n(\overline{H}(\mathbb{Q}[\sqrt{2}]))$ such that $A \cdot \mathrm{Adj}[[A]] = \mathrm{Mdet}(AA^*) \cdot I_n$, where I_n is the identity matrix. Hence, $A \cdot (\mathrm{Adj}[[A]]K) = \mathrm{Mdet}(AA^*) \cdot K$. As $\overline{H}(\mathbb{Q}[\sqrt{2}])$ is a subring, then $X_0 = \mathrm{Adj}[[A]]K$ is a column vector with coordinates in $\overline{H}(\mathbb{Q}[\sqrt{2}])$ and $AX_0 = \mathrm{Mdet}(AA^*) \cdot K$. $\blacksquare$

Theorem 6.7. *Consider the system $AX = K$ over $\overline{H}(\mathbb{Q}[\sqrt{2}])$, where $A = [a_{ij}]$ is a $m \times n$ matrix with $m \leq n$. If the system $AX = K\left(1 - \frac{\sqrt{2}}{2}j - \frac{\sqrt{2}}{2}k\right)$ has a solution over $\overline{H}(\mathbb{Q}[\sqrt{2}])$ and the matrix $s(A) = [s(a_{ij})]$ has at least one odd $m \times m$-minor, then the system $AX = K$ has a solution over $\overline{H}(\mathbb{Q}[\sqrt{2}])$.*

Proof: Multiplying the equation $AX = K\left(1 - \frac{\sqrt{2}}{2}j - \frac{\sqrt{2}}{2}k\right)$ by $1 + \frac{\sqrt{2}}{2}j + \frac{\sqrt{2}}{2}k$ from the right, we have $AX\left(1 + \frac{\sqrt{2}}{2}j + \frac{\sqrt{2}}{2}k\right) = 2K$. Therefore, the equation $AX = 2K$ has a solution over $\overline{H}(\mathbb{Q}[\sqrt{2}])$. By hypothesis, there exists an $m \times m$-submatrix $B = [b_{ij}]$ of A with an odd $\det s(B)$. From corollary 6.5, $\operatorname{Mdet}(BB^*) = S + R\sqrt{2}$ with S odd. From lemma 6.6, the system $AX = (S + R\sqrt{2})K$ has solution over $\overline{H}(\mathbb{Q}[\sqrt{2}])$. Multiplying the equation $AX = (S + R\sqrt{2})K$ by $R - S\sqrt{2}$, we have $AX(R - S\sqrt{2}) = (R^2 - 2S^2)K$. Thus the equation $AX = (R^2 - 2S^2)K$ has solution over $\overline{H}(\mathbb{Q}[\sqrt{2}])$. As $R^2 - 2S^2$ is odd, then $AX = K$ has solution over $\overline{H}(\mathbb{Q}[\sqrt{2}])$. ■

7 Main Result

Consider matrices $P = [q_{ij}] \in M_n(\mathbb{Z}[Q_{16}])$, $M(P) = [M(q_{ij})]$, $N(P) = [N(q_{ij})]$, $\varepsilon(P) = [\varepsilon(q_{ij})]$ and $\epsilon(P) = [\epsilon(q_{ij})]$, where

$$\epsilon(q_{ij}) = \begin{bmatrix} \varepsilon(q_{ij}) & 0 & 0 & \\ 0 & q_{ij}^1 & 0 & 0 \\ 0 & 0 & q_{ij}^2 & 0 \\ 0 & 0 & 0 & q_{ij}^3 \end{bmatrix}.$$

Let $\mathcal{T}$ be a diagonal block $16n \times 16n$-matrix, whose blocks of the diagonal is the matrix $[T]$, then $\mathcal{T}^{-1}$ is the diagonal block matrix, whose blocks of the diagonal are $[T]^{-1}$.Consider the matrix

$$\mathcal{T}M(P)\mathcal{T}^{-1} = [[T]M(q_{ij})[T]^{-1}] = \begin{bmatrix} \epsilon(q_{ij}) & 0 & 0 \\ 0 & N(q_{ij}) & 0 \\ 0 & 0 & Q(q_{ij}) \end{bmatrix}$$

Exchanging rows and columns of $\mathcal{T}M(P)\mathcal{T}^{-1}$, we have

$$\mathcal{M}(\mathcal{P}) = \begin{bmatrix} \epsilon(P) & 0 & 0 \\ 0 & N(P) & 0 \\ 0 & 0 & Q(P) \end{bmatrix}$$

Theorem 7.1. *The numbers* $\det M(P)$ *and* $\det \varepsilon(P)$ *have the same parity.*

Proof: The number $\det M(P)$ is equal to the number $\det \mathcal{M}(\mathcal{P})$. Now,

$$\det \mathcal{M}(\mathcal{P}) = \det \epsilon(P) \cdot \det N(P) \cdot \det Q(P)$$

i) Using the same techniques of section 4, we have $\overline{\det \epsilon(P)} = \overline{\det \varepsilon(P)}$.

ii) From lemma 4.2 and theorem 3.4, it follows $\overline{\det N(P)} = \overline{\det \varepsilon(P)}$.

iii) From theorem 6.4 and theorem 3.3, it follows $\overline{\det Q(P)} = \overline{\det \varepsilon(P)}$.

Therefore, $\overline{\det M(P)} = \overline{\det \varepsilon(P)}$. ∎

Theorem 7.2. *Let* $PX = K$ *be a linear system over* $\mathbb{Z}[Q_{16}]$*, where* $P = [p_{ij}]$ *is a* $m \times n$ *matrix with* $m \leq n$*. If* $PX = K(x-1)$ *and* $PX = K(-xy+1)$ *has a solution over* $\mathbb{Z}[Q_{16}]$ *and all* $m \times m$ *minors of* $\varepsilon(P) = [\varepsilon(p_{ij})]$ *are relatively prime, then the system* $PX = K$ *has a solution over* $\mathbb{Z}[Q_{16}]$.

Proof: First, we look at the system $\overline{P}X = \overline{K}$ over $\mathbb{Z}^4 \oplus M_2(\mathbb{Z}) \oplus \overline{H}(\mathbb{Q}[\sqrt{2}])$. Observe that this system is equivalent to the systems, $\varepsilon(P) = \varepsilon(K)$, $P^1X = K^1$, $P^2X = K^2$, $P^3X = K^3$, $P^4 = K^4$ and $P^5X = K^5$, where $P^1 = [p^1_{ij}]$, $P^2 = [p^2_{ij}]$, $P^3 = [p^3_{ij}]$, $P^4 = [p^4_{ij}]$, $P^5 = [p^5_{ij}]$ and $K^i = [k^i_1 \cdots k^i_m]^t$ for $i = 1, \ldots, 5$. The first four systems must be solved over $\mathbb{Z}$, the fifth system must be solved over $M_2(\mathbb{Z})$, and the last one over $\overline{H}(\mathbb{Q}[\sqrt{2}])$. Since the $m \times m$ minors of $\varepsilon(P)$ are relatively prime, the system $\varepsilon(P)X = \varepsilon(K)$ has a solution over $\mathbb{Z}$. Notice that

$$T(x-1) = (0, 0, -2, -2, \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \frac{\sqrt{2}}{2} - 1 + \frac{\sqrt{2}}{2}i)$$

and

$$T(-xy+1) = (0, 2, 2, 0, \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, -\frac{\sqrt{2}}{2}k - \frac{\sqrt{2}}{2}j + 1).$$

By hypothesis $PX = (x-1)K$ and $PX = (-xy+1)K$ has a solution over $\mathbb{Z}[Q_{16}]$, then the systems $P^1 = 2K^1$, $P^2X = 2K^2$, $P^3X = -2K^3$ have solutions over $\mathbb{Z}$,

$$P^4X = K^4 \cdot \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

has a solution over $M_2(\mathbb{Z})$ and

$$P^5X = K^5 \cdot \left(-\frac{\sqrt{2}}{2}k - \frac{\sqrt{2}}{2}j + 1 \right)$$

has a solution over $\overline{H}(\mathbb{Q}[\sqrt{2}])$. Since the integers $\varepsilon(p_{ij})$, p^1_{ij}, p^2_{ij}, p^3_{ij}, $\det(p^4_{ij})$ and $s(q_{ij})$ has the same parity, the matrices P^1, P^2, P^3, $D = [\det(p^4_{ij})]$ and $s(P^5)$ have at least one odd $m \times m$-minor. From lemma 2.3, the first three systems have a solution over $\mathbb{Z}$. From theorem 4.3, the system $P^4X = K^4$ has a solution over $M_2(\mathbb{Z})$. From theorem 6.7, the last system has a solution over $\overline{H}(\mathbb{Q}[\sqrt{2}])$. Hence, the system $\overline{P}X = \overline{K}$ has a solution over $\mathbb{Z}^4 \oplus M_2(\mathbb{Z}) \oplus \overline{H}(\mathbb{Q}[\sqrt{2}])$. By theorem 2.2, the system $PX = 16K$ has a solution over $\mathbb{Z}[Q_{16}]$. Now, by theorem 7.1, there exists a $16m \times 16m$ submatrix B of $M(P)$ with an odd $\det B$, from lemma 2.3 the system $PX = K$ has a solution over $\mathbb{Z}[Q_{16}]$. ∎

8 Strongly surjective map

The orbit space of the 3-sphere $\mathbb{S}^3$ with respect to the action of the quaternion group Q_{16} determined by the inclusion $Q_{16} \subseteq \mathbb{S}^3$ is a compact orientable manifold of dimension 3, denoted by $M_{Q_{16}}$. Let $\widetilde{M}_{Q_{16}}$ be its universal covering. The CW-complex structure of $M_{Q_{16}}$ has one 3-cell and two 2-cells, and the boundary operator

$$\widetilde{\partial}_{M_{Q_{16}}} : H_3(\widetilde{M}_{Q_{16}}, \widetilde{M}^2_{Q_{16}}) = \mathbb{Z}[Q_{16}] \to H_2(\widetilde{M}^2_{Q_{16}}, \widetilde{M}^1_{Q_{16}}) = \mathbb{Z}[Q_{16}] \oplus \mathbb{Z}[Q_{16}]$$

is given by $\tilde{\partial}_{M_{Q_{16}}}(a) = a.(x-1, -xy+1)$(see [6] pg. 253 and [10]).
Let W be a finite connected CW-complex of dimension 3, with m cells of dimension 3, n cells of dimension 2, and $\widetilde{W}$ its universal covering. Suppose that the boundary operator

$$\widetilde{\partial}_3 : H_3(\widetilde{W}, \widetilde{W}_2) = \oplus_{i=1}^{m} \mathbb{Z}[\Pi] \to H_2(\widetilde{W}^2, \widetilde{W}^1) = \oplus_{j=1}^{n} \mathbb{Z}[\Pi]$$

is given by the matrix

$$A = \begin{bmatrix} \widetilde{a}_{11} & \dots & \widetilde{a}_{1m} \\ \vdots & \vdots & \vdots \\ \widetilde{a}_{n1} & \dots & \widetilde{a}_{nm} \end{bmatrix}$$

with columns defined by $\widetilde{\partial}_3(\widetilde{e}^3_i) = \widetilde{a}_{1i}\widetilde{e}^2_1 + \cdots + \widetilde{a}_{ni}\widetilde{e}^2_n$ for $i = 1, \dots, m$. We have

$$H^3(W;\mathbb{Z}) = \frac{\oplus_{i=1}^{m} \mathbb{Z}}{\operatorname{Im}(\varepsilon(A)^t)}.$$

Here, $\Pi = \pi_1(W)$ and $\varepsilon : \mathbb{Z}[\Pi] \to \mathbb{Z}$ is given by $\sum_{i=1}^{p} r_i g_i \mapsto \sum_{i=1}^{p} r_i$ and $\varepsilon(A)^t$ is the transpose of $\varepsilon(A) = [\varepsilon(\widetilde{a}_{ij})]$. Given a map $f : W \to M$, consider the matrix

$$P = \begin{bmatrix} f_\#(\widetilde{a}_{11}) & \dots & f_\#(\widetilde{a}_{n1}) \\ \vdots & \vdots & \vdots \\ f_\#(\widetilde{a}_{1m}) & \dots & f_\#(\widetilde{a}_{nm}) \end{bmatrix},$$

where the ring homomorphism $f_\# : \mathbb{Z}[\Pi] \to \mathbb{Z}[\pi]$ is defined by

$$\sum_{i=1}^{p} r_i g_i \mapsto \sum_{i=1}^{p} r_i f_\#(g_i)$$

is the extension of the induced homomorphism $f_\# : \Pi \to \pi$ of fundamental groups.

Theorem 8.1. *If W is a three dimensional CW-complex with $H^3(W;\mathbb{Z}) = 0$, then there is no strongly surjective map $f : W \to M_{Q_{16}}$.*

Proof: From corollary 5.5 of [1], we have to show that the system $PX = K$ has a solution over $\mathbb{Z}[Q_{16}]$, where P is the matrix described above. From theorem 5.6 of [1], the systems $PX = K(x-1)$ and $PX = K(-xy+1)$ have solutions over $\mathbb{Z}[Q_{16}]$. The hypothesis $H^3(W;\mathbb{Z}) = 0$ implies that $m \leq n$ and all $m \times m$ minors $\{\varepsilon_1, \ldots, \varepsilon_r\}$ of $\varepsilon(P)$ are relatively prime (see chapter 3, proposition 15 of [9]). From theorem 7.2, with these hypotheses the system $PX = K$ has a solution over $\mathbb{Z}[Q_{16}]$. ∎

Finally, let $M_{Q_{32}}$ be the orbit space of 3-sphere $\mathbb{S}^3$ with respect to the action of the quaternion group Q_{32} determined by the inclusion $Q_{32} \subseteq \mathbb{S}^3$. In the same way, one can consider the problem of the existence of a strongly surjective map $f : W \to M_{Q_{32}}$. This problem is equivalent to solving the linear system $PX = K$ over $\mathbb{Z}[Q_{32}]$ satisfying the following assumptions: $PX = K(x-1)$ and $PX = K(-xy+1)$ have solutions over $\mathbb{Z}[Q_{32}]$ and all $m \times m$ minors of $\varepsilon(P)$ are relatively prime. The techniques used in this work rely heavily on the decomposition $\mathbb{Q}[Q_{16}] \cong \mathbb{Q}^4 \oplus M_2(\mathbb{Q}) \oplus H(\mathbb{Q}[\sqrt{2}])$. According to [4],

$$\mathbb{Q}[Q_{32}] \cong \mathbb{Q}^4 \oplus M_2(\mathbb{Q}) \oplus M_2(\mathbb{Q}[\sqrt{2}]) \oplus \left(\frac{2-\sqrt{2}, -1}{\mathbb{Q}[\sqrt{2+\sqrt{2+\sqrt{2}}}]}\right).$$

Here, $\left(\frac{2-\sqrt{2}, -1}{\mathbb{Q}[\sqrt{2+\sqrt{2+\sqrt{2}}}]}\right)$ is the quaternion algebra over the field $\mathbb{Q}[\sqrt{2+\sqrt{2+\sqrt{2}}}]$, generated by $i,\ j$, where $i^2 = 2-\sqrt{2}$, $j^2 = -1$ and $ij = -ji$. Therefore, the case Q_{32}, is not a simple generalization of the case Q_{16}.

Acknowledgments

I would like to thank Daciberg Lima Gonçalves and Oziride Manzoli Neto for reading and suggestions which helped to improve the manuscript.

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