Vanishing theorems of the basic harmonic forms on a complete foliated Riemannian manifold

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Abstract

A well-known result by M. Min-Oo et al. states that there are no nontrivial basic harmonic r $(0 < r < q = \mathrm{codim}\mathcal{F})$-forms on a compact foliated Riemannian manifold $(M, \mathcal{F})$. We extend this result to a complete foliated Riemannian manifold.

1 Introduction

Let $(M, g, \mathcal{F})$ be a foliated Riemannian manifold with a foliation $\mathcal{F}$ and a bundle-like metric g with respect to $\mathcal{F}$. A foliated Riemannian manifold is a Riemannian manifold with a Riemannian foliation, i.e., a foliation on a smooth manifold such that the normal bundle is endowed with a metric whose Lie derivative is zero along leaf directions (see [11]). A Riemannian metric on M is bundle-like if the leaves of the foliation $\mathcal{F}$ are locally equidistant, that is, the metric g on M induces a holonomy invariant transverse metric on the normal bundle $Q = TM/T\mathcal{F}$, where $T\mathcal{F}$ is the tangent bundle of $\mathcal{F}$. Every Riemannian foliation admits bundle-like metrics. Many researchers have studied basic forms and the basic Laplacian on foliated Riemannian manifolds. Basic forms are locally forms on the space of leaves; that is, forms ϕ satisfying $i(X)\phi = i(X)d\phi = 0$ for all $X \in T\mathcal{F}$. Basic forms are preserved by the exterior derivative and are used to define basic de-Rham cohomology groups $H_B^*(\mathcal{F})$. The basic Laplacian Δ_B for a given bundle-like metric is a version of the Laplace operator that preserves the basic forms. It is well-known [5,12] that on a closed oriented manifold M with a transversally

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oriented Riemannian foliation $\mathcal{F}$, $H^r_B(\mathcal{F}) \cong \mathcal{H}^r_B(\mathcal{F})$, where $\mathcal{H}^r_B(\mathcal{F}) = \ker\Delta_B$ is finite dimensional. And so $\chi_B(\mathcal{F}) = \sum_{r=0}^q (-1)^r \dim \mathcal{H}^r_B(\mathcal{F})$, where $\chi_B(\mathcal{F})$ is the basic Euler characteristic [2]. In 1991, M. Min-Oo et al. [8] proved that on a closed foliated Riemannian manifold M, if the transversal curvature operator of $\mathcal{F}$ is positive definite, then $H^r_B(\mathcal{F}) = 0$ $(0 < r < q)$, that is, any basic harmonic r-form is trivial.

In this paper, we study the basic r-forms on a complete foliated Riemannian manifold.

Main Theorem. *Let $(M, g, \mathcal{F})$ be a complete foliated Riemannian manifold and all leaves be compact. Assume that the mean curvature form is bounded and coclosed.*
(1) *If the transversal Ricci curvature of $\mathcal{F}$ is positive-definite, then any L^2-basic harmonic 1-forms ϕ with $\phi \in \mathcal{S}_B$ are trivial.*
(2) *If the curvature endomorphism of $\mathcal{F}$ is positive-definite, then any L^2-basic harmonic r-forms ϕ with $\phi \in \mathcal{S}_B$ are trivial.*

Here $\mathcal{S}_B$ is the Sobolev space of basic forms whose derivative belong to $L^2\Omega^*_B(\mathcal{F})$.

Note that in 1980, H. Kitahara [7] proved that under the same condition of the transversal Ricci curvature, there are no nontrivial basic Δ_T-harmonic 1-forms with finite global norms. Here Δ_T is a different operator to the basic Laplacian Δ_B. If $\mathcal{F}$ is minimal, then $\Delta_T = \Delta_B$.

2 Preliminaries

Let $(M, g, \mathcal{F})$ be a $(p+q)$-dimensional complete foliated Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g with respect to $\mathcal{F}$. Let TM be the tangent bundle of M, $T\mathcal{F}$ its integrable subbundle given by $\mathcal{F}$, and $Q = TM/T\mathcal{F}$ the corresponding normal bundle of $\mathcal{F}$. Then we have an exact sequence of vector bundles

$$0 \longrightarrow T\mathcal{F} \longrightarrow TM \underset{\sigma}{\overset{\pi}{\rightleftarrows}} Q \longrightarrow 0, \tag{2.1}$$

where $\pi : TM \to Q$ is a projection and $\sigma : Q \to T\mathcal{F}^\perp$ is a bundle map satisfying $\pi \circ \sigma = id$. Let g_Q be the holonomy invariant metric on Q induced by g, i.e., $\theta(X)g_Q = 0$ for any vector field $X \in T\mathcal{F}$, where $\theta(X)$ is the transverse Lie derivative [4]. Let R^Q and Ric^Q be the transversal curvature tensor and transversal Ricci operator of $\mathcal{F}$ with respect to the transversal Levi-Civita connection $\nabla^Q \equiv \nabla$ in Q [12], respectively. A differential form $\phi \in \Omega^r(M)$ is *basic* if $i(X)\phi = 0$ and $i(X)d\phi = 0$ for all $X \in T\mathcal{F}$. In a distinguished chart $(x_1, \cdots, x_p; y_1, \cdots, y_q)$ of $\mathcal{F}$, a basic r-form ϕ is expressed by

$$\phi = \sum_{a_1 < \cdots < a_r} \phi_{a_1 \cdots a_r} dy_{a_1} \wedge \cdots \wedge dy_{a_r},$$

where the functions $\phi_{a_1 \cdots a_r}$ are independent of x. Let $\Omega^r_B(\mathcal{F})$ be the set of all basic r-forms on M. Then $\Omega^r(M) = \Omega^r_B(\mathcal{F}) \oplus \Omega^r_B(\mathcal{F})^\perp$ [1]. Now, we recall the star operator $\bar{*} : \Omega_B^r(\mathcal{F}) \to \Omega_B^{q-r}(\mathcal{F})$ given by [5,10]

$$\bar{*}\phi = (-1)^{p(q-r)} * (\phi \wedge \chi_{\mathcal{F}}), \quad \forall \phi \in \Omega_B^r(\mathcal{F}), \tag{2.2}$$

where $\chi_{\mathcal{F}}$ is the characteristic form of $\mathcal{F}$ and $*$ is the Hodge star operator associated to g. For any basic forms $\phi, \psi \in \Omega_B^r(\mathcal{F})$, it is well-known [10] that $\phi \wedge \bar{*}\psi = \psi \wedge \bar{*}\phi$ and $\bar{*}^2\phi = (-1)^{r(q-r)}\phi$. The operator d_B is the restriction of d to the basic forms, i.e., $d_B = d|_{\Omega_B^*(\mathcal{F})}$. Let $d_T = d_B - \kappa_B \wedge$ and $\delta_T = (-1)^{q(r+1)+1}\bar{*}d_B\bar{*}$, where κ_B is the basic part of the mean curvature form κ of $\mathcal{F}$ [1]. Note that κ_B is closed, i.e., $d\kappa_B = 0$ [9,12]. The operator $\delta_B : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r-1}(\mathcal{F})$ is defined by

$$\delta_B\phi = (-1)^{q(r+1)+1}\bar{*}d_T\bar{*}\phi = \delta_T\phi + i(\kappa_B^\sharp)\phi, \tag{2.3}$$

where $(\cdot)^\sharp$ is the g_Q-dual vector field of $(\cdot)$. Generally, δ_B is not a restriction of δ on $\Omega_B^r(\mathcal{F})$, i.e., $\delta_B \neq \delta|_{\Omega_B^r(\mathcal{F})}$, where δ is the formal adjoint of d. But $\delta_B\omega = \delta\phi$ for any basic 1-form ϕ. Let $\Delta_B = d_B\delta_B + \delta_B d_B$ be a basic Laplacian. Then $\Delta^M|_{\Omega_B^0(\mathcal{F})} = \Delta_B$ [5], where Δ^M is the Laplacian on M. Let $\{E_a\}(a = 1, \cdots, q)$ be a local orthonormal basic frame of Q and θ^a a g_Q-dual 1-form to E_a. We define $\nabla_{\rm tr}^*\nabla_{\rm tr} : \Omega_B^r(\mathcal{F}) \to \Omega_B^r(\mathcal{F})$ by

$$\nabla_{\rm tr}^*\nabla_{\rm tr} = -\sum_a \nabla^2_{E_a,E_a} + \nabla_{\kappa_B^\sharp}, \tag{2.4}$$

where $\nabla^2_{X,Y} = \nabla_X\nabla_Y - \nabla_{\nabla^M_X Y}$ for any $X, Y \in TM$ and ∇^M is the Levi-Civita connection with respect to g. Then the generalized Weitzenböck type formula on $\Omega_B^r(\mathcal{F})$ is given by [3]

$$\Delta_B\phi = \nabla_{\rm tr}^*\nabla_{\rm tr}\phi + F(\phi) + A_{\kappa_B^\sharp}\phi \tag{2.5}$$

for any $\phi \in \Omega_B^r(\mathcal{F})$, where $F = \sum_{a,b=1}^q \theta^a \wedge i(E_b)R^Q(E_b, E_a)$ and

$$A_Y\phi = \theta(Y)\phi - \nabla_Y\phi. \tag{2.6}$$

In particular, for a 1-form ϕ, $F(\phi)^\sharp = {\rm Ric}^Q(\phi^\sharp)$ and $A_Y s = -\nabla_{\sigma(s)}\pi(Y)$. Let $\Omega_{B,o}^*(\mathcal{F})$ be the space of basic forms with compact supports.

Let ν be the transversal volume form, i.e., $*\nu = \chi_{\mathcal{F}}$. The pointwise inner product $\langle\cdot,\cdot\rangle$ on $\Omega_B^r(\mathcal{F})$ is given by

$$\langle\phi, \psi\rangle\nu = \phi \wedge \bar{*}\psi \tag{2.7}$$

for any basic forms $\phi, \psi \in \Omega_B^r(\mathcal{F})$. And the global inner product $\ll \cdot,\cdot \gg$ on $\Omega_B^*(\mathcal{F})$ is defined by

$$\ll \phi, \psi \gg = \int_M \langle\phi, \psi\rangle\mu_M \tag{2.8}$$

for any $\phi, \psi \in \Omega_B^r(\mathcal{F})$, one of which has compact support, where $\mu_M = \nu \wedge \chi_{\mathcal{F}}$ is the volume form with respect to g. It is well-known [3] that $\ll \nabla_{\rm tr}^*\nabla_{\rm tr}\phi, \psi \gg = \ll \nabla_{\rm tr}\phi, \nabla_{\rm tr}\psi \gg$ for any $\phi, \psi \in \Omega_{B,0}^r(\mathcal{F})$ and

$$\ll d_B\phi, \psi \gg = \ll \phi, \delta_B\psi \gg \tag{2.9}$$

for any $\phi \in \Omega^r_{B,o}(\mathcal{F}), \psi \in \Omega^{r+1}_{B,o}(\mathcal{F})$. The basic form ϕ is said to be *L^2-basic form* if ϕ has finite global norm, i.e., $\|\phi\|^2 < \infty$. Let $\mathcal{H}^r_{B,2}(\mathcal{F})$ be the space of L^2-basic harmonic forms, i.e.,

$$\mathcal{H}^r_{B,2}(\mathcal{F}) = \{\phi \in L^2\Omega^r_B(\mathcal{F}) \mid d_B\phi = \delta_B\phi = 0\}. \tag{2.10}$$

Generally, the space $\mathcal{H}^r_{B,2}(\mathcal{F})$ can have infinite dimension. And if the dimension of $\mathcal{H}^r_{B,2}(\mathcal{F})$ is finite, then it depends on the bundle-like metric. Trivially, if M is compact, then $\mathcal{H}^r_{B,2}(\mathcal{F}) \cong H^r_B(\mathcal{F})$. So we study the vanishing properties of the L^2-basic harmonic spaces on a complete foliated Riemannian manifold.

Remark 2.1. (1) The operator δ_T is the formal adjoint of d_B with respect to the global norm $(\cdot,\cdot)$, which is given by

$$(\phi,\psi) = \int_M \langle\phi,\psi\rangle \nu \wedge dx_1 \wedge \cdots \wedge dx_p. \tag{2.11}$$

Let $\Delta_T = d_B\delta_T + \delta_T d_B$ be a Laplacian. If the foliation is minimal, then $\delta_T = \delta_B$. So $\Delta_B = \Delta_T$.

(2) In 1980, H. Kitahara [7] proved that if the transversal Ricci curvature is nonnegative and positive at some point, then there are no nontrivial L^2-basic Δ_T-harmonic 1-forms.

3 Vanishing theorem

Let $(M, g, \mathcal{F})$ be a complete foliated Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g with respect to $\mathcal{F}$. Assume that all leaves of $\mathcal{F}$ are compact. Now, we consider a smooth function μ on $\mathbb{R}$ satisfying

$$(i)\ 0 \le \mu(t) \le 1 \text{ on } \mathbb{R}, \quad (ii)\ \mu(t) = 1 \text{ for } t \le 1, \quad (iii)\ \mu(t) = 0 \text{ for } t \ge 2.$$

Let x_0 be a point in M. For each point $y \in M$, we denote by $\rho(y)$ the distance between leaves through x_0 and y. For any real number $l > 0$, we define a Lipschitz continuous function ω_l on M by

$$\omega_l(y) = \mu(\rho(y)/l).$$

Trivially, ω_l is a basic function. Let $B(l) = \{y \in M | \rho(y) \le l\}$ for $l > 0$. Then ω_l satisfies the following properties:

$$\begin{array}{ll} 0 \le \omega_l(y) \le 1 & \text{for any } y \in M \\ \operatorname{supp}\omega_l \subset B(2l) & \\ \omega_l(y) = 1 & \text{for any } y \in B(l) \\ \lim_{l\to\infty}\omega_l = 1 & \\ |d_B\omega_l| \le \frac{C}{l} & \text{almost everywhere on } M, \end{array}$$

where C is a positive constant independent of l [13]. Hence $\omega_l\psi$ has compact support for any basic form $\psi \in \Omega^*_B(\mathcal{F})$ and $\omega_l\psi \to \psi$ (strongly) when $l \to \infty$.

Lemma 3.1. [6] *For any $\phi \in \Omega_B^r(\mathcal{F})$, there exists a number A depending only on μ, such that*

$$\|d_B\omega_l \wedge \phi\|^2_{B(2l)} \le \frac{qA^2}{l^2}\|\phi\|^2_{B(2l)},$$
$$\|d_B\omega_l \wedge \bar{*}\phi\|^2_{B(2l)} \le \frac{qA^2}{l^2}\|\phi\|^2_{B(2l)},$$
$$\|d_B\omega_l \otimes \phi\|^2_{B(2l)} \le \frac{qA^2}{l^2}\|\phi\|^2_{B(2l)},$$

where $\|\phi\|^2_{B(2l)} = \ll \phi, \phi \gg_{B(2l)} = \int_{B(2l)} \langle \phi, \phi \rangle \mu_M$.

Proposition 3.2. *For any L^2-basic form ψ, if $\Delta_B\psi = 0$, then $d_B\psi = 0$ and $\delta_B\psi = 0$.*

Proof. Let ψ be a L^2-basic form. Then we have

$$\ll \Delta_B\psi, \omega_l^2\psi \gg_{B(2l)} = \ll d_B\psi, d_B(\omega_l^2\psi) \gg_{B(2l)} + \ll \delta_B\psi, \delta_B(\omega_l^2\psi) \gg_{B(2l)}. \tag{3.1}$$

By a direct calculation, we have

$$d_B(\omega_l^2\psi) = \omega_l^2 d_B\psi + 2\omega_l d_B\omega_l \wedge \psi, \tag{3.2}$$
$$\delta_B(\omega_l^2\psi) = \omega_l^2\delta_B\psi + (-1)^{q(r+1)+1}\bar{*}(2\omega_l d_B\omega_l \wedge \bar{*}\psi). \tag{3.3}$$

From (3.1), (3.2) and (3.3), if $\Delta_B\psi = 0$, then

$$\begin{aligned} &\|\omega_l d_B\psi\|^2_{B(2l)} + \|\omega_l\delta_B\psi\|^2_{B(2l)} \\ &= -2 \ll \omega_l d_B\psi, d_B\omega_l \wedge \psi \gg_{B(2l)} + 2(-1)^{q(r+1)} \ll \omega_l\delta_B\psi, \bar{*}(d_B\omega_l \wedge \bar{*}\psi) \gg_{B(2l)}. \end{aligned}$$

Hence by the the Schwartz's inequality and Lemma 3.1, we have

$$\begin{aligned} &\|\omega_l d_B\psi\|^2_{B(2l)} + \|\omega_l\delta_B\psi\|^2_{B(2l)} \\ &\le \epsilon_1\|\omega_l d_B\psi\|^2_{B(2l)} + \epsilon_2\|\omega_l\delta_B\psi\|^2_{B(2l)} + \frac{B_1}{l}\|\psi\|^2_{B(2l)} \end{aligned}$$

for some positive real numbers ϵ_1, ϵ_2 and B_1. Therefore, we have

$$\|\omega_l d_B\psi\|^2_{B(2l)} + \|\omega_l\delta_B\psi\|^2_{B(2l)} \le \frac{B_2}{l}\|\psi\|^2_{B(2l)}$$

for some positive real number B_2. Since ψ is the L^2-basic form, letting $l \to \infty$, $d_B\psi = \delta_B\psi = 0$. ∎

Remark 3.3. In 1979, H. Kitahara [6] proved the corresponding result with the Laplacian Δ_T. Namely, on a complete foliated manifold, if $\Delta_T\phi = 0$, then $d_B\phi = \delta_T\phi = 0$.

Now we prove the vanishing theorem of the L^2-basic harmonic form on a complete foliated Riemannian manifold. First of all, we prepare some lemmas.

Lemma 3.4. *Let $(M, g, \mathcal{F})$ be a complete foliated Riemannian manifold whose leaves are compact. Suppose that κ_B is bounded and coclosed. Then for any L^2-basic harmonic form ϕ,*

$$\limsup_{l\to\infty} \ll A_{\kappa_B^\sharp}\phi, \omega_l^2\phi \gg_{B(2l)} = 0. \tag{3.4}$$

Proof. Let ϕ be a L^2-basic harmonic form. Since $\theta(X)\phi = d_B i(X)\phi$, from (2.6) we have

$$\ll A_{\kappa_B^\sharp}\phi, \omega_l^2\phi \gg_{B(2l)} = \ll d_B i(\kappa_B^\sharp)\phi, \omega_l^2\phi \gg_{B(2l)} - \ll \nabla_{\kappa_B^\sharp}\phi, \omega_l^2\phi \gg_{B(2l)} \tag{3.5}$$

Since $\delta_B\phi = 0$, from (3.3) and Lemma 3.1, we have

$$\begin{aligned}
| \ll d_B i(\kappa_B^\sharp)\phi, \omega_l^2\phi \gg_{B(2l)} | &= 2| \ll \omega_l i(\kappa_B^\sharp)\phi, \bar{*}(d_B\omega_l \wedge \bar{*}\phi) \gg_{B(2l)} | \\
&\le \epsilon_3 \|\omega_l i(\kappa_B^\sharp)\phi\|^2_{B(2l)} + \frac{1}{\epsilon_3}\|d_B\omega_l \wedge \bar{*}\phi\|^2_{B(2l)} \\
&\le \epsilon_3 \|\omega_l i(\kappa_B^\sharp)\phi\|^2_{B(2l)} + \frac{B_3}{l^2}\|\phi\|^2_{B(2l)}
\end{aligned}$$

for some positive real numbers ϵ_3 and B_3. By using $|i(\kappa_B^\sharp)\phi|^2 + |\kappa_B \wedge \phi|^2 = |\kappa_B|^2|\phi|^2$, we have

$$| \ll d_B i(\kappa_B^\sharp)\phi, \omega_l^2\phi \gg_{B(2l)} | \le \epsilon_3 \max(|\kappa_B|^2)\|\omega_l\phi\|^2_{B(2l)} + \frac{B_3}{l^2}\|\phi\|^2_{B(2l)} \tag{3.6}$$

On the other hand, since $\delta_B\kappa_B = 0$, by a direct calculation, we have

$$\begin{aligned}
\ll \nabla_{\kappa_B^\sharp}\phi, \omega_l^2\phi \gg_{B(2l)} &= \frac{1}{2} \ll d_B(|\omega_l\phi|^2), \kappa_B \gg_{B(2l)} - \ll \omega_l\phi, \kappa_B^\sharp(\omega_l)\phi \gg_{B(2l)} \\
&= - \ll \omega_l\phi, \kappa_B^\sharp(\omega_l)\phi \gg_{B(2l)} .
\end{aligned}$$

Hence by the Schwartz inequality, we have

$$\begin{aligned}
| \ll \nabla_{\kappa_B^\sharp}\phi, \omega_l^2\phi \gg_{B(2l)} | &= | \ll \omega_l\phi, \kappa_B^\sharp(\omega_l)\phi \gg_{B(2l)} | \\
&\le \epsilon_4 \|\omega_l\phi\|^2_{B(2l)} + \frac{B_4}{l^2}\max(|\kappa_B|^2)\|\phi\|^2_{B(2l)}
\end{aligned} \tag{3.7}$$

for a positive real numbers ϵ_4 and B_4. From (3.6) and (3.7), by letting $l \to \infty$, we have

$$\begin{aligned}
&\limsup_{l\to\infty} | \ll d_B i(\kappa_B^\sharp)\phi, \omega_l^2\phi \gg_{B(2l)} | \le \epsilon_3 \max(|\kappa_B|^2)\|\phi\|^2, \\
&\limsup_{l\to\infty} | \ll \nabla_{\kappa_B^\sharp}\phi, \omega_l^2\phi \gg_{B(2l)} | \le \epsilon_4 \|\phi\|^2.
\end{aligned}$$

Since ϵ_3 and ϵ_4 are arbitrary positive numbers, we have

$$\limsup_{l\to\infty} | \ll d_B i(\kappa_B^\sharp)\phi, \omega_l^2\phi \gg_{B(2l)} | = 0, \tag{3.8}$$

$$\limsup_{l\to\infty} | \ll \nabla_{\kappa_B^\sharp}\phi, \omega_l^2\phi \gg_{B(2l)} | = 0. \tag{3.9}$$

Hence from (3.5), (3.8) and (3.9), the proof is completed. ∎

Theorem 3.5. *Let $(M, g, \mathcal{F})$ be as in Lemma 3.4. Suppose that κ_B is bounded and coclosed. If the curvature endomorphism F of $\mathcal{F}$ is positive-definite, then any L^2- basic harmonic r-forms ϕ with $\phi \in \mathcal{S}_B$ are trivial, i.e., $\mathcal{H}^r_{B,2}(\mathcal{F}) = \{0\}$.*

Proof. Let ϕ be a L^2-basic harmonic r-form. From (2.5) and Proposition 3.2, we have

$$\langle \nabla^*_{\rm tr}\nabla_{\rm tr}\phi, \omega_l^2\phi\rangle + \langle F(\phi), \omega_l^2\phi\rangle + \langle A_{\kappa_B^\sharp}\phi, \omega_l^2\phi\rangle = 0. \tag{3.10}$$

On the other hand, a direct calculation gives

$$\ll \nabla^*_{\rm tr}\nabla_{\rm tr}\phi, \omega_l^2\phi \gg_{B(2l)} = \ll \nabla_{\rm tr}\phi, 2\omega_l d_B\omega_l \otimes \phi \gg_{B(2l)} + \|\omega_l \nabla_{\rm tr}\phi\|^2_{B(2l)}. \tag{3.11}$$

From Lemma 3.1, we have

$$| \ll \nabla_{\rm tr}\phi, 2\omega_l d_B\omega_l \otimes \phi \gg_{B(2l)} | \le \epsilon_5 \|\omega_l \nabla_{\rm tr}\phi\|^2_{B(2l)} + \frac{B_5}{l^2}\|\phi\|^2_{B(2l)}$$

for some positive constants ϵ_5 and B_5. Hence by letting $l \to \infty$, we have

$$\limsup_{l\to\infty} \ll \nabla_{\rm tr}\phi, 2\omega_l d_B\omega_l \otimes \phi \gg_{B(2l)} \le \epsilon_5 \|\nabla_{\rm tr}\phi\|^2.$$

Since ϵ_5 is arbitrary and $\phi \in \mathcal{S}_B$ (i.e., $\|\nabla_{\rm tr}\phi\|^2 < \infty$), we have

$$\limsup_{l\to\infty} \ll \nabla_{\rm tr}\phi, 2\omega_l d_B\omega_l \otimes \phi \gg_{B(2l)} = 0. \tag{3.12}$$

Hence from (3.11), (3.12) and Lemma 3.4, we have

$$\|\nabla_{\rm tr}\phi\|^2 + \limsup_{l\to\infty} \ll F(\phi), \omega_l^2\phi \gg_{B(2l)} = 0, \tag{3.13}$$

which complete the proof. ∎

Since $F(\phi^\sharp) = \mathrm{Ric}^Q(\phi^\sharp)$ for any basic 1-form ϕ, we have the following corollary.

Corollary 3.6. *Let $(M, g, \mathcal{F})$ be as in Lemma 3.4. Suppose that κ_B is bounded and coclosed. If the transversal Ricci curvature Ric^Q is positive-definite, then any L^2-basic harmonic 1-forms ϕ with $\phi \in \mathcal{S}_B$ are trivial, $\mathcal{H}^1_{B,2}(\mathcal{F}) = \{0\}$.*

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