A new proof of extreme amenability of the unitary group of the hyperfinite II_1 factor

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Abstract

We provide an alternative proof for the extreme amenability of the unitary group of the hyperfinite II_1-factor von Neumann algebra, endowed with the strong operator topology.

1 Introduction

Ever since the introduction of *rings of operators* in the groundbreaking work of Murray and von Neumann [7, 8], the hyperfinite II_1-factor $\mathcal{R}$ has played an important role in (what is nowadays called) the theory of von Neumann algebras. The first construction of $\mathcal{R}$ was given in terms of finite-dimensional matrix algebras or for example as the group von Neumann algebra of the group S_∞ of finitary permutations on the set of natural numbers. In seminal work of Connes [1], $\mathcal{R}$ was shown to be the unique injective factor of type II_1. It also followed, that $\mathcal{R}$ is isomorphic to the group von Neumann algebra $L\Gamma$ for every countable, amenable group Γ, whose non-trivial conjugacy classes are all infinite.

The direct relationship between the concept of amenability and hyperfiniteness triggered the question in what sense the unitary group $\mathrm{U}(\mathcal{R}) = \{u \in \mathcal{R} \mid uu^* = u^*u = 1\}$ of the hyperfinite II_1-factor is amenable itself as a topological group. This point was clarified by Giordano-Pestov [2] who showed that $\mathrm{U}(\mathcal{R})$ is extremely amenable. The aim of this note is to provide a new and direct proof (assuming the results on Lévy groups from [3]) of extreme amenability of the unitary group of the hyperfinite II_1 factor, endowed with the strong operator

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topology. Recall, a topological group G is said to be **extremely amenable** if every continuous action of G on a compact Hausdorff space X admits a fixed point. A nice account on the history of the subject can be found in Pestov's book [9].

A milestone in the study of extreme amenability was set by Gromov and Milman [3] - they proved that the unitary group $\mathrm{U}(\ell^2(\mathbb{N}))$ with the strong operator topology is extremely amenable. The core in their proof is to show that $\mathrm{U}(\ell^2(\mathbb{N}))$ with the strong operator topology is a Lévy group by treating $\mathrm{SU}(n)$ as a Riemannian manifold and showing that $\inf_t \mathrm{Ric}(t,t) \to \infty$ as $n \to \infty$, where t runs over all unit tangent vectors in the tangent space of $\mathrm{SU}(n)$. Together with the isoperimetric inequality this implied that $(\mathrm{SU}(n), d_n, \mu_n)$ forms a Lévy family with the respect to the unnormalized Hilbert-Schmidt metric d_n and Haar measure μ_n on $\mathrm{SU}(n)$. This rather deep fact was then used in the proof of extreme amenability of $\mathrm{U}(\mathcal{R})$ by Giordano and Pestov [2]. Note that the corresponding statement fails for the family $(\mathrm{U}(n), d_n, \mu_n)$, since the Ricci curvature vanishes on tangent vectors corresponding to the center of $\mathrm{U}(n)$.

The main purpose of this note is to give a direct argument for the fact that after normalization of the metric the unitary groups do form a Lévy family.

Theorem 1. *The family $(\mathrm{U}(n), d_n/n, \mu_n)_n$ is a Lévy family.*

In particular, we do not rely on the relationship with the isoperimetric inequality and on curvature computations. To the best of our knowledge, this direct argument was unnoticed – and still implies in a straightforward way extreme amenability of $U(\mathcal{R})$ using [9, Theorem 4.1.3].

Corollary 2 (Giordano-Pestov [2])**.** The unitary group $\mathrm{U}(\mathcal{R})$ of the hyperfinite II_1 factor, endowed with the strong operator topology, is extremely amenable.

2 Metric measure spaces

Recall that a **space with metric and measure**, or a *mm*-**space**, is a triple (X, d, μ) consisting of a set X, a metric d on X and a probability Borel measure on the metric space (X, d). The **concentration function** $\alpha_X : [0, \infty) \to [0, 1/2]$ of an *mm*-space X (introduced by Milman and Schechtman in [5, 6]) is defined as

$$\alpha_X(\varepsilon) = \begin{cases} 1/2, & \text{if } \varepsilon = 0, \\ 1 - \inf\{\mu(A_\varepsilon) \mid A \subseteq X \text{ is Borel}\,, \mu(A) \geq 1/2\}, & \text{if } \varepsilon > 0. \end{cases}$$

A family $(X_n, d_n, \mu_n)_n$ of *mm*-spaces is a **Lévy family** if $\alpha_{X_n}(\varepsilon) \to_{n\to\infty} 0$ pointwise for all $\varepsilon > 0$. This is not the original definition of a Lévy family, but it is equivalent.

A metrizable group (G, d) is called **Lévy group** if there is a family of compact subgroups $(G_n)_n$ of G, directed by inclusion, with dense union and such that $(G_n, d_n, \mu_n)_n$ forms a Lévy family, where d_n is the restriction of d to G_n and μ_n is the normalized Haar measure on G_n.

Let $d_{1,n}$ denote the normalized trace metric on the space $M_{n\times n}(\mathbb{C})$ of $n \times n$-matrices induced from the normalized trace norm $\|\cdot\|_{1,n}$, where $n \in \mathbb{N}$. That is, with tr the unnormalized trace on $M_{n\times n}(\mathbb{C})$, $d_{1,n}(u,v) = \|u-v\|_{1,n} = \frac{1}{n}\operatorname{tr}(|u-v|)$, $u,v \in M_{n\times n}(\mathbb{C})$. Here, $|a| = (a^*a)^{1/2}$ denotes the absolute value of the matrix as usual. Note that $d_{1,n}$ defines bi-invariant metric on the group $\mathrm{U}(n)$. Denote by $\operatorname{rk}(x)$ the rank of $x \in M_{n\times n}(\mathbb{C})$ and by $\|\cdot\|_{\infty,n} := \sup_{\xi\in\mathbb{C}^n,\|\xi\|_n=1}\|\cdot\xi\|_n$ the operator norm. Note that $\|xy\|_{1,n} \le \|x\|_{1,n}\|y\|_{\infty,n}$ and hence

$$\|x\|_{1,n} \le \frac{\operatorname{rk}(x)}{n}\,\|x\|_{\infty,n}\,. \tag{1}$$

Proposition 3. Let $1 \le k \le n \in \mathbb{N}$ and $u \in \mathrm{U}(k)$. Then there exists $v \in \mathrm{U}(k-1)$ such that $d_{1,n}(v \oplus 1_{n-k+1}, u \oplus 1_{n-k}) \le 4/n$.

Proof. The cases $k = 1,2$ are trivial since $d(1_n, u \oplus 1_{n-2}) \le 2/n$ for all $u \in \mathrm{U}(2)$. Consider the case $k \ge 3$ and assume without loss of generality that $k = n$. Denote by $\{e_k\}_{k=1,\dots,n}$ the standard orthonormal basis of $\mathbb{C}^n$. If $ue_n = e_n$, then $u \in \mathrm{U}(n-1)$ and we can choose $v = u \in \mathrm{U}(n-1)$. Hence assume that $ue_n = \xi \ne e_n$ and consider $X := \operatorname{span}\langle e_n, \xi\rangle \cong \mathbb{C}^2$. There exists a unitary operator $w\colon X \to X$ such that $w\xi = e_n$. Define $v := (1_{X^\perp} \oplus w)u$ with $X^\perp$ the orthogonal complement of X. Then $v \in \mathrm{U}(n-1) \oplus 1_1 \subset \mathrm{U}(n)$ and we set $x := 1 - vu^* = 0_{X^\perp} \oplus (1_X - w)$. Hence, $\operatorname{rk}(x) \le 2$, $\|x\|_{\infty,n} \le 2$ and the estimate (1) imply that $d_{1,n}(v,u) = \|1 - vu^*\|_{1,n} = \|x\|_{1,n} \le 4/n$. ∎

Assume that H is a closed subgroup of a compact group G, equipped with a bi-invariant metric d. Then the formula $\widetilde{d}(g_1H, g_2H) := \inf_{h_1,h_2\in H} d(g_1h_1, g_2h_2)$ defines a left-invariant metric on the factor space G/H, see Lemma 4.5.2 in [9]. We refer to $\widetilde{d}$ as the **factor metric**. Define the **diameter** $\operatorname{diam}(G/H)$ of the factor space G/H to be

$$\operatorname{diam}(G/H) := \sup_{g_1,g_2\in G}\ \inf_{h_1,h_2\in H} d(g_1h_1, g_2h_2).$$

3 Proof of the main result

Proposition 4. $(\mathrm{U}(n), d_{1,n}, \mu_n)_{n\in\mathbb{N}}$ forms a Lévy family, where $d_{1,n}$ denotes the normalized trace metric on $\mathrm{U}(n)$ and μ_n is the normalized Haar measure on $\mathrm{U}(n)$.

Proof. Our proof is based on [2, Theorem 2.9], a result of what is called the martingale technique, see [6, Theorem 7.8] and [4, Theorem 4.4]. Consider the compact Lie group $\mathrm{U}(n)$, $3 \le n \in \mathbb{N}$, equipped with the bi-invariant trace metric $d_{1,n}$. Embed $\mathrm{U}(k)$ in $\mathrm{U}(n)$ via $\mathrm{U}(k) \ni u \mapsto u \oplus 1_{n-k} \in \mathrm{U}(n)$, where $k \le n$, $k \in \mathbb{N}$. We calculate the diameter $a_k := \operatorname{diam}(\mathrm{U}(k)/\mathrm{U}(k-1))$ of the factor space $\mathrm{U}(k)/\mathrm{U}(k-1)$ with regard to the metric inherited from $\mathrm{U}(n)$, where $k = 1,\dots,n$. We use Proposition 3 to obtain

$$a_k = \sup_{u\in\mathrm{U}(k)}\ \inf_{v\in\mathrm{U}(k-1)} d_{1,n}(1, uv) \le \frac{4}{n}.$$

Thus [6, Theorem 7.8] and [4, Theorem 4.4] and the above calculations imply that the concentration function of the *mm*-space $(\mathrm{U}(n), d_{1,n}, \mu_n)$ satisfies

$$\alpha_{\mathrm{U}(n)}(\varepsilon) \leq 2\exp\left(-\frac{\varepsilon^2}{8\sum_{k=0}^{n-1} a_k^2}\right) \leq 2\exp\left(-\frac{n^2\varepsilon^2}{8\sum_{k=0}^{n-1} 16}\right) = 2\exp\left(-\frac{n\varepsilon^2}{128}\right).$$

Hence, $\alpha_{\mathrm{U}(n)} \to_{n\to\infty} 0$ pointwise on $(0,\infty)$ and thus $(\mathrm{U}(n), d_{1,n}, \mu_n)_n$ is a Lévy family. ∎

Actually Proposition 3 and Proposition 4 hold analogously for the orthogonal groups $\mathrm{O}(n)$, thus $(\mathrm{O}(n), d_{1,n}, \mu_n)_n$ forms a Lévy family.

Theorem 5. *The group* $\mathrm{U}(\mathcal{R})$ *with the strong operator topology is a Lévy group.*

Proof. Consider the realization of $\mathcal{R}$ as an infinite tensor product of copies of $M_2\mathbb{C}$. The inclusion $\otimes_{i=1}^n M_2\mathbb{C} \subset \mathcal{R}$ yields an inclusion $\mathrm{U}(2^n) \subset \mathrm{U}(\mathcal{R})$. The directed family $\{\mathrm{U}(2^n)\}_n$ of compact subgroups of $\mathrm{U}(\mathcal{R})$ is strongly dense in $\mathrm{U}(\mathcal{R})$. Moreover, the strong topology in $\mathrm{U}(\mathcal{R})$ is induced from the 1-norm which restricts to $d_{1,2^n}$ on each $\mathrm{U}(2^n)$. By Proposition 4 $(\mathrm{U}(2^n), d_{1,2^n}, \mu_{2^n})_n$ forms a Lévy family. ∎

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