A note on the norm of a basic elementary operator

Mohamed Boumazgour Mohamed Barraa

Abstract

Let $\mathcal{L}(E)$ be the algebra of all bounded linear operators on a Banach space E. For $A, B \in \mathcal{L}(E)$, define the basic elementary operator $M_{A,B}$ by $M_{A,B}(X) = AXB$, $(X \in \mathcal{L}(E))$. If $\mathcal{S}$ is a symmetric norm ideal of $\mathcal{L}(E)$, we denote $M_{\mathcal{S},A,B}$ the restriction of $M_{A,B}$ to $\mathcal{S}$. In this paper, the norm equality $\|I + M_{\mathcal{S},A,B}\| = 1 + \|A\|\|B\|$ is studied. In particular, we give necessary and sufficient conditions on A and B for this equality to hold in the special case when E is a Hilbert space and $\mathcal{S}$ is a Schatten p-ideal of $\mathcal{L}(E)$.

1 Introduction

Let E be a complex Banach space. We denote by $\mathcal{L}(E)$ the Banach algebra of all bounded linear operators on E. For A and B in $\mathcal{L}(E)$, define the operators L_A and R_B on $\mathcal{L}(E)$ by $L_A(X) = AX$ and $R_B(X) = XB$ $(X \in \mathcal{L}(E))$, respectively. The basic elementary operator $M_{A,B}$ induced by the operators A and B is the multiplication defined by $M_{A,B} = L_A R_B$. An elementary operator on $\mathcal{L}(E)$ is a finite sum $R = \sum_{i=1}^{n} M_{A_i,B_i}$ of basic ones. A familiar example of elementary operators is the generalized derivation $\delta_{A,B}$ defined by $\delta_{A,B} = L_A - R_B$.

Let $\mathcal{S}$ be a non-zero two-sided ideal of the algebra $\mathcal{L}(E)$. We say that $\mathcal{S}$ is a symmetric norm ideal if it is equipped with a norm $\| \,.\, \|_{\mathcal{S}}$ satisfying the following conditions:

Received by the editors in March 2015.
Communicated by F. Bastin.
2010 *Mathematics Subject Classification* : 47A12, 47A30, 47B47.
Key words and phrases : Norms, elementary operators, norm ideals, numerical range.

(i) $\mathcal{S}$ is a Banach space with respect to the norm $\| \, . \, \|_S$;

(ii) $\| X \|_{\mathcal{S}} = \|X\|$ for all $X \in \mathcal{S}$ with one-dimensional range;

(iii) $\|AXB\|_{\mathcal{S}} \leq \|A\| \, \|X\|_{\mathcal{S}} \, \|B\|$ for all $A, B \in \mathcal{L}(E)$ and $X \in \mathcal{S}$.

Familiar examples of symmetric norm ideals are the Schatten p-ideals $(C_p(H), ||.||_p)$ $(1 \leq p \leq \infty)$ of operators on a Hilbert space H (see [20, 21]). Here we denote by $C_\infty(H)$ the ideal of all compact operators on H.

Let $\mathcal{S}$ be a symmetric norm ideal of $\mathcal{L}(E)$, and let $A, B \in \mathcal{L}(E)$. Then $M_{A,B}(\mathcal{S}) \subset \mathcal{S}$, and we denote by $M_{\mathcal{S},A,B}$ the restriction of $M_{A,B}$ to $\mathcal{S}$. Since $\|AXB\|_{\mathcal{S}} \leq \|A\| \, \|X\|_{\mathcal{S}} \, \|B\|$ for all $X \in \mathcal{S}$ then obviously $M_{\mathcal{S},A,B} \in \mathcal{L}(\mathcal{S})$ and $\|M_{\mathcal{S},A,B}\| \leq \|A\| \|B\|$. In the special case when $\mathcal{S}$ is a Schatten ideal $\mathcal{C}_p(H)$, we denote $M_{\mathcal{S},A,B}$ by $M_{p,A,B}$.

Many facts about the relation between the spectrum of the operator $R = \sum_{i=1}^{n} M_{A_i,B_i}$ and spectra of its coefficients A_i and B_i are known. This is not the case with the relation between the norm of R when restricted to a norm ideal and norms of A_i and B_i. Apparently, the only elementary operators on a norm ideal for which the norm is easily computed are the basic ones. For an intensive study of norms of elementary operators on Banach spaces we refer to [3, 4, 5, 9, 13, 15, 16, 19, 22, 23, 24, 25, 26, 27].

In this paper we shall study the equation

$$\|I + M_{\mathcal{S},A,B}\| = 1 + \|A\| \|B\|, \tag{1.1}$$

where I denotes the identity operator, A and B are bounded operators on a Banach space E and $\mathcal{S}$ is a symmetric norm ideal of $\mathcal{L}(E)$. Here we note that we always have $\|I + M_{\mathcal{S},A,B}\| \leq 1 + \|A\| \|B\|$. In the particular case where $B = I$, the equation (1.1) is equivalent to the Daugavet equation

$$\|I + A\| = 1 + \|A\|. \tag{1.2}$$

For more results about the Daugavet equation and its applications we refer to [1, 11] and references therein.

In order to state our results in detail, we need to recall some notations.

Let E be a complex Banach space, and let $E^{'}$ be its dual space. For $T \in \mathcal{L}(E)$, the spatial numerical range of T, denoted by $W(T)$, is defined to be the set

$$W(T) = \Big\{ f(Tx) : x \in E, \|x\| = 1 \text{ and } f \in D(x) \Big\},$$

where

$$D(x) = \Big\{ f \in E' : f(x) = \|f\| = \|x\| \Big\}.$$

If H is a Hilbert space and $T \in \mathcal{L}(H)$, then the numerical range of T is given by

$$W(T) = \Big\{ \langle Tx, x \rangle : x \in H \text{ and } \|x\| = 1 \Big\}.$$

Let $T \in \mathcal{L}(E)$. The algebraic numerical range of T is defined by

$$V(T) = \Big\{F(T) : F \in \big(\mathcal{L}(E)\big)' \text{ and } \parallel F \parallel = F(I) = 1\Big\}.$$

It is well-known that $V(T)$ $(T \in \mathcal{L}(E))$ is a compact convex subset of the plane, and that $V(T)$ contains the spectrum of T (see [6]). Furthermore, $V(T)$ coincides with the closed convex hull of $W(T)$ whenever T is a bounded operator on a Banach space. For basic facts about numerical ranges we refer to [6, 7].

For $T \in \mathcal{L}(E)$, let $\sigma(T)$, $\sigma_{ap}(T)$, $r(T)$ and $v(T)$ denote the spectrum, approximate point spectrum, spectral radius and numerical radius of T, respectively. Recall that when $v(T) = \|T\|$ then T is said to be a normaloid operator. Given $x \in E$ and $f \in E'$, we write $x \otimes f$ to denote the rank-one bounded linear operator

$$z \longmapsto f(z)x \quad (z \in E),$$

whose norm is equal to $\|x\|\|f\|$. If λ is a complex number then we denote by $\overline{\lambda}$ its complex conjugate.

2 Main results

In this section, we shall study the Daugavet equation for a given multiplication operator when it is restricted to a symmetric norm ideal.

We begin with the following lemma.

Lemma 2.1. *Let $A \in \mathcal{L}(E)$. Then*

1. $\|I + A\| = 1 + \|A\|$ *if and only if* $\|A\| \in V(A)$,
2. $\sup_{|\lambda|=1} \|I + \lambda A\| = 1 + \|A\|$ *if and only if* $v(A) = \|A\|$.

Proof. (1): See [18, Corollary 1].

(2): By a compactness argument we can find a modulus one complex number λ_0 such that $\sup_{|\lambda|=1} \|I + \lambda A\| = \|I + \lambda_0 A\|$. The result then follows from Part (1). ∎

Recall that a Banach space E is said to be uniformly convex whenever for each sequences $\{x_n\}_n$ and $\{y_n\}_n$ in F, $\|x_n\| \leq 1$, $\|y_n\| \leq 1$ for all n and $\lim_n \|x_n + y_n\| = 2$ imply $\lim_n \|x_n - y_n\| = 0$.

There is a concept that is dual to uniform convexity. A Banach space is said to be uniformly smooth whenever for each $\epsilon > 0$, there exists some $\delta > 0$ such that $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| \geq \delta$ imply $\|x + y\| \leq \|x\| + \|y\| - \epsilon\|x - y\|$.

A Banach space is uniformly smooth (respectively, uniformly convex) if and only if its norm dual is uniformly convex (respectively, uniformly smooth), (see [14]).

As a consequence of Lemma 2.1, we have the following corollary proved in [1].

Corollary 2.2. *Suppose E is a uniformly convex or uniformly smooth Banach space. Then for $A \in \mathcal{L}(E)$, we have*

1. $\|I + A\| = 1 + \|A\|$ *if and only if* $\|A\| \in \sigma_{ap}(A)$;
2. $\sup_{|\lambda|=1} \|I + \lambda A\| = 1 + \|A\|$ *if and only if* $r(A) = \|A\|$.

Proof. This follows from Lemma 2.1 and the facts that: For every bounded linear operator T on a uniformly convex or uniformly smooth Banach space E (see for example [7]), we have

i) $\|T\| \in \sigma_{ap}(T)$ if and only of $\|T\| \in V(T)$;

ii) $r(T) = \|T\|$ if and only if $v(T) = \|T\|$. ∎

Lemma 2.3. *Let $A, B \in \mathcal{L}(E)$, and let $\mathcal{S}$ be a symmetric norm ideal of $\mathcal{L}(E)$. Then*

$$\|M_{\mathcal{S},A,B}\| = \|A\|\|B\|.$$

Proof. Let $x, y \in E$ and $f \in E'$ be such that $\|x\| = \|y\| = \|f\| = 1$. Since $x \otimes f$ lies in $\mathcal{S}$ (see [17, Lemma 4.1]), and

$$|f(By)|\|Ax\| = \|M_{A,B}(x \otimes f)(y)\| \leq \|M_{\mathcal{S},A,B}(x \otimes f)\|_{\mathcal{S}} \leq \|M_{\mathcal{S},A,B}\| \leq \|A\|\|B\|,$$

then it follows that

$$\|M_{\mathcal{S},A,B}\| = \|A\|\|B\|.$$ ∎

Remark 2.4. Let $A, B \in \mathcal{L}(E)$, and let $\mathcal{S}$ be a symmetric norm ideal of $\mathcal{L}(E)$. Since $\|M_{\mathcal{S},A,B}\| = \|A\|\|B\|$ by the above lemma, then it follows from Lemma 2.1 that $M_{\mathcal{S},A,B}$ is normaloid if and only if $\sup_{|\lambda|=1} \|I + \lambda M_{\mathcal{S},A,B}\| = 1 + \|A\|\|B\|$.

In what follows H denotes a complex separable Hilbert space.

Theorem 2.5. *Let $A, B \in \mathcal{L}(H)$, and suppose that $1 < p < \infty$. Then the following are equivalent:*

1. $\|I + M_{p,A,B}\| = 1 + \|A\|\|B\|$;
2. *There exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $\lambda\|A\| \in \sigma(A)$ and $\overline{\lambda}\|B\| \in \sigma(B)$;*
3. *There exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $\lambda\|A\| \in V(A)$ and $\overline{\lambda}\|B\| \in V(B)$.*

Proof. (1) ⇔ (2): It is well-known that, for $1 < p < \infty$, $\mathcal{C}_p(H)$ is a uniformly convex Banach space (see, e.g., [21, P. 23]). Therefore, Corollary 2.2 can be applied; it shows that $M_{p,A,B}$ satisfies the equality in (1.2) if and only if $\|A\|\|B\| \in \sigma(M_{p,A,B})$. But $\sigma(M_{p,A,B}) = \sigma(A)\sigma(B)$ (see [8]); hence we derive that there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $\lambda\|A\| \in \sigma(A)$ and $\overline{\lambda}\|B\| \in \sigma(B)$.

The equivalence (2) ⇔ (3) follows from the general fact: For any bounded operator T on H, $\|T\|$ lies in $V(T)$ if and only if $\|T\|$ lies in $\sigma_{ap}(T)$, (see [12]). This completes the proof. ∎

Remark 2.6. Let $A, B \in \mathcal{L}(H)$, and suppose that and $1 < p < \infty$. From the above theorem, it follows that $\|I + M_{p,A,B}\| = 1 + \|A\|\|B\|$ if and only if $\|I + M_{p,B,A}\| = 1 + \|A\|\|B\|$.

Lemma 2.7. *Let $A, B \in \mathcal{L}(E)$, and let $\mathcal{S}$ be a norm ideal of $\mathcal{L}(E)$. Then*

1. $V(A)\,V(B) \subseteq V(M_{\mathcal{S},A,B})$,
2. $V(AB) \subseteq V(M_{A,B})$.

Proof. (1) Let $x, y \in E$ be such that $\|x\| = \|y\| = 1$, and let $f, g \in E'$ be such that $f(x) = \|f\| = g(y) = \|g\| = 1$. Define a linear functional Φ on $\mathcal{S}$ by $\Phi(X) = g(Xx)$. We easily check that Φ is continuous with $\|\Phi\| = \Phi(y \otimes f) = 1$. Hence

$$\Phi(M_{\mathcal{S},A,B}(y \otimes f)) = f(Bx)g(Ay) \in V(M_{\mathcal{S},A,B}).$$

From this we derive that

$$V(A)\,V(B) \subseteq V(M_{\mathcal{S},A,B}).$$

(2) Let $x \in E$ and $f \in E'$ be such that $f(x) = 1$. Define a linear functional Φ on $\mathcal{L}(E)$ by $\Phi(X) = f(Xx)$. Then Φ is continuous with $\|\Phi\| = \Phi(I) = 1$. Hence

$$\Phi(M_{A,B}(I)) = f(ABx) \in V(M_{A,B}).$$

Consequently,

$$V(AB) \subseteq V(M_{A,B}).$$ ■

Remark 2.8. It follows from Lemma 2.7 that, for two operators $A, B \in \mathcal{L}(E)$, $v(A)v(B) \leq v(M_{\mathcal{S},A,B}) \leq \|A\|\|B\|$, for every norm ideal $\mathcal{S}$. Hence $M_{\mathcal{S},A,B}$ is normaloid whenever A and B are normaloid.

Theorem 2.9. *Let $A, B \in \mathcal{L}(H)$, and suppose that $1 < p < \infty$. Then $M_{p,A,B}$ is normaloid if and only if A and B are normaloid.*

Proof. To prove that the condition is sufficient recall that, for $1 < p < \infty$, the space $\mathcal{C}_p(H)$ is uniformly convex. Hence, by virtue of Corollary 2.2, (2) and Lemma 2.3, we have $r(M_{p,A,B}) = \|M_{p,A,B}\| = \|A\|\|B\|$. But $r(M_{p,A,B}) = r(A)r(B)$ see ([8]), and $r(A) \leq v(A) \leq \|A\|$ and $r(B) \leq v(B) \leq \|B\|$. Then we get $v(A) = \|A\|$ and $v(B) = \|B\|$.

The necessary condition follows from Remark 2.8. ■

Let us give an example showing that the equivalences in Theorem 2.5 and Theorem 2.9 do not hold when $\mathcal{S} = \mathcal{L}(H)$.

Example 2.10. *Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then $\|M_{A,B}\| = \|A\| = \|B\| = 1$. Since $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, it follows from Lemma 2.7, (2) that $1 \in V(M_{A,B})$. Thus $v(M_{A,B}) = 1$, and $\|I + M_{A,B}\| = 2$ because $1 + V(AB) = V(I + AB) \subseteq V(I + M_{A,B})$. However $v(A) = v(B) = \frac{1}{2}$.*

Proposition 2.11. *Let $A, B \in \mathcal{L}(H)$. Then*

$$\|I + M_{1,B,A}\| = \|I + M_{\infty,A,B}\| = \|I + M_{A,B}\|.$$

Proof. Recall that if E is a Banach space and $T \in \mathcal{L}(E)$, then $\|T\| = \|T^*\|$ (T^*: the adjoint of T). By [10, Theorem 3.13], we have $(M_{\infty,A,B})^{**} = M_{A,B}$, so $\|I + M_{\infty,A,B}\| = \|I + M_{A,B}\|$. From [10], we also have $(M_{\infty,A,B})^* = M_{1,B,A}$, so that

$$\|I + M_{1,B,A}\| = \|(I + M_{\infty,A,B})^*\| = \|I + M_{\infty,A,B}\| = \|I + M_{A,B}\|. \qquad \blacksquare$$

Let $T \in \mathcal{L}(H)$. Following [24], the maximal numerical range $W_0(T)$ of T is defined by

$$W_0(T) = \Big\{\lambda \in \mathbb{C} : \text{ there exists } \{x_n\} \subseteq H,\ \|x_n\| = 1 \text{ such that } \lim_n < Tx_n, x_n >= \lambda \text{ and } \lim_n \|Tx_n\| = \|T\|\Big\}.$$

The normalized maximal numerical range of T is given by

$$W_N(T) = \begin{cases} W_0(\frac{T}{\|T\|}) & \text{if } T \neq 0 \\ 0 & \text{if } T = 0. \end{cases}$$

Theorem 2.12. *If $A, B \in \mathcal{L}(\mathcal{H})$ then the following conditions are equivalent:*

1. $\|I + M_{1,A,B}\| = 1 + \|A\|\|B\|$;
2. $\|I + M_{\infty,A,B}\| = 1 + \|A\|\|B\|$;
3. $\|I + M_{A,B}\| = 1 + \|A\|\|B\|$;
4. $W_N(A^*) \cap W_N(B) \neq \emptyset$.

Proof. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from Proposition 2.11. The equivalence (3) $\Leftrightarrow$ (4) follows from [4, Theorem 1]. $\blacksquare$

In connection with Lemma 2.1 and Corollary 2.2, it is natural to ask when a given multiplication $M_{\mathcal{S},A,B}$ is spectraloid, that is, when its spectral radius and its numerical radius coincide. The next proposition gives necessary and sufficient conditions for the multiplication $M_{\mathcal{S},A,B}$ to be spectraloid.

Proposition 2.13. *Let $A, B \in \mathcal{L}(H)$, and let $\mathcal{S}$ be a symmetric norm ideal of $\mathcal{L}(H)$. Then $M_{\mathcal{S},A,B}$ is spectraloid if and only if A and B are spectraloid operators in H and $v(M_{\mathcal{S},A,B}) = v(A)v(B)$.*

Proof. If $M_{\mathcal{S},A,B}$ is spectraloid then

$$v(M_{\mathcal{S},A,B}) = r(M_{\mathcal{S},A,B}) = r(A)r(B) \leq v(A)v(B).$$

Since by Lemma 2.7, $v(A)v(B) \leq v(M_{\mathcal{S},A,B})$, then it follows that

$$r(A)r(B) = v(A)v(B) = v(M_{\mathcal{S},A,B}).$$

Thus $r(A) = v(A)$ and $r(B) = v(B)$.

The converse is obvious. $\blacksquare$

Acknowledgement

The research of the first author was supported by the Deanship of Scientific Research at Sattam bin Abdulaziz University under the Research Project 1787/01/2014.

References

[1] Y. A. Abramovich, C. D. Aliprantis, and O. Burkinshaw, *The Daugavet equation in uniformly convex Banach spaces*, J. Func. Anal. **97**(1991), 215-230.

[2] M. Barraa, *A formula for the numerical range of elementary operators*, ISRN Math. Anal. 2014, Art. ID 246301, 4 pp.

[3] M. Barraa and M. Boumazgour, *Inner derivations and norm equality*, Proc. Amer. Math. Soc. **130**(2002), 471-476.

[4] M. Barraa and M. Boumazgour, *Norm equality for the basic elementary operator*, J. Math. Anal. Appl. **286**(2003), 359-362.

[5] A. Blanco, M. Boumazgour and T.J. Ransford, *On the norm of elementary operators*, J. London Math. Soc. 70 (2004), 479-498.

[6] F.F. Bonsall and J. Duncan, *Numerical Ranges I*, Cambridge: Cambridge university press, 1973.

[7] F.F. Bonsall and J. Duncan, *Numerical Ranges, II*, Cambridge: Cambridge university press, 1973.

[8] M. Embry and M. Rosenblum, *Spectra, tensor products and linear operator equations*, Pacific J. Math. **53**(1974), 95-107.

[9] L. Fialkow, *Structural properties of elementary operators*, in Elementary operators and applications (M. Mathieu, ed), (Proc. Int. Workshop, Blaubeurn 1991), World Scientific, Singapore, 1992, pp. 55-113.

[10] L. Filkow, *A note on the operator* $X \to AX - XB$, Trans. Amer. Math. Soc. **243**(1978), 147-168.

[11] C. L. Lin, *The unilateral shift and norm equality for bounded linear operators*, Proc. Amer. Math. Soc. **127**(1999), 1693-1696.

[12] P.R. Halmos, *A Hilbert Space problem Book*, Van Nostrand, Princeton, 1970.

[13] B. E. Johnson, *Norms of derivations on* $\mathcal{L}(X)$, Pacific. J. Math. **38**(1971), 465-469.

[14] G. Köthe, *Topological vector spaces I*, Springer-Verlag, New York/Heidelberg, 1969.

[15] M. Mathieu, *The norm problem for elementary operators*, in Recent progress in functional analysis (Valencia, 2000), 363-368, North-Holland Math. Stud., 189, North-Holland, Amsterdam, 2001.

[16] M. Mathieu, *Elementary operators on Calkin algebras*, Ir. Math. Soc. Bull. **46**(2001), 33-42.

[17] K. Mattila, *Complex strict and uniform convexity and hyponormal operators*, Math. Proc. Cambridge Philos. Soc. **96**(1984), no. 3, 483-493.

[18] R. Nakamoto and S.E. Takahasi, *Norm equality condition in triangular inequality*, Sci. Math. Jpn. **55**(2002), no. 3, 463-466.

[19] E. Saksman and H.-O. Tylli, *Multiplications and elementary operators in the Banach space setting*. Methods in Banach space theory, 253-292, London Math. Soc. Lecture Note Ser., 337, Cambridge Univ. Press, Cambridge, 2006.

[20] R. Schatten, *Norm ideals of completely continuous operators*, Springer-Verlag, Berlin, 1960.

[21] B. Simon, *Trace ideals and their applications*, Cambridge Univ. Press, Cambridge, UK, 1979.

[22] L. L. Stacho and B. Zalar, *On the norm of Jordan elementary operators in standard operator algebras*, Publ. Math. Debrecen. **49**(1996), 127-134.

[23] L. L. Stacho and B. Zalar, *Uniform primeness of the Jordan algebra of symmetric operators*, Proc. Amer. Math. Soc. **126**(1998), 2241-2247.

[24] J. Stampfli, *The norm of a derivation*, Pac. J. Math., **33**(1970), 737-747.

[25] R. M. Timoney, *Norms of elementary operators*, Ir. Math. Soc. Bull. **46**(2001), 13-17.

[26] R. M. Timoney, *Computing the norm of elementary operators*, Illinois J. Math. **47**(2003), 1207-1226.

[27] R. M. Timoney, *Some formulae for norms of elementary operators*, J. Operator Theory **57**(2007), no. 1, 121-145.

Department of Mathematics,
Faculty of Science and Humanitarian Studies,
Sattam bin Abdulaziz University,
P.O. Box 83, Alkharj 11942, Saudi Arabia.
email: boumazgour@hotmail.com

Department of Mathematics,
Faculty of Sciences
Semlalia, P.O. Box 2390, Marrakesh, Morocco.
email :barraa@hotmail.com